SCIENCE CHINA
Mathematics

• ARTICLES •

May 2015 Vol. 58 No. 5: 1047–1062
doi: 10.1007/s11425-014-4869-0

Strichartz estimates for parabolic equations with higher order differential operators

DING Yong[1,2] & SUN XiaoChun[2,3,*]

[1] *Laboratory of Mathematics and Complex Systems (BNU), Ministry of Education, Beijing* 100875*, China;*
[2] *School of Mathematical Sciences, Beijing Normal University, Beijing* 100875*, China;*
[3] *College of Mathematics and Statistics, Northwest Normal University, Lanzhou* 730070*, China*
Email: dingy@bnu.edu.cn, sunxc@mail.bnu.edu.cn

Received June 20, 2013; accepted February 21, 2014; published online July 29, 2014

Abstract The present paper first obtains Strichartz estimates for parabolic equations with nonnegative elliptic operators of order $2m$ by using both the abstract Strichartz estimates of Keel-Tao and the Hardy-Littlewood-Sobolev inequality. Some conclusions can be viewed as the improvements of the previously known ones. Furthermore, an endpoint homogeneous Strichartz estimates on $\mathrm{BMO}_x(\mathbb{R}^n)$ and a parabolic homogeneous Strichartz estimate are proved. Meanwhile, the Strichartz estimates to the Sobolev spaces and Besov spaces are generalized. Secondly, the local well-posedness and small global well-posedness of the Cauchy problem for the semilinear parabolic equations with elliptic operators of order $2m$, which has a potential $V(t,x)$ satisfying appropriate integrable conditions, are established. Finally, the local and global existence and uniqueness of regular solutions in spatial variables for the higher order elliptic Navier-Stokes system with initial data in $L^r(\mathbb{R}^n)$ is proved.

Keywords Strichartz estimates, elliptic Navier-Stokes equations, higher order elliptic operator, potential

MSC(2010) 35K25, 35K30, 35Q30

Citation: Ding Y, Sun X C. Strichartz estimates for parabolic equations with higher order differential operators. Sci China Math, 2015, 58: 1047–1062, doi: 10.1007/s11425-014-4869-0

1 Introduction

We are concerned in this paper with Strichartz estimates for the inhomogeneous initial problem associated with the parabolic equations with the constant coefficients nonnegative elliptic operators $\mathcal{P}(\mathcal{D})$ of order $2m$,

$$\begin{cases} \partial_t u(t,x) + \mathcal{P}(\mathcal{D})u(t,x) = F(t,x), & (t,x) \in (0,\infty)\times\mathbb{R}^n, \\ u(0,x) = f(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where the unknown function $u(t,x)$ is a scalar valued or vector valued function of the time variable t and the space variable x. $F(t,x)$ and $f(x)$ are given functions. The operator $\mathcal{P}$ is defined by

$$\mathcal{P}(\mathcal{D}) := \sum_{|\alpha|\leqslant 2m} a_\alpha \mathcal{D}^\alpha,$$

where m is a positive integer, $\mathcal{D} = (\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_n)$, $\mathcal{D}_j = \frac{1}{\mathrm{i}}\frac{\partial}{\partial x_j}$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}_+^n$ and $|\alpha| = \sum_{j=1}^n \alpha_j$, $n \in \mathbb{N}$. We always regard its symbol $P(\xi)$ as $2m$ order constant real coefficients elliptic

*Corresponding author

polynomial in $\mathbb{R}^n$ with $P(\xi) > 0$ for any $\xi \neq 0$. The system (1.1) is a typical parabolic equation and it has several properties resembling to the heat equations.

We now give a brief outline of this paper. In Section 2, we first recall the definitions of some function spaces as we go along. By the Fourier transform and Duhamel's principle, the solution of (1.1) can be written as

$$u(t,x) = \mathrm{e}^{-t\mathcal{P}(\mathcal{D})}f(x) + \int_0^t \mathrm{e}^{-(t-s)\mathcal{P}(\mathcal{D})}F(s,x)ds,$$

where $\mathrm{e}^{-t\mathcal{P}(\mathcal{D})}f(x) = \mathcal{F}^{-1}(\mathrm{e}^{-tP(\xi)}\mathcal{F}f(\xi))(x) = K_t(x) * f(x)$. Here $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier and inverse Fourier transforms with respect to space variable, respectively, defined by

$$\mathcal{F}(f) = \hat{f}(\xi) = \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}x\cdot\xi}f(x)dx, \quad \mathcal{F}^{-1}(g) = \check{g}(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\cdot\xi}g(\xi)d\xi$$

for any $f, g \in \mathcal{S}$ and $\mathcal{S}$ is the Schwartz function spaces. $*$ stands for the convolution operation on the space variable.

Because the multiplier corresponds to solution operator of (1.1), $\mathrm{e}^{-tP(\xi)} \in \mathcal{M}_p^p, t > 0, 1 \leqslant p < \infty$, the operator $\mathcal{P}$ generates an analytic semigroup $\mathrm{e}^{-t\mathcal{P}(\mathcal{D})}$ on L^p $(1 \leqslant p < \infty)$. It also holds when we replace L^∞ with C_b, where C_b is the space of bounded continuous functions with the sup norm. Next, we recall the self-adjoint and some of commutative properties of the operator $\mathrm{e}^{-t\mathcal{P}(\mathcal{D})}$ with Riesz potential operator $(-\Delta)^\beta$ and Bessel potential operator $(I-\Delta)^\beta$. We also state the commutative property of the operator $\mathrm{e}^{-t\mathcal{P}(\mathcal{D})}$ with the Littlewood-Paley (or dyadic) decomposition operator Δ_j. At the end of this section, we quote the $L^r \to L^p$ estimate and previous Strichartz estimates associated with the operator $\mathrm{e}^{-t\mathcal{P}(\mathcal{D})}$.

In Section 3, we get Strichartz estimates for parabolic equations with elliptic operators of order $2m$ by using both the abstract Strichartz estimates of Keel-Tao and the Hardy-Littlewood-Sobolev inequality. Particularly, it extends Lemma 3.2 in [5] to the cases: $\beta \in \mathbb{N}$, $(q,p,r) = (2, \frac{2n}{n-2\beta}, 2)$ when $n > 2\beta$ and $(q,p,r) = (\frac{4\beta}{n}, \infty, 2)$ when $n < 2\beta$. Furthermore, we prove an endpoint homogeneous Strichartz estimates on $\mathrm{BMO}_x(\mathbb{R}^n)$ and a parabolic homogeneous Strichartz estimate for parabolic equations with elliptic operators of order $2m$ where the special case $n = 2$ was proved by Tao in [8]. Meanwhile, we generalize the Strichartz estimate to the Sobolev spaces and Besov spaces.

Section 4 is devoted to the first application of Strichartz estimates. We study the local well-posedness and small global well-posedness of the Cauchy problem for the following semilinear parabolic equations with elliptic operators of order $2m$:

$$\begin{cases} \partial_t u(t,x) + \mathcal{P}(\mathcal{D})u(t,x) + V(t,x)u(t,x) = F(t,x), & (t,x) \in \mathbb{R}_+^{1+n}, \\ u(0,x) = f(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.2}$$

where $V(t,x)$ is a time dependent potential. We prove the local well-posedness and small global well-posedness of the Cauchy problem for system (1.2). The corresponding operator semigroup $S_{\mathcal{P}}(t)$ is denoted by $S_{\mathcal{P}}(t) \triangleq \mathrm{e}^{-t\mathcal{P}(\mathcal{D})}$, whilst the proof is similar to the method of [11] by using the Banach contraction mapping principle and assuming an appropriate integrability condition in space and time on $V(t,x)$. A similar idea was used by D'Ancona et al. in [1] to get analogous estimates for the Schrödinger equations.

In Section 5, our focus is a more specific application for the higher order elliptic Navier-Stokes system on the half-space $\mathbb{R}_+^{1+n}$, $n \geqslant 2$:

$$\begin{cases} \partial_t u + \mathcal{P}(\mathcal{D})u + (u\cdot\nabla)u + \nabla p = h, & (t,x) \in \mathbb{R}_+^{1+n}, \\ \nabla \cdot u = 0, & (t,x) \in \mathbb{R}_+^{1+n}, \\ u(0,x) = g(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.3}$$

where the initial data $g(x) \in L^r(\mathbb{R}^n)$. In particular, (1.3) becomes the Navier-Stokes equations in the case $\mathcal{P}(D) = -\Delta$. We establish the global existence and uniqueness of regular solutions in spatial variables for the higher order elliptic Navier-Stokes system (1.3).

2 Lemmas

Let (X, dx) be a measure space. We write the Lebesgue norm of a function $f : X \to \mathbb{R}$ by

$$\|f\|_p \equiv \|f\|_{L^p(X)} \equiv \left(\int_X |f(x)|^p dx\right)^{1/p} < \infty.$$

Let $\psi : \mathbb{R}^n \to [0,1]$ be a smooth radial cut-off function,

$$\psi(\xi) = \begin{cases} 1, & |\xi| \leqslant 1, \\ \text{smooth}, & 1 < |\xi| < 2, \\ 0, & |\xi| \geqslant 2. \end{cases}$$

Denote $\varphi(\xi) = \psi(\xi) - \psi(2\xi)$, and we introduce the function sequence $\{\varphi_j\}_{j\in\mathbb{Z}}$, where $\varphi_j(\xi) = \varphi(2^{-j}\xi),\ j \in \mathbb{Z}$. Since $\operatorname{supp}(\varphi) \subset \{\xi : 2^{-1} \leqslant |\xi| \leqslant 2\}$, we easily see that $\operatorname{supp}(\varphi_j) \subset \{\xi : 2^{j-1} \leqslant |\xi| \leqslant 2^{j+1}\},\ j \in \mathbb{Z}$ and $\sum_{j\in\mathbb{Z}} \varphi_j = 1$ for all $\xi \neq 0$. Define $\Delta_j = \mathcal{F}^{-1}\varphi_j\mathcal{F}, j \in \mathbb{Z}$. $\{\Delta_j\}_{j\in\mathbb{Z}}$ is called Littlewood-Paley (dyadic) decomposition operator.

Let $-\infty < s < \infty, 1 \leqslant p, q \leqslant \infty$. We denote by $\dot{\mathscr{S}}'$ the dual space of

$$\dot{\mathscr{S}} = \{f \in \mathscr{S}(\mathbb{R}^n) : (D^\alpha \hat{f})(0) = 0,\ \forall\, \alpha \in (\mathbb{N} \cup \{0\})^n\}.$$

We denote the homogeneous Besov spaces

$$\dot{B}^s_{p,q}(\mathbb{R}^n) = \left\{f \in \dot{\mathscr{S}}'(\mathbb{R}^n) : \|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} := \left(\sum_{j=-\infty}^{\infty} 2^{jsp}\|\Delta_j f\|_p^q\right)^{1/q} < \infty\right\}.$$

We define the inhomogeneous Besov spaces (see [10])

$$B^s_{p,q}(\mathbb{R}^n) = \left\{f \in \mathscr{S}'(\mathbb{R}^n) : \|f\|_{B^s_{p,q}(\mathbb{R}^n)} := \left(\|\mathcal{F}^{-1}\psi\mathcal{F}f\|_p^q + \sum_{j=1}^{\infty} 2^{jsp}\|\Delta_j f\|_p^q\right)^{1/q} < \infty\right\}.$$

Note that we need replace the l^q-norm by l^∞-norm in the above definition if $q = \infty$.

