SCIENCE CHINA
Mathematics

• ARTICLES •

January 2015 Vol. 58 No. 1: 179–196
doi: 10.1007/s11425-014-4925-9

Liouville type theorems for Schrödinger systems

ZHUO Ran[1,*] & LI FengQuan[2]

[1]*Department of Mathematical Sciences, Yeshiva University, New York 10033, USA;*
[2]*School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, China*

Email: ranzhuo201@gmail.com, fqli@dlut.edu.cn

Received November 15, 2013; accepted March 2, 2014; published online November 19, 2014

Abstract We study positive solutions to the following higher order Schrödinger system with Dirichlet boundary conditions on a half space:

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = u^{\beta_1}(x) v^{\gamma_1}(x), & \text{in } R^n_+, \\ (-\Delta)^{\frac{\alpha}{2}} v(x) = u^{\beta_2}(x) v^{\gamma_2}(x), & \text{in } R^n_+, \\ u = \dfrac{\partial u}{\partial x_n} = \cdots = \dfrac{\partial^{\frac{\alpha}{2}-1} u}{\partial {x_n}^{\frac{\alpha}{2}-1}} = 0, & \text{on } \partial R^n_+, \\ v = \dfrac{\partial v}{\partial x_n} = \cdots = \dfrac{\partial^{\frac{\alpha}{2}-1} v}{\partial {x_n}^{\frac{\alpha}{2}-1}} = 0, & \text{on } \partial R^n_+, \end{cases} \tag{0.1}$$

where α is any even number between 0 and n. This PDE system is closely related to the integral system

$$\begin{cases} u(x) = \displaystyle\int_{R^n_+} G(x,y) u^{\beta_1}(y) v^{\gamma_1}(y) dy, \\ v(x) = \displaystyle\int_{R^n_+} G(x,y) u^{\beta_2}(y) v^{\gamma_2}(y) dy, \end{cases} \tag{0.2}$$

where G is the corresponding Green's function on the half space. More precisely, we show that every solution to (0.2) satisfies (0.1), and we believe that the converse is also true. We establish a Liouville type theorem — the non-existence of positive solutions to (0.2) under a very weak condition that u and v are only locally integrable. Some new ideas are involved in the proof, which can be applied to a system of more equations.

Keywords Schrödinger systems, poly-harmonic operators, Dirichlet boundary conditions, method of moving planes in integral forms, Kelvin transforms, monotonicity, rotational symmetry, non-existence

MSC(2010) 31A10, 35B45, 35B53, 35J91

Citation: Zhuo R, Li F Q. Liouville type theorems for Schrödinger systems. Sci China Math, 2015, 58: 179–196, doi: 10.1007/s11425-014-4925-9

1 Introduction

Let R^n_+ be the n-dimensional upper half Euclidean space,

$$R^n_+ = \{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_n > 0\},$$

and let α be any even number satisfying $0 < \alpha < n$.

*Corresponding author

We consider the following higher order Schrödinger system with Dirichlet boundary conditions on the half space:

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = u^{\beta_1}(x) v^{\gamma_1}(x), & \text{in } R_+^n, \\ (-\Delta)^{\frac{\alpha}{2}} v(x) = u^{\beta_2}(x) v^{\gamma_2}(x), & \text{in } R_+^n, \\ u = \dfrac{\partial u}{\partial x_n} = \cdots = \dfrac{\partial^{\frac{\alpha}{2}-1} u}{\partial {x_n}^{\frac{\alpha}{2}-1}} = 0, & \text{on } \partial R_+^n, \\ v = \dfrac{\partial v}{\partial x_n} = \cdots = \dfrac{\partial^{\frac{\alpha}{2}-1} v}{\partial {x_n}^{\frac{\alpha}{2}-1}} = 0, & \text{on } \partial R_+^n, \end{cases} \tag{1.1}$$

where β_1, γ_1, β_2 and γ_2 satisfy the condition (f_1): $0 \leqslant \beta_1, \gamma_1, \beta_2, \gamma_2 \leqslant \frac{n+\alpha}{n-\alpha}$ with $\frac{n}{n-\alpha} < \beta_1 + \gamma_1 = \beta_2 + \gamma_2 \leqslant \frac{n+\alpha}{n-\alpha}, \beta_1 \neq \beta_2, \gamma_1 \neq \gamma_2$.

In the special case when $u = v$, (1.1) is reduced to the following problem for a single equation:

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u(x) = u^p(x), & \text{in } R_+^n, \\ u = \dfrac{\partial u}{\partial x_n} = \cdots = \dfrac{\partial^{\frac{\alpha}{2}-1} u}{\partial {x_n}^{\frac{\alpha}{2}-1}} = 0, & \text{on } \partial R_+^n, \end{cases} \tag{1.2}$$

where $1 < p \leqslant \frac{n+\alpha}{n-\alpha}$.

(1.2) has been studied in [13]. Under some very mild growth conditions, they proved that (1.2) is equivalent to the following integral equation:

$$u(x) = \int_{R_+^n} G(x,y) u^p(y) dy, \tag{1.3}$$

where $G(x,y)$ is the corresponding Green's function on the half space,

$$G(x,y) = \frac{C_n}{|x-y|^{n-\alpha}} \int_0^{\frac{4x_n y_n}{|x-y|^2}} \frac{z^{\frac{\alpha}{2}-1}}{(z+1)^{\frac{n}{2}}} dz.$$

Combining the method of moving planes in integral forms with some new ideas, they proved that there was no positive solutions to the integral equation (1.3) in both subcritical and critical cases, and then partially solved an open problem posed by Reichel and Weth [27].

For $\frac{\alpha}{2} = 2$, a similar system in the whole space $\mathbb{R}^n$ has been studied by Li and Ma [19],

$$\begin{cases} -\Delta u(x) = u^\beta(x) v^\gamma(x), \\ -\Delta v(x) = v^\beta(x) u^\gamma(x), \end{cases} \tag{1.4}$$

where $n \geqslant 3$, and $1 \leqslant \beta, \gamma \leqslant \frac{n+2}{n-2}$ with $\beta + \gamma = \frac{n+2}{n-2}$. When $n = 3$ and $\beta = 2$, $\gamma = 3$, (1.4) arises from the stationary Schrödinger system with critical exponents for Bose-Einstein condensate. They proved the following proposition.

Proposition 1 (See [19]). *Assume that $1 \leqslant \beta < \gamma \leqslant \frac{n+2}{n-2}$. Then any $L^{\frac{2n}{n-2}}(\mathbb{R}^n) \times L^{\frac{2n}{n-2}}(\mathbb{R}^n)$ radially symmetric solution pair (u,v) to* (1.4) *with critical exponents is unique such that $u = v$.*

For PDE system (1.1), we study the corresponding system of integral equations in the upper half space R_+^n,

$$\begin{cases} u(x) = \displaystyle\int_{R_+^n} G(x,y) u^{\beta_1}(y) v^{\gamma_1}(y) dy, \\ v(x) = \displaystyle\int_{R_+^n} G(x,y) u^{\beta_2}(y) v^{\gamma_2}(y) dy, \end{cases} \tag{1.5}$$

where β_1, γ_1, β_2 and γ_2 satisfy the condition (f_1): $0 \leqslant \beta_1, \gamma_1, \beta_2, \gamma_2 \leqslant \frac{n+\alpha}{n-\alpha}$ with $\frac{n}{n-\alpha} < \beta_1 + \gamma_1 = \beta_2 + \gamma_2 \leqslant \frac{n+\alpha}{n-\alpha}, \beta_1 \neq \beta_2, \gamma_1 \neq \gamma_2$.

In our paper, we first prove the following theorem.

Theorem 1. *Let $(u(x), v(x))$ be a pair of locally smooth solution to* (1.5)*, then $(u(x), v(x))$ satisfies* (1.1).

We believe that the converse is also true, which is one of motivations of the present article. When $u = v$, system (1.1) reduces to (1.2), Fang and Chen [13] have established the equivalence between (1.2) and (1.3), and we will generalize the equivalence to the Shrödinger system in our future paper.

Then we study the integral system and obtain the following theorem.

Theorem 2. *For β_1, γ_1, β_2 and γ_2 satisfying (f_1), we assume that $(u(x), v(x))$ is a pair of positive solutions to* (1.5)*, and u, $v \in L^p_{\rm loc}(R^n_+)$, where $p = \frac{n(\beta_1+\gamma_1-1)}{\alpha}$. Then either one of the following holds for (u, v):*

(i) *it is monotonically increasing with respect to the variable x_n, or*

(ii) *it is rotationally symmetric about any line parallel to x_n-axis.*

Finally, based on Theorem 2, we prove the following theorem.

Theorem 3. *For β_1, γ_1, β_2 and γ_2 satisfying (f_1), if $(u(x), v(x))$ is a pair of non-negative solutions to* (1.5)*, with u, $v \in L^p_{\rm loc}(R^n_+)$, and $p = \frac{n(\beta_1+\gamma_1-1)}{\alpha}$. Then $u(x) \equiv 0$ and $v(x) \equiv 0$.*

Remark 1. In Theorems 2 and 3, α can be any real number between 0 and n.

Once we establish the equivalence between the integral system (1.5) and PDE system (1.1), then this non-existence result will be applied to the PDE system. It is well known that this type of Liouville theorems are very important in establishing a priori estimates for positive solutions of a similar family of PDEs on bounded domains of Euclidean space or on Riemannian manifolds with boundaries.

In Section 2, we apply properties of the Green's function on the half space to derive the relation between partial differential equations and integral equations. In Section 3, we cleverly combine a certain type of Kelvin transform and the method of moving planes in integral forms to prove the monotonicity and rotational symmetry. In Section 4, we establish the non-existence of positive solutions to (1.5).

For more results concerning integral equations, the method of moving planes in integral forms and Schördinger type equations, please see [1–12, 15–18, 20, 22–26, 28–30] and the references therein.

2 The relation between integral equations and PDEs

Proof of Theorem 1. Since

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} G(x,y) = \delta(x-y), & \text{in } R^n_+, \\ G = \dfrac{\partial G}{\partial x_n} = \cdots = \dfrac{\partial^{\frac{\alpha}{2}-1} G}{\partial {x_n}^{\frac{\alpha}{2}-1}} = 0, & \text{on } \partial R^n_+, \end{cases}$$

it is easy to verify that, for $k = 0, 1, \ldots, \dfrac{\alpha}{2} - 1$,

$$\frac{\partial^k u(x)}{\partial {x_n}^k} = \int_{R^n_+} \frac{\partial^k G(x,y)}{\partial {x_n}^k} u^{\beta_1}(y) v^{\gamma_1}(y) dy = 0, \quad x \in \partial R^n_+,$$

$$\frac{\partial^k v(x)}{\partial {x_n}^k} = \int_{R^n_+} \frac{\partial^k G(x,y)}{\partial {x_n}^k} u^{\beta_2}(y) v^{\gamma_2}(y) dy = 0, \quad x \in \partial R^n_+.$$

By elementary calculation, we derive that

$$\begin{aligned} (-\Delta)^{\frac{\alpha}{2}} u(x) &= \int_{R^n_+} (-\Delta)^{\frac{\alpha}{2}} G(x,y) u^{\beta_1}(y) v^{\gamma_1}(y) dy \\ &= \int_{R^n_+} \delta(x-y) u^{\beta_1}(y) v^{\gamma_1}(y) dy \\ &= u^{\beta_1}(x) v^{\gamma_1}(x) dy, \quad x \in R^n_+. \end{aligned}$$

Similarly,

$$(-\Delta)^{\frac{\alpha}{2}} v(x) = u^{\beta_2}(x) v^{\gamma_2}(x), \quad x \in R^n_+.$$

This completes the proof of Theorem 1. □

3 Monotonicity and rotationally symmetry of solutions

In this section, we will give the proof of Theorem 2.

