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Karhunen-Loeve expansions for the m-th order detrended Brownian motion

AI XiaoHui[1,*] & LI WenBo V.[2]

[1]Department of Mathematics, Northeast Forestry University, Harbin 150040, China;
[2]Department of Mathematical Sciences, University of Delaware, Newark, DE 19716, USA

Email: aixiaohui2006@gmail.com, wli@math.udel.edu

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Abstract The m-th order detrended Brownian motion is defined as the orthogonal component of projection of the standard Brownian motion onto the subspace spanned by polynomials of degree up to m. We obtain the Karhunen-Loeve expansion for the process and establish a connection with the generalized (m-th order) Brownian bridge developed by MacNeill (1978) in the study of distributions of polynomial regression. The resulting distribution identity is also verified by a stochastic Fubini approach. As applications, large and small deviation asymptotic behaviors for the L_2 norm are given.

Keywords m-th order detrended Brownian motion, Karhunen-Loeve expansions, stochastic Fubini approach, Zeilberger algorithm, large deviation, small deviation

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1 Introduction

Let $X = \{X(t), 0 \leqslant t \leqslant 1\}$ denote a mean zero Gaussian process on $C[0, 1]$ and define the covariance function by $K_X(t, s) = \mathbb{E}\, X(t)X(s), 0 \leqslant s, t \leqslant 1$. Then the well-known Karhunen-Loeve (KL) expansion is

$$X(t) = \sum_{k\geqslant 1} \eta_k \sqrt{\lambda_k} f_k(t), \tag{1.1}$$

where $\{\eta_k, k \geqslant 1\}$ is a sequence of i.i.d. $N(0, 1)$ random variables, $\{f_k(t), k \geqslant 1\}$ is the set of the eigenfunctions corresponding to $\{\lambda_k, k \geqslant 1\}$, which is the set of eigenvalues of the integral operator $T_X f(t) = \int_0^1 K_X(t, s)f(s)ds$. Furthermore, the fact that $\{f_k(t), k \geqslant 1\}$ forms an orthonormal sequence in $L_2([0, 1])$ gives the distributional identity

$$\int_0^1 X^2(t)dt \overset{\text{law}}{=} \sum_{k=1}^{\infty} \lambda_k \eta_k^2 \tag{1.2}$$

as a natural consequence of the KL expansions.

*Corresponding author

The Karhunen-Loeve expansions for the so-called demeaned (or self-centered) Gaussian process on $[0,1]$, denoted by $X_0(t)$, have been studied for various X extensively, where one defines

$$X_0(t) = X(t) - \int_0^1 X(s)ds, \quad 0 \leqslant t \leqslant 1. \tag{1.3}$$

In particular, results on the demeaned Brownian process $W_0(t)$ and the demeaned Brownian bridge $B_0(t)$ can be found in Beghin et al. [3], Deheuvels [5] and Karol et al. [10].

It is natural to view the demeaned process $X_0(t)$ in (1.3) as the orthogonal component of projection of $X(t)$ onto a constant function subspace in $L^2([0,1])$, see (1.4) below with $m=0$. This idea of projection was used in [1], where the linear detrended process $X_1(t)$ with $m=1$ in (1.4) was studied, and the Karhunen-Loeve expansion for the detrended Brownian motion $W_1(t)$ was obtained.

In this paper, we extend this projection idea into the polynomial detrended process of order m, which is defined as the centered Gaussian process $X_m(t) = X(t) - \sum_{i=1}^{m+1} a_i t^{i-1}$ such that

$$\int_0^1 X_m(t)^2 dt = \min_{a_i \in \mathbb{R}, 1 \leqslant i \leqslant m+1} \int_0^1 \left(X(t) - \sum_{i=1}^{m+1} a_i t^{i-1} \right)^2 dt, \quad m = 0,1,2,\ldots \tag{1.4}$$

We note that the process $X_m(t)$ is the orthonormal component of projection of $X(t)$ onto the subspace spanned by polynomials of degree up to m in $L_2([0,1])$.

