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Gradient estimates and coupling property for semilinear SDEs driven by jump processes

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Abstract Let L be a Lévy process with characteristic measure ν, which has an absolutely continuous lower bound w.r.t. the Lebesgue measure on $\mathbb{R}^n$. By using Malliavin calculus for jump processes, we investigate Bismut formula, gradient estimates and coupling property for the semigroups associated to semilinear SDEs forced by Lévy process L.

Keywords jump processes, Bismut formula, gradient estimates, coupling property

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1 Introduction

In stochastic analysis for diffusion processes, the Bismut formula (also called Bismut-Elworthy-Li formula [7]) and the integration by parts formula are two fundamental tools. Let, for example, $\{X_t\}_{t\geqslant 0}$ be a diffusion process on $\mathbb{R}^n$ generated by an elliptic differential operator and $\{P_t\}_{t\geqslant 0}$ be the associated Markov semigroup. The Bismut formula is of type

$$\nabla_\xi P_t f(x) = \mathbb{E}\{f(X_t^x)M_t^x\}, \quad f \in \mathscr{B}_b(\mathbb{R}^n), \quad t > 0,$$

where M_t^x is a random variable independent of f, and ∇_ξ is the directional derivative along ξ. This formula is applied to various aspects such as functional inequalities, heat kernel estimates, strong Feller properties and sensitivity analysis, see [1, 2, 7, 14] and the references therein for diffusion cases and jump-diffusion cases. In recent years, there exist some results for jump processes. For example, Norris established an integration by parts formula and obtained the heat kernel estimates in [8] for SDEs driven by jump processes; in [17] and [19], Bismut formula for linear SDEs driven by Lévy processes were derived by using coupling method; in [22], the formula was also investigated for nonlinear SDEs driven by α-stable processes. With the help of Malliavin calculus for jump processes, Dong et al. [6] obtained Bismut formula for linear SDEs driven by general Lévy processes. There are some other results about this topic, such as [3, 13, 21] and so on.

Let $\{P_t\}_{t\geqslant 0}$ and $\{P_t(x,\cdot)\}_{t\geqslant 0}$ be the semigroup and transition probability kernel for a strong Markov process on a Polish space U. If $X := \{X_t\}_{t\geqslant 0}$ and $Y := \{Y_t\}_{t\geqslant 0}$ are two processes with the same transition probability kernel $\{P_t(x,\cdot)\}_{t\geqslant 0}$, then $\{X, Y\} = \{X_t, Y_t\}_{t\geqslant 0}$ is called a coupling of the strong Markov process with coupling time $T_{x,y} := \inf\{t \geqslant 0 : X_t = Y_t\}$. The coupling is called successful if

$T_{x,y} < \infty$ a.s. The recent results for coupling and applications in SDEs were summarized in [18] for both diffusion and jump processes, see also [11, 12, 16] and the references therein.

The dimension-free Harnack inequality was first introduced in [15] by Wang using couplings constructed through Girsanov transforms for diffusion semigroups on manifolds. It can be formulated as

$$(P_t f)^\alpha(x) \leqslant P_t f^\alpha(x+y) \exp\{C_\alpha(y,t)\},$$

where $\alpha > 1$ is a constant, and $C_\alpha(y,t)$ is a positive function on $\mathbb{R}^n \times (0,\infty)$ with $C_\alpha(0,t) = 0$, which is determined by the underlying stochastic equation. Since arguments used in most of references essentially rely on special properties of the Brownian motion (see [1,2,5,16] and so on), they do not apply to the jump setting. For SDEs driven by pure jump processes, a different version of Harnack inequality was presented in [20]. In [17] and [19], with helps of coupling and new Girsanov transform on the configuration space, Wang succeeded in establishing derivative formula and dimension-free Harnack inequality for linear SDEs driven by Lévy processes.

In this paper, we aim to investigate Bismut formula, gradient estimates, dimension-free Harnack inequality and coupling property for the transition semigroups associated to a kind of semilinear SDEs driven by jump processes. From now on, for $i = 1, 2$, we use $C_b^i(\mathbb{R}^n)$ to denote the family of C^i functions f such that f and its partial derivatives up to order i are bounded. We denote the uniform norm with respect to x by $\|\cdot\|_\infty$. Let $\mathscr{B}_b(\mathbb{R}^n)$ be the class of all bounded measurable functions on $\mathbb{R}^n$. The rest of this paper is organized as follows: in the second section, we list some notation and our main results; in the last section, we shall give the proofs of main results.

2 Notation and main results

For $i = 1, 2$, let W_i be the space of all càdlàg functions from $[0,\infty)$ to $\mathbb{R}^n$ vanishing at 0, which is endowed with the Skorohod topology and the probability measure $\mathbb{P}^i$ such that the coordinate process $L_t^i(w^i) = w_t^i$ is a Lévy process. Furthermore, we assume that $L^1 := \{L_t^1\}_{t\geqslant 0}$ is a purely jump process with Lévy measure $\nu_1(dz) := \rho(z)dz$, where $\rho : \mathbb{R}_0^n \to (0,\infty)$ is a differentiable function satisfying $\int_{\mathbb{R}_0^n} (|z|^2 \wedge 1)\rho(z)dz < \infty$.

Consider the following product probability space, $(\Omega, \mathscr{F}, \mathbb{P}) := (W_1 \times W_2, \mathscr{B}(W_1) \times \mathscr{B}(W_2), \mathbb{P}^1 \times \mathbb{P}^2)$, and define $L_t = L_t^1 + L_t^2$, i.e., for $w = (w^1, w^2) \in \Omega, L_t(w) = w_t^1 + w_t^2$. Then $\{L_t\}_{t\geqslant 0}$ is a Lévy process on $(\Omega, \mathscr{F}, \mathbb{P})$ with two independent parts and its Lévy measure denoted by ν satisfies $\nu(dz) \geqslant \rho(z)dz$. Denote by $\{\mathscr{F}_t\}_{t\geqslant 0}$ the smallest filtration generated by $\{L_t\}_{t\geqslant 0}$. We use N^1 and $\widetilde{N}^1$ to denote the jump measure and martingale measure of $\{L_t^1\}_{t\geqslant 0}$. Let $\mathbb{E}$ and $\mathbb{E}^1$ be the associated expectations of $\mathbb{P}$ and $\mathbb{P}^1$ respectively.