We also denote $H^{s,p}(\mathbb{R}^n)$ and $\dot{H}^{s,p}(\mathbb{R}^n)$, $s \in \mathbb{R}, 1 \leqslant p, q \leqslant \infty$ to be the inhomogeneous and homogeneous Sobolev spaces which are the completion of all infinitely differential functions f with compact support in $\mathbb{R}^n$ with respect to the norms $\|f\|_{H^{s,p}(\mathbb{R}^n)} = \|(I-\Delta)^{s/2}f\|_{L^p(\mathbb{R}^n)}$ and $\|f\|_{\dot{H}^{s,p}(\mathbb{R}^n)} = \|(-\Delta)^{s/2}f\|_{L^p(\mathbb{R}^n)}$ respectively, where $(I-\Delta)^{s/2}f = \mathcal{F}^{-1}((1+|\xi|^2)^{s/2}\mathcal{F}f(\xi))$ and $(-\Delta)^{s/2}f = \mathcal{F}^{-1}(|\xi|^{s/2}\mathcal{F}f(\xi))$.

We have the following readily verified properties:

Lemma 2.1. *For all $t > 0$ and $\beta > 0$, we have*

(a) $\mathrm{e}^{-t\mathcal{P}(\mathcal{D})}(-\Delta)^\beta = (-\Delta)^\beta \mathrm{e}^{-t\mathcal{P}(\mathcal{D})}$, *where $(-\Delta)^\beta$ is the Riesz potential operator.*

(b) $\mathrm{e}^{-t\mathcal{P}(\mathcal{D})}(I-\Delta)^\beta = (I-\Delta)^\beta \mathrm{e}^{-t\mathcal{P}(\mathcal{D})}$, *where $(I-\Delta)^\beta$ is the Bessel potential operator.*

(c) $\mathrm{e}^{-t\mathcal{P}(\mathcal{D})}\Delta_j = \Delta_j \mathrm{e}^{-t\mathcal{P}(\mathcal{D})}$, *where Δ_j $(j \in \mathbb{Z})$ is the Littlewood-Paley (dyadic) decomposition operator.*

(d) $\langle \mathrm{e}^{-t\mathcal{P}(\mathcal{D})}f, g\rangle = \langle f, \mathrm{e}^{-t\mathcal{P}(\mathcal{D})}g\rangle$, $\forall\, f, g \in L^2(\mathbb{R}^n)$.

Proof. The proofs of (a), (b) and (c) will follow from the definitions of $\mathrm{e}^{-t\mathcal{P}(\mathcal{D})}$, $(-\Delta)^\beta$, $(I-\Delta)^\beta$ and Δ_j, $j \in \mathbb{Z}$. For (d), let $f, g \in L^2(\mathbb{R}^n)$. Using Plancherel theorem, we obtain the following result,

$$\begin{aligned}\langle \mathrm{e}^{-t\mathcal{P}(\mathcal{D})}f, g\rangle &= \int \mathcal{F}^{-1}(\mathrm{e}^{-tP(\xi)}\mathcal{F}f(\xi))(x)\overline{g(x)}dx \\ &= \int \mathrm{e}^{-tP(\xi)}\mathcal{F}f(\xi)\overline{\mathcal{F}g(\xi)}d\xi = \int \mathcal{F}f(\xi)\overline{\mathrm{e}^{-tP(\xi)}\mathcal{F}g(\xi)}d\xi.\end{aligned}$$

Using Plancherel theorem again, we obtain

$$\langle \mathrm{e}^{-t\mathcal{P}(\mathcal{D})}f, g\rangle = \int f(x)\overline{\mathcal{F}^{-1}(\mathrm{e}^{-tP(\xi)}\mathcal{F}g(\xi))(x)}dx = \langle f, \mathrm{e}^{-t\mathcal{P}(\mathcal{D})}g\rangle.$$

This finishes the proof of Lemma 2.1. □

Lemma 2.2 (See [4])**.** *Assume $P(\xi)$ is a nonnegative elliptic polynomial of order $2m$. Then there exists $c > 0$, such that $P(\xi) \geqslant c(\sum_{i=1}^n |\xi_i|^2)^m$.*

Corollary 2.3. *If $P(\xi)$ is a nonnegative elliptic polynomial of order $2m$ on $\mathbb{R}^n$, then there exists a constant $C > 0$, such that $C^{-1}|\xi|^{2m} \leqslant P(\xi) \leqslant C|\xi|^{2m}$ for all $\xi \neq 0$.*

Proof. According to Lemma 2.2, there exists a constant $C_1 > 0$, such that $C_1^{-1}|\xi|^{2m} \leqslant P(\xi)$. On the other hand, from the basic algebra inequality, we obtain that there exists $C_2 > 0$, such that $P(\xi) \leqslant C_2|\xi|^{2m}$. Taking $C = \max\{C_1, C_2\}$, the proof of Corollary 2.3 is finished. □

The following two lemmas are now a direct consequence of Corollary 2.3 together with two lemmas (Lemmas 3.1 and 3.2) of [5]. The proof will be omitted.

Lemma 2.4. *Let $1 \leqslant r \leqslant p \leqslant \infty$ and $f \in L^r(\mathbb{R}^n)$. Then*

$$\|\mathrm{e}^{-t\mathcal{P}(\mathcal{D})}f\|_{L_x^p(\mathbb{R}^n)} \lesssim t^{-\frac{n}{2m}(\frac{1}{r}-\frac{1}{p})}\|f\|_{L^r(\mathbb{R}^n)}, \quad \|\partial^\beta \mathrm{e}^{-t\mathcal{P}(\mathcal{D})}f\|_{L_x^p(\mathbb{R}^n)} \lesssim t^{-\frac{\beta}{2m}-\frac{n}{2m}(\frac{1}{r}-\frac{1}{p})}\|f\|_{L^r(\mathbb{R}^n)}.$$

Lemma 2.5. *Let (q, p, r) be any $\frac{n}{2m}$-admissible triplet satisfying*

$$p < \begin{cases} \dfrac{nr}{n-2m}, & n > 2m, \\ \infty, & n \leqslant 2m, \end{cases}$$

and let $\varphi \in L^r(\mathbb{R}^n)$. Then $\mathrm{e}^{-t\mathcal{P}(\mathcal{D})}\varphi \in L^q(I; L^p(\mathbb{R}^n))$ with the estimate

$$\|\mathrm{e}^{-t\mathcal{P}(\mathcal{D})}\varphi\|_{L_t^q(I, L_x^p(\mathbb{R}^n))} \lesssim \|\varphi\|_{L^r(\mathbb{R}^n)}.$$

We can obtain the following estimate from Lemma 2.4.

Lemma 2.6. *Let $m \geqslant 1$, $T > 0$, and p, q satisfy $p > \frac{n}{2m-1}, 2m-1 = \frac{2m}{q} + \frac{n}{p}$. Assume that $f \in L^r(\mathbb{R}^n)$ with $\frac{n}{2m-1} < r \leqslant p$. Then we have $\|\mathrm{e}^{-t\mathcal{P}(\mathcal{D})}f\|_{L_t^q([0,T], L_x^p(\mathbb{R}^n))} \lesssim T^{1-\frac{n}{2m}(\frac{1}{n}+\frac{1}{r})}\|f\|_{L^r(\mathbb{R}^n)}$.*

Proof. According to Lemma 2.4, we obtain

$$\|\mathrm{e}^{-t\mathcal{P}(\mathcal{D})}f\|_{L_t^q([0,T], L_x^p(\mathbb{R}^n))} \lesssim \left(\int_0^T t^{-\frac{nq}{2m}(\frac{1}{r}-\frac{1}{p})}\|f\|^q_{L_x^r(\mathbb{R}^n)}dt\right)^{1/q} \lesssim T^{1-\frac{n}{2m}(\frac{1}{n}+\frac{1}{r})}\|f\|_{L^r(\mathbb{R}^n)}.$$

This completes the proof of Lemma 2.6. □

3 Strichartz estimates for solution of system (1.1)

It is also known that the operator $\mathcal{P}$ generates an analytic semigroup on L^r $(1 < r < \infty)$. We can verify directly that if $u(t, x)$ is a solution of free equation associated with (1.1), then $u_\lambda(t) = u(\lambda^{2m}t, \lambda x)$ is also a solution of free equation associated with (1.1) with initial value $f(\lambda x)$. If $u(t, x) \in L^q(\mathbb{R}^+, L^p(\mathbb{R}^n))$, $1 < p,\ q < \infty$, then p, q should satisfy $\frac{n}{2m}$-admissible triplet condition. So we need to introduce the following definition on admissible triplets for the $2m$ order dissipative equation. For the corresponding definition for parabolic equations the reader may refer to [4–6].

Definition 3.1. The triplet (q, p, r) is called a σ-admissible triplet if $\frac{1}{q} = \sigma(\frac{1}{r} - \frac{1}{p})$, where $1 < r \leqslant p \leqslant \infty$ and $\sigma > 0$.

Lemma 2.5 gives us the homogeneous Strichartz estimates of (1.1) except endpoint cases. To obtain the endpoint estimates we need the abstract Strichartz estimates of Keel and Tao [3].

Lemma 3.2 (See [3])**.** *Let H be a Hilbert space and (X, dx) be a measure space. Suppose that $U(t): H \to L^2(X)$ obeys the energy estimate: $\|U(t)f\|_{L^2(X)} \lesssim \|f\|_H$ and the untruncated decay estimate, i.e., for some $\sigma > 0$, $\|U(t)(U(s))^*f\|_{L^\infty} \lesssim |t-s|^{-\sigma}\|f\|_{L^1}$, $\forall\, s \neq t$. Then the estimates*

$$\|U(t)f\|_{L_t^q L_x^p} \lesssim \|f\|_H,$$

$$\left\|\int (U(s))^* F(s) ds\right\|_H \lesssim \|F\|_{L_t^{q'} L_x^{p'}},$$
$$\left\|\int_{s<t} U(t)(U(s))^* F(s) ds\right\|_{L_t^q L_x^p} \lesssim \|F\|_{L_t^{q_1'} L_x^{p_1'}}$$

hold for all $\frac{n}{2m}$-admissible triplets $(q,p,2)$ and $(q_1,p_1,2)$ with $q,q_1 \geqslant 2$, and $(q,p,\frac{n}{2m})$ and $(q_1,p_1,\frac{n}{2m})$ being not $(2,\infty,1)$.