It is clear that the optimal constants $a_i, 1 \leqslant i \leqslant m+1$ in (1.4) satisfy

$$\frac{\partial}{\partial a_i} \int_0^1 \left(X(t) - \sum_{i=1}^{m+1} a_i t^{i-1} \right)^2 dt = 0, \quad i = 1,2,\ldots,m+1, \tag{1.5}$$

which imply

$$H(a_1, a_2, \ldots, a_{m+1})^{\mathrm{T}} = \left(\int_0^1 X(t)dt, \int_0^1 tX(t)dt, \ldots, \int_0^1 t^m X(t)dt \right)^{\mathrm{T}}, \tag{1.6}$$

where H is the well-known Hilbert matrix given by

$$H = (h_{ij})_{m+1,m+1} = \left(\frac{1}{i+j-1} \right)_{1 \leqslant i,j \leqslant m+1}. \tag{1.7}$$

The inverse matrix H^{-1} can be expressed in a closed form, see Choi [4] with integer entries

$$a_{ij} = (-1)^{i+j}(i+j-1) \binom{m+i}{m+1-j} \binom{m+j}{m+1-i} \binom{i+j-2}{i-1}^2, \tag{1.8}$$

for $1 \leqslant i,j \leqslant m+1$. It then follows from (1.4) and (1.6) that the m-th order detrended Gaussian process is given by, as the orthogonal component of the projection,

$$X_m(t) = X(t) - \sum_{i=1}^{m+1} a_i t^{i-1} = X(t) - \sum_{i,j=1}^{m+1} a_{ij} t^{i-1} \int_0^1 u^{j-1} X(u)du, \tag{1.9}$$

where a_{ij} is defined in (1.8).

The main goal of this paper is finding the Karhunen-Loeve expansion of the m-th order detrended Brownian motion $W_m(t)$ defined in (1.9) with $X(t) = W(t)$, the standard Brownian motion with the covariance function $K_W(t,s) = \mathbb{E}\, W(t)W(s) = \min(t,s)$. The covariance function for W_m is given in Lemma 2.1. Of course, one can also consider the m-th order detrended Brownian bridge $B_m(t)$ generated by the standard Brownian bridge $B(t)$ with $\mathbb{E}\, B(t)B(s) = \min(s,t) - st, 0 \leqslant s,t \leqslant 1$. However, a simply covariance computation given in (2.1) in Lemma 2.1 in the next section shows that $W_m(t)$ and $B_m(t)$ are the same process on $C[0,1]$.

Before stating our Theorem 1.1, we need some notations and facts. For $v > -1$, let $j_v(\cdot)$ denote the spherical Bessel function of the first kind with index v. The positive zeros of $j_v(\cdot)$ form an infinite sequence, denoted by $0 < z_{v,1} < z_{v,2} < \cdots$. These zeros are interlaced with zeros $0 < z_{v+1,1} < z_{v+1,2} < \cdots$ of $j_{v+1}(\cdot)$ in such a way that $0 < z_{v,1} < z_{v+1,1} < z_{v,2} < z_{v+1,2} < \cdots$ (see [14]).

Now we can state one of the main results of this paper.

Theorem 1.1. *The spectrum of the Karhunen-Loeve expansion for the m-th order detrended Brownian motion* $\{W_m(t), t \in [0,1]\}$ *is given by* (2.13)*. In particular, we have the distribution identity*

$$\int_0^1 W_m^2(t)dt \overset{\text{law}}{=} \sum_{k\geqslant 1} \frac{\eta_k^2}{4z_{m-1,k}^2} + \sum_{k\geqslant 1} \frac{{\eta^*}_k^2}{4z_{m,k}^2}, \tag{1.10}$$

where $z_{m,k}$, $k = 1, 2, \dots$ *are the m-th positive zeros of the m-th order spherical Bessel function of the first kind,* $\{\eta_k, k \geqslant 1\}$ *and* $\{{\eta^*}_k, k \geqslant 1\}$ *denote two independent sequences of i.i.d* $N(0,1)$ *random variables.*

Next, we discuss the connection with the work of MacNeill [13], which studies the limiting distribution of partial sums of regression residuals. In particular, the Karhunen-loeve expansion is found for the generalized Brownian bridge of level (order) m defined by

$$\begin{aligned}\tilde{B}_m(t) = W(t) - \sum_{n=0}^{m}(2n+1)\Bigg\{ \sum_{q=0}^{[n/2]} \frac{(-1)^q \binom{2n}{n,q,q,n-2q}}{2^{4q}\binom{n-1/2}{q}} \int_0^t (s-1/2)^{n-2q} ds \Bigg\} \\ \times \Bigg\{ \sum_{q=0}^{[n/2]} \frac{(-1)^q \binom{2n}{n,q,q,n-2q}}{2^{4q}\binom{n-1/2}{q}} \cdot \left[2^{2q-n} W(1) - (n-2q) \int_0^1 \left(s - \frac{1}{2}\right)^{n-2q-1} W(s) ds \right] \Bigg\},\end{aligned} \tag{1.11}$$

