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The relations among the three kinds of conditional risk measures

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Abstract Let $(\Omega, \mathcal{E}, P)$ be a probability space, $\mathcal{F}$ a sub-σ-algebra of $\mathcal{E}$, $L^p(\mathcal{E})$ $(1 \leqslant p \leqslant +\infty)$ the classical function space and $L^p_{\mathcal{F}}(\mathcal{E})$ the $L^0(\mathcal{F})$-module generated by $L^p(\mathcal{E})$, which can be made into a random normed module in a natural way. Up to the present time, there are three kinds of conditional risk measures, whose model spaces are $L^\infty(\mathcal{E})$, $L^p(\mathcal{E})$ $(1 \leqslant p < +\infty)$ and $L^p_{\mathcal{F}}(\mathcal{E})$ $(1 \leqslant p \leqslant +\infty)$ respectively, and a conditional convex dual representation theorem has been established for each kind. The purpose of this paper is to study the relations among the three kinds of conditional risk measures together with their representation theorems. We first establish the relation between $L^p(\mathcal{E})$ and $L^p_{\mathcal{F}}(\mathcal{E})$, namely $L^p_{\mathcal{F}}(\mathcal{E}) = H_{cc}(L^p(\mathcal{E}))$, which shows that $L^p_{\mathcal{F}}(\mathcal{E})$ is exactly the countable concatenation hull of $L^p(\mathcal{E})$. Based on the precise relation, we then prove that every $L^0(\mathcal{F})$-convex $L^p(\mathcal{E})$-conditional risk measure $(1 \leqslant p \leqslant +\infty)$ can be uniquely extended to an $L^0(\mathcal{F})$-convex $L^p_{\mathcal{F}}(\mathcal{E})$-conditional risk measure and that the dual representation theorem of the former can also be regarded as a special case of that of the latter, which shows that the study of L^p-conditional risk measures can be incorporated into that of $L^p_{\mathcal{F}}(\mathcal{E})$-conditional risk measures. In particular, in the process we find that combining the countable concatenation hull of a set and the local property of conditional risk measures is a very useful analytic skill that may considerably simplify and improve the study of L^0-convex conditional risk measures.

Keywords random normed module, countable concatenation property, $L^\infty(\mathcal{E})$-conditional risk measure, $L^p(\mathcal{E})$-conditional risk measure $(1 \leqslant p < +\infty)$, $L^p_{\mathcal{F}}(\mathcal{E})$-conditional risk measure $(1 \leqslant p \leqslant +\infty)$, extension

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1 Introduction

Random metric theory is based on the idea of randomizing the classical space theory of functional analysis. As the central part of random metric theory, random normed modules (briefly, RN modules) and random locally convex modules (briefly, RLC modules) together with their random conjugate spaces have been deeply studied under the (ε, λ)-topology in the direction of functional analysis, cf. [15–17, 20] and the related references of these papers. In 2009, Filipović et al. [8] presented a kind of new topology — the locally L^0-convex topology for random normed modules and random locally convex modules and, for the first time, applied random normed modules to the study of conditional risk measures, cf. [8, 16].

Classical convex analysis (see [7]) is the analytic foundation for convex risk measures, cf. [1, 3, 10–12]. However, it is no longer well suited to L^0-convex (or conditional convex) conditional risk measures (in

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particular, those defined on the model spaces of unbounded financial positions). Just to overcome the obstacle, Filipović et.al [8, 9] presented the module approach to conditional risk. Let $(\Omega, \mathcal{E}, P)$ be a probability space, $\mathcal{F}$ a sub-σ-algebra of $\mathcal{E}$, $L^0(\mathcal{F})$ $(\bar{L}^0(\mathcal{F}))$ the set of real (extended real)-valued $\mathcal{F}$-measurable random variables on Ω, $L^p(\mathcal{E})$ $(1 \leqslant p \leqslant +\infty)$ the classical function space and $L^p_{\mathcal{F}}(\mathcal{E})$ the $L^0(\mathcal{F})$-module generated by $L^p(\mathcal{E})$, which can be made into a random normed module in a natural way, see Example 2.4 of this paper for the construction of the random normed module $L^p_{\mathcal{F}}(\mathcal{E})$. The so-called module approach is to choose $L^p_{\mathcal{F}}(\mathcal{E})$ as the model space, namely define an $L^0(\mathcal{F})$-convex conditional risk measure to be a proper $L^0(\mathcal{F})$-convex cash-invariant and monotone function from $L^p_{\mathcal{F}}(\mathcal{E})$ to $\bar{L}^0(\mathcal{F})$ and further develop conditional risk measures under the definition, which also leads to the development of random convex analysis, cf. [8, 9, 16, 17, 20]. Based on these advances, this paper further studies the relations among the three kinds of conditional risk measures.

It is easy to observe that $L^p(\mathcal{E})$ is dense in $L^p_{\mathcal{F}}(\mathcal{E})$ with respect to the (ε, λ)-topology on $L^p_{\mathcal{F}}(\mathcal{E})$ (clearly, this is not true with respect to the locally L^0-convex topology). The simple fact motivates us to further study the precise relations among the three kinds of conditional risk measures. The first kind was introduced independently by Detlefsen and Scandolo [5] and Bion-Nadal [2] as a monotone and cash-invariant function from $L^\infty(\mathcal{E})$ to $L^\infty(\mathcal{F})$ (briefly, an L^∞-conditional risk measure). The second and third kinds were introduced by Filipović et al. [9] as monotone and cash-invariant functions from $L^p(\mathcal{E})$ to $L^r(\mathcal{F})$ $(1 \leqslant r \leqslant p < +\infty)$ and from $L^p_{\mathcal{F}}(\mathcal{E})$ to $\bar{L}^0(\mathcal{F})$ $(1 \leqslant p \leqslant +\infty)$ (briefly, L^p- and $L^p_{\mathcal{F}}(\mathcal{E})$-conditional risk measures, respectively). We show that an L^∞-conditional risk measure can be uniquely extended to an $L^\infty_{\mathcal{F}}(\mathcal{E})$-conditional risk measure and the conditional convex dual representation theorem for the former can be regarded as a special case of that for the latter. We further show that an L^0-convex L^p-conditional risk measure can be uniquely extended to an L^0-convex $L^p_{\mathcal{F}}(\mathcal{E})$-conditional risk measure $(1 \leqslant p < +\infty)$ and the conditional convex dual representation theorem for the former can also be regarded as a special case of that for the latter. Hence, this paper shows that the two vector space approaches to conditional risk can be incorporated into the module approach. The second extension theorem is not very easy, whose proof is constructive, since an L^0-convex L^p-conditional risk measure is not necessarily uniformly continuous with respect to the relative topology when $L^p(\mathcal{E})$ is regarded as a subspace of $L^p_{\mathcal{F}}(\mathcal{E})$ which is endowed with the (ε, λ)-topology. It is to establish the extension theorem that we find the relation between $L^p(\mathcal{E})$ and $L^p_{\mathcal{F}}(\mathcal{E})$, namely $L^p_{\mathcal{F}}(\mathcal{E}) = H_{cc}(L^p(\mathcal{E}))$, which shows that $L^p_{\mathcal{F}}(\mathcal{E})$ is exactly the countable concatenation hull of $L^p(\mathcal{E})$. In particular, in the process we find that combining the countable concatenation hull of a set and the local property of conditional risk measures is a very useful analytic skill that may considerably simplify and improve the study of L^0-convex conditional risk measures.

Most of the main ideas and results of this paper were first announced in Guo's survey paper [17] without the detailed proofs or at most with a sketch of proofs of a few of illustrative results, in fact, this paper is just the third part of our manuscript [20]. Besides, the main results of this paper strengthen the corresponding versions announced in [17] since we observe the two interesting results — Lemma 2.16 and Proposition 3.5.

The rest of this paper is organized as follows. Section 2 recalls the Fenchel-Moreau duality theorem in random normed modules together with the dual representation of $L^p_{\mathcal{F}}(\mathcal{E})$-conditional risk measures; Sections 3 and 4 present and prove our main results as stated above in Section 1 of this paper.

Throughout this paper, we always use the following notation and terminologies:

K : the scalar field $\mathbb{R}$ of real numbers or $\mathbb{C}$ of complex numbers.

$(\Omega, \mathcal{F}, P)$: a probability space.

$L^0(\mathcal{F}, K)$ = the algebra of equivalence classes of K-valued $\mathcal{F}$-measurable random variables on $(\Omega, \mathcal{F}, P)$.

$L^0(\mathcal{F}) = L^0(\mathcal{F}, \mathbb{R})$.

$\bar{L}^0(\mathcal{F})$ = the set of equivalence classes of extended real-valued $\mathcal{F}$-measurable random variables on $(\Omega, \mathcal{F}, P)$.

As usual, $\bar{L}^0(\mathcal{F})$ is partially ordered by $\xi \leqslant \eta$ iff $\xi^0(\omega) \leqslant \eta^0(\omega)$ for P-almost all $\omega \in \Omega$ (briefly, a.s.), where ξ^0 and η^0 are arbitrarily chosen representatives of ξ and η, respectively. Then $(\bar{L}^0(\mathcal{F}), \leqslant)$ is a complete lattice, $\bigvee H$ and $\bigwedge H$ denote the supremum and infimum of a subset H, respectively.

$(L^0(\mathcal{F}), \leqslant)$ is a conditionally complete lattice. Please refer to [6] or [16, p. 3026] for the rich properties of the supremum and infimum of a set in $\bar{L}^0(\mathcal{F})$.

Let ξ and η be in $\bar{L}^0(\mathcal{F})$. $\xi < \eta$ is understood as usual, namely $\xi \leqslant \eta$ and $\xi \neq \eta$. In this paper we also use "$\xi < \eta$ (or $\xi \leqslant \eta$) on A" for "$\xi^0(\omega) < \eta^0(\omega)$ (resp., $\xi^0(\omega) \leqslant \eta^0(\omega)$) for P-almost all $\omega \in A$", where $A \in \mathcal{F}$, ξ^0 and η^0 are a representative of ξ and η, respectively.