This paper is concerned with the following stochastic differential equation (SDE) with jumps,

$$\begin{cases} dX_t = b(X_t)dt + \sigma_t dL_t, \\ X_0 = x, \end{cases} \tag{2.1}$$

where $b : \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : [0,\infty) \to \mathbb{R}^n \otimes \mathbb{R}^n$ are measurable. It is well known that there exists a unique solution of (2.1) provided the coefficient b satisfies Lipschitz conditions and the solution can be formulated as

$$X_t^x = x + \int_0^t b(X_s)ds + \int_0^t \sigma_s dL_s, \qquad t \geqslant 0. \tag{2.2}$$

We gather here the hypotheses which will be made on (2.1).

(H1) $b \in C^1(\mathbb{R}^n)$ with ∇b bounded and Lipschitz continuous. There exists a constant $\beta > 0$ such that $|\sigma_s^{-1}| \leqslant \beta$ for any $s > 0$.

(H2) There is a constant $K > 0$, such that $\langle b(x) - b(y), x - y\rangle \leqslant -K|x-y|^2$ for any $x, y \in \mathbb{R}^n$.

As we know, $\lambda_0 := \int_{\mathbb{R}_0^n} \rho(z)dz$ is either finite or infinite. If $\lambda_0 = \infty$, then we aim to investigate the Bismut formula, gradient estimate and coupling property for the associated semigroup

$$P_t f(x) := \mathbb{E} f(X_t^x), \quad t \geqslant 0, \quad x \in \mathbb{R}^n, \quad f \in \mathscr{B}_b(\mathbb{R}^n).$$

Otherwise, we consider P_t^1 instead of P_t,

$$P_t^1 f(x) := \mathbb{E}\{f(X_t^x) I_{[N_t \geqslant 1]}\}, \quad t \geqslant 0, \quad x \in \mathbb{R}^n, \quad f \in \mathscr{B}_b(\mathbb{R}^n),$$

where $N_t = N^1([0,t] \times \mathbb{R}^n)$.

Recall that the solution has successful coupling if and only if (see [4])

$$\lim_{t\to\infty} \|P_t(x,\cdot) - P_t(y,\cdot)\|_{\mathrm{Var}} = 0, \quad x, y \in \mathbb{R}^n,$$

where $P_t(x,\cdot)$ is the transition probability of X_t^x and $\|\cdot\|_{\mathrm{Var}}$ denotes the total variation norm.

Let J_t be the derivative of X_t^x w.r.t. the initial value x. We have the following main results.

Theorem 2.1. *Let* (H1) *hold and* $\rho \in C^1(\mathbb{R}^n)$ *with* $\int_{\mathbb{R}^n} |\nabla \rho(z)|dz < \infty$. *For* $t > 0$, $\xi \in \mathbb{R}^n$ *and* $f \in C_b(\mathbb{R}^n)$, *we have*

$$\nabla_\xi P_t^1 f(x) = -\mathbb{E}\bigg\{ f(X_t^x) \frac{I_{[N_t \geqslant 1]}}{N_t} \int_0^t \int_{\mathbb{R}^n} \nabla \log \rho(z) \cdot (\sigma_s^{-1} J_s \xi) \widetilde{N^1}(dz, ds) \bigg\}.$$

Furthermore,

$$\|\nabla P_t^1 f\|_\infty \leqslant \frac{4\beta \mathrm{e}^{t\|\nabla b\|_\infty}}{\lambda_0} (1 - \mathrm{e}^{-\lambda_0 t} - \mathrm{e}^{-\lambda_0 t} \lambda_0 t) \|f\|_\infty \int_{\mathbb{R}^n} |\nabla \rho(z)| dz.$$

Theorem 2.2. *Let* (H1)–(H2) *hold and* $\rho \in C^1(\mathbb{R}^n)$ *with* $\int_{\mathbb{R}^n} |\nabla \rho(z)|dz < \infty$. *Then for* $t > \frac{K+\lambda_0}{K}$,

$$\|P_t(x,\cdot) - P_t(y,\cdot)\|_{\mathrm{Var}} \leqslant \bigg\{ \frac{4\beta}{K\lambda_0} \int_{\mathbb{R}^n} |\nabla \rho(z)| dz |x-y| + 2 \bigg\} \mathrm{e}^{-\frac{K\lambda_0}{K+\lambda_0} t}.$$

The above two results are under the condition that $\rho(z)dz$ is a finite measure. Before we introduce the main results in the σ-finite case, we would like to list some notation. Denote

$$L^i = \bigg\{ h : \mathbb{R}^n \to \mathbb{R} \,\bigg|\, \int_{\mathbb{R}_0^n} |h(z)|^i \nu_1(dz) < \infty \bigg\}, \quad i = 1, 2;$$
$$L_+^1 = \{h \mid h \geqslant 0 \text{ and } h \in L^1\};$$
$$\mathscr{C} = \{h : \mathbb{R}^n \to \mathbb{R} \mid h \text{ is differential and has compact support in } \mathbb{R}_0^n\}.$$

For $h \in \mathscr{C}$, we define a weighted norm as

$$\|h\|_\rho = \bigg\{ \int_{\mathbb{R}_0^n} |\nabla h(z)|^2 \nu_1(dz) \bigg\}^{\frac{1}{2}} + \bigg\{ \int_{\mathbb{R}_0^n} h^2(z) |\nabla \log \rho(z)|^2 \nu_1(dz) \bigg\}^{\frac{1}{2}}.$$

Let $\overline{\mathscr{C}}^{\|\cdot\|_\rho}$ be the closure of $\mathscr{C}$ under $\|\cdot\|_\rho$. Denote $\mathcal{H}_\rho = \{h \in L_+^1 \cap L^2 \mid h \in \overline{\mathscr{C}}^{\|\cdot\|_\rho} \text{ and } \nabla h \text{ is bounded}\}$.

We have the following results.