To prove the theorem, we need the following lemmas.

For a given positive real number λ, when deriving (i) in Theorem 2, we denote

$$\Sigma_\lambda = \{x = (x_1, x_2, \dots, x_n) \in R^n_+ \mid 0 < x_n < \lambda\},$$
$$T_\lambda = \{x \in R^n_+ \mid x_n = \lambda\},$$

and let

$$x^\lambda = (x_1, x_2, \dots, 2\lambda - x_n)$$

be the reflection of the point $x = (x_1, x_2, \dots, x_n)$ about the plant T_λ, and

$$u_\lambda(x) = u(x^\lambda), \quad v_\lambda(x) = v(x^\lambda),$$
$$\Sigma^c_\lambda = R^n_+ \setminus \Sigma_\lambda, \quad \tilde{\Sigma}_\lambda = \{x^\lambda \mid x \in \Sigma_\lambda\}.$$

Lemma 1 (See [14])**.** (i) *For any $x, y \in \Sigma_\lambda$, $x \neq y$, we have*

$$G(x^\lambda, y^\lambda) > \max\{G(x^\lambda, y), G(x, y^\lambda)\},$$

and

$$G(x^\lambda, y^\lambda) - G(x, y) > |G(x^\lambda, y) - G(x, y^\lambda)|.$$

(ii) *For any $x \in \Sigma_\lambda$, $y \in \Sigma^c_\lambda$, it holds*

$$G(x^\lambda, y) > G(x, y).$$

Lemma 2. *For any $x \in \Sigma_\lambda$, it holds*

$$u(x) - u_\lambda(x) \leqslant \int_{\Sigma_\lambda} [G(x^\lambda, y^\lambda) - G(x, y^\lambda)][u^{\beta_1}(y)v^{\gamma_1}(y) - u^{\beta_1}_\lambda(y)v^{\gamma_1}_\lambda(y)]dy,$$
$$v(x) - v_\lambda(x) \leqslant \int_{\Sigma_\lambda} [G(x^\lambda, y^\lambda) - G(x, y^\lambda)][u^{\beta_2}(y)v^{\gamma_2}(y) - u^{\beta_2}_\lambda(y)v^{\gamma_2}_\lambda(y)]dy.$$

Proof. Obviously,

$$\begin{aligned} u(x) &= \int_{\Sigma_\lambda} G(x, y)u^{\beta_1}(y)v^{\gamma_1}(y)dy + \int_{\Sigma_\lambda} G(x, y^\lambda)u^{\beta_1}_\lambda(y)v^{\gamma_1}_\lambda(y)dy \\ &\quad + \int_{\Sigma^c_\lambda \setminus \tilde{\Sigma}_\lambda} G(x, y)u^{\beta_1}(y)v^{\gamma_1}(y)dy, \\ u_\lambda(x) &= \int_{\Sigma_\lambda} G(x^\lambda, y)u^{\beta_1}(y)v^{\gamma_1}(y)dy + \int_{\Sigma_\lambda} G(x^\lambda, y^\lambda)u^{\beta_1}_\lambda(y)v^{\gamma_1}_\lambda(y)dy \\ &\quad + \int_{\Sigma^c_\lambda \setminus \tilde{\Sigma}_\lambda} G(x^\lambda, y)u^{\beta_1}(y)v^{\gamma_1}(y)dy. \end{aligned}$$

By Lemma 3.1, it is easy to see

$$\begin{aligned} u(x) - u_\lambda(x) &= \int_{\Sigma_\lambda} [G(x^\lambda, y^\lambda) - G(x, y^\lambda)]u^{\beta_1}(y)v^{\gamma_1}(y)dy \\ &\quad + \int_{\Sigma_\lambda} [G(x, y^\lambda) - G(x^\lambda, y^\lambda)]u^{\beta_1}_\lambda(y)v^{\gamma_1}_\lambda(y)dy \\ &\quad + \int_{\Sigma^c_\lambda \setminus \tilde{\Sigma}_\lambda} [G(x, y) - G(x, y^\lambda)]u^{\beta_1}(y)v^{\gamma_1}(y)dy \\ &\leqslant \int_{\Sigma_\lambda} [G(x^\lambda, y^\lambda) - G(x, y^\lambda)][u^{\beta_1}(y)v^{\gamma_1}(y) - u^{\beta_1}_\lambda(y)v^{\gamma_1}_\lambda(y)]dy. \end{aligned}$$

Similarly, we have

$$v(x)-v_\lambda(x)\leqslant\int_{\Sigma_\lambda}[G(x^\lambda,y^\lambda)-G(x,y^\lambda)][u^{\beta_2}(y)v^{\gamma_2}(y)-u_\lambda^{\beta_2}(y)v_\lambda^{\gamma_2}(y)]dy.$$

Lemma 3 (An equivalent form of the Hardy-Littlewood-Sobolev inequality)**.** *Let* $g\in L^{\frac{nr}{n+\alpha r}}(R^n)$ *for* $\frac{n}{n-\alpha}<r<\infty$. *Define*

$$Tg(x)=\int_{R^n}\frac{1}{|x-y|^{n-\alpha}}g(y)dy.$$

Then

$$\|Tg\|_{L^r(R^n)}\leqslant C(n,r,\alpha)\|g\|_{L^{\frac{nr}{n+\alpha r}}(R^n)}.$$

This can be derived directly from the classical Hardy-Littlewood-Sobolev inequality, and for the proof, please see [6, 21].

Proof of Theorem 2. Because these is no global integrability assumption on the solution (u,v), we will combine the Kelvin transform with the method of moving planes to derive the monotonicity and rotational symmetry.

First, we introduce the Kelvin transform.

For $z^0\in\partial R^n_+$, let

$$\hat{u}(x)=\frac{1}{|x-z^0|^{n-\alpha}}u\bigg(\frac{x-z^0}{|x-z^0|^2}+z^0\bigg),\tag{3.1}$$

$$\hat{v}(x)=\frac{1}{|x-z^0|^{n-\alpha}}v\bigg(\frac{x-z^0}{|x-z^0|^2}+z^0\bigg)\tag{3.2}$$

be the Kelvin transform of u, v centered at point z^0. We consider two possible cases.

Case 1. If there is a $z^0=(z^0_1,z^0_2,\dots,z^0_{n-1},0)\in\partial R^n_+$ such that $(\hat{u}(x),\hat{v}(x))$ are not singular at z^0, by (3.1) and (3.2), we get

$$u(x)=\frac{1}{|x-z^0|^{n-\alpha}}\hat{u}\bigg(\frac{x-z^0}{|x-z^0|^2}+z^0\bigg),$$

$$v(x)=\frac{1}{|x-z^0|^{n-\alpha}}\hat{v}\bigg(\frac{x-z^0}{|x-z^0|^2}+z^0\bigg).$$

It is easy to deduce that

$$\lim_{|y|\to\infty}|y|^{n-\alpha}u(y)=\hat{u}(z^0),$$

$$\lim_{|y|\to\infty}|y|^{n-\alpha}v(y)=\hat{v}(z^0).$$

Hence,

$$u(x)=O\bigg(\frac{1}{|x|^{n-\alpha}}\bigg),\quad v(x)=O\bigg(\frac{1}{|x|^{n-\alpha}}\bigg),\quad\text{for large }|x|.\tag{3.3}$$

These imply the global integrability of the solution $(u(x),v(x))$.

In this case, we move the plane T_λ along the direction of x_n-axis to show that the solution is monotonically increasing with respect to the variable x_n. The proof consists of two steps.

Step 1 (Prepare to move the plane form near $x_n=0$)**.**

Now we compare the values of $u(x)$ and $u_\lambda(x)$, $v(x)$ and $v_\lambda(x)$. For sufficiently small positive λ, we will show that

$$u_\lambda(x)-u(x)\geqslant 0,\quad v_\lambda(x)-v(x)\geqslant 0,\quad\forall\, x\in\Sigma_\lambda.\tag{3.4}$$

Let

$$w_\lambda(x)=u_\lambda(x)-u(x),\quad g_\lambda(x)=v_\lambda(x)-v(x),$$

and define

$$\Sigma_\lambda^u = \{x \in \Sigma_\lambda \mid w_\lambda(x) < 0\}, \quad \Sigma_\lambda^v = \{x \in \Sigma_\lambda \mid g_\lambda(x) < 0\}.$$

We will prove that Σ_λ^u and Σ_λ^v are of measure zero. In fact, by the mean value theorem and Lemma 3.2, we obtain

$$\begin{aligned}
u(x) - u_\lambda(x) &\leqslant \int_{\Sigma_\lambda} [G(x^\lambda, y^\lambda) - G(x, y^\lambda)][u^{\beta_1}(y)v^{\gamma_1}(y) - u_\lambda^{\beta_1}(y)v_\lambda^{\gamma_1}(y)]dy \\
&\leqslant \int_{\Sigma_\lambda^u} [G(x^\lambda, y^\lambda) - G(x, y^\lambda)]\{u^{\beta_1}(y)[v^{\gamma_1}(y) - v_\lambda^{\gamma_1}(y)] \\
&\quad + v_\lambda^{\gamma_1}(y)[u^{\beta_1}(y) - u_\lambda^{\beta_1}(y)]\}dy \\
&\quad + \int_{\Sigma_\lambda \setminus \Sigma_\lambda^u} [G(x^\lambda, y^\lambda) - G(x, y^\lambda)]\{u^{\beta_1}(y)[v^{\gamma_1}(y) - v_\lambda^{\gamma_1}(y)]\}dy \\
&\leqslant \int_{\Sigma_\lambda^u} [G(x^\lambda, y^\lambda) - G(x, y^\lambda)]\{v_\lambda^{\gamma_1}(y)[u^{\beta_1}(y) - u_\lambda^{\beta_1}(y)]\}dy \\
&\quad + \int_{\Sigma_\lambda^v} [G(x^\lambda, y^\lambda) - G(x, y^\lambda)]\{u^{\beta_1}(y)[v^{\gamma_1}(y) - v_\lambda^{\gamma_1}(y)]\}dy \\
&\leqslant \int_{\Sigma_\lambda^u} G(x^\lambda, y^\lambda)v_\lambda^{\gamma_1}(y)[u^{\beta_1}(y) - u_\lambda^{\beta_1}(y)]dy \\
&\quad + \int_{\Sigma_\lambda^v} G(x^\lambda, y^\lambda)u^{\beta_1}(y)[v^{\gamma_1}(y) - v_\lambda^{\gamma_1}(y)]dy \\
&\leqslant \int_{\Sigma_\lambda^u} \beta_1 G(x^\lambda, y^\lambda)v_\lambda^{\gamma_1}(y)u^{\beta_1-1}(y)[u(y) - u_\lambda(y)]dy \\
&\quad + \int_{\Sigma_\lambda^v} \gamma_1 G(x^\lambda, y^\lambda)u^{\beta_1}(y)v^{\gamma_1-1}(y)[v(y) - v_\lambda(y)]dy,
\end{aligned} \tag{3.5}$$

where $\psi_\lambda(y)$ is valued between $v_\lambda(y)$ and $v(y)$, and $\varphi_\lambda(y)$ is valued between $u_\lambda(y)$ and $u(y)$. Therefore on Σ_λ^u and Σ_λ^v, we have

$$0 \leqslant u_\lambda(y) \leqslant \varphi_\lambda(y) \leqslant u(y), \quad 0 \leqslant v_\lambda(y) \leqslant \psi_\lambda(y) \leqslant v(y).$$