$\bar{L}^0_+(\mathcal{F}) = \{\xi \in \bar{L}^0(\mathcal{F}) \mid \xi \geqslant 0\}$.

$L^0_+(\mathcal{F}) = \{\xi \in L^0(\mathcal{F}) \mid \xi \geqslant 0\}$.

$\bar{L}^0_{++}(\mathcal{F}) = \{\xi \in \bar{L}^0(\mathcal{F}) \mid \xi > 0 \text{ on } \Omega\}$.

$L^0_{++}(\mathcal{F}) = \{\xi \in L^0(\mathcal{F}) \mid \xi > 0 \text{ on } \Omega\}$.

Besides, $\tilde{I}_A$ always denotes the equivalence class of I_A, where $A \in \mathcal{F}$ and I_A is the characteristic function of A. When $\tilde{A}$ denotes the equivalence class of $A (\in \mathcal{F})$, namely $\tilde{A} = \{B \in \mathcal{F} \mid P(A \triangle B) = 0\}$ (here, $A \triangle B = (A \setminus B) \cup (B \setminus A)$), we also use $I_{\tilde{A}}$ for $\tilde{I}_A$.

2 Fenchel-Moreau duality theorem in random normed modules together with the dual representation of $L^p_{\mathcal{F}}(\mathcal{E})$-conditional risk measures

Definition 2.1 (See [15, 16])**.** An ordered pair $(E, \|\cdot\|)$ is called a random normed space (briefly, an RN space) over K with base $(\Omega, \mathcal{F}, P)$ if E is a linear space over K and $\|\cdot\|$ is a mapping from E to $L^0_+(\mathcal{F})$ such that the following are satisfied:

(RN-1) $\|\alpha x\| = |\alpha| \|x\|, \forall\, \alpha \in K$ and $x \in E$;

(RN-2) $\|x\| = 0$ implies $x = \theta$ (the null element of E);

(RN-3) $\|x + y\| \leqslant \|x\| + \|y\|, \forall\, x, y \in E$.

Here, $\|\cdot\|$ is called the random norm on E and $\|x\|$ the random norm of $x \in E$ (if $\|\cdot\|$ only satisfies (RN-1) and (RN-3) above, it is called a random seminorm on E).

Furthermore, if, in addition, E is a left module over the algebra $L^0(\mathcal{F}, K)$ (briefly, an $L^0(\mathcal{F}, K)$-module) such that

(RNM-1) $\|\xi x\| = |\xi| \|x\|, \forall\, \xi \in L^0(\mathcal{F}, K)$ and $x \in E$.

Then $(E, \|\cdot\|)$ is called a random normed module (briefly, an RN module) over K with base $(\Omega, \mathcal{F}, P)$, the random norm $\|\cdot\|$ with the property (RNM-1) is also called an L^0-norm on E (a mapping only satisfying (RN-3) and (RNM-1) above is called an L^0-seminorm on E).

Definition 2.2 (See [15])**.** Let $(E, \|\cdot\|)$ be an RN space over K with base $(\Omega, \mathcal{F}, P)$. A linear operator f from E to $L^0(\mathcal{F}, K)$ is said to be an a.s. bounded random linear functional if there is $\xi \in L^0_+(\mathcal{F})$ such that $\|f(x)\| \leqslant \xi \|x\|, \forall\, x \in E$. Denote by E^* the linear space of a.s. bounded random linear functionals on E, define $\|\cdot\| : E^* \to L^0_+(\mathcal{F})$ by $\|f\| = \bigwedge\{\xi \in L^0_+(\mathcal{F}) \mid \|f(x)\| \leqslant \xi \|x\| \text{ for all } x \in E\}$ for all $f \in E^*$, then it is easy to check that $(E^*, \|\cdot\|)$ is also an RN module over K with base $(\Omega, \mathcal{F}, P)$, called the random conjugate space of E.

Example 2.3. Let $L^0(\mathcal{F}, B)$ be the $L^0(\mathcal{F}, K)$-module of equivalence classes of $\mathcal{F}$-random variables (or, strongly $\mathcal{F}$-measurable functions) from $(\Omega, \mathcal{F}, P)$ to a normed space $(B, \|\cdot\|)$ over K. $\|\cdot\|$ induces an L^0-norm (still denoted by $\|\cdot\|$) on $L^0(\mathcal{F}, B)$ by $\|x\| :=$ the equivalence class of $\|x^0(\cdot)\|$ for all $x \in L^0(\mathcal{F}, B)$, where $x^0(\cdot)$ is a representative of x. Then $(L^0(\mathcal{F}, B), \|\cdot\|)$ is an RN module over K with base $(\Omega, \mathcal{F}, P)$. Specially, $L^0(\mathcal{F}, K)$ is an RN module, the L^0-norm $\|\cdot\|$ on $L^0(\mathcal{F}, K)$ is still denoted by $|\cdot|$.

Filipović et al. [8] constructed important RN modules $L^p_{\mathcal{F}}(\mathcal{E})$ $(1 \leqslant p \leqslant +\infty)$. In this paper, we will prove that $L^p_{\mathcal{F}}(\mathcal{E})$ plays the role of universal model spaces for L^0-convex conditional risk measures.

Example 2.4. Let $(\Omega, \mathcal{E}, P)$ be a probability space and $\mathcal{F}$ a sub-σ-algebra of $\mathcal{E}$. Define $|||\cdot|||_p \colon L^0(\mathcal{E}) \to \bar{L}^0_+(\mathcal{F})$ by

$$|||x|||_p = \begin{cases} E[|x|^p|\mathcal{F}]^{\frac{1}{p}}, & \text{when } 1 \leqslant p < \infty, \\ \bigwedge\{\xi \in \bar{L}^0_+(\mathcal{F}) \mid |x| \leqslant \xi\}, & \text{when } p = +\infty, \end{cases}$$

for all $x \in L^0(\mathcal{E})$.

Denote $L^p_{\mathcal{F}}(\mathcal{E}) = \{x \in L^0(\mathcal{E}) \mid |||x|||_p \in L^0_+(\mathcal{F})\}$, then $(L^p_{\mathcal{F}}(\mathcal{E}), |||\cdot|||_p)$ is an RN module over R with base $(\Omega, \mathcal{F}, P)$ and $L^p_{\mathcal{F}}(\mathcal{E}) = L^0(\mathcal{F}) \cdot L^p(\mathcal{E}) = \{\ \xi x \mid \xi \in L^0(\mathcal{F}) \text{ and } x \in L^p(\mathcal{E})\}$.

Definition 2.5 (See [15])**.** Let $(E, \|\cdot\|)$ be an RN space over K with base $(\Omega, \mathcal{F}, P)$. For any positive numbers ε and λ with $0 < \lambda < 1$, let $N_\theta(\varepsilon, \lambda) = \{x \in E \mid P\{\omega \in \Omega \mid \|x\|(\omega) < \varepsilon\} > 1 - \lambda\}$, then $\{N_\theta(\varepsilon, \lambda) \mid \varepsilon > 0, 0 < \lambda < 1\}$ forms a local base at θ of some Hausdorff linear topology on E, called the (ε, λ)-topology induced by $\|\cdot\|$.

From now on, we always denote by $\mathcal{T}_{\varepsilon,\lambda}$ the (ε, λ)-topology for every RN space if there is no possible confusion. Clearly, the (ε, λ)-topology for the special class of RN modules $L^0(\mathcal{F}, B)$ is exactly the ordinary topology of convergence in measure, and $(L^0(\mathcal{F}, K), \mathcal{T}_{\varepsilon,\lambda})$ is a topological algebra over K. It is also easy to check that $(E, \mathcal{T}_{\varepsilon,\lambda})$ is a topological module over $(L^0(\mathcal{F}, K), \mathcal{T}_{\varepsilon,\lambda})$ when $(E, \|\cdot\|)$ is an RN module over K with base $(\Omega, \mathcal{F}, P)$, namely the module multiplication operation is jointly continuous.

Let $(E, \|\cdot\|)$ be an RN module over K with base $(\Omega, \mathcal{F}, P)$. For any $\varepsilon \in L^0_{++}(\mathcal{F})$, let $U(\varepsilon) = \{x \in E \mid \|x\| \leqslant \varepsilon\}$. A subset G of E is $\mathcal{T}_c$-open if for each fixed $x \in G$ there is some $\varepsilon \in L^0_{++}(\mathcal{F})$ such that $x + U(\varepsilon) \subset G$. Denote by $\mathcal{T}_c$ the family of $\mathcal{T}_c$-open subsets of E, then $\mathcal{T}_c$ is a Hausdorff topology on E, called the locally L^0-convex topology induced by $\|\cdot\|$. Filipović et al. [8] first observed this kind of topology. In this paper, for any RN module $(E, \|\cdot\|)$ we always use $\mathcal{T}_c$ for the L^0-convex topology on E if there is no possible confusion.

$(L^0(\mathcal{F}, K), \mathcal{T}_c)$ is a topological ring, namely the addition and multiplication operations are jointly continuous, and further [8] pointed out that $\mathcal{T}_c$ is not necessarily a linear topology since the mapping $\alpha \mapsto \alpha x$ (x is fixed) is no longer continuous in general. These observations led [8] to the study of a class of topological modules over the topological ring $(L^0(\mathcal{F}, K), \mathcal{T}_c)$, namely the locally L^0-convex modules, it is proved that $(E, \mathcal{T}_c)$ is a Hausdorff locally L^0-convex module for any random normed module $(E, \|\cdot\|)$, cf. [8].