for $0 \leqslant t \leqslant 1$, where $W(t)$ is the standard Brownian motion. Note the centered Gaussian process $\tilde{B}_m(t)$ has the properties of $\tilde{B}_m(0) = \tilde{B}_m(1) = 0$, $\mathbb{E}\{\tilde{B}_m(t)\} = 0$, $\int_0^1 \tilde{B}_m(t)dt = 0$. Its covariance function is

$$\mathbb{E}\,\tilde{B}_m(t)\tilde{B}_m(s) = t \wedge s - \sum_{n=0}^{m} (2n+1) g_n(t) g_n(s), \quad s,t \in [0,1], \tag{1.12}$$

where

$$g_n(t) = \sum_{q=0}^{[n/2]} \frac{(-1)^q \binom{2n}{n,q,q,n-2q}}{2^{4q}\binom{n-1/2}{q}} \int_0^t (u-1/2)^{n-2q} du. \tag{1.13}$$

It is clear that $\tilde{B}_m(t)$ and $W_m(t)$ are two different centered Gaussian processes by comparing their covariance functions. However, they have the same set of eigenvalues in their KL expansion by comparing our Theorem 1.1 for $W_m(t)$ with the KL expansion given in [13] for $\tilde{B}_m(t)$. Thus as a consequence, we have

Corollary 1.2.

$$\int_0^1 W_m^2(t)dt \overset{\text{law}}{=} \int_0^1 \tilde{B}_m^2(t)dt \overset{\text{law}}{=} \sum_{k\geqslant 1} \frac{\eta_k^2}{4z_{m-1,k}^2} + \sum_{k\geqslant 1} \frac{{\eta^*}_k^2}{4z_{m,k}^2}. \tag{1.14}$$

In Section 3, we give a direct proof of Corollary 1.2 by using the stochastic Fubini approach developed by Donati-Martin and Yor [7]. This has the advantage of relative simplicity but it is hard to know at the beginning that two processes $W_m(t)$ and $\tilde{B}_m(t)$ have the same spectrum in their KL expansions. Note also that the first distribution identity in Corollary 1.2 implies that $W_m(t)$ and $\tilde{B}_m(t)$ have the same spectrum, see [6]. So we indeed have two different proofs for Corollary 1.2. One is just discussed above and the other is based on KL expansion for $W_m(t)$ which we found first and only sketched in Section 2 since certain complicated combinatorial identities have to be proved rigorously based on Zeilberger's algorithm of finding proper recurrent relations. We also find the associated eigenfunctions which provide additional information in the KL expansion. Because both approaches have their merits and so we present both in a way that they complement each other. The rest of the paper is organized as follows. In Section 2, we give the proof of Theorem 1.1 by direct KL expansion. The stochastic Fubini approach is in Section 3. As applications of Theorem 1.1, we provide the standard treatment for the characteristic function, large deviations and small deviations of the m-th order detrended Brownian motion $W_m(t)$ in Section 4.

2 The Karhunen-Loeve expansion for $W_m(t), 0 \leqslant t \leqslant 1$

We start with the explicit covariance function for the m-th order detrended Brownian motion $W_m(t)$ defined in (1.9) with $X(t) = W(t)$.

Lemma 2.1. *For $0 \leqslant s, t \leqslant 1$, we have the covariance function*

$$\begin{aligned} K_{W_m}(s,t) &= \mathbb{E}\, W_m(t)W_m(s) \\ &= t \wedge s - t - s + ts + \sum_{p,q=1}^{m+1} \frac{a_{pq}}{q(q+1)} s^{p-1} t^{q+1} \\ &\quad + \sum_{i,j=1}^{m+1} \frac{a_{ij}}{j(j+1)} t^{i-1} s^{j+1} + \sum_{i,j=1}^{m+1} \sum_{p,q=1}^{m+1} \frac{a_{ij} a_{pq} t^{i-1} s^{p-1}}{(q+1)(j+1)(q+j+1)}, \end{aligned} \tag{2.1}$$

where a_{ij} is defined in (1.8).