Theorem 2.3. *Let* (H1) *hold and* $\rho \in C^1(\mathbb{R}_0^n)$. *If* $\theta := \liminf_{x\to\infty} \frac{\nu_1([h \geqslant x^{-1}])}{\log x} > 0$ *for some* $h \in \mathcal{H}_\rho$, *then for* $t > \frac{8}{(\theta \wedge 1)(1-\mathrm{e}^{-1})}$, $\xi \in \mathbb{R}^n$ *and* $f \in C_b(\mathbb{R}^n)$,

$$\begin{aligned}\nabla_\xi P_t f(x) = -\mathbb{E}\bigg\{ f(X_t^x) \bigg[& H_t^{-1} \int_0^t \int_{\mathbb{R}_0^n} \frac{\nabla(\rho(z)h(z))}{\rho(z)} \cdot \sigma_s^{-1} J_s \xi \widetilde{N^1}(dz, ds) \\ & + H_t^{-2} \int_0^t \int_{\mathbb{R}_0^n} \nabla h(z) \cdot (\sigma_s^{-1} J_s h(z) \xi) N^1(dz, ds) \bigg] \bigg\},\end{aligned}$$

where $H_t = \int_0^t \int_{\mathbb{R}_0^n} h(z) N^1(dz, ds)$. *Furthermore,*

$$\|\nabla P_t f\|_\infty \leqslant C\left(1 + \frac{1}{t - \frac{8}{(\theta \wedge 1)(1-\mathrm{e}^{-1})}}\right)\beta \|f\|_\infty \exp\{\|\nabla b\|_\infty t\} \times \left\{\left\|\frac{|\nabla(h\rho)|}{\rho}\right\|_{L^2} + 2\|\nabla h\|_\infty \sqrt{\|h\|_{L^2}^2 + \|h\|_{L^1}^2}\right\},$$

where C *is a constant independent of* t.

Theorem 2.4. *Let* (H1)–(H2) *hold and* $\rho \in C^1(\mathbb{R}_0^n)$. *If* $\liminf_{x\to\infty} \frac{\nu_1([h \geqslant x^{-1}])}{\log x} > 0$ *for some* $h \in \mathcal{H}_\rho$, *then for any* $\alpha \in (0,1)$, *there exists a constant* $C(\alpha)$ *independent of* t *such that*

$$\|P_t(x,\cdot) - P_t(y,\cdot)\|_{\mathrm{Var}} \leqslant \frac{C(\alpha)}{K} \mathrm{e}^{-\alpha K t}.$$

Remark 2.5. Under either the conditions of Theorem 2.2 or Theorem 2.4, if there exists at least one invariant measure for $\{X_t^x\}_{t\geqslant 0}$ and each invariant measure is integrable, then $\{X_t^x\}_{t\geqslant 0}$ is exponentially ergodic.

We shall use Malliavin calculus for jump processes to prove the above results. So it is necessary for us to recall the notion of L^1-derivative and an integration by parts formula for jump processes. For $T > 0$, let $V = \{V(s,z)\}_{s\leqslant T, z\in\mathbb{R}_0^n}$ be a predictable process satisfying $\mathbb{E}|\int_0^T \int_{\mathbb{R}_0^n} V(s,z) N^1(dz, ds)| < \infty$. Define a perturbed random measure $N^{1,\epsilon}$ by

$$N^{1,\epsilon}(B \times [0,t]) = \int_0^t \int_{\mathbb{R}_0^n} I_B(z + \epsilon V(s,z)) N^1(dz, ds), \tag{2.3}$$

where $\int_B \rho(z)dz < \infty$. Let $\{L_t^\epsilon\}_{0\leqslant t\leqslant T}$ be the associated Lévy process perturbed by ϵV, i.e.,

$$L_t^\epsilon = L_t + \epsilon \int_0^t \int_{\mathbb{R}_0^n} V(s,z) N^1(dz, ds).$$

Definition 2.6 (See [3])**.** A function $F_t(L) := F(\{L_s\}_{s\leqslant t})$ is called to have an L^1-derivative in the direction V, if there exists an integrable random variable denoted by $D_V F_t(L)$, such that

$$\lim_{\epsilon\to 0} \mathbb{E}\left|\frac{F_t(L^\epsilon) - F_t(L)}{\epsilon} - D_V F_t(L)\right| = 0.$$

With this notion, we have an integration by parts formula. Denote

$$\mathcal{V} = \{V : \Omega \times [0,T] \times \mathbb{R}_0^n \to \mathbb{R}^n \mid V \text{ is predictable with } V \text{ and } D_z V \text{ bounded}, \exists\, U_0 \subset \mathbb{R}_0^n \text{ compact, s.t. } \mathrm{Supp} V \subset [0,T] \times U_0\}.$$

Proposition 2.7. *Let* $\rho \in C^1(\mathbb{R}_0^n)$. *If a bounded function* $F_t(L)$ *has an* L^1*-derivative* $D_V F_t(L)$ *for* $V \in \mathcal{V}$, *then*

$$\mathbb{E}\{D_V F_t(L)\} = -\mathbb{E}\{F_t(L) R_t\}, \tag{2.4}$$

where

$$R_t = \int_0^t \int_{\mathbb{R}_0^n} \frac{\mathrm{div}(\rho(z) V(s,z))}{\rho(z)} \widetilde{N^1}(dz, ds).$$

This proof can be obtained by the same argument of Proposition 2.2 in [6].

Remark 2.8. Denote

$$\widetilde{\mathcal{V}} = \{V : \Omega \times [0,T] \times \mathbb{R}_0^n \to \mathbb{R}^n \mid V \text{ is predictable with } V \text{ and } D_z V \text{ bounded}, \exists\, U \subset \mathbb{R}^n \text{ compact, s.t. } \mathrm{Supp} V \subset [0,T] \times U\}.$$

If the condition $\rho \in C^1(\mathbb{R}_0^n)$ is replaced by $\rho \in C^1(\mathbb{R}^n)$, which implies $\rho(z)dz$ is a finite measure, then (2.4) also holds for $V \in \widetilde{\mathcal{V}}$.