We can verify that

$$\begin{aligned}
G(x, y) &= \frac{C_n}{|x-y|^{n-\alpha}} \int_0^{\frac{4x_n y_n}{|x-y|^2}} \frac{z^{\frac{\alpha}{2}-1}}{(1+z)^{\frac{n}{2}}} dz \\
&\leqslant \frac{C}{|x-y|^{n-\alpha}}.
\end{aligned} \tag{3.6}$$

Applying (3.6) to (3.5),

$$\begin{aligned}
|u(x) - u_\lambda(x)| &\leqslant \int_{\Sigma_\lambda^u} \frac{C}{|x-y|^{n-\alpha}} |v_\lambda^{\gamma_1}(y)u^{\beta_1-1}(y)||u(y) - u_\lambda(y)|dy \\
&\quad + \int_{\Sigma_\lambda^v} \frac{C}{|x-y|^{n-\alpha}} |u^{\beta_1}(y)v^{\gamma_1-1}(y)||v(y) - v_\lambda(y)|dy.
\end{aligned} \tag{3.7}$$

Now, combining Lemma 3.2, Hölder's inequality and (3.7), we obtain, for $p > \frac{n}{n-\alpha}$,

$$\begin{aligned}
\|w_\lambda\|_{L^p(\Sigma_\lambda^u)} &\leqslant C_1 \|u^{\beta_1} v^{\gamma_1-1} g_\lambda\|_{L^{\frac{np}{n+\alpha p}}(\Sigma_\lambda^v)} \\
&\quad + C_2 \|v_\lambda^{\gamma_1} u^{\beta_1-1} w_\lambda\|_{L^{\frac{np}{n+\alpha p}}(\Sigma_\lambda^u)} \\
&\leqslant C_1 \|u\|_{L^p(\Sigma_\lambda^v)}^{\beta_1} \|v\|_{L^p(\Sigma_\lambda^v)}^{\gamma_1-1} \|g_\lambda\|_{L^p(\Sigma_\lambda^v)} \\
&\quad + C_2 \|u\|_{L^p(\Sigma_\lambda^u)}^{\beta_1-1} \|v_\lambda\|_{L^p(\Sigma_\lambda^u)}^{\gamma_1} \|w_\lambda\|_{L^p(\Sigma_\lambda^u)}.
\end{aligned} \tag{3.8}$$

Similarly, we have

$$\|g_\lambda\|_{L^p(\Sigma_\lambda^v)} \leqslant C_3\|u\|^{\beta_2-1}_{L^p(\Sigma_\lambda^u)}\|v_\lambda\|^{\gamma_2}_{L^p(\Sigma_\lambda^u)}\|w_\lambda\|_{L^p(\Sigma_\lambda^u)} + C_4\|u\|^{\beta_2}_{L^p(\Sigma_\lambda^v)}\|v\|^{\gamma_2-1}_{L^p(\Sigma_\lambda^v)}\|g_\lambda\|_{L^p(\Sigma_\lambda^v)}. \tag{3.9}$$

Since $u \in L^p_{\text{loc}}(R^n_+)$ and $v \in L^p_{\text{loc}}(R^n_+)$, by (3.3), we have

$$\int_{R^n_+} u^p(y)dy < \infty, \quad \int_{R^n_+} v^p(y)dy < \infty,$$

hence we choose sufficiently small positive λ such that

$$C_1\|u\|^{\beta_1}_{L^p(\Sigma_\lambda^v)}\|v\|^{\gamma_1-1}_{L^p(\Sigma_\lambda^v)} \leqslant \frac{1}{4},$$
$$C_2\|u\|^{\beta_1-1}_{L^p(\Sigma_\lambda^u)}\|v_\lambda\|^{\gamma_1}_{L^p(\Sigma_\lambda^u)} \leqslant \frac{1}{4},$$
$$C_3\|u\|^{\beta_2-1}_{L^p(\Sigma_\lambda^u)}\|v_\lambda\|^{\gamma_2}_{L^p(\Sigma_\lambda^u)} \leqslant \frac{1}{4},$$
$$C_4\|u\|^{\beta_2}_{L^p(\Sigma_\lambda^v)}\|v\|^{\gamma_2-1}_{L^p(\Sigma_\lambda^v)} \leqslant \frac{1}{4}.$$

Combining (3.8), (3.9) with the above inequalities, we derive

$$\|w_\lambda\|_{L^p(\Sigma_\lambda^u)} = 0, \quad \|g_\lambda\|_{L^p(\Sigma_\lambda^v)} = 0.$$

Therefore, Σ_λ^u and Σ_λ^v must be of measure zero. We conclude that, for sufficiently small positive λ,

$$w_\lambda(x) \geqslant 0, \quad g_\lambda(x) \geqslant 0, \quad \text{a.e. } x \in \Sigma_\lambda. \tag{3.10}$$

Step 2 (Move the plane to the limiting position to derive monotonicity and rotationally symmetry)**.**

Step 1 provides a starting point to move the plane T_λ. Now, we start to move the plane T_λ along the x_n direction as long as (3.10) holds.

Define

$$\lambda_0 = \sup\{\lambda \mid w_\mu(x) \geqslant 0, g_\mu(x) \geqslant 0, \mu \leqslant \lambda, \forall\, x \in \Sigma_\mu\}.$$

In fact, we will show that $\lambda = \infty$. If $\lambda_0 < \infty$, we will prove that $(u(x), v(x))$ is symmetric about T_{λ_0}, i.e.,

$$w_{\lambda_0}(x) \equiv 0, \quad g_{\lambda_0}(x) \equiv 0, \quad \forall\, x \in \Sigma_{\lambda_0}. \tag{3.11}$$

If (3.11) does not hold, then we must have

$$w_{\lambda_0}(x) > 0, \quad g_{\lambda_0}(x) > 0, \quad \forall\, x \in \Sigma_{\lambda_0}. \tag{3.12}$$

Next we prove (3.12). Indeed, from the proof of Lemmas 3.1 and 3.2, we have

$$\begin{aligned} u_{\lambda_0}(x) - u(x) &\geqslant \int_{\Sigma_{\lambda_0}} [G(x^{\lambda_0}, y^{\lambda_0}) - G(x, y^{\lambda_0})][u^{\beta_1}_{\lambda_0}(y)v^{\gamma_1}_{\lambda_0}(y) - u^{\beta_1}(y)v^{\gamma_1}(y)]dy \\ &\quad + \int_{\Sigma^c_{\lambda_0}\setminus\tilde{\Sigma}_{\lambda_0}} [G(x^{\lambda_0}, y) - G(x, y)]u^{\beta_1}(y)v^{\gamma_1}(y)dy \\ &\geqslant \int_{\Sigma^c_{\lambda_0}\setminus\tilde{\Sigma}_{\lambda_0}} [G(x^{\lambda_0}, y) - G(x, y)]u^{\beta_1}(y)v^{\gamma_1}(y)dy. \end{aligned} \tag{3.13}$$

Similarly, we have

$$v_{\lambda_0}(x) - v(x) \geqslant \int_{\Sigma^c_{\lambda_0}\setminus\tilde{\Sigma}_{\lambda_0}} [G(x^{\lambda_0}, y) - G(x, y)]u^{\beta_2}(y)v^{\gamma_2}(y)dy.$$

If (3.12) is violated, then there exists some point $x^0 \in \Sigma_{\lambda_0}$ such that

$$u(x^0) = u_{\lambda_0}(x^0) \quad \text{or} \quad v(x^0) = v_{\lambda_0}(x^0).$$

Then, by (3.13), we must have, either

$$u^{\beta_1}(y)v^{\gamma_1}(y)dy = 0, \quad \forall\, y \in \Sigma_{\lambda_0}^c \setminus \tilde{\Sigma}_{\lambda_0},$$

or

$$u^{\beta_2}(y)v^{\gamma_2}(y) = 0, \quad \forall\, y \in \Sigma_{\lambda_0}^c \setminus \tilde{\Sigma}_{\lambda_0}.$$

Obviously,

$$u(y) = 0, \quad \text{or} \quad v(y) = 0, \quad \forall\, y \in \Sigma_{\lambda_0}^c \setminus \tilde{\Sigma}_{\lambda_0}.$$

This is a contradiction with our assumption that $u > 0$ and $v > 0$, and hence (3.12) holds.

Now based on (3.12), we will show that the plane T_λ can be moved further while (3.10) still holds. This would contradict the definition of λ_0.

For any small $\eta > 0$, we can choose R sufficiently large, so that

$$\int_{R_+^n \setminus B_R(0))} u^p(x)dx < \eta, \quad \int_{R_+^n \setminus B_R(0))} v^p(x)dx < \eta. \tag{3.14}$$

For any $\tau > 0$, define

$$E_\tau = \{x \in \Sigma_{\lambda_0} \mid w_{\lambda_0} > \tau\}, \quad F_\tau = \{\Sigma_{\lambda_0} \setminus B_R(0)\} \setminus E_\tau.$$

Obviously,

$$\lim_{\tau \to 0} \mu(F_\tau) = 0.$$

For $\lambda > \lambda_0$, let

$$D_\tau = (\Sigma_\lambda \setminus \Sigma_{\lambda_0}) \cap B_R(0).$$

It is easy to prove that

$$\{\Sigma_\lambda^u \cap B_R(0)\} \subset (\Sigma_\lambda^u \cap E_\tau) \cup F_\tau \cup D_\tau. \tag{3.15}$$

Apparently, $\mu(D_\tau)$ is small for λ close to λ_0. We will show that $\mu(\Sigma_\lambda^u \cap E_\tau)$ is sufficiently small as λ is close to λ_0.