Proof. For $0 \leqslant s, t \leqslant 1$, $j, q = 1, 2, \ldots, m+1$, we have

$$\mathbb{E}\left(W(s) \int_0^1 u^{j-1} W(u) du\right) = \int_0^1 u^{j-1} (u \wedge s) du = \frac{s}{j} - \frac{s^{j+1}}{j(j+1)}$$

and

$$\begin{aligned} \mathbb{E}\left(\int_0^1 u^{j-1} W(u) du \int_0^1 v^{q-1} W(v) dv\right) &= \int_0^1 v^{q-1} \mathbb{E}\left(W(v) \int_0^1 u^{j-1} W(u) du\right) dv \\ &= \int_0^1 v^{q-1} \left(\frac{v}{j} - \frac{v^{j+1}}{j(j+1)}\right) dv = \frac{j+q+2}{(j+1)(q+1)(j+q+1)}. \end{aligned}$$

Substituting the above equations into the product expansion

$$\begin{aligned} \mathbb{E}\, W_m(t)W_m(s) = \mathbb{E}&\left(W(t) - \sum_{i,j=1}^{m+1} a_{ij} t^{i-1} \int_0^1 u^{j-1} W(u) du\right) \\ &\times \left(W(s) - \sum_{p,q=1}^{m+1} a_{pq} s^{p-1} \int_0^1 v^{q-1} W(v) dv\right), \end{aligned}$$

we obtain (2.1) by using the facts that $(a_{ij}) = H^{-1}$ and for the Hilbert matrix H defined in (1.7), the first row is $1/j$ and the second row is $1/(j+1)$ with $1 \leqslant j \leqslant m+1$. □

Next, we list four combinatorial identities, which are used in the proof of Theorem 1.1.

Lemma 2.2. *For any positive integer k, we have*

$$\begin{aligned} &\sum_{p=2k+2}^{n} \frac{(-1)^p (p+n-1)(p+n-3)!}{(n-p)!(p-1)!(p-2k-2)!} = \frac{(-1)^n (2k+n-1)!(n+2k+(n-1)^2+2k(n-1))}{(n-1)(2k+1)!(n-2k-2)!}, \\ &\sum_{p=2k+1}^{n} \frac{(-1)^p (n+p-1)(n+p-3)!}{(n-p)!(p-1)!(p-2k-1)!} = \frac{(-1)^n (n+2k-2)!(2k(n-1)+(n-1)^2+2k)}{(n-1)(2k)!(n-1-2k)!}, \\ &\sum_{p=2k+1}^{n} \frac{(-1)^p (p+n-2)!}{(n-p)!(p-1)!(p-2k-1)!} = \frac{(-1)^n (2k+n-1)!}{(2k)!(n-2k-1)!}, \end{aligned}$$

$$\sum_{p=2k+2}^{n} \frac{(-1)^p (p+n-2)!}{(n-p)!(p-1)!(p-2k-2)!} = \frac{(-1)^n (2k+n)!}{(2k+1)!(n-2k-2)!}.$$

Proof. We only indicate the evaluation of the first identity by rewriting the left-hand side as

$$f(n) = \sum_{p=2k+2}^{n} F(n,p), \quad F(n,p) = \frac{(-1)^p (p+n-1)(p+n-3)!}{(n-p)!(p-1)!(p-2k-2)!}.$$

Then by using the Zeilberger's algorithm, see [16, 17], we find the recurrence relation

$$\begin{aligned} G(n,p+1) - G(n,p) &= (n^4 - 4k^2n^2 - 2kn - 2n^3 + 2n^2 - n)F(n+1,p) \\ &\quad + (4k^2n^2 - 2kn + n^4 - n - 2k + 4kn^3 - 4k^2)F(n,p), \end{aligned}$$

where $G(n,p)$ is defined by

$$\frac{(-1)^{p+1}G(n,p)}{(n+p-3)!} = \frac{4kn^3 - 4n^3 + 5n^2 + n - 1 + 2n^4 - 4kn^2 + 2k + 2pn^3 - 2pn^2 + 4pkn^2 - 2pk}{(p-2)!(n+1-p)!(p-2k-3)!}.$$

Summing over p and then rearranging terms, we find $f(n)$ satisfies the recurrence

$$f(n+1) = -\frac{4k^2n^2 - 2kn + n^4 - n - 2k + 4kn^3 - 4k^2}{n^4 - 4k^2n^2 - 2kn - 2n^3 + 2n^2 - n} f(n), \tag{2.2}$$

which allows us to check easily for the conclusion of the first identity.