3 Proofs of main results

In this section, we would like to give the proofs of the theorems listed in Section 2. Before we move on, it is necessary for us to show the existence of L^1-derivative of (2.2). For $T>0$, denote

$$L^2([0,T])=\left\{k:[0,T]\to[0,\infty)\,\middle|\,\int_0^T k^2(s)ds<\infty\right\}.$$

Lemma 3.1. *Assume $b\in C^2(\mathbb{R}^n)$ with bounded derivatives. Let V be a predictable process satisfying*

$$|V(s,z)|\leqslant k(s)h(z),\quad \forall\, s\in[0,T],\quad \forall\, z\in\mathbb{R}_0^n,$$

for a nondecreasing function $k\in L^2([0,T])$ and $h\in L^{1+}\cap L^2$. Then X_t has an L^1-derivative. Moreover, the L^1-derivative satisfies the following equations,

$$\begin{cases} dD_VX_t=\nabla b(X_t)D_VX_tdt+\sigma_t\displaystyle\int_{\mathbb{R}_0^n}V(t,z)N^1(dz,dt),\\ D_VX_0=0.\end{cases}\tag{3.1}$$

We omit this proof since it is standard and can be found in [3].

Let $J_t=\nabla_xX_t^x$. Then J_t satisfies

$$\begin{cases} dJ_t=\nabla b(X_t^x)J_tdt,\\ J_0=I.\end{cases}\tag{3.2}$$

The inverse matrix J_t^{-1} solves the following equations,

$$\begin{cases} dJ_t^{-1}=-J_t^{-1}\nabla b(X_t^x)dt,\\ J_0^{-1}=I.\end{cases}\tag{3.3}$$

Combining (3.1)–(3.3), we can obtain

$$D_VX_t=J_t\int_0^t\int_{\mathbb{R}_0^n}J_s^{-1}\sigma_sV(s,z)N^1(dz,ds).\tag{3.4}$$

Proof of Theorem 2.1. Due to an approximation argument, the proof can be divided into two steps.

Step1. $b\in C^2(\mathbb{R}^n)$ with bounded derivatives. For $k\in\mathbb{N}$, take $U_k=\{z\in\mathbb{R}^n\mid |z|\leqslant k\}$ and

$$\phi_k(z)=\begin{cases}1, & z\in U_k,\\ \dfrac{1}{2}(1+\cos(\pi(|z|-k))), & z\in U_{k+1}\backslash U_k,\\ 0, & z\in\mathbb{R}^n\backslash U_{k+1}.\end{cases}\tag{3.5}$$

Then $I_{U_k}\leqslant\phi_k\leqslant I_{U_{k+1}}$ and $|\nabla\phi_k|\leqslant\frac{\pi}{2}I_{\mathbb{R}^n\backslash U_{k+1}}$. For any $\xi\in\mathbb{R}^n$, set $V_k(s,z)=\sigma_s^{-1}J_s\xi\phi_k(z)$ and $V(s,z)=\sigma_s^{-1}J_s\xi$. Let $\chi:[0,\infty)\to[0,\infty)$ be a smooth function satisfying $\chi(0)=0$ and $\chi(i)=1$ for all $i\in\mathbb{N}$. Then $\chi(N_t)=I_{[N_t\geqslant1]}$. In the following steps, we can use $\chi(N_t)$ to replace $I_{[N_t\geqslant1]}$ if necessary. It follows from (2.4) and Remark 2.2 that

$$\begin{aligned}\mathbb{E}\left\{D_{V_k}\left(f(X_t^x)\frac{I_{[N_t\geqslant1]}}{N_t}\right)\right\}&=-\mathbb{E}\left\{f(X_t^x)\frac{I_{[N_t\geqslant1]}}{N_t}\int_0^t\int_{\mathbb{R}^n}\frac{\mathrm{div}(\rho(z)V_k(s,z))}{\rho(z)}\widetilde{N^1}(dz,ds)\right\}\\ &=-\mathbb{E}\left\{f(X_t^x)\frac{I_{[N_t\geqslant1]}}{N_t}\int_0^t\int_{\mathbb{R}^n}(\nabla\log\rho(z)\cdot(\sigma_s^{-1}J_s\xi\phi_k(z))\right.\\ &\qquad\left.+\sigma_s^{-1}J_s\xi\cdot\nabla\phi_k(z))\widetilde{N^1}(dz,ds)\right\}.\end{aligned}\tag{3.6}$$

Note that

$$\begin{aligned}
&\mathbb{E}\bigg|D_{V_k}\bigg(f(X_t^x)\frac{I_{[N_t\geqslant 1]}}{N_t}\bigg)-D_V\bigg(f(X_t^x)\frac{I_{[N_t\geqslant 1]}}{N_t}\bigg)\bigg|\\
&\quad\leqslant \|\nabla f\|_\infty\mathbb{E}|D_{V_k}X_t^x-D_VX_t^x|\\
&\quad\leqslant t\mathrm{e}^{\|\nabla b\|_\infty t}\|\nabla f\|_\infty\int_{\mathbb{R}^n}(1-\phi_k(z))\rho(z)dz\to 0,\quad k\to\infty,\\
&\mathbb{E}\bigg|\int_0^t\int_{\mathbb{R}^n}\nabla\log\rho(z)\cdot(\sigma_s^{-1}J_s\xi)(1-\phi_k(z))\widetilde{N^1}(dz,ds)\bigg|\to 0,\quad k\to\infty,
\end{aligned}$$

and

$$\mathbb{E}\bigg|\int_0^t\int_{\mathbb{R}^n}\sigma_s^{-1}J_s\xi\cdot\nabla\phi_k(z)\widetilde{N^1}(dz,ds)\bigg|\to 0,\quad k\to\infty.$$

Let $k\to\infty$ in (3.6), then we obtain

$$\mathbb{E}\bigg\{D_V\bigg(f(X_t^x)\frac{I_{[N_t\geqslant 1]}}{N_t}\bigg)\bigg\}=-\mathbb{E}\bigg\{f(X_t^x)\frac{I_{[N_t\geqslant 1]}}{N_t}\int_0^t\int_{\mathbb{R}^n}\nabla\log\rho(z)\cdot(\sigma_s^{-1}J_s\xi)\widetilde{N^1}(dz,ds)\bigg\}. \tag{3.7}$$

It follows from (3.4) that $D_VX_t^x=N_tJ_t\xi$, then

$$\frac{I_{[N_t\geqslant 1]}}{N_t}D_VX_t^x=J_t\xi I_{[N_t\geqslant 1]}. \tag{3.8}$$