Proposition 2.6 (See [13, 16])**.** *Let $(E, \|\cdot\|)$ be an RN module over K with base $(\Omega, \mathcal{F}, P)$ and f a linear operator from E to $L^0(\mathcal{F}, K)$. Then $f \in E^*$ iff f is a continuous module homomorphism from $(E, \mathcal{T}_{\varepsilon,\lambda})$ to $(L^0(\mathcal{F}, K), \mathcal{T}_{\varepsilon,\lambda})$ iff f is a continuous module homomorphism from $(E, \mathcal{T}_c)$ to $(L^0(\mathcal{F}, K), \mathcal{T}_c)$, in which case $\|f\| = \bigvee\{|f(x)| \mid x \in E \text{ and } \|x\| \leqslant 1\}$.*

Let $(E, \|\cdot\|)$ be an RN module over K with base $(\Omega, \mathcal{F}, P)$, $E^*_{\varepsilon,\lambda}$ denote the $L^0(\mathcal{F}, K)$-module of continuous module homomorphisms from $(E, \mathcal{T}_{\varepsilon,\lambda})$ to $(L^0(\mathcal{F}, K), \mathcal{T}_{\varepsilon,\lambda})$ and E^*_c the $L^0(\mathcal{F}, K)$-module of continuous module homomorphisms from $(E, \mathcal{T}_c)$ to $(L^0(\mathcal{F}, K), \mathcal{T}_c)$, then Proposition 2.6 shows that $E^* = E^*_{\varepsilon,\lambda} = E^*_c$.

Proposition 2.7 below is the representation theorem of the random conjugate spaces of $L^p_{\mathcal{F}}(\mathcal{E})$.

Proposition 2.7 (See [8, 16])**.** *Let $1 \leqslant p < +\infty$ and q the Hölder conjugate number of p. Then for each $f \in (L^p_{\mathcal{F}}(\mathcal{E}))^*$, there exists a unique $y \in L^q_{\mathcal{F}}(\mathcal{E})$ such that $f(x) = E(xy \mid \mathcal{F})$ for any $x \in L^p_{\mathcal{F}}(\mathcal{E})$ and $\|f\| = |||y|||_q$, namely $(L^p_{\mathcal{F}}(\mathcal{E}))^*$ is isometrically isomorphic to $L^q_{\mathcal{F}}(\mathcal{E})$ under the canonical mapping.*

Let E be an $L^0(\mathcal{F})$-module and f a function from E to $\bar{L}^0(\mathcal{F})$. The effective domain of f is denoted by $\mathrm{dom}(f) := \{x \in E \mid |f(x)| < +\infty \text{ on } \Omega\}$ and the epigraph of f by $\mathrm{epi}(f) := \{(x, r) \in E \times L^0(\mathcal{F}) \mid f(x) \leqslant r\}$. f is proper if $\mathrm{dom}(f) \neq \emptyset$ and $f(x) > -\infty$ on Ω. f is L^0-convex if $f(\xi x + (1 - \xi)y) \leqslant \xi f(x) + (1 - \xi)f(y)$ for all $x,\ y \in E$ and $\xi \in L^0_+(\mathcal{F})$ with $0 \leqslant \xi \leqslant 1$, where the following convention is adopted: $0 \cdot (\pm\infty) = 0$ and $+\infty \pm (\pm\infty) = +\infty$. $f\colon E \to \bar{L}^0(\mathcal{F})$ is said to be local (or, to have the local property) if $\tilde{I}_A f(x) = \tilde{I}_A f(\tilde{I}_A x)$ for all $x \in E$ and $A \in \mathcal{F}$. In [9], it is proved that an L^0-convex function is local.

Definition 2.8. Let $(E, \|\cdot\|)$ be an RN module over R with base $(\Omega, \mathcal{F}, P)$ and $f : E \to \bar{L}^0(\mathcal{F})$ a proper L^0-convex function. f is $\mathcal{T}_{\varepsilon,\lambda}$-lower semicontinuous if $\mathrm{epi}(f)$ is closed in $(E, \mathcal{T}_{\varepsilon,\lambda}) \times (L^0(\mathcal{F}), \mathcal{T}_{\varepsilon,\lambda})$.

Let $(E, \|\cdot\|)$ be an RN module over R with base $(\Omega, \mathcal{F}, P)$ and $f : E \to \bar{L}^0(\mathcal{F})$ a proper L^0-convex function. $f^* : E^* \to \bar{L}^0(\mathcal{F})$ is defined by $f^*(g) = \bigvee\{g(x) - f(x) \mid x \in E\}$, called the random conjugate function of f, $f^{**} : E \to \bar{L}^0(\mathcal{F})$ is defined by $f^{**}(x) = \bigvee\{g(x) - f^*(g) \mid g \in E^*\}$, called the random bi-conjugate function of f. Then we have the following random version under the (ε, λ)-topology of the classical Fenchel-Moreau dual representation theorem.

Proposition 2.9 (See [20]). *Let $(E, \|\cdot\|)$ be an RN module over R with base $(\Omega, \mathcal{F}, P)$ and $f : E \to \bar{L}^0(\mathcal{F})$ a proper $\mathcal{T}_{\varepsilon,\lambda}$-lower semicontinuous L^0-convex function. Then $f^{**} = f$.*

Definition 2.10 (See [8]). Let $(E, \|\cdot\|)$ be an RN module over R with base $(\Omega, \mathcal{F}, P)$ and $f : E \to \bar{L}^0(\mathcal{F})$ a proper L^0-convex function. f is $\mathcal{T}_c$-lower semicontinuous if $\{x \in E \mid f(x) \leqslant r\}$ is $\mathcal{T}_c$-closed for each $r \in L^0(\mathcal{F})$.

To introduce the random version under the locally L^0-convex topology of the classical Fenchel-Moreau dual representation theorem, let us first recall the notion of countable concatenation property of a set or an $L^0(\mathcal{F}, K)$-module. The introducing of the notion utterly results from the study of the locally L^0-convex topology, the reader will see that this notion is ubiquitous in the theory of the locally L^0-convex topology. From now on, we always suppose that all the $L^0(\mathcal{F}, K)$-modules E involved in this paper have the property that for any $x,\ y \in E$, if there is a countable partition $\{A_n, n \in \mathbb{N}\}$ of Ω to $\mathcal{F}$ such that $\tilde{I}_{A_n} x = \tilde{I}_{A_n} y$ for each $n \in \mathbb{N}$ then $x = y$. Guo [16] already pointed out that all random locally convex modules possess this property, so the assumption is not too restrictive.

Definition 2.11 (See [16]). Let E be an $L^0(\mathcal{F}, K)$-module. A sequence $\{x_n, n \in \mathbb{N}\}$ in E is countably concatenated in E with respect to a countable partition $\{A_n, n \in \mathbb{N}\}$ of Ω to $\mathcal{F}$ if there is $x \in E$ such that $\tilde{I}_{A_n} x = \tilde{I}_{A_n} x_n$ for each $n \in \mathbb{N}$, in which case we define $\sum_{n=1}^{\infty} \tilde{I}_{A_n} x_n$ as x. A subset G of E is said to have the countable concatenation property if each sequence $\{x_n, n \in \mathbb{N}\}$ in G is countably concatenated in E with respect to an arbitrary countable partition $\{A_n, n \in \mathbb{N}\}$ of Ω to $\mathcal{F}$ and $\sum_{n=1}^{\infty} \tilde{I}_{A_n} x_n \in G$.

From now on, let E be an $L^0(\mathcal{F}, K)$-module with the countable concatenation property and G a subset of E. $H_{cc}(G)$ always denotes the countable concatenation hull of G in E, namely $H_{cc}(G) = \{\sum_{n=1}^{\infty} \tilde{I}_{A_n} g_n : \{A_n, n \in \mathbb{N}\}$ is a countable partition of Ω to $\mathcal{F}$ and $\{g_n, n \in \mathbb{N}\}$ is a sequence in $G\}$. Furthermore, if $x = \sum_{n=1}^{\infty} \tilde{I}_{A_n} x_n$ for some countable partition $\{A_n, n \in \mathbb{N}\}$ of Ω to $\mathcal{F}$ and some sequence $\{x_n, n \in \mathbb{N}\}$ in E, then $\sum_{n=1}^{\infty} \tilde{I}_{A_n} x_n$ is called a canonical representation of x.

When an RN module E has the countable concatenation property, in [20] we proved that a proper L^0-convex function f on E is $\mathcal{T}_{\varepsilon,\lambda}$-lower semicontinuous iff it is $\mathcal{T}_c$-lower semicontinuous. So we have the following corollary, namely the random version under the locally L^0-convex topology of the classical Fenchel-Moreau dual representation theorem:

Corollary 2.12 (See [20]). *Let $(E, \|\cdot\|)$ be an RN module over R with base $(\Omega, \mathcal{F}, P)$ such that E has the countable concatenation property and $f : E \to \bar{L}^0(\mathcal{F})$ a proper $\mathcal{T}_c$-lower semicontinuous L^0-convex function. Then $f^{**} = f$.*

The so-called module approach to conditional risk is to develop conditional risk measures under the following definition:

Definition 2.13 (See [9]). Let $1 \leqslant p \leqslant +\infty$. A proper function $f : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ is said to be

(1) monotone if $f(x) \leqslant f(y)$ for all $x, y \in L^p_{\mathcal{F}}(\mathcal{E})$ such that $x \geqslant y$;

(2) cash invariant if $f(x + y) = f(x) - y$ for all $x \in L^p_{\mathcal{F}}(\mathcal{E})$ and $y \in L^0(\mathcal{F})$.

Furthermore, a proper, monotone and cash invariant function f from $L^p_{\mathcal{F}}(\mathcal{E})$ to $\bar{L}^0(\mathcal{F})$ is called an $L^p_{\mathcal{F}}(\mathcal{E})$-conditional risk measure.

Since $L^p_{\mathcal{F}}(\mathcal{E})$ has the countable concatenation property, Filipović et al. [9] essentially used Corollary 2.12 as well as the typical techniques from conditional risk measures to obtain the following representation theorem:

Theorem 2.14 (See [9]). *Let $1 \leqslant p < +\infty$ and $1 < q \leqslant +\infty$ be a pair of Hölder conjugate numbers and $f : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ a $\mathcal{T}_c$- (or equivalently, $\mathcal{T}_{\varepsilon,\lambda}$-) lower semicontinuous $L^0(\mathcal{F})$-convex $L^p_{\mathcal{F}}(\mathcal{E})$-conditional risk measure. Then $f(x) = \bigvee\{E(xy \mid \mathcal{F}) - f^*(y) \mid y \in L^q_{\mathcal{F}}(\mathcal{E}),\ y \leqslant 0$ and $E(y \mid \mathcal{F}) = -1\}$ for all $x \in L^p_{\mathcal{F}}(\mathcal{E})$.*

Just as the notion of a random normed module is a proper random generalization of that of a normed space, one can have the notion of a random locally convex module as a random generalization of that of a Hausdorff locally convex space. A random locally convex module over K with base $(\Omega, \mathcal{F}, P)$ is an order pair $(E, \mathcal{P})$, where E is an $L^0(\mathcal{F}, K)$-module and $\mathcal{P}$ is a separating family of L^0-seminorms on E.