Actually,

$$\begin{aligned} w_\lambda(x) &= u_\lambda(x) - u(x) \\ &= u_\lambda(x) - u_{\lambda_0}(x) + u_{\lambda_0}(x) - u(x) \\ &< 0, \quad \forall\, x \in \Sigma_\lambda^u \cap E_\tau. \end{aligned}$$

Therefore,

$$u_{\lambda_0}(x) - u_\lambda(x) > w_{\lambda_0}(x) > \tau, \quad \forall\, x \in \Sigma_\lambda^u \cap E_\tau.$$

So, we have

$$(\Sigma_\lambda^u \cap E_\tau) \subset H_\tau = \{x \in B_R(0) \mid u_{\lambda_0}(x) - u_\lambda(x) > \tau\}. \tag{3.16}$$

Applying Chebyshev inequality, we get

$$\begin{aligned} \mu(H_\tau) &\leqslant \frac{1}{\tau^{s+1}} \int_{H_\tau} |u_{\lambda_0}(y) - u_\lambda(y)|^{s+1} dy \\ &\leqslant \frac{1}{\tau^{s+1}} \int_{B_R(0)} |u_{\lambda_0}(y) - u_\lambda(y)|^{s+1} dy. \end{aligned} \tag{3.17}$$

For each fixed τ, the right-hand side of (3.17) can be sufficiently small, when λ is sufficiently close to λ_0. Therefore, by (3.15) and (3.16), we can easily see that $\mu(\Sigma_\lambda^u \cap B_R(0))$ can be sufficiently small. Similarly, we have $\mu(\Sigma_\lambda^v \cap B_R(0))$ can be sufficiently small.

Combining this with (3.14), we have

$$
\begin{aligned}
C_1\|u\|^{\beta_1}_{L^p(\Sigma^v_\lambda)}\|v\|^{\gamma_1-1}_{L^p(\Sigma^v_\lambda)} &\leqslant \frac{1}{4},\\
C_2\|u\|^{\beta_1-1}_{L^p(\Sigma^u_\lambda)}\|v\|^{\gamma_1}_{L^p(\Sigma^u_\lambda)} &\leqslant \frac{1}{4},\\
C_3\|u\|^{\beta_2-1}_{L^p(\Sigma^u_\lambda)}\|v_\lambda\|^{\gamma_2}_{L^p(\Sigma^u_\lambda)} &\leqslant \frac{1}{4},\\
C_4\|u\|^{\beta_2}_{L^p(\Sigma^v_\lambda)}\|v\|^{\gamma_2-1}_{L^p(\Sigma^v_\lambda)} &\leqslant \frac{1}{4}.
\end{aligned}
$$

Together with (3.8) and (3.9), we obtain

$$\|w_\lambda\|_{L^p(\Sigma^u_\lambda)} = 0, \quad \|g_\lambda\|_{L^p(\Sigma^v_\lambda)} = 0.$$

Hence, for $\lambda > \lambda_0$ and sufficiently close to λ_0, we have

$$w_\lambda(x) \geqslant 0, \quad g_\lambda(x) \geqslant 0, \quad \forall\, x \in \Sigma_\lambda.$$

This is a contradiction with the definition of λ_0. Therefore, $(u(x), v(x))$ is symmetric about T_{λ_0}. If $\lambda_0 < \infty$, we have $u(x) = u_{\lambda_0}(x) = 0$ for $x \in \partial R^n_+$. This would contradict with our assumption that $u > 0$. Therefore the solution $(u(x), v(x))$ is monotonically increasing with respect to the variable x_n.

Case 2. At least one of $\hat{u}(x)$ and $\hat{v}(x)$ is singular at all

$$z^0 = (z^0_1, z^0_2, \ldots, z^0_{n-1}, 0) \in \partial R^n_+.$$

Without loss of generality, we may assume that $\hat{u}(x)$ is singular at z^0. We will show that $(\hat{u}(x), \hat{v}(x))$ is rotationally symmetric about the line parallel to the x_n-axis and passing through z^0, we have

$$
\begin{aligned}
\hat{u}(x) &= \frac{1}{|x-z^0|^{n-\alpha}} u\left(\frac{x-z^0}{|x-z^0|^2} + z^0\right)\\
&= \frac{1}{|x-z^0|^{n-\alpha}} \int_{R^n_+} G\left(\frac{x-z^0}{|x-z^0|^2} + z^0, y\right) u^{\beta_1}(y) v^{\gamma_1}(y)dy,
\end{aligned}
$$

let

$$y = \frac{z-z^0}{|z-z^0|^2} + z^0,$$

then,

$$
\begin{aligned}
\hat{u}(x) &= \frac{1}{|x-z^0|^{n-\alpha}} \int_{R^n_+} G\left(\frac{x-z^0}{|x-z^0|^2} + z^0, \frac{z-z^0}{|z-z^0|^2} + z^0\right)\\
&\quad \times u^{\beta_1}\left(\frac{z-z^0}{|z-z^0|^2} + z^0\right) v^{\gamma_1}\left(\frac{z-z^0}{|z-z^0|^2} + z^0\right) \frac{1}{|z-z^0|^{2n}} dz\\
&= \frac{1}{|x-z^0|^{n-\alpha}} \int_{R^n_+} G\left(\frac{x-z^0}{|x-z^0|^2} + z^0, \frac{z-z^0}{|z-z^0|^2} + z^0\right)\\
&\quad \times \hat{u}^{\beta_1}(z)\hat{v}^{\gamma_1}(z) \frac{1}{|z-z^0|^{2n}} |z-z^0|^{(n-\alpha)\beta_1} |z-z^0|^{(n-\alpha)\gamma_1} dz\\
&= \int_{R^n_+} \frac{G(\frac{x-z^0}{|x-z^0|^2}, \frac{z-z^0}{|z-z^0|^2})}{|x-z^0|^{(n-\alpha)}|z-z^0|^{(n-\alpha)}}\\
&\quad \times \frac{1}{|z-z^0|^{2n-(n-\alpha)(\beta_1+\gamma_1+1)}} \hat{u}^{\beta_1}(z)\hat{v}^{\gamma_1}(z)dz\\
&= \int_{R^n_+} G(x,z) \frac{1}{|z-z^0|^{2n-(n-\alpha)(\beta_1+\gamma_1+1)}} \hat{u}^{\beta_1}(z)\hat{v}^{\gamma_1}(z)dz. \qquad (3.18)
\end{aligned}
$$

Similarly,

$$\hat{v}(x) = \int_{R^n_+} G(x,z)\frac{1}{|z-z^0|^{2n-(n-\alpha)(\beta_2+\gamma_2+1)}}\hat{u}^{\beta_2}(z)\hat{v}^{\gamma_2}(z)dz.$$

In the following, we discuss the critical and subcritical cases separately.

(A_1) In the critical case when $\beta_1+\gamma_1=\beta_2+\gamma_2=\frac{n+\alpha}{n-\alpha}$, if $(u(x),v(x))$ is a solution to (1.5), then $(\hat{u}(x),\hat{v}(x))$ satisfies

$$\begin{cases} \hat{u}(x) = \int_{R^n_+} G(x,y)\hat{u}^{\beta_1}(y)\hat{v}^{\gamma_1}(y)dy, \\ \hat{v}(x) = \int_{R^n_+} G(x,y)\hat{u}^{\beta_2}(y)\hat{v}^{\gamma_2}(y)dy. \end{cases} \tag{3.19}$$

Since $u\in L^p_{\rm loc}(R^n_+)$ and $v\in L^p_{\rm loc}(R^n_+)$, for any domain Ω that is a positive distance away from z^0, we have

$$\int_\Omega \hat{u}^p(x)dx<\infty, \quad \int_\Omega \hat{v}^p(x)dx<\infty.$$

Now, we apply the method of moving planes to $(\hat{u},\hat{v})$.

In this case, for a given real number λ, we define

$$\hat{\Sigma}_\lambda = \{x=(x_1,x_2,\ldots,x_n)\in R^n_+ \mid x_1\leqslant\lambda\},$$
$$\hat{T}_\lambda = \{x\in R^n_+ \mid x_1=\lambda\},$$

and let

$$x^\lambda=(2\lambda-x_1,x_2,\ldots,x_n).$$

For $x,y\in\hat{\Sigma}_\lambda$, $x\neq y$, we have

$$G(x,y)=G(x^\lambda,y^\lambda), \quad G(x^\lambda,y)=G(x,y^\lambda), \quad G(x^\lambda,y^\lambda)>G(x^\lambda,y). \tag{3.20}$$

In this case, we move the plane $\hat{T}_\lambda$ along the direction of x_1-axis until $\lambda=z^0_1$ to show that the solution is rotationally symmetric about the line passing through z^0 and parallel to the x_n-axis. The proof consists of two steps.

Step 1 (Prepare to move the plane from near $x_1=-\infty$)**.**

Define

$$\hat{\Sigma}^u_\lambda = \{x\in\hat{\Sigma}_\lambda\setminus B_\epsilon((z^0)^\lambda) \mid \hat{w}_\lambda(x)<0\},$$
$$\hat{\Sigma}^v_\lambda = \{x\in\hat{\Sigma}_\lambda\setminus B_\epsilon((z^0)^\lambda) \mid \hat{g}_\lambda(x)<0\},$$
$$\hat{w}_\lambda(x)=\hat{u}_\lambda(x)-\hat{u}(x), \quad \hat{g}_\lambda(x)=\hat{v}_\lambda(x)-\hat{v}(x).$$

In this step, we will show that for sufficiently negative λ, and sufficiently small $\epsilon>0$,

$$\hat{w}_\lambda(x)\geqslant 0, \quad \hat{g}_\lambda(x)\geqslant 0, \quad \forall\, x\in\hat{\Sigma}_\lambda\setminus B_\epsilon((z^0)^\lambda). \tag{3.21}$$

Obviously, we have

$$\hat{u}(x) = \int_{\hat{\Sigma}_\lambda} G(x,y)\hat{u}^{\beta_1}(y)\hat{v}^{\gamma_1}(y)dy + \int_{\hat{\Sigma}_\lambda} G(x,y^\lambda)\hat{u}^{\beta_1}_\lambda(y)\hat{v}^{\gamma_1}_\lambda(y)dy,$$
$$\hat{u}_\lambda(x) = \int_{\hat{\Sigma}_\lambda} G(x^\lambda,y)\hat{u}^{\beta_1}(y)\hat{v}^{\gamma_1}(y)dy + \int_{\hat{\Sigma}_\lambda} G(x^\lambda,y^\lambda)\hat{u}^{\beta_1}_\lambda(y)\hat{v}^{\gamma_1}_\lambda(y)dy.$$