By (3.7) and (3.8) one arrives at

$$\begin{aligned}
\nabla_\xi\mathbb{E}\{f(X_t^x)I_{[N_t\geqslant 1]}\}&=\mathbb{E}\{\nabla f(X_tx)J_t\xi I_{[N_t\geqslant 1]}\}=\mathbb{E}\bigg\{\nabla f(X_t^x)\frac{I_{[N_t\geqslant 1]}}{N_t}D_VX_t^x\bigg\}\\
&=\mathbb{E}\bigg\{D_Vf(X_t^x)\frac{I_{[N_t\geqslant 1]}}{N_t}\bigg\}=\mathbb{E}\bigg\{D_V\bigg(f(X_t^x)\frac{I_{[N_t\geqslant 1]}}{N_t}\bigg)\bigg\}\\
&=-\mathbb{E}\bigg\{f(X_t^x)\frac{I_{[N_t\geqslant 1]}}{N_t}\int_0^t\int_{\mathbb{R}^n}\nabla\log\rho(z)\cdot(\sigma_s^{-1}J_s\xi)\widetilde{N^1}(dz,ds)\bigg\}.
\end{aligned}\tag{3.9}$$

Note that

$$\begin{aligned}
\mathbb{E}\bigg\{\frac{I_{[N_t\geqslant 1]}}{N_t}\bigg\}&=\mathrm{e}^{-\lambda_0t}\sum_{k=1}^\infty\frac{1}{k}\frac{(\lambda_0t)^k}{k!}=\frac{\mathrm{e}^{-\lambda_0t}}{\lambda_0t}\sum_{k=1}^\infty\frac{k+1}{k}\frac{(\lambda_0t)^{k+1}}{(k+1)!}\\
&\leqslant 2\frac{\mathrm{e}^{-\lambda_0t}}{\lambda_0t}\sum_{k=1}^\infty\frac{(\lambda_0t)^{k+1}}{(k+1)!}=\frac{2}{\lambda_0t}(1-\mathrm{e}^{-\lambda_0t}-\mathrm{e}^{-\lambda_0t}\lambda_0t).
\end{aligned}$$

Then,

$$\begin{aligned}
|\nabla_\xi P_t^1f(x)|&=|\nabla_\xi\mathbb{E}\{f(X_t^x)I_{[N_t\geqslant 1]}\}|\\
&=\bigg|\mathbb{E}\bigg\{f(X_t^x)\frac{I_{[N_t\geqslant 1]}}{N_t}\int_0^t\int_{\mathbb{R}^n}\nabla\log\rho(z)\cdot(\sigma_s^{-1}J_s\xi)\widetilde{N^1}(dz,ds)\bigg\}\bigg|\\
&\leqslant 2\|f\|_\infty\mathrm{e}^{t\|\nabla b\|_\infty}\beta|\xi|t\int_{\mathbb{R}^n}|\nabla\rho(z)|dz\mathbb{E}\frac{I_{[N_t\geqslant 1]}}{N_t}\\
&\leqslant\frac{4}{\lambda_0}\|f\|_\infty\mathrm{e}^{t\|\nabla b\|_\infty}\beta|\xi|(1-\mathrm{e}^{-\lambda_0t}-\mathrm{e}^{-\lambda_0t}\lambda_0t)\int_{\mathbb{R}^n}|\nabla\rho(z)|dz.
\end{aligned}\tag{3.10}$$

Step 2. $b\in C^1(\mathbb{R}^n)$ with ∇b bounded and Lipschitz continuous. Now we can choose a sequence of functions $\{b_k\}_{k\geqslant 1}\subset C^2(\mathbb{R}^n)$ such that $b_k\to b$ and $\nabla b_k\to\nabla b$ in pointwise sense as $k\to\infty$ and $\sup_{k\geqslant 1}\|\nabla b_k\|_\infty<\infty$. For each $k\geqslant 1$, consider the following equations,

$$\begin{cases}dX_t^k=b_k(X_t^k)dt+\sigma_tdL_t,\\ X_0^k=x.\end{cases}\tag{3.11}$$

Let $\{X_t^k\}_{t\geqslant 0}$ be the solution of (3.11) and $\{J_t^k\}_{t\geqslant 0}$ be the associated derivative with respect to x. It is easy to prove that $\lim_{k\to\infty}\mathbb{E}\{\sup_{s\leqslant t}|X_t^k-X_t|^2\}=0$ and $\lim_{k\to\infty}\mathbb{E}\{\sup_{s\leqslant t}|J_t^k-J_t|\}=0$. So with the help of dominated convergence theorem, we can obtain that (3.9) holds with $b\in C^1(\mathbb{R}^n)$ and ∇b bounded and Lipschitz continuous. Furthermore,

$$\|\nabla P_t^1 f\|_\infty \leqslant \frac{4\mathrm{e}^{t\|\nabla b\|_\infty}\beta}{\lambda_0}(1-\mathrm{e}^{-\lambda_0 t}-\mathrm{e}^{-\lambda_0 t}\lambda_0 t)\|f\|_\infty\int_{\mathbb{R}^n}|\nabla\rho(z)|dz. \qquad \square$$

Proof of Theorem 2.2. By (H2) we can get

$$d|X_t^x-X_t^y|^2=2\langle b(X_t^x)-b(X_t^y),X_t^x-X_t^y\rangle dt\leqslant -2K|X_t^x-X_t^y|^2dt.$$

Gronwall's inequality implies $|X_t^x-X_t^y|\leqslant \mathrm{e}^{-Kt}|x-y|$, which can derive

$$|J_t|\leqslant \mathrm{e}^{-Kt}. \tag{3.12}$$

(3.10) and (3.12) yield

$$\begin{aligned}|\nabla_\xi P_t^1 f(x)| &= \left|\mathbb{E}\left\{f(X_t^x)\frac{I_{[N_t\geqslant 1]}}{N_t}\int_0^t\int_{\mathbb{R}^n}\nabla\log\rho(z)\cdot(\sigma_s^{-1}J_s\xi)\widetilde{N^1}(dz,ds)\right\}\right|\\ &\leqslant 2\|f\|_\infty\beta|\xi|\int_0^t|J_s|ds\int_{\mathbb{R}^n}|\nabla\rho(z)|dz\mathbb{E}\left\{\frac{I_{[N_t\geqslant 1]}}{N_t}\right\}\\ &\leqslant \frac{4\beta|\xi|}{K\lambda_0 t}\int_{\mathbb{R}^n}|\nabla\rho(z)|dz\|f\|_\infty.\end{aligned}$$