Theorem 3.3. *Let $(q,p,2)$ be $\frac{n}{2m}$-admissible. If $q \geqslant 2$ and $(q,p,\frac{n}{2m})$ is not $(2,\infty,1)$, then*

$$\|e^{-t\mathcal{P}(\mathcal{D})} f\|_{L_t^q(I;L_x^p(\mathbb{R}^n))} \lesssim \|f\|_{L^2(\mathbb{R}^n)}. \tag{3.1}$$

Proof. We only need to prove (3.1) for $I=[0,\infty)$ since the proofs for other cases are similar. Assume that $(q,p,2)$ is an $\frac{n}{2m}$-admissible triplet with $q \geqslant 2$ and $(q,p,\frac{n}{2m})$ is not $(2,\infty,1)$. It follows from Lemma 2.4 that we have the energy estimate

$$\|e^{-t\mathcal{P}(\mathcal{D})} f\|_{L_x^2(\mathbb{R}^n)} \lesssim \|f\|_{L^2(\mathbb{R}^n)}, \quad \forall\, t>0, \tag{3.2}$$

and untruncated decay estimate

$$\|e^{-(t+s)\mathcal{P}(\mathcal{D})} f\|_{L^\infty(\mathbb{R}^n)} \lesssim |t+s|^{-\frac{n}{2m}} \|f\|_{L^1(\mathbb{R}^n)} \lesssim |t-s|^{-\frac{n}{2m}} \|f\|_{L^1(\mathbb{R}^n)}, \quad \forall\, s \neq t,\ s,t \in (0,\infty). \tag{3.3}$$

By (3.2), (3.3) and Lemma 2.1, we can apply Lemma 3.2 with $U(t)=e^{-t\mathcal{P}(\mathcal{D})}$ for $t>0$, $H=L^2(\mathbb{R}^n)$ and $X=\mathbb{R}^n$ to obtain (3.1). □

Theorem 3.3 extends Lemma 3.2 in [5] to the cases: $\beta \in \mathbb{N}$, $(q,p,r)=(2,\frac{2n}{n-2\beta},2)$ when $n>2\beta$ and $(q,p,r)=(\frac{4\beta}{n},\infty,2)$ when $n<2\beta$.

Now we establish the inhomogeneous Strichartz estimate under the conditions in the following theorem which is weaker than the $\frac{n}{2m}$-admissibility of $(q,p,2)$ and $(q_1,p_1,2)$.

Theorem 3.4. *Let $1 \leqslant p_1' < p \leqslant \infty$ and $1 < q_1' < q < \infty$, $p_1' = \frac{p_1}{p_1-1}, q_1' = \frac{q_1}{q_1-1}$. If (q,p) and (q_1,p_1) satisfy*

$$\left(\frac{1}{q_1'}-\frac{1}{q}\right)+\frac{n}{2m}\left(\frac{1}{p_1'}-\frac{1}{p}\right)=1, \tag{3.4}$$

then

$$\left\|\int_0^t e^{-(t-s)\mathcal{P}(\mathcal{D})} F(s,x) ds\right\|_{L_t^q(I;L_x^p(\mathbb{R}^n))} \lesssim \|F\|_{L_t^{q_1'}(I;L_x^{p_1'}(\mathbb{R}^n))}, \tag{3.5}$$

where I is either $[0,\infty)$ or $[0,T]$ for some $0<T<\infty$.

Proof. We only need to prove (3.5) for $I=[0,\infty)$ and the proofs for other cases are similar. Assume that $(q,p,2)$ and $(q_1,p_1,2)$ satisfy $1 \leqslant p_1' < p \leqslant \infty$, $1<q_1'<q<\infty$ and $\frac{1}{q_1'}+\frac{n}{2m}(\frac{1}{p_1'}-\frac{1}{p})=1+\frac{1}{q}$. It follows from Lemma 2.4 that

$$\|e^{-(t-s)\mathcal{P}(\mathcal{D})} F(s,x)\|_{L_x^p(\mathbb{R}^n)} \lesssim |t-s|^{-\frac{n}{2m}(\frac{1}{p_1'}-\frac{1}{p})} \|F(s,x)\|_{L_x^{p_1'}(\mathbb{R}^n)}, \quad \forall\, s<t.$$

Then the Hardy-Littlewood-Sobolev inequality implies that

$$\begin{aligned}\left\|\int_0^t e^{-(t-s)\mathcal{P}(\mathcal{D})} F(s,x) ds\right\|_{L_t^q(I;L_x^p(\mathbb{R}^n))} &\lesssim \left\|\int_0^t \|e^{-(t-s)\mathcal{P}(\mathcal{D})} F(s,x)\|_{L_x^p(\mathbb{R}^n)} ds\right\|_{L_t^q(I)}\\ &\lesssim \left\|\int_0^t |t-s|^{-\frac{n}{2m}(\frac{1}{p_1'}-\frac{1}{p})} \|F(s,x)\|_{L_x^{p_1'}(\mathbb{R}^n)} ds\right\|_{L_t^q(I)}\\ &\lesssim \|F(s,x)\|_{L_t^{q_1'}(I;L_x^{p_1'}(\mathbb{R}^n))}.\end{aligned}$$

This finishes the proof of (3.5). □

Remark 3.5. Since $e^{-t\mathcal{P}(\mathcal{D})}$ commutes with $(-\Delta)^\beta$ and $(I-\Delta)^\beta$ for $\beta > 0$, if (q,p) satisfies the assumption of Theorem 3.3, then (3.1) holds with $\|\cdot\|_{L^p(\mathbb{R}^n)}$ replaced by either $\|\cdot\|_{\dot{H}^{\beta,p}(\mathbb{R}^n)}$ or $\|\cdot\|_{H^{\beta,p}(\mathbb{R}^n)}$. Similarly, if (q,p) and (q_1,p_1) satisfy the assumption of Theorem 3.4, then (3.5) holds with the same replacement.

A locally integrable function f will be said to belong to $\mathrm{BMO}(\mathbb{R}^n)$ if the semi-norm

$$\|f\|_{\mathrm{BMO}(\mathbb{R}^n)} = \left(\sup_Q \mathcal{L}(Q)^{-n}\int_Q |f(x)-f_Q|^2 dx\right)^{1/2} < \infty,$$

where Q is a cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes, $\mathcal{L}(Q)$ is the sidelength of Q and $f_Q = \mathcal{L}(Q)^{-n}\int_Q f(x)dx$ denotes the mean value of f over the cube Q.

Theorem 3.6. *Let $n = 2m$. Then*

$$\|e^{-t\mathcal{P}(\mathcal{D})} f\|_{L^2_t((0,\infty);\mathrm{BMO}_x(\mathbb{R}^n))} \lesssim \|f\|_{L^2(\mathbb{R}^n)}. \tag{3.6}$$

Proof. Define $\varphi \in C^\infty(\mathbb{R})$ with $\mathrm{supp}(\varphi) \subseteq (1/2, 2)$, $\varphi(x) = 1$ for $x \in (\frac{3}{4}, \frac{9}{8})$ and $\sum_{k\in\mathbb{Z}} \varphi(2^{-k}t) = 1$ for all $t > 0$.

Let $\varphi_k(t) = \varphi(2^{-k}t)$. Define $P_k f = \mathcal{F}^{-1}(\varphi_k(|\cdot|)\mathcal{F}f(\cdot))$ be a Littlewood-Paley decomposition with respect to φ_k (see [7]). Since $\mathrm{BMO}(\mathbb{R}^n) = \dot{F}^{0,2}_\infty(\mathbb{R}^n)$ (see [2]),

$$\|g\|_{\mathrm{BMO}(\mathbb{R}^n)} \approx \left\|\left(\sum_{K\in\mathbb{Z}} |P_k g|^2\right)^{1/2}\right\|_{L^\infty(\mathbb{R}^n)}.$$

Then we have

$$\begin{aligned}\|e^{-t\mathcal{P}(\mathcal{D})} f\|^2_{L^2_t((0,\infty);\mathrm{BMO}_x(\mathbb{R}^n))} &= \int_0^\infty \|e^{-t\mathcal{P}(\mathcal{D})} f\|^2_{\mathrm{BMO}_x(\mathbb{R}^n)} dt \\ &\lesssim \int_0^\infty \sup_x \sum_{k\in\mathbb{Z}} |e^{-t\mathcal{P}(\mathcal{D})} P_k f|^2 dt \\ &\lesssim \sum_{k\in\mathbb{Z}} \|e^{-t\mathcal{P}(\mathcal{D})} P_k f\|^2_{L^2_t L^\infty_x}.\end{aligned}$$

Take $\psi \in C^\infty(\mathbb{R})$ with $\mathrm{supp}(\psi) \subseteq (1/4, 4)$ and $\psi(x)\varphi(x) = \varphi(x)$. Define $\tilde{P}_k f = \mathcal{F}^{-1}(\psi_k(\mathcal{F}f))$. Then we have $\|e^{-t\mathcal{P}(\mathcal{D})} f\|^2_{L^2_t((0,\infty);\mathrm{BMO}_x(\mathbb{R}^n))} \lesssim \sum_k \|e^{-t\mathcal{P}(\mathcal{D})} P_k \tilde{P}_k f\|^2_{L^2_t L^\infty_x(\mathbb{R}^n)}$.

If we can show that $\|e^{-t\mathcal{P}(\mathcal{D})} P_k f\|^2_{L^2_t L^\infty_x(\mathbb{R}^n)} \lesssim \|f\|^2_{L^2(\mathbb{R}^n)}$, then we obtain

$$\|e^{-t\mathcal{P}(\mathcal{D})} f\|^2_{L^2_t((0,\infty);\mathrm{BMO}_x(\mathbb{R}^n))} \lesssim \sum_k \|e^{-t\mathcal{P}(\mathcal{D})} P_k \tilde{P}_k f\|^2_{L^2_t L^\infty_x(\mathbb{R}^n)} \lesssim \sum_k \|\tilde{P}_k f\|^2_{L^2(\mathbb{R}^n)} \lesssim \|f\|^2_{L^2(\mathbb{R}^n)}.$$

Next, we show that $\|e^{-t\mathcal{P}(\mathcal{D})} P_k f\|^2_{L^2_t L^\infty_x(\mathbb{R}^n)} \lesssim \|f\|^2_{L^2(\mathbb{R}^n)}$. Let $M_k = B(0, 2^{k+1}) \backslash B(0, 2^{k-1})$ and χ_{M_k} be its characteristic function. Since φ is supported on $(1/2, 2)$ and $n = 2m$, we have

$$\begin{aligned}\|e^{-t\mathcal{P}(\mathcal{D})} P_k f\|^2_{L^2_t L^\infty_x(\mathbb{R}^n)} &\lesssim \int_0^\infty \sup_x \left|\int_{\mathbb{R}^n} e^{-tP(\xi)} e^{i\langle \xi, x\rangle} \hat{f}(\xi)\varphi(2^{-k}|\xi|) d\xi\right|^2 dt \\ &\lesssim \int_0^\infty \int_{\mathbb{R}^n} \chi_{M_k}(\xi) d\xi \sup_x \int_{\mathbb{R}^n} |e^{-tP(\xi)} e^{i\langle\xi,x\rangle} \hat{f}(\xi)\varphi(2^{-k}|\xi|)|^2 d\xi dt \\ &\lesssim 2^{(k-1)n}(2^{2n}-1) \int_0^\infty \int_{M_k} e^{-2tP(\xi)} |\hat{f}(\xi)\varphi(2^{-k}|\xi|)|^2 d\xi dt \\ &\lesssim 2^{(k-1)n}(2^{2n}-1) \int_0^\infty \int_{M_k} e^{-2t|\xi|^n} |\hat{f}(\xi)\varphi(2^{-k}|\xi|)|^2 d\xi dt \\ &\lesssim 2^{(k-1)n}(2^{2n}-1) \int_0^\infty e^{-t2^{(k-1)n+1}} dt \|f\|^2_{L^2(\mathbb{R}^n)} \\ &\lesssim (2^{2n-1} - 1/2)\|f\|^2_{L^2(\mathbb{R}^n)} \lesssim \|f\|^2_{L^2(\mathbb{R}^n)}.\end{aligned}$$