Similarly, the remaining three identities can be checked respectively by taking $G(n,p)$ as

$$\frac{(-1)^{p+1}(p+n-3)!\Delta}{(p-2k-2)!(p-2)!(n+1-p)!},$$

with $\Delta = n - 6n^3 + 2n^4 + 7n^2 + 4kn^3 - 4kn^2 - 2 + 2k + 2pn^3 - 4pn^2 + 4pkn^2 + p - 2pk$;

$$\frac{2(-1)^p(p-1)(2k-p+1)(p+n-2)!}{(n+1-p)(n-p)!(p-1)!(p-2k-1)!} \quad \text{and} \quad \frac{2(-1)^p(p-1)(2k-p+2)(p+n-2)!}{(n+1-p)(n-p)!(p-1)!(p-2k-2)!}.$$

The corresponding recurrence relations are, respectively,

$$\begin{aligned} &(4kn^3 - 4kn^2 + 4k^2n^2 + 2k - 4k^2 - 2n^3 + n^2 - 2kn + n^4)F(n,p) \\ &\quad + (n^2 - 2n^3 + 4kn^2 - 2kn + n^4 - 4k^2n^2)F(n+1,p) = G(n,p+1) - G(n,p), \\ &(2k+n)F(n,p) + (n-2k)F(n+1,p) = G(n,p+1) - G(n,p), \\ &(n+1+2k)F(n,p) + (n-2k-1)F(n+1,p) = G(n,p+1) - G(n,p). \end{aligned}$$

Proof of Theorem 1.1. We first compute the eigenvalues of $W_m(t)$ by substituting $K_{W_m}(s,t)$ of (2.1) into

$$T_{W_m}f(t) = \int_0^1 K_{W_m}(s,t)f(s)ds = \lambda f(t).$$

In order to handle the $t \wedge s = \min(t,s)$ term, we split the integration range and obtain

$$\begin{aligned} \lambda f(t) &= \int_0^t sf(s)ds + t\int_t^1 f(s)ds - t\int_0^1 f(s)ds - \int_0^1 sf(s)ds + t\int_0^1 sf(s)ds \\ &\quad + \sum_{p,q=1}^{m+1} \frac{a_{pq}}{q(q+1)} t^{q+1} \int_0^1 s^{p-1}f(s)ds + \sum_{i,j=1}^{m+1} \frac{a_{ij}}{j(j+1)} t^{i-1} \int_0^1 s^{j+1}f(s)ds \\ &\quad + \sum_{i,j=1}^{m+1} \sum_{p,q=1}^{m+1} \frac{a_{ij}a_{pq}}{(q+1)(j+1)(q+j+1)} t^{i-1} \int_0^1 s^{p-1}f(s)ds. \end{aligned} \tag{2.3}$$

By differentiating both sides of (2.3) with respect to t twice, we have

$$\lambda f''(t) = -f(t) + \sum_{p,q=1}^{m+1} a_{pq} t^{q-1} \int_0^1 s^{p-1} f(s)ds + \sum_{i=3,j=1}^{m+1} \frac{(i-1)(i-2)a_{ij}t^{i-3}}{j(j+1)} \int_0^1 s^{j+1} f(s)ds$$
$$+ \sum_{i=3,j=1}^{m+1} \sum_{p,q=1}^{m+1} \frac{(i-1)(i-2)a_{ij}a_{pq}t^{i-3}}{(q+1)(j+1)(q+j+1)} \int_0^1 s^{p-1} f(s)ds. \tag{2.4}$$

We can reorder the nonhomogeneous item of the differential equation (2.4) to obtain

$$\lambda f''(t) + f(t) - \sum_{k=1}^{m+1} b_k t^{k-1} = 0, \tag{2.5}$$

where for $k = 1, 2, \dots, m-1$,

$$b_k = \sum_{p=1}^{m+1} a_{pk} \int_0^1 s^{p-1} f(s)ds + \sum_{j=1}^{m+1} \frac{k(k+1)a_{k+2,j}}{j(j+1)} \int_0^1 s^{j+1} f(s)ds$$
$$+ \sum_{j=1}^{m+1} \sum_{p,q=1}^{m+1} \frac{k(k+1)a_{k+2,j}a_{pq}}{(q+1)(j+1)(q+j+1)} \int_0^1 s^{p-1} f(s)ds,$$
$$b_m = \sum_{p=1}^{m+1} a_{p,m} \int_0^1 s^{p-1} f(s)ds, \tag{2.6}$$
$$b_{m+1} = \sum_{p=1}^{m+1} a_{p,m+1} \int_0^1 s^{p-1} f(s)ds. \tag{2.7}$$