By (3.20), we calculate

$$\begin{aligned}\hat{u}(x)-\hat{u}_\lambda(x) &= \int_{\hat{\Sigma}_\lambda}[G(x^\lambda,y^\lambda)-G(x,y^\lambda)][\hat{u}^{\beta_1}(y)\hat{v}^{\gamma_1}-\hat{u}^{\beta_1}_\lambda(y)\hat{v}^{\gamma_1}_\lambda(y)]dy \\ &\leqslant \int_{\hat{\Sigma}^u_\lambda}[G(x^\lambda,y^\lambda)-G(x,y^\lambda)]\{\hat{u}^{\beta_1}(y)[\hat{v}^{\gamma_1}(y)-\hat{v}^{\gamma_1}_\lambda(y)]\end{aligned}$$

$$
\begin{aligned}
&+ \hat{v}_\lambda^{\gamma_1}(y)[\hat{u}^{\beta_1}(y) - \hat{u}_\lambda^{\beta_1}(y)]\}dy \\
&+ \int_{\hat{\Sigma}_\lambda \setminus \hat{\Sigma}_\lambda^u} [G(x^\lambda, y^\lambda) - G(x, y^\lambda)]\{\hat{u}^{\beta_1}(y)[\hat{v}^{\gamma_1}(y) - \hat{v}_\lambda^{\gamma_1}(y)]\}dy \\
\leqslant &\int_{\hat{\Sigma}_\lambda^u} [G(x^\lambda, y^\lambda) - G(x, y^\lambda)]\{\hat{v}_\lambda^{\gamma_1}(y)[\hat{u}^{\beta_1}(y) - \hat{u}_\lambda^{\beta_1}(y)]\}dy \\
&+ \int_{\hat{\Sigma}_\lambda^v} [G(x^\lambda, y^\lambda) - G(x, y^\lambda)]\{\hat{u}^{\beta_1}(y)[\hat{v}^{\gamma_1}(y) - \hat{v}_\lambda^{\gamma_1}(y)]\}dy \\
\leqslant &\int_{\hat{\Sigma}_\lambda^u} G(x^\lambda, y^\lambda)\hat{v}_\lambda^{\gamma_1}(y)[\hat{u}^{\beta_1}(y) - \hat{u}_\lambda^{\beta_1}(y)]dy \\
&+ \int_{\hat{\Sigma}_\lambda^v} G(x^\lambda, y^\lambda)\hat{u}^{\beta_1}(y)[\hat{v}^{\gamma_1}(y) - \hat{v}_\lambda^{\gamma_1}(y)]dy \\
\leqslant &\int_{\hat{\Sigma}_\lambda^u} \beta_1 G(x^\lambda, y^\lambda)\hat{v}_\lambda^{\gamma_1}(y)\hat{u}^{\beta_1-1}(y)[\hat{u}(y) - \hat{u}_\lambda(y)]dy \\
&+ \int_{\hat{\Sigma}_\lambda^v} \gamma_1 G(x^\lambda, y^\lambda)\hat{u}^{\beta_1}(y)\hat{v}^{\gamma_1-1}(y)[\hat{v}(y) - \hat{v}_\lambda(y)]dy.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\hat{v}(x) - \hat{v}_\lambda(x) \leqslant &\int_{\hat{\Sigma}_\lambda^u} \beta_2 G(x^\lambda, y^\lambda)\hat{v}_\lambda^{\gamma_2}(y)\hat{u}^{\beta_2-1}(y)[\hat{u}(y) - \hat{u}_\lambda(y)]dy \\
&+ \int_{\hat{\Sigma}_\lambda^v} \gamma_2 G(x^\lambda, y^\lambda)\hat{u}^{\beta_2}(y)\hat{v}^{\gamma_2-1}(y)[\hat{v}(y) - \hat{v}_\lambda(y)]dy,
\end{aligned}
$$

we show that for sufficiently negative λ and for sufficiently small $\epsilon > 0$, $\hat{\Sigma}_\lambda^u$ and $\hat{\Sigma}_\lambda^v$ must be of measure zero. In fact, the proof of the step is similar to Step 1 in Case 1.

Step 2 (Move the plane to the limiting position to derive symmetry)**.**

Step 1 provides a starting point to move the plane $\hat{T}_\lambda$. Now we start to move the plane $\hat{T}_\lambda$ along the x_1 direction as long as (3.21) holds to the limiting position.

Define

$$
\lambda_0 = \sup\{\lambda \leqslant z_1^0 \mid \hat{w}_\mu(x) \geqslant 0, \hat{g}_\mu(x) \geqslant 0, \mu \leqslant \lambda, \forall\, x \in \hat{\Sigma}_\mu\}.
$$

We will show that $\lambda_0 = z_1^0$. Suppose on the contrary, if $\lambda_0 < z_1^0$, we will prove that $(\hat{u}(x), \hat{v}(x))$ is symmetric about T_{λ_0}, i.e.,

$$
\hat{w}_{\lambda_0}(x) \equiv 0, \quad \hat{g}_{\lambda_0}(x) \equiv 0, \quad \text{a.e. } \forall\, x \in \hat{\Sigma}_{\lambda_0} \setminus B_\epsilon((z^0)^{\lambda_0}). \tag{3.22}
$$

If (3.22) does not hold, we will show that

$$
w_{\lambda_0} > 0, \quad g_{\lambda_0} > 0, \quad \forall\, x \in \hat{\Sigma}_{\lambda_0} \setminus B_\epsilon((z^0)^{\lambda_0}). \tag{3.23}
$$

By elementary calculation, we obtian

$$
\begin{aligned}
\hat{u}_{\lambda_0}(x) - \hat{u}(x) &= \int_{\hat{\Sigma}_{\lambda_0}} [G(x^{\lambda_0}, y^{\lambda_0}) - G(x, y^{\lambda_0})][\hat{u}_{\lambda_0}^{\beta_1}(y)\hat{v}_{\lambda_0}^{\gamma_1}(y) - \hat{u}^{\beta_1}(y)\hat{v}^{\gamma_1}]dy \\
&= \int_{\hat{\Sigma}_{\lambda_0}} [G(x^{\lambda_0}, y^{\lambda_0}) - G(x, y^{\lambda_0})][\hat{u}_{\lambda_0}^{\beta_1}(y)(\hat{v}_{\lambda_0}^{\gamma_1}(y) - \hat{v}^{\gamma_1}(y)) \\
&\quad + \hat{v}^{\gamma_1}(y)(\hat{u}_{\lambda_0}^{\beta_1}(y) - \hat{u}^{\beta_1}(y))]dy.
\end{aligned} \tag{3.24}
$$

If (3.23) is violated, then there exists some point $x^0 \in \hat{\Sigma}_{\lambda_0} \setminus B_\epsilon((z^0)^{\lambda_0})$ such that

$$
\hat{u}(x^0) = \hat{u}_{\lambda_0}(x^0) \quad \text{or} \quad \hat{v}(x^0) = \hat{v}_{\lambda_0}(x^0).
$$

Without loss of generality, we may assume $\hat{u}(x^0) = \hat{u}_{\lambda_0}(x^0)$. Combining this with (3.24), we have

$$\hat{u}_{\lambda_0}^{\beta_1}(y)\hat{v}_{\lambda_0}^{\gamma_1}(y) = \hat{u}^{\beta_1}(y)\hat{v}^{\gamma_1}(y), \quad \forall\, y \in \hat{\Sigma}_{\lambda_0}. \tag{3.25}$$

By the definition of λ_0, we have

$$\hat{u}_{\lambda_0}(y) \geqslant \hat{u}(y), \quad \forall\, y \in \hat{\Sigma}_{\lambda_0}.$$

By (3.25), we deduce

$$\hat{v}_{\lambda_0}^{\gamma_1}(y) \leqslant \hat{v}^{\gamma_1}(y), \quad \forall\, y \in \hat{\Sigma}_{\lambda_0}.$$

Hence we derive that

$$\hat{v}_{\lambda_0}^{\gamma_1}(y) = \hat{v}^{\gamma_1}(y), \quad \forall\, y \in \hat{\Sigma}_{\lambda_0}.$$

By (3.25), it is easy to see that

$$\hat{u}_{\lambda_0}^{\gamma_1}(y) = \hat{u}^{\gamma_1}(y), \quad \forall\, y \in \hat{\Sigma}_{\lambda_0}.$$

This is a contradiction with our assumptions, and therefore (3.23) must hold.

Next based on (3.23), we show that the plane can be moved further to the right, i.e., for $\lambda > \lambda_0$ and sufficiently close to λ_0,

$$\hat{w}_\lambda(x) \geqslant 0, \quad \hat{g}_\lambda(x) \geqslant 0, \quad \text{a.e.}\ \forall\, x \in \hat{\Sigma}_\lambda \setminus B_\epsilon((z^0)^\lambda). \tag{3.26}$$

The proof is similar to that in Step 2 in Case 1. We only need to use $\hat{\Sigma}_\lambda \setminus B_\epsilon((z^0)^\lambda)$ instead of Σ_λ and $\hat{\Sigma}_{\lambda_0} \setminus B_\epsilon((z^0)^\lambda)$ instead of Σ_{λ_0}. So, (3.26) is a contradiction with the definition of λ_0. Moreover, (3.22) must hold, i.e., if $\lambda_0 < z_1^0$, for $\forall \epsilon > 0$, then

$$\hat{w}_{\lambda_0} \equiv 0, \quad \hat{g}_{\lambda_0} \equiv 0, \quad \text{a.e.}\ \forall\, x \in \hat{\Sigma}_{\lambda_0} \setminus B_\epsilon((z^0)^{\lambda_0}).$$

Since $\hat{u}$ is singular at z^0, $\hat{u}$ must be singular at $(z^0)^\lambda$. This is impossible. We get

$$\hat{w}_{z_1^0}(x) \geqslant 0, \quad \text{a.e.}\ \forall\, x \in \hat{\Sigma}_{z_1^0}.$$

Similarly, we can move the plane from $x_1 = +\infty$ to the left and prove that $\hat{w}_{z_1^0}(x) \leqslant 0$. Hence, we obtain $\hat{w}_{z_1^0}(x) \equiv 0$, a.e. $\forall\, x \in \hat{\Sigma}_{z_1^0}$. Hence, it is easy to see $\lambda_0 = z_1^0$.

(A_2) In the subcritical case when $\frac{n}{n-\alpha} < \beta_1 + \gamma_1 = \beta_2 + \gamma_2 < \frac{n+\alpha}{n-\alpha}$, if $(u(x), v(x))$ is the solution to (0.2), then $(\hat{u}(x), \hat{v}(x))$ satisfies

$$\begin{cases} \hat{u}(x) = \displaystyle\int_{R_+^n} \frac{1}{|y - z^0|^a} G(x,y)\hat{u}^{\beta_1}(y)\hat{v}^{\gamma_1}(y)dy, \\ \hat{v}(x) = \displaystyle\int_{R_+^n} \frac{1}{|y - z^0|^b} G(x,y)\hat{u}^{\beta_2}(y)\hat{v}^{\gamma_2}(y)dy, \end{cases} \tag{3.27}$$

where $a = 2n - (n-\alpha)(\beta_1 + \gamma_1 + 1) > 0$, and $b = 2n - (n-\alpha)(\beta_2 + \gamma_2 + 1) > 0$.