The proof is completed. ∎

Theorem 3.7 (Parabolic Strichartz estimate). (a) *Let* $1 \leqslant r \leqslant p \leqslant \infty$ *and* $0 < T < \infty$. *If* $n < 2m$, *then*

$$\int_0^T s^{-\frac{nr}{2pm}} \|e^{-s\mathcal{P}(\mathcal{D})} f\|_{L_x^p(\mathbb{R}^n)}^r ds \lesssim T^{1-\frac{n}{2m}} \|f\|_{L^r(\mathbb{R}^n)}^r. \tag{3.7}$$

(b) *Let* $2 < p \leqslant \infty$. *If* $n = 2m$, *then*

$$\int_0^\infty s^{-2/p} \|e^{-s\mathcal{P}(\mathcal{D})} f\|_{L_x^p(\mathbb{R}^n)}^2 ds \lesssim \|f\|_{L^2(\mathbb{R}^n)}^2. \tag{3.8}$$

Proof. (a) Let $1 \leqslant r \leqslant p \leqslant \infty$ and $n < 2m$. It follows from Lemma 2.4 that

$$s^{-\frac{nr}{2pm}} \|e^{-s\mathcal{P}(\mathcal{D})} f\|_{L_x^p(\mathbb{R}^n)}^r \lesssim s^{-\frac{n}{2m}} \|f\|_{L_x^r(\mathbb{R}^n)}^r.$$

Furthermore, $n < 2m$ implies that $\int_0^T s^{-\frac{n}{2m}} ds = \frac{2m}{2m-n} T^{1-\frac{n}{2m}}$. Thus (3.7) holds.

(b) The following proof is essentially the same as the proof in Tao [8] and Zhai [11]. For the sake of completeness, it is provided here. We use the TT^* method. Thus, by duality and the self-adjointness of $e^{-t\mathcal{P}(\mathcal{D})}$ it suffices to verify

$$\left\| \int_0^\infty s^{-\frac{1}{p}} e^{-s\mathcal{P}(\mathcal{D})} F(s,x) ds \right\|_{L^2(\mathbb{R}^n)}^2 \lesssim \int_0^\infty \|F(s,x)\|_{L_x^{p'}(\mathbb{R}^n)}^2 ds \tag{3.9}$$

for all test functions F, where $p' := p/(p-1)$ is the dual exponent. The left-hand side of (3.9) can be written as $\int_0^\infty \int_0^\infty s_1^{-\frac{1}{p}} s^{-\frac{1}{p}} \langle e^{-\frac{s+s_1}{2}\mathcal{P}(\mathcal{D})} F(s,x), e^{-\frac{s+s_1}{2}\mathcal{P}(\mathcal{D})} F(s_1,x) \rangle_x ds ds_1$.

Applying Lemma 2.4 and writing $g(s) = \|F(s,x)\|_{L_x^{p'}(\mathbb{R}^n)}$, we have

$$|\langle e^{-\frac{s+s_1}{2}\mathcal{P}(\mathcal{D})} F(s,x), e^{-\frac{s+s_1}{2}\mathcal{P}(\mathcal{D})} F(s_1,x) \rangle_x| \lesssim (s+s_1)^{-2(\frac{1}{p'}-\frac{1}{2})} g(s) g(s_1).$$

Hence, it suffices to prove that

$$\int_0^\infty \int_0^\infty \frac{g(s)g(s_1) ds ds_1}{(s+s_1)^{1-2/p} s^{1/p} s_1^{1/p}} \lesssim \int_0^\infty g(s)^2 ds. \tag{3.10}$$

By symmetry we can reduce to the region where $s_1 \leqslant s$. If one decomposes into the dyadic ranges $2^{-k}s \leqslant s_1 \leqslant 2^{-k+1}s$, we can bound the left-hand side of (3.10) by

$$\begin{aligned} &\int_0^\infty \sum_{k=1}^\infty \int_{2^{-k}s}^{2^{-k+1}s} \frac{g(s)g(s_1)}{s^{1-2/p} s^{1/p} (2^{-k}s)^{1/p}} ds_1 ds \\ &\quad \lesssim \sum_{k=1}^\infty 2^{k/p} \int_0^\infty \int_{2^{-k}s \leqslant s_1 \leqslant 2^{-k+1}s} \frac{g(s)g(s_1)}{s} ds_1 ds \\ &\quad \lesssim \sum_{k=1}^\infty 2^{k(\frac{1}{p}-\frac{1}{2})} \int_0^\infty g(s)^2 ds \lesssim \int_0^\infty g(s)^2 ds \end{aligned}$$

with the second inequality using the boundedness of L^2 for Hardy-Littlewood maximum functions. In fact, we have

$$\begin{aligned} \int_0^\infty \int_{2^{-k}s}^{2^{-k+1}s} \frac{g(s)g(s_1)}{s} ds_1 ds &\lesssim 2^{-k} \int_0^\infty g(s) M(g)(2^{-k}s) ds \\ &\lesssim 2^{-k} \|g(s)\|_{L^2(\mathbb{R})} \|M(g)(2^{-k}s)\|_{L^2(\mathbb{R})} \lesssim 2^{-\frac{k}{2}} \|g(s)\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

The proof is completed. □

We can refer to (3.8) as a parabolic homogeneous Strichartz estimate. The special case $n = 2$ of (3.8) was proved by Tao in [8].

Theorem 3.8. *Let $n > 2m > 0$, $p \in [1, 2)$, $q \in (1, 2)$. If $\frac{1}{q} + \frac{n}{2m}(\frac{1}{p} - \frac{1}{2}) = \frac{3}{2}$ then*

$$\left\| \int_0^t \mathrm{e}^{-(t-s)\mathcal{P}(\mathcal{D})} F(s,x) ds \right\|_{L_t^2(I; L_x^{\frac{2n}{n-2m}}(\mathbb{R}^n))} \lesssim \|F\|_{L_t^q(I;Z)} \tag{3.11}$$

holds with $Z = \dot{H}_x^{m,p}(\mathbb{R}^n)$ or $H_x^{m,p}(\mathbb{R}^n)$.

Proof. We only need to prove (3.11) for $Z = \dot{H}_x^{m,p}(\mathbb{R}^n)$. Assuming that p, q and m satisfy the conditions given by the theorem, we have $\frac{n}{2m}(\frac{1}{p} - \frac{1}{2}) \in (1/2, 1)$. According to the imbedding of $\dot{H}^{m,2}(\mathbb{R}^n)$ into $L^{\frac{2n}{n-2m}}(\mathbb{R}^n)$, Lemmas 2.1 and 2.4 and the Hardy-Littelewood-Sobolev inequality, we obtain

$$\begin{aligned}
\left\| \int_0^t \mathrm{e}^{-(t-s)\mathcal{P}(\mathcal{D})} F(s,x) ds \right\|_{L_t^2(I; L_x^{\frac{2n}{n-2m}}(\mathbb{R}^n))} &\lesssim \left\| \int_0^t \mathrm{e}^{-(t-s)\mathcal{P}(\mathcal{D})} F(s,x) ds \right\|_{L_t^2(I; \dot{H}_x^{m,2}(\mathbb{R}^n))} \\
&\lesssim \left\| (-\Delta)^{m/2} \int_0^t \mathrm{e}^{-(t-s)\mathcal{P}(\mathcal{D})} F(s,x) ds \right\|_{L_t^2(I; L_x^2(\mathbb{R}^n))} \\
&\lesssim \left\| \int_0^t \mathrm{e}^{-(t-s)\mathcal{P}(\mathcal{D})} ((-\Delta)^{m/2} F(s,x)) ds \right\|_{L_t^2(I; L_x^2(\mathbb{R}^n))} \\
&\lesssim \left\| \int_0^t \|\mathrm{e}^{-(t-s)\mathcal{P}(\mathcal{D})} ((-\Delta)^{m/2} F(s,x))\|_{L_x^2(\mathbb{R}^n)} ds \right\|_{L_t^2(I)} \\
&\lesssim \left\| \int_0^t |t-s|^{-\frac{n}{2m}(\frac{1}{p}-\frac{1}{2})} \|(-\Delta)^{m/2} F(s,x)\|_{L_x^p(\mathbb{R}^n)} ds \right\|_{L_t^2(I)} \\
&\lesssim \|(-\Delta)^{m/2} F(t,x)\|_{L_t^q(I; L_x^p(\mathbb{R}^n))} \lesssim \|F\|_{L_t^q(I; \dot{H}_x^{m,p}(\mathbb{R}^n))}.
\end{aligned}$$

This finishes the proof of (3.11). □

Using the Littlewood-Paley decomposition, we establish the following estimates in Besov spaces.

Theorem 3.9. (a) *Let $(q, p, 2)$ be $\frac{n}{2m}$-admissible. If $q \geqslant 2$ and $(q, p, \frac{n}{2m})$ is not $(2, \infty, 1)$, then*

$$\|\mathrm{e}^{-t\mathcal{P}(\mathcal{D})} f\|_{L_t^q(I;X_1)} \lesssim \|f\|_{X_2} \tag{3.12}$$

holds with $(X_1, X_2) = (B_{p,2}^s(\mathbb{R}^n), B_{2,2}^s(\mathbb{R}^n))$ or $(\dot{B}_{p,2}^s(\mathbb{R}^n), \dot{B}_{2,2}^s(\mathbb{R}^n))$.

(b) *Let $1 \leqslant p_1' < p \leqslant \infty$ and $1 < q_1' \leqslant 2 < q < \infty$. If (q, p) and (q_1, p_1) satisfy (3.4) and $q_1 \geqslant 2$, then*

$$\left\| \int_0^t \mathrm{e}^{-(t-s)\mathcal{P}(\mathcal{D})} F(s,x) ds \right\|_{L_t^q(I;Y_1)} \lesssim \|F\|_{L_t^{q_1'}(I,Y_2)} \tag{3.13}$$

holds with $(Y_1, Y_2) = (B_{p,2}^s(\mathbb{R}^n), B_{p_1',2}^s(\mathbb{R}^n))$ or $(\dot{B}_{p,2}^s(\mathbb{R}^n), \dot{B}_{p_1',2}^s(\mathbb{R}^n))$.