The general solution to the inhomogeneous second order differential equation (2.5) is

$$f(t) = C_2 \sin(\lambda^{-1/2}t) + C_1 \cos(\lambda^{-1/2}t) + b_1 + b_2 t$$
$$+ \sum_{k=1}^{[m/2]} \sum_{i=1}^{k+1} (-1)^{k+i-1} \frac{(2k)!}{(2i-2)!} \lambda^{k-i+1} b_{2k+1} t^{2i-2}$$
$$+ \sum_{k=1}^{[(m-1)/2]} \sum_{i=1}^{k+1} (-1)^{k+i-1} \frac{(2k+1)!}{(2i-1)!} \lambda^{k-i+1} b_{2k+2} t^{2i-1}, \tag{2.8}$$

where C_1 and C_2 are constants. Substitute (2.8) into the equations (2.6) and (2.7), we obtain the following equations:

$$C_2 \sum_{p=1}^{m+1} \int_0^1 a_{p,m} t^{p-1} \sin(\lambda^{-1/2}t)dt + C_1 \sum_{p=1}^{m+1} \int_0^1 a_{p,m} t^{p-1} \cos(\lambda^{-1/2}t)dt = 0 \tag{2.9}$$

and

$$C_2 \sum_{p=1}^{m+1} \int_0^1 a_{p,m+1} t^{p-1} \sin(\lambda^{-1/2}t)dt + C_1 \sum_{p=1}^{m+1} \int_0^1 a_{p,m+1} t^{p-1} \cos(\lambda^{-1/2}t)dt = 0. \tag{2.10}$$

In order to have constants C_1 and C_2 such that $C_1^2 + C_2^2 \neq 0$, we need the cross terms from (2.9) and (2.10) to be the same. So we have the relation below for the eigenvalue λ,

$$\sum_{p=1}^{m+1} \int_0^1 a_{p,m} t^{p-1} \sin(\lambda^{-1/2}t)dt \cdot \sum_{p=1}^{m+1} \int_0^1 a_{p,m+1} t^{p-1} \cos(\lambda^{-1/2}t)dt$$
$$= \sum_{p=1}^{m+1} \int_0^1 a_{p,m} t^{p-1} \cos(\lambda^{-1/2}t)dt \cdot \sum_{p=1}^{m+1} \int_0^1 a_{p,m+1} t^{p-1} \sin(\lambda^{-1/2}t)dt.$$

After carrying out the four integrations above and simplifying by using the four identities in Lemma 2.2, we obtain

$$j_{m-1}(2^{-1}\lambda^{-1/2})j_m(2^{-1}\lambda^{-1/2}) = 0, \tag{2.11}$$

where $j_m(x)$ is the (unmodified) spherical Bessel function given by the formula (see [14])

$$\begin{aligned} j_m(x) = \sin(x - m\pi/2) &\sum_{k=0}^{[m/2]} (-1)^k \frac{(m+2k)!}{2^{2k}(2k)!(m-2k)!x^{2k+1}} \\ + \cos(x - m\pi/2) &\sum_{k=0}^{[(m-1)/2]} (-1)^k \frac{(m+2k+1)!}{2^{2k+1}(2k+1)!(m-2k-1)!x^{2k+2}}. \end{aligned} \tag{2.12}$$

Thus the solutions of (2.11) are

$$\lambda_{2k-1} = (2z_{m-1,k})^{-2}, \quad \lambda_{2k} = (2z_{m,k})^{-2}, \quad k = 1, 2, \ldots, \tag{2.13}$$

where $z_{m,k}$, $k = 1, 2, \ldots$ are as given in the Theorem 1.1. Finally, the standard KL expansion theory allows us to complete the proof. □

3 Stochastic Fubini approach to Corollary 1.2

A large class of identities in law between two Brownian quadratic functionals are the consequences of the following Fubini type identity between double Wiener integrals (see [7]): Let $\phi : [0,1]^2 \to \mathbb{R}$ be a deterministic function such that

$$\int_0^1 \int_0^1 \phi^2(u,s) ds du < \infty.$$

Then the identity in law holds

$$\int_0^1 \left(\int_0^1 \phi(u,s) dW(s) \right)^2 du \overset{\text{law}}{=} \int_0^1 \left(\int_0^1 \phi(s,u) dW(s) \right)^2 du. \tag{3.1}$$

It is not hard to check $\int_0^1 \phi_m(u,s)dW(s) = \tilde{B}_m(u)$ from (1.11) by integration by parts, where

$$\phi_m(s,u) = 1_{\{u \leqslant s\}} - \sum_{n=0}^{m} (2n+1) g_n(s) f_n(u), \tag{3.2}$$

where $g_n(t)$ is given in (1.13) and $f_n(t) = g_n'(t)$. So we only need to check

$$\int_0^1 \phi_m(s,u) dW(s) = -W_m(u). \tag{3.3}$$

This seems very complicated for the function $g_n(t)$ given in (1.13). However, there is an alternative and simpler formula

$$g_n(t) = \sum_{i=0}^{n} (-1)^{n+i} (i+1)^{-1} \binom{n+i}{i} \binom{n}{i} t^{i+1} \tag{3.4}$$

from [9], which allows us to prove Corollary 1.2 via stochastic Fubini approach. Once again, just like the KL expansion arguments, we need some combinatorial identities stated in the new two lemmas.