In this case, we move the plane $\hat{T}_\lambda$ along the direction of x_1-axis. The proof consists of two steps.

Step 1 (Prepare to move the plane from near $x_1 = -\infty$)**.** Define

$$\hat{\Sigma}_\lambda^u = \{x \in \hat{\Sigma}_\lambda \setminus B_\epsilon((z^0)^\lambda) \mid \hat{w}_\lambda(x) < 0\},$$
$$\hat{\Sigma}_\lambda^v = \{x \in \hat{\Sigma}_\lambda \setminus B_\epsilon((z^0)^\lambda) \mid \hat{g}_\lambda(x) < 0\},$$
$$\hat{w}_\lambda(x) = \hat{u}_\lambda(x) - \hat{u}(x), \quad \hat{g}_\lambda(x) = \hat{v}_\lambda(x) - \hat{v}(x).$$

In this step, we will show that for sufficiently negative λ,

$$\hat{w}_\lambda(x) \geqslant 0, \quad \hat{g}_\lambda(x) \geqslant 0, \quad \forall\, x \in \hat{\Sigma}_\lambda \setminus B_\epsilon((z^0)^\lambda). \tag{3.28}$$

We show that for sufficiently negative λ, $\hat{\Sigma}_\lambda^u$ and $\hat{\Sigma}_\lambda^v$ are of measure zero. In fact, we have

$$\begin{aligned}\hat{u}(x) = &\int_{\hat{\Sigma}_\lambda} \frac{1}{|y-z^0|^a} G(x,y)\hat{u}^{\beta_1}(y)\hat{v}^{\gamma_1}(y)dy \\ &+ \int_{\hat{\Sigma}_\lambda} \frac{1}{|y^\lambda-z^0|^a} G(x,y^\lambda)\hat{u}_\lambda^{\beta_1}(y)\hat{v}_\lambda^{\gamma_1}(y)dy, \\ \hat{u}_\lambda(x) = &\int_{\hat{\Sigma}_\lambda} \frac{1}{|y-z^0|^a} G(x^\lambda,y)\hat{u}^{\beta_1}(y)\hat{v}^{\gamma_1}(y)dy \\ &+ \int_{\hat{\Sigma}_\lambda} \frac{1}{|y^\lambda-z^0|^a} G(x^\lambda,y^\lambda)\hat{u}_\lambda^{\beta_1}(y)\hat{v}_\lambda^{\gamma_1}(y)dy.\end{aligned}$$

By (3.20), we calculate

$$\begin{aligned}\hat{u}(x)-\hat{u}_\lambda(x) &= \int_{\hat{\Sigma}_\lambda} [G(x^\lambda,y^\lambda)-G(x,y^\lambda)]\left[\frac{1}{|y-z^0|^a}\hat{u}^{\beta_1}(y)\hat{v}^{\gamma_1} - \frac{1}{|y^\lambda-z^0|^a}\hat{u}_\lambda^{\beta_1}(y)\hat{v}_\lambda^{\gamma_1}(y)\right]dy \\ &= \int_{\hat{\Sigma}_\lambda} [G(x^\lambda,y^\lambda)-G(x,y^\lambda)]\left\{\frac{1}{|y^\lambda-z^0|^a}\left[\hat{u}^{\beta_1}(y)\hat{v}^{\gamma_1}(y)-\hat{u}_\lambda^{\beta_1}(y)\hat{v}_\lambda^{\gamma_1}(y)\right]\right. \\ &\quad \left. + \hat{u}^{\beta_1}(y)\hat{v}^{\gamma_1}(y)\left(\frac{1}{|y-z^0|^a}-\frac{1}{|y^\lambda-z^0|^a}\right)\right\}dy \\ &\leqslant \int_{\hat{\Sigma}_\lambda} [G(x^\lambda,y^\lambda)-G(x,y^\lambda)]\left\{\frac{1}{|y^\lambda-z^0|^a}[\hat{u}^{\beta_1}(y)\hat{v}^{\gamma_1}(y)-\hat{u}_\lambda^{\beta_1}(y)\hat{v}_\lambda^{\gamma_1}(y)]\right\}dy \\ &\leqslant \int_{\hat{\Sigma}_\lambda^v} \frac{1}{|y^\lambda-z^0|^a}[G(x^\lambda,y^\lambda)-G(x,y^\lambda)]\{\hat{u}^{\beta_1}(y)[\hat{v}^{\gamma_1}(y)-\hat{v}_\lambda^{\gamma_1}(y)]\}dy \\ &\quad + \int_{\hat{\Sigma}_\lambda^u} \frac{1}{|y^\lambda-z^0|^a}[G(x^\lambda,y^\lambda)-G(x,y^\lambda)]\{\hat{v}_\lambda^{\gamma_1}(y)[\hat{u}^{\beta_1}(y)-\hat{u}_\lambda^{\beta_1}(y)]\}dy \\ &\leqslant \int_{\hat{\Sigma}_\lambda^v} \frac{1}{|y^\lambda-z^0|^a}G(x^\lambda,y^\lambda)\{\hat{u}^{\beta_1}(y)[\hat{v}^{\gamma_1}(y)-\hat{v}_\lambda^{\gamma_1}(y)]\}dy \\ &\quad + \int_{\hat{\Sigma}_\lambda^u} \frac{1}{|y^\lambda-z^0|^a}G(x^\lambda,y^\lambda)\{\hat{v}_\lambda^{\gamma_1}(y)[\hat{u}^{\beta_1}(y)-\hat{u}_\lambda^{\beta_1}(y)]\}dy.\end{aligned}$$

By the mean value theorem, we have

$$\begin{aligned}\hat{u}(x)-\hat{u}_\lambda(x) \leqslant &\int_{\hat{\Sigma}_\lambda^v} \gamma_1 \frac{1}{|y^\lambda-z^0|^a}G(x^\lambda,y^\lambda)\hat{u}^{\beta_1}(y)\hat{v}^{\gamma_1-1}(y)[\hat{v}(y)-\hat{v}_\lambda(y)]dy \\ &+ \int_{\hat{\Sigma}_\lambda^u} \beta_1 \frac{1}{|y^\lambda-z^0|^a}G(x^\lambda,y^\lambda)\hat{u}^{\beta_1-1}(y)\hat{v}_\lambda^{\gamma_1}(y)[\hat{u}(y)-\hat{u}_\lambda(y)]dy.\end{aligned} \tag{3.29}$$

For $0<\alpha<n$, by (3.6), we also have

$$G(x,y) \leqslant \frac{C}{|x-y|^{n-\alpha}}. \tag{3.30}$$

By (3.29) and (3.30), we have

$$\begin{aligned}|\hat{u}(x)-\hat{u}_\lambda(x)| \leqslant &\int_{\hat{\Sigma}_\lambda^v} \frac{C}{|x-y|^{n-\alpha}}\frac{1}{|y^\lambda-z^0|^a}|\hat{u}^{\beta_1}(y)\hat{v}^{\gamma_1-1}(y)||\hat{v}(y)-\hat{v}_\lambda(y)|dy \\ &+ \int_{\hat{\Sigma}_\lambda^u} \frac{C}{|x-y|^{n-\alpha}}\frac{1}{|y^\lambda-z^0|^a}|\hat{u}^{\beta_1-1}(y)\hat{v}_\lambda^{\beta_1}(y)||\hat{u}(y)-\hat{u}_\lambda(y)|dy.\end{aligned} \tag{3.31}$$

We apply Lemma 3.3 and Hölder's inequality to (3.31) to obtain, for $p>\frac{n}{n-\alpha}$,

$$\|\hat{w}_\lambda\|_{L^p(\hat{\Sigma}_\lambda^u)} \leqslant \hat{C}_1\left\|\frac{1}{|y^\lambda-z^0|^a}\hat{u}^{\beta_1}\hat{v}^{\gamma_1-1}\hat{g}_\lambda\right\|_{L^{\frac{np}{n+\alpha p}}(\hat{\Sigma}_\lambda^v)} + \hat{C}_2\left\|\frac{1}{|y^\lambda-z^0|^a}\hat{v}_\lambda^{\gamma_1}\hat{u}^{\beta_1-1}\hat{w}_\lambda\right\|_{L^{\frac{np}{n+\alpha p}}(\hat{\Sigma}_\lambda^u)}$$

$$
\begin{aligned}
&\leqslant \hat{C}_1 \left\| \frac{1}{|y^\lambda - z^0|^a} \hat{u}^{\beta_1} \hat{v}^{\gamma_1 - 1} \right\|_{L^{\frac{n}{\alpha}}(\hat{\Sigma}_\lambda^v)} \|\hat{g}_\lambda\|_{L^p(\hat{\Sigma}_\lambda^v)} \\
&\quad + \hat{C}_2 \left\| \frac{1}{|y^\lambda - z^0|^a} \hat{u}^{\beta_1 - 1} \hat{v}_\lambda^{\gamma_1} \right\|_{L^{\frac{n}{\alpha}}(\hat{\Sigma}_\lambda^u)} \|\hat{w}_\lambda\|_{L^p(\hat{\Sigma}_\lambda^u)} \\
&\leqslant \hat{C}_1 \left\| \frac{1}{|y^\lambda - z^0|^{a_1}} \hat{u}^{\beta_1} \right\|_{L^{\frac{p}{\beta_1}}(\hat{\Sigma}_\lambda^v)} \left\| \frac{1}{|y^\lambda - z^0|^{a_2}} \hat{v}^{\gamma_1 - 1} \right\|_{L^{\frac{p}{\gamma_1 - 1}}(\hat{\Sigma}_\lambda^v)} \|\hat{g}_\lambda\|_{L^p(\hat{\Sigma}_\lambda^v)} \\
&\quad + \hat{C}_2 \left\| \frac{1}{|y^\lambda - z^0|^{a_3}} \hat{u}^{\beta_1 - 1} \right\|_{L^{\frac{p}{\beta_1 - 1}}(\hat{\Sigma}_\lambda^u)} \left\| \frac{1}{|y^\lambda - z^0|^{a_4}} v_\lambda^{\gamma_1} \right\|_{L^{\frac{p}{\gamma_1}}(\hat{\Sigma}_\lambda^u)} \|\hat{w}_\lambda\|_{L^p(\hat{\Sigma}_\lambda^u)}, \qquad (3.32)
\end{aligned}
$$