Proof. We only check (3.12) provided that $(X_1, X_2) = (\dot{B}_{p,2}^s(\mathbb{R}^n), \dot{B}_{2,2}^s(\mathbb{R}^n))$ and (3.13) provided that $(Y_1, Y_2) = (\dot{B}_{p,2}^s(\mathbb{R}^n), \dot{B}_{p_1',2}^s(\mathbb{R}^n))$ because the proofs of other cases are similar.

If $q = \infty$, then $p = 2$, we obtain

$$\|\mathrm{e}^{-t\mathcal{P}(\mathcal{D})} f\|_{L_t^\infty(I, B_{2,2}^s)} = \sup_{t \in I} \|\mathrm{e}^{-t\mathcal{P}(\mathcal{D})} (-\Delta)^{\frac{s}{2}} f\|_{L^2} = \|\mathrm{e}^{-t\mathcal{P}(\mathcal{D})} (-\Delta)^{\frac{s}{2}} f\|_{L_t^\infty L_x^2} \leqslant \|(-\Delta)^{\frac{s}{2}} f\|_{L^2} = \|f\|_{B_{2,2}^s}.$$

Next, we assume that $q < \infty$. Define $u(t) = \mathrm{e}^{-t\mathcal{P}(\mathcal{D})} f$. Then using Lemma 2.1(c), we have $\Delta_j(u) = \mathrm{e}^{-t\mathcal{P}(\mathcal{D})} \Delta_j f$. Hence

$$\|u\|_{L_t^q(I; \dot{B}_{p,2}^s(\mathbb{R}^n))} = \left(\int_I \left(\sum_j 2^{2sj} \|\mathrm{e}^{-t\mathcal{P}(\mathcal{D})} (\Delta_j f)\|_{L^p(\mathbb{R}^n)}^2 \right)^{q/2} dt \right)^{2/q}.$$

Letting $A_j(t) = 2^{2sj} \|\mathrm{e}^{-t\mathcal{P}(\mathcal{D})} (\Delta_j f)\|_{L^p(\mathbb{R}^n)}^2$ and $k = q/2 \geqslant 1$, we have

$$\|u\|_{L_t^q(I; \dot{B}_{p,2}^s(\mathbb{R}^n))} = \left(\int_I \left(\sum_j A_j(t) \right)^k dt \right)^{1/k} = \left\| \sum_j A_j(\cdot) \right\|_{L^k(I)} = \sum_j 2^{2sj} \|\mathrm{e}^{-t\mathcal{P}(\mathcal{D})} (\Delta_j f)\|_{L^q(I; L^p(\mathbb{R}^n))}^2.$$

Using Theorem 3.3, we deduce $\|u\|_{L^q_t(I;\dot{B}^s_{p,2}(\mathbb{R}^n))} \lesssim (\sum_j 2^{2sj}\|\Delta_j f\|^2_{L^2(\mathbb{R}^n)})^{1/2} \lesssim \|f\|_{\dot{B}^s_{2,2}(\mathbb{R}^n)}$. Therefore, (3.12) holds.

We assume that $p<\infty$ since the case $p=\infty$ is similar.

Let $u(t)=\int_0^t \mathrm{e}^{-(t-s)\mathcal{P}(\mathcal{D})}F(s,x)ds$. Then

$$\begin{aligned} 2^{sj}\Delta_j(u) &= 2^{sj}\mathcal{F}^{-1}\int_0^t \psi_j\mathcal{F}(\mathrm{e}^{-(t-s')\mathcal{P}(\mathcal{D})}F(s',x))ds' \\ &= 2^{sj}\mathcal{F}^{-1}\int_0^t \mathrm{e}^{-(t-s')P(\xi)}\psi_j\mathcal{F}(F(s',\xi))ds' \\ &= 2^{sj}\int_0^t \mathcal{F}^{-1}(\mathrm{e}^{-(t-s')P(\xi)}\psi_j\mathcal{F}(F(s',\xi)))ds' \\ &= \int_0^t \mathrm{e}^{-(t-s')\mathcal{P}(\mathcal{D})}(2^{sj}\mathcal{F}^{-1}(\psi_j\mathcal{F}(F(s',\xi))))ds' \\ &= \int_0^t \mathrm{e}^{-(t-s')\mathcal{P}(\mathcal{D})}v_j(s')ds', \end{aligned}$$

where $v_j(t)=2^{sj}\mathcal{F}^{-1}(\psi_j\mathcal{F}(F(t,\xi)))$. Thus

$$\|u\|^2_{L^q_t(I;\dot{B}^s_{p,2}(\mathbb{R}^n))} \lesssim \left(\int_I\left(\sum_j\left\|\int_0^t \mathrm{e}^{-(t-s')\mathcal{P}(\mathcal{D})}v_j(s')ds'\right\|^2_{L^p(\mathbb{R}^n)}\right)^{q/2}dt\right)^{2/q}.$$

In a similar manner to the proof of (3.12), we have

$$\begin{aligned} \|u\|^2_{L^q_t(I;\dot{B}^s_{p,2}(\mathbb{R}^n))} &\lesssim \left(\int_I\left(\sum_j\left\|\int_0^t \mathrm{e}^{-(t-s')\mathcal{P}(\mathcal{D})}v_j(s')ds'\right\|^2_{L^p}\right)^{\frac{q}{2}}dt\right)^{\frac{2}{q}} \\ &\lesssim \sum_j\left(\int_I\left(\left\|\int_0^t \mathrm{e}^{-(t-s')\mathcal{P}(\mathcal{D})}v_j(s')ds'\right\|_{L^p}\right)^q dt\right)^{\frac{2}{q}} \\ &\lesssim \sum_j\left\|\int_0^t \mathrm{e}^{-(t-s')\mathcal{P}(\mathcal{D})}v_j(s')ds'\right\|^2_{L^q_t(I;L^p(\mathbb{R}^n))}. \end{aligned}$$

Applying Theorem 3.4, we get $\|u\|^2_{L^q_t(I;\dot{B}^s_{p,2}(\mathbb{R}^n))} \lesssim \sum_j \|v_j\|^2_{L^{q_1'}_t(I;L^{p_1'}(\mathbb{R}^n))} \lesssim \sum_j(\int_I B_j(t)dt)^k$, where $B_j(t)=\|v_j(t)\|^{q_1'}_{L^{p_1'}(\mathbb{R}^n)}$ and $k=2/q_1'\geqslant 1$. An application of Minkowski inequality yields

$$\|u\|^{2/k}_{L^q_t(I;\dot{B}^s_{p,2}(\mathbb{R}^n))} \lesssim \left\|\int_I B_j(t)dt\right\|_{l^k(\mathbb{Z})} \lesssim \int_I\|B_j(t)\|_{l^k(\mathbb{Z})}dt \lesssim \|F\|^{q_1'}_{L^{q_1'}_t(I;\dot{B}^s_{p_1',2}(\mathbb{R}^n))}.$$

Thus (3.13) holds. □

4 Well-posedness for the system (1.2)

Theorem 4.1. *Let $n\geqslant 2m$, $I=[0,T]$ or $[0,\infty)$. Suppose V is a real potential and*

$$V\in L^r_t(I;L^s_x(\mathbb{R}^n)),\quad \frac{1}{r}+\frac{n}{2ms}=1,$$

for some fixed $r\in(1,2)\cup(2,\infty)$. Let $F\in L^{q_1'}_t(I;L^{p_1'}_x(\mathbb{R}^n))$ for some $\frac{n}{2m}$-admissible triplet $(q_1,p_1,2)$ with $p_1'\in[1,2)$ and $q_1'\in(1,2)$. Then (1.2) has a unique solution $v(t,x)$ satisfying

$$\|v\|_{L^q_t(I;L^p_x(\mathbb{R}^n))} \lesssim \|f\|_{L^2(\mathbb{R}^n)}+\|F\|_{L^{q_1'}_t(I;L^{p_1'}_x(\mathbb{R}^n))}, \tag{4.1}$$

for all $\frac{n}{2m}$-admissible triplets $(q,p,2)$ with $2\leqslant q\leqslant\infty$ and $(q,p,\frac{n}{2m})\neq(2,\infty,1)$.

Proof. We shall prove this theorem for $n > 2m$. In the case $n = 2m$, we can replace in the sequel the space $L_t^2(J; L_x^{\frac{2n}{n-2m}}(\mathbb{R}^n))$ by any $L_t^q(J; L_x^p(\mathbb{R}^n))$ for 1-admissible $(q, p, 2)$ with p arbitrarily large.

We consider the following two cases.

Case 1. $r \in (2, \infty)$. Let $(q, p, 2)$ $(2 \leqslant q \leqslant \infty)$ be $\frac{n}{2m}$-admissible. Let $J = [0, \varepsilon]$, where $\varepsilon > 0$ will be determined later and $(k, l, 2)$ is $\frac{n}{2m}$-admissible with $q \leqslant k \leqslant \infty$, and set

$$X = L_t^k(J; L_x^l(\mathbb{R}^n)) \cap L_t^2(J; L_x^{\frac{2n}{n-2m}}(\mathbb{R}^n))$$

with $\|v\|_X := \max\{\|v\|_{L_t^k(J;L_x^l(\mathbb{R}^n))}, \|v\|_{L_t^2(J;L_x^{\frac{2n}{n-2m}}(\mathbb{R}^n))}\}$.

By interpolation (see Triebel [9]), X can be embedded into $L_t^q(J; L_x^p(\mathbb{R}^n))$ for each $\frac{n}{2m}$-admissible triplet $(q, p, 2)$ with $2 \leqslant q \leqslant k$. Define $T(v)$ on X by

$$T(v) = \mathrm{e}^{-t\mathcal{P}(\mathcal{D})} f + \int_0^t \mathrm{e}^{-(t-s)\mathcal{P}(\mathcal{D})}(F(s,x) - V(s,x)v(s,x))ds, \quad \forall\, v = v(t,x) \in X.$$

Applying Theorems 3.3 and 3.4, we have

$$\|Tv\|_{L_t^q(J;L_x^p(\mathbb{R}^n))} \leqslant C\|f\|_{L^2(\mathbb{R}^n)} + C\|F\|_{L_t^{q_1'}(J;L_x^{p_1'}(\mathbb{R}^n))} + C\|Vv\|_{L_t^{q_2'}(J;L_x^{p_2'}(\mathbb{R}^n))},$$

for all $\frac{n}{2m}$-admissible triplets $(q, p, 2)$, $(q_1, p_1, 2)$, and $(q_2, p_2, 2)$ satisfying

$$2 \leqslant q \leqslant k, \quad q_1' \in (1,2), \quad q_2' \in (1,2), \quad 1 \leqslant p_1' < 2 \leqslant p \leqslant \infty, \quad 1 < p_2' < 2.$$