Similarly, $\mathcal{P}$ can induce the two kinds of topologies, namely the (ε,λ)-topology and the locally L^0-convex topology, and the two random versions for a random locally convex module of the classical Fenchel-Moreau dual representation theorem also hold under the two kinds of topologies, which are proved in [20] and will be used in Theorem 2.15 below.

Similarly to the theory of classical duality pair, we have developed the theory of random duality pair under the framework of a random locally convex module so that one can understand the two kinds of random w^*-topologies, see [20] for details. We consider the natural random duality pair $\langle L^1_{\mathcal{F}}(\mathcal{E}), L^\infty_{\mathcal{F}}(\mathcal{E})\rangle$, namely $\langle\cdot,\cdot\rangle : L^1_{\mathcal{F}}(\mathcal{E}) \times L^\infty_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$ is given by $\langle x,y\rangle = E(xy \mid \mathcal{F})$ for any $x \in L^1_{\mathcal{F}}(\mathcal{E})$ and $y \in L^\infty_{\mathcal{F}}(\mathcal{E})$, since $L^1_{\mathcal{F}}(\mathcal{E})$ and $L^\infty_{\mathcal{F}}(\mathcal{E})$ both have the countable concatenation property, a proper L^0-convex function f from $L^\infty_{\mathcal{F}}(\mathcal{E})$ to $\bar{L}^0(\mathcal{F})$ is $\sigma_{\varepsilon,\lambda}(L^\infty_{\mathcal{F}}(\mathcal{E}), L^1_{\mathcal{F}}(\mathcal{E}))$-lower semicontinuous iff it is $\sigma_c(L^\infty_{\mathcal{F}}(\mathcal{E}), L^1_{\mathcal{F}}(\mathcal{E}))$-lower semicontinuous. Furthermore, for the random conjugate spaces of $L^\infty_{\mathcal{F}}(\mathcal{E})$ under the two kinds of random w^*-topologies, we have that $(L^\infty_{\mathcal{F}}(\mathcal{E}),\ \sigma(L^\infty_{\mathcal{F}}(\mathcal{E}), L^1_{\mathcal{F}}(\mathcal{E})))^*_{\varepsilon,\lambda} = (L^\infty_{\mathcal{F}}(\mathcal{E}), \sigma(L^\infty_{\mathcal{F}}(\mathcal{E}),\ L^1_{\mathcal{F}}(\mathcal{E})))^*_c = L^1_{\mathcal{F}}(\mathcal{E})$, cf. [18, 20]. Thus by the random version in random locally convex modules of the classical Fenchel-Moreau dual representation theorem and the similar techniques in [9], we can obtain Theorem 2.15 below.

Theorem 2.15. *Let $f : L^\infty_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ be a $\sigma_{\varepsilon,\lambda}(L^\infty_{\mathcal{F}}(\mathcal{E}), L^1_{\mathcal{F}}(\mathcal{E}))$ (or equivalently, $\sigma_c(L^\infty_{\mathcal{F}}(\mathcal{E}), L^1_{\mathcal{F}}(\mathcal{E}))$-lower semicontinuous $L^0(\mathcal{F})$-convex $L^\infty_{\mathcal{F}}(\mathcal{E})$-conditional risk measure). Then $f(x) = \bigvee\{E(xy \mid \mathcal{F}) - f^*(y) \mid y \in L^1_{\mathcal{F}}(\mathcal{E}), y \leqslant 0$ and $E(y \mid \mathcal{F}) = -1\}$ for all $x \in L^\infty_{\mathcal{F}}(\mathcal{E})$.*

In the final part of this section, let us give Lemma 2.16 below, which is almost obvious but frequently used in the proofs of the main results of this paper.

Lemma 2.16. *Let E be an $L^0(\mathcal{F})$-module with the countable concatenation property. Then we have the following statements*:

(1) *Let $f : E \to \bar{L}^0(\mathcal{F})$ have the local property and $x = \Sigma^\infty_{n=1}\tilde{I}_{A_n}x_n$ for some countable partition $\{A_n, n \in \mathbb{N}\}$ of Ω to $\mathcal{F}$ and some sequence $\{x_n, n \in \mathbb{N}\}$ in E, then $f(x) = \Sigma^\infty_{n=1}\tilde{I}_{A_n}f(x_n)$.*

(2) *Let $f : E \to \bar{L}^0(\mathcal{F})$ have the local property and $G \subset E$ be a nonempty subset, then $\bigvee\{f(x) \mid x \in G\} = \bigvee\{f(x) \mid x \in H_{cc}(G)\}$.*

(3) *Let f and g be any two functions from E to $\bar{L}^0(\mathcal{F})$ such that they both have the local property and $G \subset E$ a nonempty subset. If $f(x) = g(x)$ for all $x \in G$, then $f(x) = g(x)$ for all $x \in H_{cc}(G)$.*

(4) *Let $\{f_\alpha, \alpha \in \Gamma\}$ be a family of functions from E to $\bar{L}^0(\mathcal{F})$ such that each f_α has the locally property, then $f : E \to \bar{L}^0(\mathcal{F})$ defined by $f(x) = \bigvee\{f_\alpha(x) \mid \alpha \in \Gamma\}$ for all $x \in E$, also has the local property.*

3 The relations between $L^p(\mathcal{E})$ and $L^p_{\mathcal{F}}(\mathcal{E})$

In the sequel of this paper, $(\Omega, \mathcal{E}, P)$ always denotes a probability space, $\mathcal{F}$ a sub-σ-algebra of $\mathcal{E}$ and $L^p(\mathcal{E}) := L^p(\Omega, \mathcal{E}, P)$ $(1 \leqslant p \leqslant +\infty)$.

Let us first give a useful proposition.

Proposition 3.1 (See [14, 19])**.** *Let $(E, \|\cdot\|)$ be an RN module over K with base $(\Omega, \mathcal{F}, P)$ and $1 \leqslant p \leqslant +\infty$. Let $L^p(E) = \{x \in E \mid \|x\|_p < +\infty\}$, where $\|\cdot\|_p : E \to [0, +\infty]$ is defined by*

$$\|x\|_p = \begin{cases} \left(\int_\Omega \|x\|^p dP\right)^{\frac{1}{p}}, & \text{when } 1 \leqslant p < +\infty, \\ \inf\{M \in [0, +\infty] \mid \|x\| \leqslant M\}, & \text{when } p = +\infty, \end{cases}$$

for all $x \in E$.

Then $(L^p(E), \|\cdot\|_p)$ is a normed space and $L^p(E)$ is $\mathcal{T}_{\varepsilon,\lambda}$-dense in E.

When we take $E = L^p_{\mathcal{F}}(\mathcal{E})$ in Proposition 3.1, it is obvious that $L^p(E) = L^p(\mathcal{E})$, so we have the following:

Corollary 3.2. *$L^p(\mathcal{E})$, regarded as a subspace of the RN module $(L^p_{\mathcal{F}}(\mathcal{E}), |||\cdot|||_p)$, is $\mathcal{T}_{\varepsilon,\lambda}$-dense in $L^p_{\mathcal{F}}(\mathcal{E})$.*

Proposition 3.3 below is a stronger proposition than Corollary 3.2, which will be needed in the proofs of Theorems 4.3, 4.4, 4.7 and 4.10 below.

Proposition 3.3. *Let $1 \leqslant p \leqslant +\infty$, then $L^p_{\mathcal{F}}(\mathcal{E}) = H_{cc}(L^p(\mathcal{E}))$.*

Proof. Since $L^p_{\mathcal{F}}(\mathcal{E})$ has the countable concatenation property, $H_{cc}(L^p(\mathcal{E})) \subset L^p_{\mathcal{F}}(\mathcal{E})$ is obvious. Conversely, since $L^p_{\mathcal{F}}(\mathcal{E}) = L^0(\mathcal{F}) \cdot L^p(\mathcal{E}) := \{\xi g \mid \xi \in L^0(\mathcal{F}) \text{ and } g \in L^p(\mathcal{E})\}$, for any $x = \xi g \in L^p_{\mathcal{F}}(\mathcal{E})$ for some $\xi \in L^0(\mathcal{F})$ and $g \in L^p(\mathcal{E})$, let ξ^0 be any chosen representative of ξ and $A_n = \{\omega \in \Omega \mid n-1 \leqslant |\xi^0(\omega)| < n\}$ for each $n \in \mathbb{N}$, then it is clear that $\xi = \Sigma_{n=1}^{\infty} \tilde{I}_{A_n}\xi$, and hence $x = \xi g = (\Sigma_{n=1}^{\infty}\tilde{I}_{A_n}\xi)g = \Sigma_{n=1}^{\infty}\tilde{I}_{A_n}(\tilde{I}_{A_n}\xi g) \in H_{cc}(L^p(\mathcal{E}))$ by an easy observation that each $\tilde{I}_{A_n}\xi g \in L^p(\mathcal{E})$. □

Remark 3.4. Let $x \in L^p_{\mathcal{F}}(\mathcal{E})$, $\{A_n, n \in \mathbb{N}\}$ be a countable partition of Ω to $\mathcal{F}$ and $\{x_n, n \in \mathbb{N}\}$ a sequence in $L^p(\mathcal{E})$ such that $x = \Sigma_{n=1}^{\infty}\tilde{I}_{A_n}x_n$. Since it is obvious that $\Sigma_{n=1}^{\infty}\tilde{I}_{A_n}|||x_n|||_p$ converges in probability measure P, $\Sigma_{n=1}^{\infty}\tilde{I}_{A_n}x_n$ both converges in $\mathcal{T}_{\varepsilon,\lambda}$ to x and unconditionally converges in $\mathcal{T}_{\varepsilon,\lambda}$ to x in $(L^p_{\mathcal{F}}(\mathcal{E}), |||\cdot|||_p)$.