Furthermore,

$$\begin{aligned}|P_tf(x)-P_tf(y)| &\leqslant |P_t^1f(x)-P_t^1f(y)|+|\mathbb{E}f(X_t^x)I_{[N_t=0]}-\mathbb{E}f(X_t^y)I_{[N_t=0]}|\\ &\leqslant \|\nabla P_t^1 f\|_\infty|x-y|+2\|f\|_\infty\mathbb{P}(N_t=0)\\ &\leqslant \frac{4\beta|\xi|}{K\lambda_0 t}\int_{\mathbb{R}^n}|\nabla\rho(z)|dz\|f\|_\infty|x-y|+2\|f\|_\infty\mathrm{e}^{-\lambda_0 t}.\end{aligned}$$

Combining this with (3.12) and using Markov property, we have, for $t>s>0$,

$$\begin{aligned}|P_tf(x)-P_tf(y)| &\leqslant \mathbb{E}|P_s(X_{t-s}^x)-P_s(X_{t-s}^y)|\\ &\leqslant \frac{4\beta}{K\lambda_0 s}\int_{\mathbb{R}^n}|\nabla\rho(z)|dz\|f\|_\infty|X_{t-s}^x-X_{t-s}^y|+2\|f\|_\infty\mathrm{e}^{-\lambda_0 s}\\ &\leqslant \frac{4\beta}{K\lambda_0 s}\int_{\mathbb{R}^n}|\nabla\rho(z)|dz\|f\|_\infty\mathrm{e}^{-K(t-s)}|x-y|+2\|f\|_\infty\mathrm{e}^{-\lambda_0 s}.\end{aligned}$$

Let $t>\frac{K+\lambda_0}{K}$ and $s=\frac{Kt}{K+\lambda_0}$. Then

$$|P_tf(x)-P_tf(y)|\leqslant\left\{\frac{4\beta}{K\lambda_0}\int_{\mathbb{R}^n}|\nabla\rho(z)|dz|x-y|+2\right\}\|f\|_\infty\mathrm{e}^{-\frac{K\lambda_0}{K+\lambda_0}t}.$$

Furthermore,

$$\|P_t(x,\cdot)-P_t(y,\cdot)\|_{\mathrm{Var}}\leqslant\left\{\frac{4\beta}{K\lambda_0}\int_{\mathbb{R}^n}|\nabla\rho(z)|dz|x-y|+2\right\}\mathrm{e}^{-\frac{K\lambda_0}{K+\lambda_0}t}.$$

The proof is complete. $\square$

Due to Theorem 2.1 and Young's inequality, we have further estimates.

Proposition 3.2. *Let* (H1) *hold and* $\rho\in C^1(\mathbb{R}^n)$ *with* $|\nabla\log\rho|$ *bounded. For any* $t>0$, $\xi\in\mathbb{R}^n$ *and a positive function* $f\in C_b(\mathbb{R}^n)$, *we have* $\nabla_\xi\log P_t^1f(x)\leqslant\beta t\mathrm{e}^{t\|\nabla b\|_\infty}\|\nabla\log\rho\|_\infty|\xi|(1+\lambda_0)$.

Proof. As in the proof of Theorem 2.1, we also take $V(s,z)=\sigma_s^{-1}J_s\xi$. Since $D_V N_t=0$, we have $D_V(\frac{I_{[N_t\geqslant 1]}}{N_t})=0$. Moreover,

$$\mathbb{E}\bigg\{\frac{I_{[N_t\geqslant 1]}}{N_t}R_t\bigg\}=0, \tag{3.13}$$

where $R_t=\int_0^t\int_{\mathbb{R}^n}\nabla\log\rho(z)\cdot(\sigma_s^{-1}J_s\xi)\widetilde{N^1}(dz,ds)$. For some $\epsilon>0$ and $f\in C_b^2(\mathbb{R}^n)$, by (3.9), (3.13) and Young's inequality (see [2]) we have

$$\begin{aligned}\nabla_\xi P_t^1 f(x) &= -\mathbb{E}\bigg\{f(X_t^x)\bigg(\frac{I_{[N_t\geqslant 1]}}{N_t}R_t\bigg)\bigg\}=\mathbb{E}\bigg\{(f(X_t^x)I_{[N_t\geqslant 1]}+\epsilon)\bigg(-\frac{I_{[N_t\geqslant 1]}}{N_t}R_t\bigg)\bigg\}\\ &\leqslant \delta\mathbb{E}\{(f(X_t^x)I_{[N_t\geqslant 1]}+\epsilon)\log\delta(f(X_t^x)I_{[N_t\geqslant 1]}+\epsilon)\}\\ &\quad-\delta\mathbb{E}\{f(X_t^x)I_{[N_t\geqslant 1]}+\epsilon\}\log\delta\mathbb{E}\{f(X_t^x)I_{[N_t\geqslant 1]}+\epsilon\}\\ &\quad+\delta\mathbb{E}\{f(X_t^x)I_{[N_t\geqslant 1]}+\epsilon\}\log\mathbb{E}\exp\bigg\{-\frac{I_{[N_t\geqslant 1]}}{\delta N_t}R_t\bigg\}.\end{aligned} \tag{3.14}$$

Note that

$$\begin{aligned}\mathbb{E}\exp\bigg\{-\frac{I_{[N_t\geqslant 1]}}{\delta N_t}R_t\bigg\} &\leqslant \mathbb{E}\exp\bigg\{\frac{I_{[N_t\geqslant 1]}}{\delta N_t}\int_0^t\int_{\mathbb{R}^n}|\nabla\log\rho(z)\cdot\sigma_s^{-1}J_s\xi|N^1(dz,ds)\\ &\quad+\frac{I_{[N_t\geqslant 1]}}{\delta N_t}\int_0^t\int_{\mathbb{R}^n}|\nabla\log\rho(z)\cdot\sigma_s^{-1}J_s\xi|\rho(z)dzds\bigg\}\\ &\leqslant \exp\bigg\{\frac{\beta t\mathrm{e}^{t\|\nabla b\|_\infty}}{\delta}\|\nabla\log\rho\|_\infty|\xi|(1+\lambda_0)\bigg\}.\end{aligned} \tag{3.15}$$