Here and later $C > 0$ is a constant. Clearly, Hölder's inequality implies

$$\|Tv\|_{L_t^q(J;L_x^p(\mathbb{R}^n))} \leqslant C\|f\|_{L^2(\mathbb{R}^n)} + C\|F\|_{L_t^{q_1'}(J;L_x^{p_1'}(\mathbb{R}^n))} + C\|V\|_{L_t^r(J;L_x^s(\mathbb{R}^n))}\|v\|_{L_t^2(J;L_x^{\frac{2n}{n-2m}}(\mathbb{R}^n))}$$

provided $\frac{1}{q_2} = \frac{1}{2} - \frac{1}{r}$, $\frac{1}{p_2} = \frac{n+2m}{2n} - \frac{1}{s}$. This and the assumption on r and s imply that $q_2' \in (1,2)$, $p_2' \in (1,2)$ and

$$\frac{1}{q_2} + \frac{n}{2m}\frac{1}{p_2} = \frac{1}{2} + \frac{n}{2m}\frac{n+2m}{2n} - \left(\frac{n}{2m}\frac{1}{s} + \frac{1}{r}\right) = \frac{n}{4m}.$$

Taking $(q, p, 2)$ to be $(k, l, 2)$ and $(2, \frac{2n}{n-2m}, 2)$, we get

$$\|T(v)\|_X \leqslant C\|f\|_2 + C\|F\|_{L_t^{q_1'}(J;L_x^{p_1'}(\mathbb{R}^n))} + C\|V\|_{L_t^r(J;L_x^s(\mathbb{R}^n))}\|v\|_X.$$

Hence $T(v) \in X$ and T is an operator from X to X. Since $r < \infty$, we may choose such an $\varepsilon > 0$ that

$$C\|V\|_{L_t^r(J;L_x^s(\mathbb{R}^n))} \leqslant \frac{1}{2}. \tag{4.2}$$

This fact yields that $\|T(v_1) - T(v_2)\|_X \leqslant \frac{1}{2}\|v_1 - v_2\|_X, \forall\, v_1, v_2 \in X$. Thus T is a contraction operator on X, and T has a unique fixed point $v(t,x)$ which is the unique solution of (1.2) and v satisfies $\|v\|_X \lesssim \|f\|_2 + \|F\|_{L_t^{q_1'}(J;L_x^{p_1'}(\mathbb{R}^n))}$.

Since X is embedded in $L_t^q(J; L_x^p(\mathbb{R}^n))$, one finds $\|v\|_{L_t^q(J;L_x^p(\mathbb{R}^n))} \lesssim \|f\|_2 + \|F\|_{L_t^{q_1'}(J;L_x^{p_1'}(\mathbb{R}^n))}$. Now, we can apply the previous arguments to any subinterval $J = [t_1, t_2]$ on which a condition like (4.2) holds and obtain

$$\|v\|_{L_t^q(J;L_x^p(\mathbb{R}^n))} \lesssim \|v(t_1)\|_{L^2(\mathbb{R}^n)} + \|F\|_{L_t^{q_1'}(J;L_x^{p_1'}(\mathbb{R}^n))}. \tag{4.3}$$

Note that (4.3) implies

$$\|v(t_2)\|_{L^2(\mathbb{R}^n)} \lesssim \|v(t_1)\|_{L^2(\mathbb{R}^n)} + \|F\|_{L_t^{q_1'}(J;L_x^{p_1'}(\mathbb{R}^n))}. \tag{4.4}$$

If $I = [0, T]$ for $0 < T < \infty$, we can partition I into a finite number of subintervals on which the condition (4.2) holds. If $I = [0, \infty)$, since $V \in L_t^r(I; L_x^s(\mathbb{R}^n))$, we can find $T_1 > 0$ such that

$C\|V\|_{L_t^r((T_1,\infty);L_x^s(\mathbb{R}^n))} < \frac{1}{2}$; and partition $[0,T_1]$ similarly. Thus we can prove (4.1) by inductively applying (4.3) and (4.4).

Case 2. $r \in (1,2)$. Since $(r, \frac{2s}{s+2})$ is the dual of $(r', \frac{2s}{s-2})$, our assumption on r and s implies

$$\frac{1}{r'} + \frac{n}{2m}\frac{s-2}{2s} = \frac{n}{2ms} + \frac{n}{2m}\frac{s-2}{2s} = \frac{n}{2m}.$$

Thus $(r', \frac{2s}{s-2})$ is $\frac{n}{2m}$-admissible with $r \in (1,2)$. In a fashion analogous to handling Case 1, we use Theorems 3.3 and 3.4 to obtain $\|Tv\|_{L_t^q(J;L_x^p(\mathbb{R}^n))} \leqslant C\|f\|_{L^2(\mathbb{R}^n)} + C\|F\|_{L_t^{q_1'}(J;L_x^{p_1'}(\mathbb{R}^n))} + C\|Vv\|_{L_t^r(J;L_x^{\frac{2s}{s+2}}(\mathbb{R}^n))}$.

Again, by Hölder's inequality we have

$$\|Tv\|_{L_t^q(J;L_x^p(\mathbb{R}^n))} \leqslant C\|f\|_{L^2(\mathbb{R}^n)} + C\|F\|_{L_t^{q_1'}(J;L_x^{p_1'}(\mathbb{R}^n))} + C\|V\|_{L_t^r(J;L_x^s(\mathbb{R}^n))}\|v\|_{L_t^\infty(J;L_x^2(\mathbb{R}^n))}.$$

Similarly, taking $(q,p,2)$ to be $(\infty,2,2)$ and $(2,\frac{2n}{n-2m},2)$, we have

$$\|Tv\|_X \leqslant C\|f\|_{L^2(\mathbb{R}^n)} + C\|F\|_{L_t^{q_1'}(J;L_x^{p_1'}(\mathbb{R}^n))} + C\|V\|_{L_t^r(J;L_x^s(\mathbb{R}^n))}\|v\|_X.$$

The rest of the proof is similar to that of the first case. □

5 Well-posedness for the Navier-Stokes system (1.3)

We can prove the following estimate by estimating kernel function $K_t(x)$ in mixed norm spaces.

Lemma 5.1. *Let $m \geqslant 1$, $0 < T < \infty$, $1 \leqslant p_1' < p \leqslant \infty$, $1 \leqslant q_1' < q \leqslant \infty$, $\frac{1}{r} = \frac{1}{p} + \frac{1}{p_1}$ and $\frac{1}{h} = \frac{1}{q} + \frac{1}{q_1}$. If $0 < \frac{nh}{2m}\left(1-\frac{1}{r}\right) < 1$, then*

$$\left\| \int_0^t \mathrm{e}^{-(t-s)\mathcal{P}(\mathcal{D})} F(s,x)ds \right\|_{L_t^q([0,T);X)} \lesssim T^{\frac{1}{h}-\frac{n}{2m}(1-\frac{1}{r})}\|F\|_{L_t^{q_1'}([0,T);Y)} \tag{5.1}$$

holds with $(X,Y) = (L_x^p(\mathbb{R}^n), L_x^{p_1'}(\mathbb{R}^n))$, $(\dot{H}_x^{\beta,p}(\mathbb{R}^n), \dot{H}_x^{\beta,p_1'}(\mathbb{R}^n))$ or $(H_x^{\beta,p}(\mathbb{R}^n), H_x^{\beta,p_1'}(\mathbb{R}^n))$ for all $\beta > 0$.

Proof. We only prove the case $(X,Y) = (L_x^p(\mathbb{R}^n), L_x^{p_1'}(\mathbb{R}^n))$ since similar arguments apply to other cases. Assume that $T \in (0,\infty), 1 \leqslant p_1' < p \leqslant \infty$, $1 \leqslant q_1' < q \leqslant \infty$, $\frac{1}{r} = \frac{1}{p} + \frac{1}{p_1}$, $\frac{1}{h} = \frac{1}{q} + \frac{1}{q_1}$ and $\frac{nh}{2m}(1-\frac{1}{r}) \in (0,1)$. According to the Young's inequality and the definition of $\mathrm{e}^{-t\mathcal{P}(\mathcal{D})}$(see Appendix), we have

$$\begin{aligned}\left\| \int_0^t \mathrm{e}^{-(t-s)\mathcal{P}(\mathcal{D})} F(s,x)ds \right\|_{L_t^q(I;L_x^p(\mathbb{R}^n))} &\lesssim \left\| \int_0^t \|K_{t-s}(x) *_x F(s,x)\|_{L_x^p(\mathbb{R}^n)}ds \right\|_{L_t^q(I)} \\ &\lesssim \left\| \int_0^t \|K_{t-s}(x)\|_{L_x^r(\mathbb{R}^n)}\|F(s,x)\|_{L_x^{p_1'}(\mathbb{R}^n)}ds \right\|_{L_t^q(I)} \\ &\lesssim \|K_t(x)\|_{L_t^h(I;L_x^r(\mathbb{R}^n))}\|F(s,x)\|_{L_t^{q_1'}(I;L_x^{p_1'}(\mathbb{R}^n))}.\end{aligned}$$

Thus it suffices to prove $\|K_t(x)\|_{L_t^h(I;L_x^r(\mathbb{R}^n))} \lesssim T^{\frac{1}{h}-\frac{n}{2m}(1-\frac{1}{r})}$. In fact, it follows from Lemma A.1 of Appendix that $K_t(x) \in L^k(\mathbb{R}^n)$ for all $1 \leqslant k \leqslant \infty$. Since $\frac{1}{r} = \frac{1}{p} + \frac{1}{p_1}$ and $p_1' < p$ imply that $r > 1$, $K_t(x) \in L^r(\mathbb{R}^n)$. Hence

$$\begin{aligned}\|K_t(x)\|_{L_t^h(I;L_x^r(\mathbb{R}^n))} &\lesssim \left(\int_0^T t^{-\frac{nh}{2m}} \left(\int_{\mathbb{R}^n} \frac{t^{n/2m}}{(1+|x|)^{(n+2m)r}}dx \right)^{\frac{h}{r}} dt \right)^{1/h} \lesssim \left[\int_0^T t^{-\frac{nh}{2m}(1-\frac{1}{r})}dt \right]^{1/h} \\ &\lesssim T^{\frac{1}{h}-\frac{n}{2m}(1-\frac{1}{r})}.\end{aligned}$$

This finishes the proof of Lemma 5.1. □

In the rest of this paper, we use the notation L^p indiscriminately for scalar and vector valued functions. Because the mild solutions for the system (1.3) are

$$u(t,x) = \mathrm{e}^{-t\mathcal{P}(\mathcal{D})}g(x) + \int_0^t \mathrm{e}^{-(t-s)\mathcal{P}(\mathcal{D})}(\mathbb{P}h - \mathbb{P}\nabla(u\otimes u)(s,x))ds,$$

where $\mathbb{P}$ (Larey projection) is the orthogonal projection operator onto the divergence free vector field defined as follows.

We denote the Riesz transforms by $R_j = \frac{\partial}{\partial x_j}(-\Delta)^{-1/2}, j = 1,2,\dots,n$. For an arbitrary vector field $u(x) = (u_1(x), u_2(x), \dots, u_n(x))$ on $\mathbb{R}^n$, we set $z(x) = \sum_{k=1}^n (R_k u_k)(x)$ and define the operator $\mathbb{P}$ by

$$\widehat{(\mathbb{P}u)}(\xi) = \sum_{k=1}^n \left(\delta_{jk} - \frac{\xi_j\xi_k}{|\xi|^2}\right)\widehat{u_k}(\xi), \quad j = 1,2,\dots,n$$

with $\delta_{j,k}$ being the Kronecker symbol.