Proposition 3.5 below is also crucial for the proofs of Theorems 4.3, 4.4, 4.7 and 4.10 below.

Proposition 3.5. *Let $1 \leqslant q \leqslant +\infty$ and γ be any positive number. Then we have the following statements*:

(1) *Let $G_1 = \{y \in L^q_{\mathcal{F}}(\mathcal{E}) \mid y \leqslant 0$ and $E(y \mid \mathcal{F}) = -1\}$ and $G_2 = \{y \in L^q(\mathcal{E}) \mid y \leqslant 0$ and $E(y \mid \mathcal{F}) = -1\}$, then $G_1 = H_{cc}(G_2)$.*

(2) *Let G_1 be the same as in* (1) *above, $G_3 = \{y \in L^q(\mathcal{E}) \mid y \leqslant 0, E(y \mid \mathcal{F}) = -1$ and $E(|y|^q \mid \mathcal{F}) \in L^{\infty}(\mathcal{F})\}$ and $G_4 = \{y \in L^q(\mathcal{E}) \mid y \leqslant 0, E(y \mid \mathcal{F}) = -1$ and $E(|y|^q \mid \mathcal{F}) \in L^{\gamma}(\mathcal{F})\}$, then $G_1 = H_{cc}(G_3) = H_{cc}(G_4)$.*

Proof. (1) It is obvious that G_1 has the countable concatenation property and $G_2 \subset G_1$, so $H_{cc}(G_2) \subset G_1$. Conversely, let y be a given element in G_1, ξ^0 any chosen representative of $|||y|||_q$, $A_n = \{\omega \in \Omega \mid n-1 \leqslant \xi^0(\omega) < n\}$ and $y_n = \tilde{I}_{A_n}y + \tilde{I}_{A_n^c}(-1)$ for all $n \in \mathbb{N}$. Then it is clear that each $y_n \in G_2$ and $y = (\Sigma_{n=1}^{\infty}\tilde{I}_{A_n})y = \Sigma_{n=1}^{\infty}\tilde{I}_{A_n}y = \Sigma_{n=1}^{\infty}\tilde{I}_{A_n}y_n$, so $y \in H_{cc}(G_2)$. Thus $G_1 = H_{cc}(G_2)$.

(2) It is obvious that $G_3 \subset G_4 \subset G_1$, so $H_{cc}(G_3) \subset H_{cc}(G_4) \subset G_1$. It remains to prove that $G_1 \subset H_{cc}(G_3)$ as follows: Let $y \in G_1$, ξ^0, A_n and y_n be the same as in the proof of (1) for each $n \in \mathbb{N}$, then it is very easy to observe that each y_n also belongs to G_3, so $y = \Sigma_{n=1}^{\infty}\tilde{I}_{A_n}y_n \in H_{cc}(G_3)$, which shows that $G_1 \subset H_{cc}(G_3)$. □

4 Extension theorem for conditional risk measures

4.1 Extension theorem for L^{∞}-conditional risk measures

Definition 4.1 (See [2,5]). A function $f : L^{\infty}(\mathcal{E}) \to L^{\infty}(\mathcal{F})$ is said to be

(1) monotone if $f(x) \leqslant f(y)$ for all $x, y \in L^{\infty}(\mathcal{E})$ such that $x \geqslant y$;

(2) cash invariant if $f(x+y) = f(x) - y$ for all $x \in L^{\infty}(\mathcal{E})$ and $y \in L^{\infty}(\mathcal{F})$;

(3) $\mathcal{F}$-local if $\tilde{I}_A f(x) = \tilde{I}_A f(\tilde{I}_A x)$ for any $A \in \mathcal{F}$ and $x \in L^{\infty}(\mathcal{E})$;

(4) $L^0(\mathcal{F})$-convex if $f(\xi x + (1-\xi)y) \leqslant \xi f(x) + (1-\xi)f(y)$ for any $\xi \in L^0_+(\mathcal{F})$ with $0 \leqslant \xi \leqslant 1$ and $x, y \in L^{\infty}(\mathcal{E})$.

Furthermore, a monotone and cash invariant function from $L^{\infty}(\mathcal{E})$ to $L^{\infty}(\mathcal{F})$ is called an L^{∞}-conditional risk measure.

Let $\mathcal{P}$ be the set of all the probability measures Q on $(\Omega, \mathcal{E})$ such that Q is absolutely continuous with respect to P on $\mathcal{E}$ and $\mathcal{P}_{\mathcal{F}} = \{Q \in \mathcal{P} \mid Q = P \text{ on } \mathcal{F}\}$.

For an L^{∞}-conditional risk measure $f : L^{\infty}(\mathcal{E}) \to L^{\infty}(\mathcal{F})$, $\alpha : \mathcal{P}_{\mathcal{F}} \to \bar{L}^0(\mathcal{F})$ defined by $\alpha(Q) = \bigvee\{E_Q(-x \mid \mathcal{F}) - f(x) : x \in L^{\infty}(\mathcal{E})\}$ for all $Q \in \mathcal{P}_{\mathcal{F}}$, is called the random penalty function of f, where $E_Q(\cdot \mid \mathcal{F})$ denotes the conditional expectation given $\mathcal{F}$ under Q. The following representation theorem is known.

Theorem 4.2 (See [2,5]). *Let $f: L^\infty(\mathcal{E}) \to L^\infty(\mathcal{F})$ be an $L^0(\mathcal{F})$-convex L^∞-conditional risk measure. Then the following statements are equivalent*:

(1) *f is continuous from above, namely $f(x_n) \nearrow f(x)$ whenever $x_n \searrow x$*;

(2) *f has the "Fatou property": For any bounded sequence $\{x_n, n \in \mathbb{N}\}$ which converges P-a.s. to some x, then $f(x) \leqslant \underline{lim}_n f(x_n)$*;

(3) *$f(x) = \bigvee\{E_Q(-x \mid \mathcal{F}) - \alpha(Q) \mid Q \in \mathcal{P}_\mathcal{F}\}$ for all $x \in L^\infty(\mathcal{E})$.*

For an $L^0(\mathcal{F})$-convex L^∞-conditional risk measure $f: L^\infty(\mathcal{E}) \to L^\infty(\mathcal{F})$, $f^*: L^1_\mathcal{F}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ defined by $f^*(y) = \bigvee\{E(xy \mid \mathcal{F}) - f(x) \mid x \in L^\infty(\mathcal{E})\}$ for all $y \in L^1_\mathcal{F}(\mathcal{E})$, is called the random conjugate function of f, where $E(\cdot \mid \mathcal{F})$ denotes the conditional expectation given $\mathcal{F}$ under P.

Identifying each $Q \in \mathcal{P}_\mathcal{F}$ with the Radon-Nikodým derivative $\frac{dQ}{dP}$, $\mathcal{P}_\mathcal{F}$ and $\{y \mid y \in L^1_+(\mathcal{E}), E(y \mid \mathcal{F}) = 1\}$ are identified, then (3) of Theorem 4.2 amounts to the following (4).

(4) $f(x) = \bigvee\{E(xy \mid \mathcal{F}) - f^*(y) \mid y \in L^1(\mathcal{E}), y \leqslant 0$ and $E(y \mid \mathcal{F}) = -1\}$.

Theorem 4.3. *Let $f: L^\infty(\mathcal{E}) \to L^\infty(\mathcal{F})$ be an L^∞-conditional risk measure. Then there is a unique $L^\infty_\mathcal{F}(\mathcal{E})$-conditional risk measure $\bar{f}: L^\infty_\mathcal{F}(\mathcal{E}) \to L^0(\mathcal{F})$ such that $|\bar{f}(x) - \bar{f}(y)| \leqslant |||x-y|||_\infty$ for all $x, y \in L^\infty_\mathcal{F}(\mathcal{E})$ and $\bar{f}|_{L^\infty(\mathcal{E})} = f$.*

Proof. Let us first recall the L^0-norm $|||\cdot|||_\infty : L^\infty_\mathcal{F}(\mathcal{E}) \to L^0_+(\mathcal{F})$, which is defined by $|||x|||_\infty = \bigwedge\{\xi \in \bar{L}^0_+(\mathcal{F}) \mid |x| \leqslant \xi\}$ for all $x \in L^\infty_\mathcal{F}(\mathcal{E})$, then it is obvious that $|||x|||_\infty \in L^\infty_+(\mathcal{F})$ for all $x \in L^\infty(\mathcal{E})$.

Since $x = y + x - y \leqslant y + |x-y| \leqslant y + |||x-y|||_\infty$ for all $x, y \in L^\infty(\mathcal{E})$, $f(x) \geqslant f(y + |||x-y|||_\infty) = f(y) - |||x-y|||_\infty$, namely $f(y) - f(x) \leqslant |||x-y|||_\infty$, which shows that $|f(y) - f(x)| \leqslant |||x-y|||_\infty$ for all $x, y \in L^\infty(\mathcal{E})$.