where

$$
\begin{aligned}
a_1 &= [2n - p(n-\alpha)] \frac{p}{\beta_1}, \\
a_2 &= [2n - p(n-\alpha)] \frac{p}{\gamma_1 - 1}, \\
a_3 &= [2n - p(n-\alpha)] \frac{p}{\beta_1 - 1}, \\
a_4 &= [2n - p(n-\alpha)] \frac{p}{\gamma_1}.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\|\hat{g}_\lambda\|_{L^p(\Sigma_\lambda^v)} &\leqslant \hat{C}_3 \left\| \frac{1}{|y^\lambda - z^0|^b} \hat{u}^{\beta_2 - 1} \hat{v}_\lambda^{\gamma_2} \right\|_{L^{\frac{n}{\alpha}}(\hat{\Sigma}_\lambda^u)} \|w_\lambda\|_{L^r(\Sigma_\lambda^u)} \\
&\quad + \hat{C}_4 \left\| \frac{1}{|y^\lambda - z^0|^b} \hat{u}^{\beta_2} \hat{v}^{\gamma_2 - 1} \right\|_{L^{\frac{n}{\alpha}}(\hat{\Sigma}_\lambda^v)} \|g_\lambda\|_{L^r(\Sigma_\lambda^v)} \\
&\leqslant \hat{C}_3 \left\| \frac{1}{|y^\lambda - z^0|^{b_1}} \hat{u}^{\beta_2 - 1} \right\|_{L^{\frac{p}{\beta_2 - 1}}(\hat{\Sigma}_\lambda^u)} \left\| \frac{1}{|y^\lambda - z^0|^{b_2}} \hat{v}_\lambda^{\gamma_2} \right\|_{L^{\frac{p}{\gamma_2}}(\hat{\Sigma}_\lambda^u)} \|\hat{w}_\lambda\|_{L^p(\hat{\Sigma}_\lambda^u)} \\
&\quad + \hat{C}_4 \left\| \frac{1}{|y^\lambda - z^0|^{b_3}} \hat{u}^{\beta_2} \right\|_{L^{\frac{p}{\beta_2}}(\hat{\Sigma}_\lambda^v)} \left\| \frac{1}{|y^\lambda - z^0|^{b_4}} \hat{v}^{\gamma_2 - 1} \right\|_{L^{\frac{p}{\gamma_2 - 1}}(\hat{\Sigma}_\lambda^v)} \|\hat{g}_\lambda\|_{L^p(\hat{\Sigma}_\lambda^v)}, \qquad (3.33)
\end{aligned}
$$

where

$$
\begin{aligned}
b_1 &= [2n - p(n-\alpha)] \frac{p}{\beta_2 - 1}, \\
b_2 &= [2n - p(n-\alpha)] \frac{p}{\gamma_2}, \\
b_3 &= [2n - p(n-\alpha)] \frac{p}{\beta_2}, \\
b_4 &= [2n - p(n-\alpha)] \frac{p}{\gamma_2 - 1}.
\end{aligned}
$$

Since u, $v \in L^p_{\mathrm{loc}}(R^n_+)$, for any domain Ω that is a positive distance away form z^0, we have

$$
\left\| \frac{1}{|y^\lambda - z^0|^{a_1}} \hat{u}^{\beta_1} \right\|_{L^{\frac{p}{\beta_1}}(\Omega)} < \infty, \quad \left\| \frac{1}{|y^\lambda - z^0|^{a_2}} \hat{v}^{\gamma_1 - 1} \right\|_{L^{\frac{p}{\gamma_1 - 1}}(\Omega)} < \infty, \qquad (3.34)
$$

$$
\left\| \frac{1}{|y^\lambda - z^0|^{a_3}} \hat{u}^{\beta_1 - 1} \right\|_{L^{\frac{p}{\beta_1 - 1}}(\Omega)} < \infty, \quad \left\| \frac{1}{|y^\lambda - z^0|^{a_4}} v_\lambda^{\gamma_1} \right\|_{L^{\frac{p}{\gamma_1}}(\Omega)} < \infty, \qquad (3.35)
$$

$$
\left\| \frac{1}{|y^\lambda - z^0|^{b_1}} \hat{u}^{\beta_2 - 1} \right\|_{L^{\frac{p}{\beta_2 - 1}}(\Omega)} < \infty, \quad \left\| \frac{1}{|y^\lambda - z^0|^{b_2}} \hat{v}_\lambda^{\gamma_2} \right\|_{L^{\frac{p}{\gamma_2}}(\Omega)} < \infty, \qquad (3.36)
$$

$$
\left\| \frac{1}{|y^\lambda - z^0|^{b_3}} \hat{u}^{\beta_2} \right\|_{L^{\frac{p}{\beta_2}}(\Omega)} < \infty, \quad \left\| \frac{1}{|y^\lambda - z^0|^{b_4}} \hat{v}^{\gamma_2 - 1} \right\|_{L^{\frac{p}{\gamma_2 - 1}}(\Omega)} < \infty. \qquad (3.37)
$$

Next, we will prove that the sets $\hat{\Sigma}_\lambda^u$ and $\hat{\Sigma}_\lambda^v$ are empty for sufficiently negative λ. So, we can derive

$$\hat{w}_\lambda(x) \geqslant 0, \quad \hat{g}_\lambda(x) \geqslant 0, \quad \text{a.e. in } \Sigma_\lambda.$$

By (3.34)–(3.36), and (3.37), the proof is similar to the proof of (3.10).

Step 2 (Move the plane to the limiting position to derive symmetry)**.** Step 1 provides a starting point to move the plane $\hat{T}_\lambda$. Now we start to move the plane $\hat{T}_\lambda$ along the x_1 direction as long as (3.28) holds to the limiting position.

Define

$$\lambda_0 = \sup\{\lambda \mid \hat{w}_\mu(x) \geqslant 0, \hat{g}_\mu(x) \geqslant 0, \mu \leqslant \lambda, \forall\, x \in \hat{\Sigma}_\mu\},$$

we will prove that $\hat{u}(x)$ is symmetric about T_{λ_0}, i.e.,

$$\hat{w}_{\lambda_0}(x) \equiv 0, \quad \hat{g}_{\lambda_0}(x) \equiv 0, \quad \forall\, x \in \Sigma_\lambda.$$

The proof is similar to that in Step 2 in Case 2(A_1), and we derive that λ_0 must be z_1^0.

Thus we complete the proof of Theorem 2.

4 Non-existence of the positive solution of integral equations

In this section, we will complete the proof of Theorem 3.

Proof of Theorem 3. By virtue of Theorem 2, we present the proof in two cases.

For Case 1, we have shown that solutions $(u(x), v(x))$ are monotonically increasing with respect to the variable x_n, this contradicts with asymptotic behavior of $(u(x), v(x))$ at infinity. Therefore, we conclude that the positive solution of (1.5) does not exist.

For Case 2, let x^1 and x^2 be any two points in R_+^n,

$$x^1 = (x_1^1, x_2^1, \dots, x_{n-1}^1, x_n), \quad x^2 = (x_1^2, x_2^2, \dots, x_{n-1}^2, x_n).$$

Let z^0 be the projection of the midpoint $x^0 = \frac{x^1+x^2}{2}$, where $z^0 \in \partial R_+^n$. In the proof of Case 2, we know $(\hat{u}(x), \hat{v}(x))$ is axially symmetric with respect to $\overline{x^0 z^0}$. Set

$$y^1 = \frac{x^1 - z^0}{|x^1 - z^0|^2} + z^0, \quad y^2 = \frac{x^2 - z^0}{|x^2 - z^0|^2} + z^0,$$

it is easy to see $\hat{u}(y^1) = \hat{u}(y^2)$, hence $u(x^1) = u(x^2)$. This implies that $u(x)$ only depends on the x_n-variable. Similarly, we derive that $v(x)$ only depends on the x_n-variable. Next, we will prove the non-existence of the positive solution of integral equations (1.5).

For $x = (x', x_n) \in R^{n-1} \times [0, +\infty)$, we fix $x \in R_+^n$, let $|x_n - y_n|^2 = a^2$, $|x' - y'|^2 = r^2$, and R be large enough. By using polar coordinates, we have

$$\begin{aligned}
+\infty > u(x) &= u(x_n) \\
&= \int_{R_+^n} \frac{C_n}{|x-y|^{n-\alpha}} \int_0^{\frac{4x_n y_n}{|x-y|^2}} \frac{z^{\frac{\alpha}{2}-1}}{(z+1)^{\frac{n}{2}}} dz u^{\beta_1}(y) v^{\gamma_1}(y) dy \\
&\sim C \int_{R_+^n} \frac{1}{|x-y|^{n-\alpha}} \int_0^{\frac{4x_n y_n}{|x-y|^2}} z^{\frac{\alpha}{2}-1} dz u^{\beta_1}(y) v^{\gamma_1}(y) dy \\
&\geqslant C x_n^{\frac{\alpha}{2}} \int_0^\infty u^{\beta_1}(y_n) v^{\gamma_1}(y_n) y_n^{\frac{\alpha}{2}} \int_{R^{n-1}} \frac{1}{(|x'-y'|^2 + |x_n - y_n|^2)^{\frac{n}{2}}} dy' dy_n \\
&\geqslant C x_n^{\frac{\alpha}{2}} \int_0^\infty u^{\beta_1}(y_n) v^{\gamma_1}(y_n) y_n^{\frac{\alpha}{2}} \int_0^\infty \frac{r^{n-2}}{(r^2 + a^2)^{\frac{n}{2}}} dr dy_n \\
&= C x_n^{\frac{\alpha}{2}} \int_0^\infty u^{\beta_1}(y_n) v^{\gamma_1}(y_n) y_n^{\frac{\alpha}{2}} \frac{1}{|x_n - y_n|} \int_0^\infty \frac{\tau^{n-2}}{(\tau^2 + 1)^{\frac{n}{2}}} d\tau dy_n
\end{aligned}$$

$$\sim Cx_n^{\frac{\alpha}{2}} \int_0^\infty u^{\beta_1}(y_n)v^{\gamma_1}(y_n)y_n^{\frac{\alpha}{2}-1}dy_n. \tag{4.1}$$

By similar calculation, we get

$$\begin{aligned}
+\infty > v(x) &= v(x_n) \\
&= \int_{R_+^n} \frac{C_n}{|x-y|^{n-\alpha}} \int_0^{\frac{4x_n y_n}{|x-y|^2}} \frac{z^{\frac{\alpha}{2}-1}}{(z+1)^{\frac{n}{2}}} dz u^{\beta_2}(y)v^{\gamma_2}(y)dy \\
&\sim C \int_{R_+^n} \frac{1}{|x-y|^{n-\alpha}} \int_0^{\frac{4x_n y_n}{|x-y|^2}} z^{\frac{\alpha}{2}-1} dz u^{\beta_2}(y)v^{\gamma_2}(y)dy \\
&\geqslant Cx_n^{\frac{\alpha}{2}} \int_0^\infty \frac{u^{\beta_2}(y_n)v^{\gamma_2}(y_n)y_n^{\frac{\alpha}{2}}}{|x_n-y_n|} \int_0^\infty \frac{\tau^{n-2}}{(\tau^2+1)^{\frac{n}{2}}} d\tau dy_n \\
&\sim Cx_n^{\frac{\alpha}{2}} \int_0^\infty u^{\beta_2}(y_n)v^{\gamma_2}(y_n)y_n^{\frac{\alpha}{2}-1}dy_n.
\end{aligned} \tag{4.2}$$