Lemma 5.2. *Let $m \geqslant 1$ and $T > 0$. Assume that $u, v \in L_t^q([0,T]; L_x^p(\mathbb{R}^n))$ with p, q satisfying*

$$\max\left\{\frac{n}{2m-1}, 2\right\} < p < \infty, \quad 2m-1 = \frac{2m}{q} + \frac{n}{p}.$$

Then $B(u,v) = \int_0^t \mathrm{e}^{-(t-s)\mathcal{P}(\mathcal{D})}\mathbb{P}\nabla(u\otimes v)ds$ is bounded from $L_t^q([0,T]; L_x^p(\mathbb{R}^n)) \times L_t^q([0,T]; L_x^p(\mathbb{R}^n))$ to $L_t^q([0,T]; L_x^p(\mathbb{R}^n))$ with $\|B(u,v)\|_{L_t^q([0,T];L_x^p(\mathbb{R}^n))} \lesssim \|u\|_{L_t^q([0,T];L_x^p(\mathbb{R}^n))}\|v\|_{L_t^q([0,T];L_x^p(\mathbb{R}^n))}$.

Proof. By Lemma 2.4 and L^p-boundedness of Riesz transform, we have

$$\begin{aligned}\|B(u,v)\|_{L_x^p(\mathbb{R}^n)} &\lesssim \int_0^t \|\nabla \mathrm{e}^{-(t-s)\mathcal{P}(\mathcal{D})}\mathbb{P}(u(s,\cdot)\otimes v(s,\cdot))\|_{L_x^p(\mathbb{R}^n)}ds \\ &\lesssim \int_0^t \frac{1}{|t-s|^{\frac{1}{2m}+\frac{n}{2m}(\frac{2}{p}-\frac{1}{p})}}\|(u(s,\cdot)\otimes v(s,\cdot))\|_{L_x^{p/2}(\mathbb{R}^n)}ds \\ &\lesssim \int_0^t \frac{1}{|t-s|^{\frac{1}{2m}+\frac{n}{2mp}}}\|u(s,\cdot)\|_{L_x^p(\mathbb{R}^n)}\|v(s,\cdot)\|_{L_x^p(\mathbb{R}^n)}ds.\end{aligned}$$

Since $m \geqslant 1$ and $p > \frac{n}{2m-1}$, $0 < \frac{1}{2m} + \frac{n}{2pm} < 1$. It follows from $2m-1 = \frac{2m}{q} + \frac{n}{p}$ and the Hardy-Littlewood-Sobolev inequality that

$$\begin{aligned}\|B(u,v)\|_{L_t^q([0,T];L_x^p(\mathbb{R}^n))} &\lesssim \|(\|u(s,\cdot)\|_{L_x^p(\mathbb{R}^n)}\|v(s,\cdot)\|_{L_x^p(\mathbb{R}^n)})\|_{L_t^{q/2}([0,T])} \\ &\lesssim \|u\|_{L_t^q([0,T];L_x^p(\mathbb{R}^n))}\|v\|_{L_t^q([0,T];L_x^p(\mathbb{R}^n))}.\end{aligned}$$

This completes the proof of Lemma 5.2. □

Applying Theorem 3.4 and Lemmas 5.1, 5.2 and 2.6, we obtain the global existence and uniqueness of solutions for system (1.3).

Theorem 5.3. *Let $m \in \mathbb{N}, 1 \leqslant m < \frac{1}{2} + \frac{n}{4}$, $0 < T < \infty$, $p > \frac{n}{2m-1}$ and $\frac{n}{p} + \frac{2m}{q} = 2m-1$.*

(a) *Assume that $\frac{n}{2m-1} < r \leqslant p$, $1 \leqslant p_1' < p \leqslant \infty$, $1 \leqslant q_1' < q \leqslant \infty$, $g \in L^r(\mathbb{R}^n)$ with $\nabla \cdot g = 0$ and $h \in L_t^{q_1'}([0,T]; L_x^{p_1'}(\mathbb{R}^n))$, $0 < \frac{n}{2m}(\frac{1}{q} + \frac{1}{q_1})(1 - \frac{1}{p} - \frac{1}{p_1}) < 1$. If there exists a suitable constant $C > 0$ such that*

$$T^{1-\frac{n}{2m}(\frac{1}{n}+\frac{1}{r})}\|g\|_{L^r(\mathbb{R}^n)} + T^{\frac{1}{q}+\frac{1}{q_1}-\frac{n}{2m}(\frac{1}{p_1'}-\frac{1}{p})}\|h\|_{L_t^{q_1'}([0,T];L_x^{p_1'}(\mathbb{R}^n))} \lesssim C, \tag{5.2}$$

then (1.3) has a unique strong solution $v \in L_t^q([0,T]; L_x^p(\mathbb{R}^n))$ in the sense of

$$v = \mathrm{e}^{-t\mathcal{P}(\mathcal{D})}g(x) + \int_0^t \mathrm{e}^{-t\mathcal{P}(\mathcal{D})}[\mathbb{P}h(s,x) - \mathbb{P}\nabla\cdot(v\otimes v)(s,x)]ds.$$

(b) *Assume that* $g \in L^{\frac{n}{2m-1}}(\mathbb{R}^n)$ *with* $\nabla \cdot g = 0$ *and* $h \in L_t^{q_1'}([0,\infty); L_x^{p_1'}(\mathbb{R}^n))$ *with* q_1' *and* p_1' *satisfying* $1 < q_1' < q < \infty$,

$$1 \leqslant p_1' < p < \begin{cases} \dfrac{n^2}{(n-2m)(2m-1)}, & 2m < n, \\ \infty, & 2m \geqslant n, \end{cases} \quad \text{and} \quad \frac{n}{p_1'} + \frac{2m}{q_1'} = 4m - 1.$$

If $\|g\|_{L^{\frac{n}{2m-1}}(\mathbb{R}^n)} + \|h\|_{L_t^{q_1'}([0,\infty);L_x^{p_1'}(\mathbb{R}^n))}$ *is small enough, then* (1.3) *has a unique strong solution* $v \in L_t^q([0,\infty); L_x^p(\mathbb{R}^n))$.

Proof. (a) Under the assumption of (a), let $X = L^q([0,T]; L^p(\mathbb{R}^n))$. Define

$$Tu = \mathrm{e}^{-t\mathcal{P}(\mathcal{D})}g(x) + \int_0^t \mathrm{e}^{-(t-s)\mathcal{P}(\mathcal{D})}(\mathbb{P}h(s,x) - \mathbb{P}\nabla(u \otimes u)(s,x))ds.$$

We will prove that if $a := T^{1-\frac{1}{2m}(\frac{1}{n}+\frac{1}{r})}\|g\|_{L^r(\mathbb{R}^n)} + T^{\frac{1}{q}+\frac{1}{q_1}-\frac{n}{2m}(\frac{1}{p_1'}-\frac{1}{p})}\|h\|_{L^{q_1'}([0,T];L^{p_1'}(\mathbb{R}^n))}$ is bounded by an appropriate constant, then T is a contraction operator on the ball B_R in X with radius $R = 2a$. For any $u_1, u_2 \in B_R$, we have

$$\begin{aligned} \|Tu_1 - Tu_2\|_X &= \left\| \int_0^t \mathrm{e}^{-(t-s)\mathcal{P}(\mathcal{D})}\mathbb{P}\nabla(u_1 \otimes u_1)ds - \int_0^t \mathrm{e}^{-(t-s)\mathcal{P}(\mathcal{D})}\mathbb{P}\nabla(u_2 \otimes u_2)ds \right\|_X \\ &= \|B(u_1 - u_2, u_1) + B(u_2, u_1 - u_2)\|_X \\ &\leqslant \|B(u_1 - u_2, u_1)\|_X + \|B(u_2, u_1 - u_2)\|_X, \end{aligned}$$

where $B(u,v) = \int_0^t \mathrm{e}^{-(t-s)\mathcal{P}(\mathcal{D})}\mathbb{P}\nabla(u \otimes v)ds$.

It follows from Lemma 5.2 that B is bounded on X. Hence

$$\|Tu_1 - Tu_2\|_X \leqslant C\|u_1 - u_2\|_X\|u_1\|_X + C\|u_2\|_X\|u_1 - u_2\|_X,$$

where $C > 0$ only depends on m, p and q. Thus

$$\|Tu_1 - Tu_2\|_X \leqslant C(\|u_1\|_X + \|u_2\|_X)\|u_1 - u_2\|_X \leqslant CR\|u_1 - u_2\|_X.$$

To estimate $\|Tu\|_X$ for $u \in B_R$, we use $T(0) = \mathrm{e}^{-t\mathcal{P}(\mathcal{D})}g(x) + \int_0^t \mathrm{e}^{-(t-s)\mathcal{P}(\mathcal{D})}\mathbb{P}h(s,x)ds$ to obtain $\|T(0)\|_X \leqslant a$ according to Theorem 5.1 and Lemma 2.6. Consequently,

$$\|T(u)\|_X = \|T(u) - T(0) + T(0)\|_X \leqslant \|T(u-0)\|_X + \|T(0)\|_X \leqslant CR\|u\|_X + a.$$

Since a is bounded by a suitable constant, we have $\|T(u_1) - T(u_2)\|_X \leqslant \frac{1}{2}\|u_1 - u_2\|_X$ and $\|T(u)\|_X \leqslant R$. It follows from the Banach contraction mapping principle that there exists a unique strong solution $u \in X = L_t^q([0,T]; L_x^p(\mathbb{R}^n))$.

(b) Note that $\frac{n}{p} + \frac{2m}{q} = 2m - 1$ implies that $(q, p, \frac{n}{2m-1})$ is $\frac{n}{2m}$ -admissible. By Lemma 2.5, we get $\|\mathrm{e}^{-t\mathcal{P}(\mathcal{D})}g\|_{L_t^q([0,\infty);L_x^p(\mathbb{R}^n))} \lesssim \|g\|_{L^{\frac{n}{2m-1}}(\mathbb{R}^n)}$. On the other hand, Theorem 3.4 implies

$$\left\| \int_0^t \mathrm{e}^{-(t-s)\mathcal{P}(\mathcal{D})}h(s,x)ds \right\|_{L_t^q([0,\infty);L_x^p(\mathbb{R}^n))} \lesssim \|h\|_{L_t^{q_1'}([0,\infty);L_x^{p_1'}(\mathbb{R}^n))}.$$

Applying Lemma 5.2 for $T = \infty$ and the Banach contraction mapping principle, we can prove (b) since $\|g\|_{L^{\frac{n}{2m-1}}(\mathbb{R}^n)} + \|h\|_{L_t^{q_1'}([0,\infty);L_x^{p_1'}(\mathbb{R}^n))}$ is small enough. □

We show that the solution established in Theorem 5.3 is smooth in spatial variables. For a non-negative multi-index $k = (k_1, \ldots, k_n)$ we define $D^k = (\frac{\partial}{\partial x_1})^{k_1} \cdots (\frac{\partial}{\partial x_n})^{k_n}$ and $|k| = k_1 + \cdots + k_n$.