$L^\infty(\mathcal{E})$ is $\mathcal{T}_{\varepsilon,\lambda}$-dense in $L^\infty_\mathcal{F}(\mathcal{E})$ by Corollary 3.2. Furthermore, it is clear that f is uniformly $\mathcal{T}_{\varepsilon,\lambda}$-continuous from $(L^\infty(\mathcal{E}), |||\cdot|||_\infty)$ to $(L^\infty(\mathcal{F}), |\cdot|)$ (namely $L^\infty(\mathcal{E})$ is regarded as a subspace of $(L^\infty_\mathcal{F}(\mathcal{E}), |||\cdot|||_\infty)$ and $L^\infty(\mathcal{F})$ as a subspace of $(L^0(\mathcal{F}), |\cdot|)$). Thus f has a unique extension $\bar{f}: L^\infty_\mathcal{F}(\mathcal{E}) \to L^0(\mathcal{F})$ such that $|\bar{f}(x) - \bar{f}(y)| \leqslant |||x-y|||_\infty$ for all $x, y \in L^\infty_\mathcal{F}(\mathcal{E})$. Since f has the $\mathcal{F}$-local property, $\bar{f}$ must have this property. Further, by Proposition 3.3 every $x \in L^\infty_\mathcal{F}(\mathcal{E})$ can be expressed as $x = \Sigma_{n=1}^\infty \tilde{I}_{A_n} x_n$ for some countable partition $\{A_n, n \in \mathbb{N}\}$ of Ω to $\mathcal{F}$ and some sequence $\{x_n, n \in \mathbb{N}\}$ in $L^\infty(\mathcal{E})$ and every $y \in L^0(\mathcal{F})$ as $y = \Sigma_{n=1}^\infty \tilde{I}_{B_n} y_n$ for some countable partition $\{B_n, n \in \mathbb{N}\}$ of Ω to $\mathcal{F}$ and some sequence $\{y_n, n \in \mathbb{N}\}$ in $L^\infty(\mathcal{F})$, where the first series converges in $\mathcal{T}_{\varepsilon,\lambda}$ in $(L^\infty(\mathcal{E}), |||\cdot|||_\infty)$ and the second does in $(L^0(\mathcal{F}), |\cdot|)$. Thus one can easily see that $\bar{f}$ is monotone and cash invariant in the sense of Definition 2.13. □

In Theorem 4.3, when f is $L^0(\mathcal{F})$-convex, it is clear that $\bar{f}$ is also $L^0(\mathcal{F})$-convex.

Theorem 4.4. *Let $f: L^\infty(\mathcal{E}) \to L^\infty(\mathcal{F})$ be an $L^0(\mathcal{F})$-convex L^∞-conditional risk measure and $\bar{f}: L^\infty_\mathcal{F}(\mathcal{E}) \to L^0(\mathcal{F})$ the unique extension as determined in Theorem 4.3. Then the following statements are equivalent*:

(1) *f is continuous from above*;

(2) *f has the Fatou property*;

(3) *$f(x) = \bigvee\{E(xy \mid \mathcal{F}) - f^*(y) \mid y \in L^1(\mathcal{E}), y \leqslant 0$ and $E(y \mid \mathcal{F}) = -1\}$ for all $x \in L^\infty(\mathcal{E})$*;

(4) *$f(x) = \bigvee\{E(xy \mid \mathcal{F}) - f^*(y) \mid y \in L^1_\mathcal{F}(\mathcal{E}), y \leqslant 0$ and $E(y \mid \mathcal{F}) = -1\}$ for all $x \in L^\infty(\mathcal{E})$*;

(5) *$\bar{f}(x) = \bigvee\{E(xy \mid \mathcal{F}) - \bar{f}^*(y) \mid y \in L^1_\mathcal{F}(\mathcal{E}), y \leqslant 0$ and $E(y \mid \mathcal{F}) = -1\}$ for all $x \in L^\infty_\mathcal{F}(\mathcal{E})$*;

(6) *$\bar{f}$ is $\sigma_{\varepsilon,\lambda}(L^\infty_\mathcal{F}(\mathcal{E}), L^1_\mathcal{F}(\mathcal{E}))$ (or equivalently, $\sigma_c(L^\infty_\mathcal{F}(\mathcal{E}), L^1_\mathcal{F}(\mathcal{E}))$)-lower semicontinuous.*

Proof. (1)⇔(2)⇔(3) is just Theorem 4.2.

(3)⇔(4). The equivalence relation is easily seen by applying Proposition 3.5(1) for $q = 1$ and Lemma 2.16(2) since $E(xy \mid \mathcal{F}) - f^*(y)$ is a local function of y for each fixed $x \in L^\infty(\mathcal{E})$.

(5)⇒(4). Before the proof, let us first notice that $f^*(y) = \bar{f}^*(y)$ for all $y \in L^1_\mathcal{F}(\mathcal{E})$: since $f^*(y) = \bigvee\{E(xy \mid \mathcal{F}) - f(x) \mid x \in L^\infty(\mathcal{E})\}$, $\bar{f}^*(y) = \bigvee\{E(xy \mid \mathcal{F}) - \bar{f}(x) \mid x \in L^\infty_\mathcal{F}(\mathcal{E})\}$ and $L^\infty_\mathcal{F}(\mathcal{E}) = H_{cc}(L^\infty(\mathcal{E}))$ by Proposition 3.3, then $\bar{f}^*(y) = \bigvee\{E(xy \mid \mathcal{F}) - f(x) \mid x \in L^\infty(\mathcal{E})\} = f^*(y)$ by noticing that both $E(xy \mid \mathcal{F})$ and $\bar{f}(x)$ are local functions of $x \in L^\infty_\mathcal{F}(\mathcal{E})$ for each fixed $y \in L^1_\mathcal{F}(\mathcal{E})$ and applying Lemma 2.16(2) to the local function $g: L^\infty_\mathcal{F}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ defined by $g(x) = E(xy \mid \mathcal{F}) - \bar{f}(x)$.

(4)⇒(5). by Lemma 2.16(4), it is clear that $\bar{g}: L^\infty_\mathcal{F}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ defined by $\bar{g}(x) = \bigvee\{E(xy \mid \mathcal{F}) - \bar{f}^*(y) \mid y \in L^1_\mathcal{F}(\mathcal{E}), y \leqslant 0$ and $E(y \mid \mathcal{F}) = -1\}$ for any $x \in L^\infty_\mathcal{F}(\mathcal{E})$ is local since $E(xy \mid \mathcal{F}) - \bar{f}^*(y)$

is a local function of x for each fixed y. By applying Lemma 2.16(3) to $\bar{f}$ and $\bar{g}$ one can see that $\bar{f}=\bar{g}$ since $L^\infty_{\mathcal{F}}(\mathcal{E})=H_{cc}(L^\infty(\mathcal{E}))$ and it is just (4) that $\bar{f}(x)=\bar{g}(x)$ for all $x\in L^\infty(\mathcal{E})$.

(5)⇒(6) is clear.

(6)⇒(5) is by Theorem 2.15. □

4.2 Extension theorem for L^p-conditional risk measures

Motivated by the work in [21,22], Filipović et al. [9] introduced the following L^p-conditional risk measures as follows:

Definition 4.5 (See [9]). Let $1\leqslant r\leqslant p<+\infty$ and f be a function from $L^p(\mathcal{E})$ to $L^r(\mathcal{F})$. f is an L^p-conditional risk measure if the following two conditions are satisfied:

(1) $f(x)\leqslant f(y)$ for all $x,y\in L^p(\mathcal{E})$ with $x\geqslant y$;

(2) $f(x+y)=f(x)-y$ for all $x\in L^p(\mathcal{E})$ and $y\in L^p(\mathcal{F})$.

Similar to Definition 4.1, one can have the notions of $\mathcal{F}$-local and $L^0(\mathcal{F})$-convex L^p-conditional risk measures. The following representation result is known:

Proposition 4.6 (See [9]). *Let f be an $L^0(\mathcal{F})$-convex L^p-conditional risk measure from $L^p(\mathcal{E})$ to $L^r(\mathcal{F})$ and $1\leqslant r\leqslant p<+\infty$. If f is continuous from $(L^p(\mathcal{E}),\|\cdot\|_p)$ to $(L^r(\mathcal{E}),\|\cdot\|_r)$, then*

$$f(x)=\bigvee\{E(xy\mid F)-f^*(y)\mid y\in L^q(\mathcal{E}),y\leqslant 0,E(|y|^q\mid\mathcal{F})\in L^{\frac{r(p-1)}{p-r}}(\mathcal{F})\text{ and }E(y\mid\mathcal{F})=-1\}$$

for all $x\in L^p(\mathcal{E})$, where $\frac{1}{p}+\frac{1}{q}=1$ and $\frac{r(p-1)}{p-r}=+\infty$ when $p=r$ and $f^*:L^q_{\mathcal{F}}(\mathcal{E})\to\bar{L}^0(\mathcal{F})$ is defined by $f^*(y)=\bigvee\{E(xy\mid\mathcal{F})-f(x)\mid x\in L^p(\mathcal{E})\}$ for all $y\in L^q_{\mathcal{F}}(\mathcal{E})$.

The aim of this subsection is to give an extension theorem for any $L^0(\mathcal{F})$-convex L^p-conditional risk measure, in particular, in this process we also improve Proposition 4.6 in that we can give a necessary and sufficient condition that any $L^0(\mathcal{F})$-convex L^p-conditional risk measure can be represented as in Proposition 4.6, in fact, a new and shorter proof of Proposition 4.6 will be also given.

An important special case of $L^0(\mathcal{F})$-convex L^p-conditional risk measures is the following conditional risk measure derived from the solution to backward stochastic differential equations (BSDE, for short).

Let R^d be the d-dimensional Euclidean space of d-tuples of real numbers, whose elements are described in terms of column vectors, $(B_t)_{t\geqslant 0}$ a standard d-dimensional Brown motion defined on a probability space $(\Omega,\mathcal{F},P)$ and $(\mathcal{F}_t)_{t\geqslant 0}$ the augmented filtration generated by $(B_t)_{t\geqslant 0}$.

Let a function $g:\mathbb{R}^+\times\Omega\times\mathbb{R}^d\to\mathbb{R}$, $(t,\omega,z)\to g(t,\omega,z)$ (briefly,$g(t,z)$) satisfy the following conditions:

(A) g is Lipschitz in z, i.e., there exists a constant $\mu>0$ such that we have $dt\times dP$-a.s., for any $z_0,z_1\in\mathbb{R}^d$, $|g(t,z_0)-g(t,z_1)|\leqslant\mu\|z_0-z_1\|$.

(B) For all $z\in\mathbb{R}^d$, $g(\cdot,z)$ is a predictable process such that for any $T>0$, $E[\int_0^T g(t,\omega,z)^2dt]<+\infty$ for any $z\in\mathbb{R}^d$.

(C) $dt\times dP$-a.s., $g(t,0)=0$.

(D) g is convex in z: $\forall\,\alpha\in[0,1]$, $\forall\,z_0,z_1\in\mathbb{R}^d$, $dt\times dP$-a.s., $g(t,\alpha z_0+(1-\alpha)z_1)\leqslant\alpha g(t,z_0)+(1-\alpha)g(t,z_1)$.