Combining this with (3.14), one can derive

$$\begin{aligned}\nabla_\xi P_t^1 f(x) &\leqslant \delta\mathbb{E}\{(f(X_t^x)I_{[N_t\geqslant 1]}+\epsilon)\log\delta(f(X_t^x)I_{[N_t\geqslant 1]}+\epsilon)\}\\ &\quad-\delta\mathbb{E}\{f(X_t^x)I_{[N_t\geqslant 1]}+\epsilon\}\log\delta\mathbb{E}\{f(X_t^x)I_{[N_t\geqslant 1]}+\epsilon\}\\ &\quad+\mathbb{E}\{f(X_t^x)I_{[N_t\geqslant 1]}+\epsilon\}\{\beta t\mathrm{e}^{t\|\nabla b\|_\infty}\|\nabla\log\rho\|_\infty|\xi|(1+\lambda_0)\}.\end{aligned} \tag{3.16}$$

By dominated convergence theorem and letting $\delta\to 0+$, one arrives at

$$\nabla_\xi P_t^1 f(x)\leqslant \mathbb{E}\{f(X_t^x)I_{[N_t\geqslant 1]}+\epsilon\}\{\beta t\mathrm{e}^{t\|\nabla b\|_\infty}\|\nabla\log\rho\|_\infty|\xi|(1+\lambda_0)\}.$$

Let $\epsilon\to 0+$, then $\nabla_\xi P_t^1 f(x)\leqslant \beta t\mathrm{e}^{t\|\nabla b\|_\infty}\|\nabla\log\rho\|_\infty|\xi|(1+\lambda_0)P_t^1 f(x)$. □

With the help of the proof of Proposition 3.2, the following dimension-free Harnack inequality for P_t^1 can be derived.

Theorem 3.3. *Let* (H1) *hold and* $\rho\in C^1(\mathbb{R}^n)$ *with* $|\nabla\log\rho|$ *bounded. For* $t>0$, $\xi\in\mathbb{R}^n$, $\alpha>1$ *and a positive function* $f\in C_b(\mathbb{R}^n)$, *we have*

$$(P_t^1 f)^\alpha(x)\leqslant P_t^1 f^\alpha(x+y)\exp\bigg\{\beta t\mathrm{e}^{t\|\nabla b\|_\infty}\|\nabla\log\rho\|_\infty|y|(1+\lambda_0)\frac{\ln\alpha}{\alpha-1}\bigg\}.$$

Proof. Set $\theta(s)=1+(\alpha-1)s$, then $\theta(0)=1$ and $\theta(1)=\alpha$. Due to (3.15), we have

$$\begin{aligned}&\frac{d}{ds}(\log(P_t^1 f^{\theta(s)})^{\frac{\alpha}{\theta(s)}}(x+sy))\\ &\quad=\frac{\alpha(\alpha-1)\{P_t^1(f^{\theta(s)}\log f^{\theta(s)})-P_t^1 f^{\theta(s)}\log P_t^1 f^{\theta(s)}\}}{\theta(s)^2 P_t^1 f^{\theta(s)}}(x+sy)-\frac{\alpha\nabla_y P_t^1 f^{\theta(s)}}{\theta(s)P_t^1 f^{\theta(s)}}(x+sy)\\ &\quad=\frac{\alpha}{\theta(s)}\bigg\{\frac{\alpha-1}{\theta(s)}\frac{\{P_t^1(f^{\theta(s)}\log f^{\theta(s)})-P_t^1 f^{\theta(s)}\log P_t^1 f^{\theta(s)}\}}{P_t^1 f^{\theta(s)}}(x+sy)-\frac{\nabla_y P_t^1 f^{\theta(s)}}{P_t^1 f^{\theta(s)}}(x+sy)\bigg\}\\ &\quad\geqslant -\frac{\alpha(\alpha-1)}{\theta^2(s)}\log\mathbb{E}\exp\bigg\{\frac{-\theta(s)}{\alpha-1}\frac{I_{[N_t\geqslant 1]}}{N_t}R_t\bigg\}\geqslant -\frac{\alpha}{\theta(s)}\beta t\mathrm{e}^{t\|\nabla b\|_\infty}\|\nabla\log\rho\|_\infty|y|(1+\lambda_0).\end{aligned} \tag{3.17}$$

Taking integral over $[0,1]$ w.r.t. ds, we arrive at

$$\log(P_t^1 f^\alpha)(x+y) - \log(P_t^1 f)^\alpha(x) \geqslant -\beta t \mathrm{e}^{t\|\nabla b\|_\infty}\|\nabla \log\rho\|_\infty |y|(1+\lambda_0)\int_0^1 \frac{\alpha}{\theta(s)}ds$$
$$= -\beta t \mathrm{e}^{t\|\nabla b\|_\infty}\|\nabla \log\rho\|_\infty |y|(1+\lambda_0)\frac{\ln\alpha}{\alpha-1}.$$

Therefore, $(P_t^1 f)^\alpha(x) \leqslant P_t^1 f^\alpha(x+y)\exp\{\beta t\mathrm{e}^{t\|\nabla b\|_\infty}\|\nabla\log\rho\|_\infty|y|(1+\lambda_0)\frac{\ln\alpha}{\alpha-1}\}$. □

As an immediate consequence of Theorem 3.3, the following result can be obtained.

Corollary 3.4. *Under the conditions of Theorem* 3.3*, the following estimate holds:*

$$(P_t^1 f)(x) \leqslant P_t^1 f(x+y)\exp\{\beta t\mathrm{e}^{t\|\nabla b\|_\infty}\|\nabla\log\rho\|_\infty|y|(1+\lambda_0)\}.$$

We would like to give an example to illustrate that conditions for ρ in Theorem 2.1 can be satisfied.

Example 3.5. Take $\rho(z) = \mathrm{e}^{-S(|z|)}$, where $S:(0,\infty)\to[0,\infty)$ is differentiable and satisfies:

(1) $\lim_{r\to+\infty}\frac{r}{S(r)} \leqslant C$ for $C \geqslant 0$;

(2) $\int_0^\infty S'(r)\mathrm{e}^{-r}dr < \infty$.