Corollary 5.4. *Under the hypothesis of Theorem* 5.3 *we assume further that for a non-negative multi-index* k, $D^k g \in L^r(\mathbb{R}^n)$ *and* $D^k h \in L_t^{q_1'}([0,T]; L_x^{p_1'}(\mathbb{R}^n))$. *Then the solution* v *established in Theorem* 5.3 *satisfies*

$$D^j v \in L^q([0,T]; L^p(\mathbb{R}^n)), \tag{5.3}$$

for any non-negative multi-index j *with* $|j| \leqslant |k|$.

Proof. The proof is similar to that of Theorem 5.3. We only demonstrate the case $|j| = 1$, since similar arguments apply to the cases $|j| = 2, 3, \ldots, |k|$. Define

$$\overline{T}(Du) = \mathrm{e}^{-t\mathcal{P}(\mathcal{D})}(Dg) + \int_0^t \mathrm{e}^{-(t-s)\mathcal{P}(\mathcal{D})} P(Dh) ds - B(Dv, v) - B(v, Dv). \tag{5.4}$$

Consider the integral equation $Dv = \overline{T}(Dv)$. Then $\overline{T}$ is a mapping of the space X of function v with

$$v \in L^q([0,T]; L^p(\mathbb{R}^n)) \quad \text{and} \quad Dv \in L^q([0,T]; L^p(\mathbb{R}^n)).$$

The norm in X is defined by $\|v\|_X = \|v\|_{L^q([0,T];L^p(\mathbb{R}^n))} + \|Dv\|_{L^q([0,T];L^p(\mathbb{R}^n))}$. The assumption on Dg and Dh implies that the first two terms on the right-hand side of (5.4) are bounded in X. The boundness of the other terms follows from Lemma 5.2. So T is a contraction mapping of X into itself and has a unique fixed point in X. Therefore, the solution v established in Theorem 5.3 satisfies $Dv \in L^q([0,T]; L^p(\mathbb{R}^n))$. □

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant Nos. 11371057, 11261051 and 11161042), Doctoral Fund of Ministry of Education of China (Grant No. 20130003110003) and the Fundamental Research Funds for the Central Universities (Grant No. 2012CXQT09). The authors thank the anonymous referees for their helpful suggestions and valuable comments on the original version of this manuscript.

References

1 D'Ancona P, Pierfelice V, Visciglia N. Some remarks on the Schrödinger equation with a potential in $L_t^r L_x^s$. Math Ann, 2005, 333: 271–290
2 Frazier M, Jawerth B, Weiss G. Littlewood-Paley Theory and the Study of Function Spaces. CBMS Reg Conf Ser Math. Providence, RI: Amer Math Soc, 1991
3 Keel M, Tao T. Endpoint Strichartz estimates. Amer J Math, 1998, 120: 955–980
4 Miao C, Gu Y. Space-time estimates for parabolic type operator and application to nonlinear parabolic equations. J Partial Differential Equations, 1998, 11: 301–312
5 Miao C, Yuan B, Zhang B. Well-posedness of the Cauchy problem for the fractional power dissipative equations. Nonlinear Anal, 2008, 68: 461–484
6 Miao C, Zhang B. The Cauchy problem for semilinear parabolic equations in Besov spaces. Houston J Math, 2004, 30: 829–878
7 Stein E M. Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals. Princeton: Princeton University Press, 1993
8 Tao T. Global regularity of wave maps IV: Absence of stationary or self-similar solutions in the energy class. ArXiv: 0806.3592v2, 2009
9 Triebel H. Interpolation Theory, Function Spaces, Differential Operators. Amsterdam-New York-Oxford: North-Holland Publishing Co, 1978
10 Triebel H. Theory of Function Spaces. Basel: Birkhäuser, 1983
11 Zhai Z. Strichartz type estimates for fractional heat equations. J Math Anal Appl, 2009, 356: 642-658

Appendix

We consider the linear semigroup $S_{\mathcal{P}}(t) \triangleq \mathrm{e}^{-t\mathcal{P}(\mathcal{D})}$ generated by the following linear parabolic equations with elliptic operators $\mathcal{P}(\mathcal{D})$ of order $2m$:

$$\begin{cases} \partial_t u(t,x) + \mathcal{P}(\mathcal{D}) u(t,x) = 0, & (t,x) \in (0,\infty) \times \mathbb{R}^n, \\ u(0,x) = f(x), & x \in \mathbb{R}^n. \end{cases} \tag{A.1}$$

We show that the kernel function of the operator semigroup $S_{\mathcal{P}}(t)$ generates a bounded linear operator on $L^p(\mathbb{R}^n)$ for $p \in [0,\infty]$.

Consider the Cauchy problem for system (A.1). By the Fourier transform and Duhamel's principle, the solution of (A.1) can be written as $u(t,x) = \mathrm{e}^{-t\mathcal{P}(\mathcal{D})}f(x) = \mathcal{F}^{-1}(\mathrm{e}^{-tP(\xi)}\mathcal{F}f(\xi))(x) = K_t(x) * f(x)$.

From the above and Young's inequality it is seen that, to guarantee the $L^p \to L^p$ boundedness of the linear operator $S_{\mathcal{P}}(t)$ one only needs that the kernel function $K_t(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\xi}\mathrm{e}^{-tP(\xi)}d\xi$ is bounded on L^1. Notice that $P(\xi)$ has constant real coefficients and $P(\xi) > 0$ for all $\xi \neq 0$. It is obvious that $\mathrm{e}^{-tP(\xi)} \in L^1(\mathbb{R}^n)$, so

$$K_t(x) \in L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n), \quad |K_t(x)| \leqslant Ct^{-\frac{n}{2m}} \quad \text{for all } t > 0, \tag{A.2}$$

and by the Riemann-Lebesgue theorem, $\lim_{|x|\to\infty} K_t(x) = 0$.

Lemma A.1. *The kernel function $K_t(x)$ has the following point-wise estimate,*

$$|K_t(x)| \leqslant Ct^{-\frac{n}{2m}}(1 + |t^{-\frac{1}{2m}}x|)^{-n-2m}, \quad x \in \mathbb{R}^n,$$

for $m \in \mathbb{N}$ and $0 < t < \infty$. Consequently one has $K_t(x) \in L^p(\mathbb{R}^n)$ for any $1 \leqslant p \leqslant \infty$ and $0 < t < \infty$.

Proof. Define the invariant derivative operator $L(x,D) = \frac{x\cdot\nabla_\xi}{\mathrm{i}|x|^2}$. Then we have $L(x,D)\mathrm{e}^{\mathrm{i}x\xi} = \mathrm{e}^{\mathrm{i}x\xi}$. The conjugate operator is $L^*(x,D) = -\frac{x\cdot\nabla_\xi}{\mathrm{i}|x|^2}$. Thus we may write $K_t(x)$ as

$$\begin{aligned} K_t(x) &= (2\pi)^{-n}\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\xi}L^*(\mathrm{e}^{-tP(\xi)})d\xi \\ &= (2\pi)^{-n}\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\xi}\rho\left(\frac{\xi}{\delta}\right)L^*(\mathrm{e}^{-tP(\xi)})d\xi + (2\pi)^{-n}\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\xi}\left(1 - \rho\left(\frac{\xi}{\delta}\right)\right)L^*(\mathrm{e}^{-tP(\xi)})d\xi \\ &\triangleq I + II, \end{aligned}$$

where $\delta > 0$ is to be chosen later and $\rho(\xi)$ is a $C_c^\infty(\mathbb{R}^n)$-function satisfying

$$\rho(\xi) = \begin{cases} 1, & |\xi| \leqslant 1, \\ 0, & |\xi| > 2. \end{cases}$$

It is clear that

$$|I| \leqslant \frac{Ct}{|x|}\int_{|\xi|\leqslant 2\delta} P_{2m-1}(\xi)d\xi \leqslant \frac{Ct}{|x|}\int_{|\xi|\leqslant 2\delta} |\xi|^{2m-1}d\xi \leqslant Ct|x|^{-1}\delta^{2m+n-1}.$$

To estimate II, take a sufficiently large natural number $N > 2m + n$ and integrate by parts to obtain

$$\begin{aligned} |II| &\leqslant (2\pi)^{-n}\int_{\mathbb{R}^n}\left|\mathrm{e}^{\mathrm{i}x\xi}(L^*)^{N-1}\left(\left(1-\rho\left(\frac{\xi}{\delta}\right)\right)L^*(\mathrm{e}^{-tP(\xi)})\right)\right|d\xi \\ &\leqslant C|x|^{-N}\int_{|\xi|\geqslant\delta}\sum_{l=1}^{N} t^l|\xi|^{2lm-N}\mathrm{e}^{-tP(\xi)}d\xi \\ &\quad + C|x|^{-N}\sum_{k=1}^{N-1} C_k\delta^{-k}\int_{\delta\leqslant|\xi|\leqslant 2\delta}\sum_{l=1}^{N-k} C_l t^l|\xi|^{2lm-(N-k)}\mathrm{e}^{-tP(\xi)}d\xi \\ &\leqslant C|x|^{-N}\int_{|\xi|\geqslant\delta} t|\xi|^{2m-N}\mathrm{e}^{-tP(\xi)}d\xi + C|x|^{-N}\int_{|\xi|\geqslant\delta} t|\xi|^{2m-N}t^{N-1}|\xi|^{2m(N-1)}\mathrm{e}^{-tP(\xi)}d\xi \\ &\quad + C|x|^{-N}\sum_{k=1}^{N-1}\int_{\delta\leqslant|\xi|\leqslant 2\delta}(t|\xi|^{2m-N}\mathrm{e}^{-tP(\xi)} + t^{N-k}|\xi|^{2m(N-k)-N}\mathrm{e}^{-tP(\xi)})d\xi. \end{aligned}$$

In view of the facts that

$$t^{N-1}|\xi|^{2m(N-1)}\mathrm{e}^{-tP(\xi)} \leqslant t^{N-1}|\xi|^{2m(N-1)}\mathrm{e}^{-t|\xi|^{2m}} \leqslant C,$$

$$t^{N-k-1}|\xi|^{2m(N-k-1)}\mathrm{e}^{-tP(\xi)} \leqslant t^{N-k-1}|\xi|^{2m(N-k-1)}\mathrm{e}^{-t|\xi|^{2m}} \leqslant C$$

for $k = 1, 2, \ldots, N-1$, it is found that $|II|$ is dominated by

$$Ct|x|^{-N}\left(\int_{|\xi|\geqslant\delta}|\xi|^{2m-N}d\xi + \int_{\delta\leqslant|\xi|\leqslant 2\delta}\delta^{2m-N}d\xi\right) \leqslant Ct|x|^{-N}\delta^{2m-N+n}.$$

Taking $\delta = |x|^{-1}$ gives $|K_t(x)| \leqslant Ct|x|^{-n-2m}$. This together with the boundedness of $K_t(x)$ (see (A.2)) completes the proof of the lemma. □