Lemma 3.1. *For the function $g_n(t)$ in (3.4), we have $g_n(1) = 0$, i.e.,*

$$\sum_{i=0}^{n} (-1)^i (i+1)^{-1} \binom{n+i}{i} \binom{n}{i} = 0, \quad n = 1, 2, \ldots$$

Proof. Similar to the proof given in Lemma 2.2, we set

$$F(n,i)=\frac{(-1)^i(n+i)!}{i!(i+1)!(n-i)!}, \tag{3.5}$$

and find by the Wilf-Zeilberger method [15] that $F(n,i)=G(n,i)-G(n,i+1)$, where

$$G(n,i)=\frac{(-1)^i(n+i)!}{n(n+1)(i-1)!i!(n-i)!}. \tag{3.6}$$

Thus the result follows. □

Lemma 3.2. *For $f_n(t)=g_n'(t)$ and $g_n(t)$ in (3.4),*

$$\sum_{n=0}^{m}(2n+1)f_n(u)\int_0^1 f_n(s)W(s)ds=\sum_{i,j=1}^{m+1}a_{ij}u^{i-1}\int_0^1 s^{j-1}W(s)ds.$$

Proof. Substitute the expression for $f_n(t)$, we only need to check the following combinatorial identity:

$$\begin{aligned}\sum_{n=1}^{m+1}(2n-1)\sum_{i,j=1}^{n}(-1)^{i+j}\binom{n+i-2}{i-1}\binom{n-1}{i-1}\binom{n+j-2}{j-1}\binom{n-1}{j-1}u^{i-1}s^{j-1}\\ =\sum_{i,j=1}^{m+1}(-1)^{i+j}(i+j-1)\binom{m+i}{m+1-j}\binom{m+j}{m+1-i}\binom{i+j-2}{i-1}^2u^{i-1}s^{j-1}.\end{aligned} \tag{3.7}$$

By using the elementary combinational identity

$$\binom{n+i-2}{i-1}\binom{n-1}{i-1}=\binom{n+i-2}{2i-2}\binom{2i-2}{i-1}, \tag{3.8}$$

we obtain

$$\begin{aligned}\sum_{n=1}^{m+1}(2n-1)\sum_{i,j=1}^{n}(-1)^{i+j}\binom{n+i-2}{i-1}\binom{n-1}{i-1}\binom{n+j-2}{j-1}\binom{n-1}{j-1}u^{i-1}s^{j-1}\\ =\sum_{n=1}^{m+1}(2n-1)\sum_{i,j=1}^{n}(-1)^{i+j}\binom{n+i-2}{2i-2}\binom{2i-2}{i-1}\binom{n+j-2}{2j-2}\binom{2j-2}{j-1}u^{i-1}s^{j-1}.\end{aligned} \tag{3.9}$$

Changing the order of the summations in i,j and n, and making use of [8, (23)], (3.9) is equal to

$$\begin{aligned}&\sum_{i,j=1}^{m+1}(-1)^{i+j}\binom{2i-2}{i-1}\binom{2j-2}{j-1}u^{i-1}s^{j-1}\sum_{n=\max\{i,j\}}^{m+1}(2n-1)\binom{n+i-2}{2i-2}\binom{n+j-2}{2j-2}\\ &=\sum_{i,j=1}^{m+1}(-1)^{i+j}\binom{2i-2}{i-1}\binom{2j-2}{j-1}u^{i-1}s^{j-1}\frac{1}{j+i-1}\binom{m+i-1}{2i-2}\binom{m+j-1}{2j-2}(m+i)(m+j)\\ &=\sum_{i,j=1}^{m+1}\frac{(-1)^{i+j}(m+i)!(m+j)!}{(j+i-1)[(i-1)!(j-1)!]^2(m+1-i)!(m+1-j)!}u^{i-1}s^{j-1}\\ &=\sum_{i,j=1}^{m+1}(-1)^{i+j}(i+j-1)\binom{m+i}{m+1-j}\binom{m+j}{m+1-i}\binom{i+j-2}{i-1}^2u^{i-1}s^{j-1},\end{aligned}$$

which gives (3.7). □

Now we can give an alternative argument by using stochastic Fubini approach, see (3.1).

Proof of Corollary 1.2. The distributional identity (1.14) is obtained by taking $\phi_m(s,u)$ as given in (3.2). So as mentioned at the beginning of this section, we only need to prove (3.3).