Obviously, there exists a sequence $\{y_n^k\}$ such that

$$u^{\beta_1}(y_n^k)v^{\gamma_1}(y_n^k)(y_n^k)^{\frac{\alpha}{2}} \to 0, \quad \text{as} \quad y_n^k \to \infty, \tag{4.3}$$
$$u^{\beta_2}(y_n^k)v^{\gamma_2}(y_n^k)(y_n^k)^{\frac{\alpha}{2}} \to 0, \quad \text{as} \quad y_n^k \to \infty. \tag{4.4}$$

Let $x_n = R$ be sufficiently large, by (4.1),

$$\begin{aligned}
+\infty > u(x_n) &\geqslant Cx_n^{\frac{\alpha}{2}} \int_0^1 \frac{u^{\beta_1}(y_n)v^{\gamma_1}(y_n)y_n^{\frac{\alpha}{2}}}{|x_n-y_n|} dy_n \\
&\geqslant \frac{C}{R} R^{\frac{\alpha}{2}} \int_0^1 u^{\beta_1}(y_n)v^{\gamma_1}(y_n)y_n^{\frac{\alpha}{2}} dy_n \\
&\geqslant CR^{\frac{\alpha}{2}-1} = Cx_n^{\frac{\alpha}{2}-1}.
\end{aligned} \tag{4.5}$$

Similarly,

$$\begin{aligned}
+\infty > v(x_n) &\geqslant Cx_n^{\frac{\alpha}{2}} \int_0^1 \frac{u^{\beta_2}(y_n)v^{\gamma_2}(y_n)y_n^{\frac{\alpha}{2}}}{|x_n-y_n|} dy_n \\
&\geqslant \frac{C}{R} R^{\frac{\alpha}{2}} \int_0^1 u^{\beta_2}(y_n)v^{\gamma_2}(y_n)y_n^{\frac{\alpha}{2}} dy_n \\
&\geqslant CR^{\frac{\alpha}{2}-1} = Cx_n^{\frac{\alpha}{2}-1}.
\end{aligned} \tag{4.6}$$

For sufficiently large $x_n = R$, applying (4.5) and (4.6),

$$\begin{aligned}
u(x_n) &\geqslant Cx_n^{\frac{\alpha}{2}} \int_{\frac{R^2}{2}}^{R^2} (C{y_n}^{\beta_1(\frac{\alpha}{2}-1)})(C{y_n}^{\gamma_1(\frac{\alpha}{2}-1)}) \frac{y_n^{\frac{\alpha}{2}}}{|x_n-y_n|} dy_n \\
&= Cx_n^{\frac{\alpha}{2}} \int_{\frac{R^2}{2}}^{R^2} (C{y_n}^{(\beta_1+\gamma_1)(\frac{\alpha}{2}-1)}) \frac{y_n^{\frac{\alpha}{2}}}{|x_n-y_n|} dy_n \\
&\geqslant CR^{\frac{\alpha}{2}} (R^2)^{(\beta_1+\gamma_1)(\frac{\alpha}{2}-1)} \frac{1}{R^2} \int_{\frac{R^2}{2}}^{R^2} y_n^{\frac{\alpha}{2}} dy_n \\
&= CR^{2(\beta_1+\gamma_1)(\frac{\alpha}{2}-1)+\frac{3\alpha}{2}} \\
&= Cx_n^{2(\beta_1+\gamma_1)(\frac{\alpha}{2}-1)+\frac{3\alpha}{2}} \\
&= Cx_n^{2p_1(\frac{\alpha}{2}-1)+\frac{3\alpha}{2}},
\end{aligned} \tag{4.7}$$

$$v(x_n) \geqslant Cx_n^{2(\beta_2+\gamma_2)(\frac{\alpha}{2}-1)+\frac{3\alpha}{2}} = Cx_n^{2p_1(\frac{\alpha}{2}-1)+\frac{3\alpha}{2}}, \tag{4.8}$$

where $p_1 = \beta_1 + \gamma_1 = \beta_2 + \gamma_2$.

For $x_n = R$, repeating this way m times, we obtain

$$u(x_n) \geqslant C{x_n}^{(2p_1)^m(\alpha-1)+\frac{(2p_1)^m-1}{2p_1-1}\frac{3\alpha}{2}}, \tag{4.9}$$

$$v(x_n) \geqslant C{x_n}^{(2p_1)^m(\alpha-1)+\frac{(2p_1)^m-1}{2p_1-1}\frac{3\alpha}{2}}. \tag{4.10}$$

By (4.9) and (4.10), we have

$$u^{\beta_1}(x_n)v^{\gamma_1}(x_n)(x_n)^{\frac{\alpha}{2}} \geqslant Cx_n^{f(2p_1)}, \tag{4.11}$$

where

$$f(t) = \left[t^m(\alpha-1) + \frac{t^m-1}{t-1}\frac{3\alpha}{2}\right]\frac{t}{2} + \frac{\alpha}{2}.$$

If $f(2p_1) > 0$, for sufficiently large x_n, we can derive the contradiction with (4.3). Next, we will prove $f(2p_1) > 0$ for some choice of m.

Let

$$g(t) = f(t)(t-1).$$

It is easy to see

$$g'(t) = \frac{t^m}{2}\left[(m+2)\left(\frac{\alpha}{2}-1\right)t + (m+1)(\alpha+1)\right] - \frac{\alpha}{4}.$$

For $1 < t < \frac{2(n+\alpha)}{n-\alpha}$, we will show that $g'(t) > 0$. We only need to verify

$$(m+2)\left(\frac{\alpha}{2}-1\right)t + (m+1)(\alpha+1) \geqslant \frac{\alpha}{2}.$$

For $\frac{\alpha}{2} - 1 < 0$, $n \geqslant 3$, and $\frac{2n}{n-\alpha} < t \leqslant \frac{2(n+\alpha)}{n-\alpha}$, it suffices to show

$$(m+2)(\alpha-1)\frac{2(3+\alpha)}{3-\alpha} + (m+1)(\alpha+1) \geqslant \frac{\alpha}{2},$$

this only requires

$$m \geqslant \left\lceil \frac{-3\alpha^2-5\alpha+18}{6(\alpha-1)} \right\rfloor + 1,$$

where $\lceil a \rfloor$ is the integer part of a.

For $\frac{\alpha}{2} - 1 \geqslant 0$ and $\frac{2n}{n-\alpha} < t \leqslant \frac{2(n+\alpha)}{n-\alpha}$, we only need to show

$$(m+2)(\alpha-1) + (m+1)(\alpha+1) \geqslant \frac{\alpha}{2},$$

which is obviously true since $m > 0$.

This completes the proof of Theorem 3. □

Acknowledgements This work was supported by China Scholarship Council (Grant No. 201206060010).

References

1 Berestycki H, Nirenberg L. On the method of moving planes and the sliding method. Bol Soc Brazs Mat, 1991, 1: 1–37
2 Caffarelli L, Gidas B, Spruck J. Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth. Comm Pure Appl Math, 1989, XLII: 271–297
3 Chen D Z, Ma L. A Liouville type theorem for an integral system. Comm Pure Appl Anal, 2006, 5: 855–859
4 Chen W X, Li C M. A priori estimates for prescribing scalar curvature equations. Ann of Math, 1997, 145: 547–564
5 Chen W X, Li C M. Regularity of solutions for a system of integral equations. Comm Pure Appl Anal, 2005, 4: 1–8

6 Chen W X, Li C M. The best constant in a weighted Hardy-Littlewood-Sobolev inequality. Proc Amer Math Soc, 2008, 136: 955–962
7 Chen W X, Li C M. An integral system and the Lane-Emden conjecture. Discrete Contin Dyn Syst, 2009, 4: 1167–1184
8 Chen W X, Li C M. Methods on Nonlinear Elliptic Equations. New York: American Institute of Mathematical Sciences, 2010
9 Chen W X, Li C M, Ou B. Qualitative properties of solutions for an integral equation. Discrete Contin Dyn Syst, 2005, 12: 347–354
10 Chen W X, Li C M, Ou B. Classification of solutions for a system of integral equations. Comm PDEs, 2005, 30: 59–65
11 Chen W X, Li C M, Ou B. Classification of solutions for an integral equation. Comm Pure Appl Math, 2006, 59: 330–343
12 Chen W X, Zhu J Y. Radial symmetry and regularity of solutions for poly-harmonic Dirichlet problems. J Math Anal Appl, 2011, 377: 744–753
13 Fang Y Q, Chen W X. A Liouville type theorem for poly-harmonic Dirichlet problem in a half space. Adv Math, 2012, 229: 2835–2867
14 Fang Y Q, Zhang J H. Nonexistence of positive solution for an integral equation on a half-space R^n_+. Comm Pure Appl Anal, 2013, 12: 663–678
15 Gidas B, Ni W M, Nirenberg L. Symmetry of Positive Solutions of Nonlinear Elliptic Equations in R^n. New York: Academic Press, 1981
16 Jin C, Li C M. Symmetry of solutions to some systems of integral equations. Proc Amer Math Soc, 2006, 134: 1661–1670
17 Kanna T, Lakshmanan M. Exact soliton solutions, shape changing collisions and partially coherent solitons in coupled nonlinear Schröedinger equations. Phys Rev Lett, 2001, 86: 5043–5046
18 Li C M, Lim J. The singularity analysis of solutions to some integral equations. Comm Pure Appl Anal, 2007, 6: 453–464
19 Li C M, Ma L. Uniqueness of positive bound states to Schrödinger systems with critical exponents. SIAM J Math Anal, 2008, 40: 1049–1057
20 Li D Y, Zhuo R. An integral equation on half space. Proc Amer Math Soc, 2010, 138: 2779–2791
21 Lieb E. Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities. Ann of Math, 1983, 118: 349–374
22 Lin T C, Wei J C. Spikes in two coupled nonlinear Schrödinger equations. Ann Inst H Poincaré Anal Non Linéaire, 2005, 22: 403–439
23 Ma C, Chen W X, Li C M. Regularity of solutions for an integral system of Wolff type. Adv Math, 2011, 226: 2676–2699
24 Ma L, Chen D Z. Radial symmetry and monotonicity for an integral equation. J Math Anal Appl, 2008, 342: 943–949
25 Ma L, Zhao L. Sharp thresholds of blow-up and global existence for the coupled nonlinear Schrödinger system. J Math Phys, 2008, doi: 10.1063/1.2939238
26 Ou B. A remark on a singular integral equation. Houston J Math, 1999, 25: 181–184
27 Reichel W, Weth T. A priori bounds and a Liouville theorem on a half-space for higher-order elliptic Dirichlet problems. Math Z, 2009, 261: 805–827
28 Serrin J. A symmetry problem in potential theory. Arch Rational Mech Anal, 1971, 43: 304–318
29 Wei J C, Xu X W. Classification of solutions of higher order conformally invariant equations. Math Ann, 1999, 313: 207–228
30 Zhuo R, Li D Y. A system of integral equations on half space. J Math Anal Appl, 2011, 381: 392–401