Then, for any $T>0$, the following BSDE:

$$\begin{cases}-dY_t=g(t,Z_t)dt-Z_t'dB_t,\\ Y_T=\xi,\end{cases}$$

where $\xi\in L^2(\Omega,\mathcal{F}_T,P)$ and Z_t' is the transpose of Z_t, has a unique solution $(Y_t,Z_t)_{t\in[0,T]}$ consisting of predictable stochastic processes such that $E[\int_0^T Y_t^2dt]<+\infty$ and $E[\int_0^T\|Z_t\|^2dt]<+\infty$, cf. [4,21]. Peng [21] defined the conditional g-expectation of ξ at time t as

$$\mathcal{E}_g(\xi\mid\mathcal{F}_t):=Y_t.$$

Now, for any fixed $t\in[0,T]$, define $\rho_t^g(\cdot):L^2(\mathcal{F}_T)\to L^2(\mathcal{F}_t)$ by $\rho_t^g(x)=\mathcal{E}_g(-x\mid\mathcal{F}_t)$ for all $x\in L^2(\mathcal{F}_T)$, then ρ_t^g is an $L^0(\mathcal{F}_t)$-convex L^2-conditional risk measure. By [21, Theorem 3.2], ρ_t^g is continuous from

$(L^2(\mathcal{F}_T), \|\cdot\|_2)$ to $(L^2(\mathcal{F}_t), \|\cdot\|_2)$. Again by [21, Theorem 3.2], $|\rho_t^g(x) - \rho_t^g(y)| \leqslant c(E[|x-y|^2|\mathcal{F}_t])^{\frac{1}{2}}$ for all $x, y \in L^2(\mathcal{F}_T)$, where $c = \mathrm{e}^{8(1+\mu^2)(T-t)}$, namely when $L^2(\mathcal{F}_T)$ is regarded as a subspace of the RN module $(L^2_{\mathcal{F}_t}(\mathcal{F}_T), |||\cdot|||_2)$ and $L^2(\mathcal{F}_t)$ is regarded as a subspace of the RN module $(L^0(\mathcal{F}_t), |\cdot|)$, ρ_t^g is Lipschitz with respect to the L^0-norms. So, ρ_t^g can be uniquely extended to an $L^0(\mathcal{F}_t)$-convex $L^2_{\mathcal{F}_t}(\mathcal{F}_T)$-conditional risk measure $\bar{\rho}_t^g$ such that $|\bar{\rho}_t^g(x) - \bar{\rho}_t^g(y)| \leqslant c(E[|x-y|^2|\mathcal{F}_t])^{\frac{1}{2}}$ for all $x, y \in L^2_{\mathcal{F}_t}(\mathcal{F}_T)$ since $L^2(\mathcal{F}_T)$ is $\mathcal{T}_{\varepsilon,\lambda}$-dense in $(L^2_{\mathcal{F}_t}(\mathcal{F}_T), |||\cdot|||_2)$, the proof is the same as that of Theorem 4.3.

Since a general $L^0(\mathcal{F})$-convex L^p-conditional risk measure from $L^p(\mathcal{E})$ to $L^r(\mathcal{F})$ may not necessarily $\mathcal{T}_{\varepsilon,\lambda}$-uniformly continuous when $L^p(\mathcal{E})$ is regarded a subspace of $(L^p_{\mathcal{F}}(\mathcal{E}), |||\cdot|||_p)$ and $L^r(\mathcal{F})$ as a subspace of $(L^0(\mathcal{F}), |\cdot|)$, we are forced to use a constructive way to obtain a unique extension, namely Theorem 4.7 below.

Theorem 4.7. *Let $f : L^p(\mathcal{E}) \to L^r(\mathcal{F})$ be an $L^0(\mathcal{F})$-convex L^p-conditional risk measure. Then there is a unique $L^0(\mathcal{F})$-convex $L^p_{\mathcal{F}}(\mathcal{E})$-conditional risk measure $\bar{f} : L^p_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$ such that $\bar{f}|_{L^p(\mathcal{E})} = f$.*

Proof. Define $\bar{f} : L^p_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$ by $\bar{f}(x) = \Sigma_{n=1}^{\infty} \tilde{I}_{A_n} f(x_n)$ for any canonical representation $\Sigma_{n=1}^{\infty} \tilde{I}_{A_n} x_n$ of x,

First, $\bar{f}$ is well defined. In fact, for any two canonical representations $\Sigma_{n=1}^{\infty} \tilde{I}_{A_n} x_n$ and $\Sigma_{n=1}^{\infty} \tilde{I}_{B_n} y_n$ of x, $\Sigma_{n=1}^{\infty} \tilde{I}_{A_n} f(x_n) = \Sigma_{i,j=1}^{\infty} \tilde{I}_{A_i \cap B_j} f(x_i) = \Sigma_{i,j=1}^{\infty} \tilde{I}_{A_i \cap B_j} f(y_j) = \Sigma_{n=1}^{\infty} \tilde{I}_{B_n} f(y_n)$.

Second, $\bar{f}$ is $L^0(\mathcal{F})$-convex: Let $\xi \in L^0_+(\mathcal{F})$ such that $0 \leqslant \xi \leqslant 1$ and $x, y \in L^p_{\mathcal{F}}(\mathcal{E})$ have the canonical representations $\Sigma_{n=1}^{\infty} \tilde{I}_{A_n} x_n$ and $\Sigma_{n=1}^{\infty} \tilde{I}_{B_n} y_n$, respectively. Then $x = \Sigma_{i,j=1}^{\infty} \tilde{I}_{A_i \cap B_j} x_i$ and $y = \Sigma_{i,j=1}^{\infty} \tilde{I}_{A_i \cap B_j} y_j$, and thus $\bar{f}(\xi x + (1-\xi) y) = \bar{f}(\Sigma_{i,j=1}^{\infty} \tilde{I}_{A_i \cap B_j} (\xi x_i + (1-\xi) y_j)) = \Sigma_{i,j=1}^{\infty} \tilde{I}_{A_i \cap B_j} f(\xi x_i + (1-\xi) y_j) \leqslant \xi \Sigma_{i,j=1}^{\infty} \tilde{I}_{A_i \cap B_j} f(x_i) + (1-\xi) \Sigma_{i,j=1}^{\infty} \tilde{I}_{A_i \cap B_j} f(y_j) = \xi \bar{f}(x) + (1-\xi) \bar{f}(y)$.

Similarly, $\bar{f}$ is also monotone. Furthermore, by Proposition 3.3 and the local property of $\bar{f}$, $\bar{f}$ is also cash invariant in the sense of Definition 2.13.

Finally, any $L^0(\mathcal{F})$-convex $L^p_{\mathcal{F}}(\mathcal{E})$-conditional risk measure g with $g|_{L^p(\mathcal{E})} = f$ must coincide with $\bar{f}$ since g has the local property by $L^p_{\mathcal{F}}(\mathcal{E}) = H_{cc}(L^p(\mathcal{E}))$ and applying Lemma 2.16(3) to $\bar{f}$ and g, which proves the uniqueness. □

Theorem 4.10 below shows that the continuity of f in Proposition 4.6 can be weakened to that $\bar{f}$ is $\mathcal{T}_{\varepsilon,\lambda}$ (or equivalently, $\mathcal{T}_c$)-lower semicontinuous, whereas the implication of the continuity of f is reflected to some extent by Theorem 4.8 below.

Theorem 4.8. *Let f and $\bar{f}$ be the same as in Theorem 4.7. If f is continuous from $(L^p(\mathcal{E}), \|\cdot\|_p)$ to $(L^r(\mathcal{F}), \|\cdot\|_r)$, then $\bar{f}$ is $\mathcal{T}_{\varepsilon,\lambda}$-continuous from $(L^p_{\mathcal{F}}(\mathcal{E}), |||\cdot|||_p)$ to $(L^0(\mathcal{F}), |\cdot|)$.*

Proof. When $L^p(\mathcal{E})$ is regard as a subspace $(L^p_{\mathcal{F}}(\mathcal{E}), |||\cdot|||_p)$ and $(L^r(\mathcal{F}), \|\cdot\|_r)$ is regarded as a subspace of $(L^0(\mathcal{F}), |\cdot|)$, we first prove that f is $\mathcal{T}_{\varepsilon,\lambda}$-continuous from $(L^p(\mathcal{E}), |||\cdot|||_p)$ to $(L^r(\mathcal{F}), |\cdot|)$. For this, we only need to prove that, for each fixed $x_0 \in L^p(\mathcal{E})$ and each sequence $\{x_n, n \in \mathbb{N}\}$ in $L^p(\mathcal{E})$ such that $\{E[|x_n - x_0|^p \mid \mathcal{F}]^{\frac{1}{p}} : n \in \mathbb{N}\}$ converges in probability measure P to 0, there exists a subsequence $\{x_{n_k}, k \in \mathbb{N}\}$ of $\{x_n, n \in \mathbb{N}\}$ such that $\{f(x_{n_k}), k \in \mathbb{N}\}$ converges in probability measure P to $f(x_0)$. Since f is $\mathcal{F}$-local, we only need to prove that, for any positive number δ, there exist an $\mathcal{F}$-measurable subset H_δ of Ω and a subsequence $\{x_{n_k}, k \in \mathbb{N}\}$ of $\{x_n, n \in \mathbb{N}\}$ such that $P(\Omega \setminus H_\delta) > 1 - \delta$ and $\{f(x_{n_k}), k \in \mathbb{N}\}$ converges in probability measure P to $f(x_0)$ on $\Omega \setminus H_\delta$. In fact, by the Egoroff theorem there are such H_δ and $\{x_{n_k}, k \in \mathbb{N}\}$ such that $\{E[|x_{n_k} - x_0|^p|\mathcal{F}]^{\frac{1}{p}}, k \in \mathbb{N}\}$ converges uniformly to 0 on $\Omega \setminus H_\delta$, so that $\{\tilde{I}_{\Omega\setminus H_\delta} x_{n_k}, k \in \mathbb{N}\}$ converges to $\tilde{I}_{\Omega\setminus H_\delta} x_0$ in the usual L^p-norm $\|\cdot\|_p$ by the Lebesgue domination convergence theorem, hence $\{\tilde{I}_{\Omega\setminus H_\delta} f(x_{n_k}), k \in \mathbb{N}\}$ converges in the L^r-norm to $\tilde{I}_{\Omega\setminus H_\delta} f(x_0)$, which implies that $\{f(x_{n_k}), k \in \mathbb{N}\}$ converges in probability measure P to $f(x_0)$ on $\Omega \setminus H_\delta$.