By using the facts that $g_n(1)=0$ for $n\geqslant 1$ and $g_0(1)f_0(u)=1$, we have

$$\begin{aligned}\int_0^1 \phi_m(s,u)dW(s) &= \int_0^1 \left[1_{\{u\leqslant s\}} - \sum_{n=0}^m (2n+1)g_n(s)f_n(u)\right]dW(s)\\ &= W(1)-W(u)-\sum_{n=0}^m (2n+1)\left[g_n(1)W(1)-\int_0^1 W(s)f_n(s)ds\right]f_n(u)\\ &= -W(u)+\sum_{n=0}^m (2n+1)f_n(u)\int_0^1 f_n(s)W(s)ds\\ &= -W_m(t),\end{aligned}$$

where we used Lemmas 3.1 and 3.2 in the last equality. The proof is finished. □

It is worthwhile to mention that the distributional identity [1, (1.10)] follows from (1.14). To be more precise, we have the distribution identity

$$\int_0^1 W_1(u)^2du \overset{\text{law}}{=} \int_0^1 \tilde{B}_1^2(u)du, \tag{3.10}$$

where the detrended Brownian motion is

$$W_1(u)=W(u)+(6u-4)\int_0^1 W(s)ds+(6-12u)\int_0^1 sW(s)ds \tag{3.11}$$

and the process

$$\begin{aligned}\tilde{B}_1(u) &= W(u)-uW(1)+3u(1-u)\left(W(1)-2\int_0^1 W(s)ds\right)\\ &= B(u)-6u(1-u)\int_0^1 B(v)dv, \quad 0\leqslant t\leqslant 1\end{aligned} \tag{3.12}$$

is the second level (order) Brownian bridge.

4 Applications

In this section, we describe some applications of the Karhunen-Loeve expansions given in Theorem 1.1. First, the characteristic function for $\int_0^1 W_m^2(t)dt$ is known from that of $\int_0^1 B_m^2 dt$ in [13].

Proposition 4.1. *The characteristic function of $\int_0^1 W_m^2(t)dt$ is*

$$\mathbb{E}\exp\left\{\left(\mathrm{i}\theta\int_0^1 W_m^2(t)dt\right)\right\} = \left(\frac{4\Gamma(m+1/2)\Gamma(m+3/2)}{\pi((\mathrm{i}\theta/2)^{1/2}/2)^{2m-1}} j_{m-1}(\mathrm{i}\theta/2)^{1/2} j_m(\mathrm{i}\theta/2)^{1/2}\right)^{-1/2}, \tag{4.1}$$

where $j_m(\cdot)$ is the m-th order spherical Bessel function of the first kind.

Next, we give the large deviation probability for L_2-norm of the m-th order detrended Brownian motion.

Proposition 4.2. *Let $W_m(t)$ be an m-th order detrended Brownian motion, then as $x\to\infty$,*

$$\mathbb{P}\left(\int_0^1 W_m(t)^2dt > x\right) = (c+o(1))x^{-1/2}\exp(-2z_{m-1/2,1}^2 x), \tag{4.2}$$

where $c=\pi^{-3/2}2^{3m+1/2}z_{m-1,1}^{m-2}\{\Gamma(m+1/2)\Gamma(m+3/2)(j_{m-1}(z_{m-1,1})j_{m+1}(z_{m-1,1})+j_m^2(z_{m-1,1}))\}^{-1/2}$.

This can be proved by following the arguments in [1, Proposition 3.2].

Finally, we briefly describe the small deviation probability for L_2-norm of the m-th order detrended Brownian motion.

Proposition 4.3. *There exists some constant $c_m > 0$ such that as $\varepsilon \to 0$,*

$$\mathbb{P}\bigg(\int_0^1 W_m(t)^2 dt \leqslant \varepsilon \bigg) = (c_m + o(1))\varepsilon^{1/2-m} \exp\bigg(-\frac{1}{8\varepsilon} \bigg).$$

This can be proved by following the arguments in [1, Proposition 3.3]. The additional facts needed are $z_{m-1,k} = (k + m/2 - 3/4)\pi + O(k^{-1})$, $z_{m,k} = (k + m/2 - 1/4)\pi + O(k^{-1})$ as $k \to \infty$ (see [2]) and for $d > -1$, $\mathbb{P}(\sum_{k\geqslant 1}(k+d)^{-2}\xi_k^2 \leqslant \varepsilon\pi^2) = (1 + o(1))c_m\varepsilon^{-d} \exp(-\frac{1}{8\varepsilon})$ given in [12]. The constant c_m can be determined by using the comparison theorem in [11].

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