We can now prove that $\bar{f}$ is $\mathcal{T}_{\varepsilon,\lambda}$-continuous. Let $\{x_k, k \in \mathbb{N}\}$ be a sequence in $L^p_{\mathcal{F}}(\mathcal{E})$ convergent in $\mathcal{T}_{\varepsilon,\lambda}$ to $x \in L^p_{\mathcal{F}}(\mathcal{E})$, where $\mathcal{T}_{\varepsilon,\lambda}$ is the (ε,λ)-topology on $L^p_{\mathcal{F}}(\mathcal{E})$ induced by $|||\cdot|||_p$, then for any canonical representation $\Sigma_{n=1}^{\infty} \tilde{I}_{A_n} x_n$ of x, we only need to prove that $\{\bar{f}(x_k), k \in \mathbb{N}\}$ converges in probability P to $\bar{f}(x)$ on each fixed A_n. Now, let $n_0 \in \mathbb{N}$ be fixed. For each given canonical representation $\Sigma_{n=1}^{\infty} \tilde{I}_{A_n^k} x_n^k$ of x_k, we choose $m_k \in \mathbb{N}$ such that $P(\Sigma_{n \geqslant m_k} A_n^k) < \frac{1}{k}$, then $\{\Sigma_{n=1}^{m_k} \tilde{I}_{A_n^k} x_n^k, k \in \mathbb{N}\}$ still converges in $\mathcal{T}_{\varepsilon,\lambda}$ to x, hence $\{\tilde{I}_{A_{n_0}} \Sigma_{n=1}^{m_k} \tilde{I}_{A_n^k} x_n^k, k \in \mathbb{N}\}$, of course, converges in $\mathcal{T}_{\varepsilon,\lambda}$ to $\tilde{I}_{A_{n_0}} x$ $(= \tilde{I}_{A_{n_0}} x_{n_0})$. By what we have proved, $\{f(\tilde{I}_{A_{n_0}} \Sigma_{n=1}^{m_k} \tilde{I}_{A_n^k} x_n^k), k \in \mathbb{N}\}$ converges in the probability measure P to $f(\tilde{I}_{A_{n_0}} x_{n_0})$. By the

$\mathcal{F}$-local property of f, $\tilde{I}_{A_{n_0}} f(\Sigma_{n=1}^{m_k} \tilde{I}_{A_n^k} x_n^k) = \tilde{I}_{A_{n_0}} f(\tilde{I}_{A_{n_0}} \Sigma_{n=1}^{m_k} \tilde{I}_{A_n^k} x_n^k)$ and $\tilde{I}_{A_{n_0}} f(x_{n_0}) = \tilde{I}_{A_{n_0}} f(\tilde{I}_{A_{n_0}} x_{n_0})$, then $\{f(\tilde{I}_{A_{n_0}} \Sigma_{n=1}^{m_k} \tilde{I}_{A_n^k} x_n^k), k \in \mathbb{N}\}$ converges in the probability measure P to $f(x_{n_0})$ on A_{n_0}.

Finally, since $\bar{f}(x_k) = \tilde{I}_{\bigcup_{n=1}^{m_k} A_n^k} f(\Sigma_{n=1}^{m_k} \tilde{I}_{A_n^k} x_n^k)) + \Sigma_{n \geqslant m_k}^{\infty} \tilde{I}_{A_n^k} f(x_n^k)$ and $\tilde{I}_{A_{n_0}} f(x) = \tilde{I}_{A_{n_0}} \bar{f}(x) = \tilde{I}_{A_{n_0}} \bar{f}(\tilde{I}_{A_{n_0}} x) = \tilde{I}_{A_{n_0}} f(x_{n_0})$, we have that $\{\bar{f}(x_k), k \in \mathbb{N}\}$ converges in the probability measure P to $\bar{f}(x)$ on A_{n_0} by noticing that $P(\Sigma_{n \geqslant m_k}^{\infty} A_n^k) \to 0$ when $k \to \infty$. □

Lemma 4.9. *Let $f : L^p(\mathcal{E}) \to L^r(\mathcal{F})$ be an $L^0(\mathcal{F})$-convex L^p-conditional risk measure and $\bar{f} : L^p_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$ its unique extension. Then we have that $f^*(y) = \bar{f}^*(y)$ for all $y \in L^q_{\mathcal{F}}(\mathcal{E})$.*

Proof. Let us recall: $f^*(y) = \bigvee\{E(xy \mid \mathcal{F}) - f(x) \mid x \in L^p(\mathcal{E})\}$ and $\bar{f}^*(y) = \bigvee\{E(xy \mid \mathcal{F}) - \bar{f}(x) \mid x \in L^p_{\mathcal{F}}(\mathcal{E})\}$. By Proposition 3.3, $L^p_{\mathcal{F}}(\mathcal{E}) = H_{cc}(L^p(\mathcal{E}))$, which, together with the local property of $g : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ defined by $g(x) = E(xy \mid \mathcal{F}) - \bar{f}(x)$ for all $x \in L^p_{\mathcal{F}}(\mathcal{E})$ implies the $f^*(y) = \bar{f}^*(y)$ for all $y \in L^q_{\mathcal{F}}(\mathcal{E})$ by applying Lemma 2.16(2) to the local function g and $G := L^p(\mathcal{E})$. □

Theorem 4.10. *Let $1 \leqslant r \leqslant p < +\infty$, q be the Hölder conjugate number of p, $f : L^p(\mathcal{E}) \to L^r(\mathcal{F})$ an $L^0(\mathcal{F})$-convex L^p-conditional risk measure and $\bar{f} : L^p_{\mathcal{F}}(\mathcal{E}) \to L^0(\mathcal{F})$ the unique extension of f determined by Theorem 4.7. Then the following statements are equivalent*:

(1) *$f(x) = \bigvee\{E(xy \mid \mathcal{F}) - f^*(y) \mid y \in L^q(\mathcal{E}), y \leqslant 0, E(|y|^q \mid \mathcal{F}) \in L^\infty(\mathcal{F})$ and $E(y \mid \mathcal{F}) = -1\}$ for all $x \in L^p(\mathcal{E})$.*

(2) *For any given positive number γ, $f(x) = \bigvee\{E(xy \mid \mathcal{F}) - f^*(y) \mid y \in L^q(\mathcal{E}), y \leqslant 0, E(|y|^q \mid \mathcal{F}) \in L^\gamma(\mathcal{F})$ and $E(y \mid \mathcal{F}) = -1\}$ for all $x \in L^p(\mathcal{E})$.*

(3) *$f(x) = \bigvee\{E(xy \mid \mathcal{F}) - f^*(y) \mid y \in L^q(\mathcal{E}), y \leqslant 0, E(|y|^q \mid \mathcal{F}) \in L^{\frac{r(p-1)}{p-r}}(\mathcal{F})$ and $E(y \mid \mathcal{F}) = -1\}$ for all $x \in L^p(\mathcal{E})$.*

(4) *$f(x) = \bigvee\{E(xy \mid \mathcal{F}) - f^*(y) \mid y \in L^q_{\mathcal{F}}(\mathcal{E}), y \leqslant 0$ and $E(y \mid \mathcal{F}) = -1\}$ for all $x \in L^p(\mathcal{E})$.*

(5) *$\bar{f}(x) = \bigvee\{E(xy \mid \mathcal{F}) - \bar{f}^*(y) \mid y \in L^q_{\mathcal{F}}(\mathcal{E}), y \leqslant 0$ and $E(y \mid \mathcal{F}) = -1\}$ for all $x \in L^p_{\mathcal{F}}(\mathcal{E})$.*

(6) *$\bar{f}$ is $\mathcal{T}_{\varepsilon,\lambda}$- (or equivalently, $\mathcal{T}_c$-) lower semicontinuous.*

(7) *$\bar{f}$ is continuous from $(L^p_{\mathcal{F}}(\mathcal{E}), \mathcal{T}_c)$ to $(L^0(\mathcal{F}), \mathcal{T}_c)$.*

Proof. For any fixed $x \in L^p(\mathcal{E})$, let $g(y) = E(xy \mid \mathcal{F}) - f^*(y)$ for any $y \in L^q_{\mathcal{F}}(\mathcal{E})$, then g has the local property. By applying Lemma 2.16(2) and Proposition 3.5(2) one can easily see that (1)⇔(2)⇔(3)⇔(4).

By Lemma 4.9, $f^* = \bar{f}^*$, so (5)⇒(4) is clear. If (4) is true, let $\bar{g} : L^p_{\mathcal{F}}(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$ be defined by

$$\bar{g}(x) = \bigvee\{E(xy \mid \mathcal{F}) - \bar{f}^*(y) \mid y \in L^q_{\mathcal{F}}(\mathcal{E}), y \leqslant 0 \text{ and } E(y \mid \mathcal{F}) = -1\}$$

for any $x \in L^p_{\mathcal{F}}(\mathcal{E})$, then by Lemma 2.16(3) we have that $\bar{f} = \bar{g}$ since (4) just shows that $\bar{f}(x) = \bar{g}(x)$ for any $x \in L^p(\mathcal{E})$, which shows (4)⇒(5).

(5)⇒(6) is clear.

(6)⇒(5) is by Proposition 4.6.

(6)⇒(7) is by [20, Theorem 3.44].

(7)⇒(6) is clear. □

Theorem 4.10 not only gives a very short proof of Filipović et al. [9, Proposition 4.6], whose original proof in [9] is somewhat complicated, but also improves Proposition 4.6 in that we give a necessary and sufficient condition for Theorem 4.10(3) to hold, namely that $\bar{f}$ is $\mathcal{T}_{\varepsilon,\lambda}$- (or $\mathcal{T}_c$-) lower semicontinuous. Besides, maybe Theorem 4.10(1) is more interesting.

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