A simple choice of S is $S(r) = r^k$ for $k \geqslant 1$.

As we know, in order to obtain strong Feller property of P_t, it is crucial for Lévy measure to be an infinite measure, see [9] and [19]. Let us first give the following result about the pure jump processes. Let $\{M_t\}_{t\geqslant 0}$ be a Lévy process on $\mathbb{R}^n$ with jump measure $N_M(dz,dt)$ and characteristic measure κ. Let $h:\mathbb{R}^n\to[0,\infty)$ be a function satisfying $\int_{\mathbb{R}^n_0} h(z)\kappa(dz) < \infty$. Define $H_{M,t} := \int_0^t\int_{\mathbb{R}^n_0} h(z)N_M(dz,ds)$.

Proposition 3.6. *If* $\theta := \liminf_{x\to\infty}\frac{\kappa([h\geqslant x^{-1}])}{\log x} > 0$*, then for* $p\geqslant 1$ *and* $t > \frac{2p}{(\theta\wedge 1)(1-\mathrm{e}^{-1})}$ *there exists a constant* $C(p)$ *independent of* t *such that*

$$\mathbb{E}H_{M,t}^{-p} \leqslant C(p)\left(1+\frac{1}{t-\frac{2p}{(\theta\wedge1)(1-\mathrm{e}^{-1})}}\right).$$

Proof. Since $\theta > 0$, there exists $x_0 > 1$ such that $\kappa([h\geqslant x^{-1}]) \geqslant \frac{1}{2}(\theta\wedge 1)\log x$ for any $x \geqslant x_0$. Therefore,

$$\int_{\mathbb{R}^n_0}(1-\mathrm{e}^{-xh(z)})\kappa(dz) \geqslant \int_{[h\geqslant x^{-1}]}(1-\mathrm{e}^{-1})\kappa(dz) \geqslant \frac{\theta\wedge 1}{2}(1-\mathrm{e}^{-1})\log x. \tag{3.18}$$

Let $F_t(dx)$ be the distribution function of $H_{M,t}$ and $\Gamma(x)$ denote the Γ-function. Recall the formula

$$x^{-p} = \frac{1}{\Gamma(p)}\int_0^\infty r^{p-1}\mathrm{e}^{-rx}dr, \quad x>0. \tag{3.19}$$

By (3.18) and (3.19), we have

$$\begin{aligned}
\mathbb{E}H_{M,t}^{-p} &= \int_0^\infty x^{-p}F_t(dx) = \int_0^\infty\left(\frac{1}{\Gamma(p)}\int_0^\infty r^{p-1}\mathrm{e}^{-rx}dr\right)F_t(dx)\\
&= \frac{1}{\Gamma(p)}\int_0^\infty\left(\int_0^\infty \mathrm{e}^{-rx}F_t(dx)\right)r^{p-1}dr\\
&= \frac{1}{\Gamma(p)}\int_0^\infty r^{p-1}\exp\left\{-t\int_{\mathbb{R}^n_0}(1-\mathrm{e}^{-rh(z)})\kappa(dz)\right\}dr\\
&\leqslant \frac{x_0^p}{p\Gamma(p)} + \frac{1}{\Gamma(p)}\int_{x_0}^\infty r^{p-1}\exp\left\{-t\frac{\theta\wedge1}{2}(1-\mathrm{e}^{-1})\log r\right\}dr\\
&= \frac{x_0^p}{p\Gamma(p)} + \frac{1}{\Gamma(p)}\frac{x_0^{p-\frac{\theta\wedge1}{2}(1-\mathrm{e}^{-1})t}}{\frac{\theta\wedge1}{2}(1-\mathrm{e}^{-1})t-p} \leqslant C(p)\left(1+\frac{1}{t-\frac{2p}{(\theta\wedge1)(1-\mathrm{e}^{-1})}}\right).
\end{aligned} \tag{3.20}$$

The proof is complete. □

Proof of Theorem 2.3. For $\xi \in \mathbb{R}^n$ and $h \in \mathcal{H}_\rho$, we set $V(s,z) = \sigma_s^{-1} J_s h(z)\xi$. By (3.4), one can obtain

$$D_V X_t = H_t J_t \xi, \tag{3.21}$$

where $H_t := \int_0^t \int_{\mathbb{R}_0^n} h(z) N^1(dz, ds)$. Choose $\{h_k\} \subset \mathscr{C}$ such that $\|h_k - h\|_\rho \to 0$ as $k \to \infty$. Let $V_k(s,z) = \sigma_s^{-1} J_s h_k(z)\xi$. Then (2.4) holds for V_k. By the same argument as the beginning of the proof of Theorem 2.1, we can obtain that (2.4) holds for V. Therefore, for $f \in C_b^2(\mathbb{R}^n)$, (2.4) and (3.21) yield

$$\begin{aligned}\nabla_\xi \mathbb{E} f(X_t^x) &= \mathbb{E}\{\nabla f(X_t^x) J_t \xi\} = \mathbb{E}\{\nabla f(X_t^x) H_t^{-1} D_V X_t^x\} = \mathbb{E}\{H_t^{-1} D_V f(X_t^x)\} \\ &= \mathbb{E}\{D_V\{H_t^{-1} f(X_t^x)\} - D_V(H_t^{-1}) f(X_t^x)\} = \mathbb{E}\{f(X_t^x)\{-H_t^{-1} R_t - D_V(H_t^{-1})\}\} \\ &= -\mathbb{E}\bigg\{f(X_t^x)\bigg[H_t^{-1}\int_0^t \int_{\mathbb{R}_0^n} \frac{\nabla(\rho(z)h(z))}{\rho(z)} \cdot (\sigma_s^{-1} J_s \xi) \widetilde{N^1}(dz, ds) \\ &\quad + H_t^{-2} \int_0^t \int_{\mathbb{R}_0^n} \nabla h(z) \cdot (\sigma_s^{-1} J_s h(z)\xi) N^1(dz, ds)\bigg]\bigg\}. \end{aligned}\tag{3.22}$$

By Proposition 3.6, for $t > \frac{8}{(\theta\wedge 1)(1-\mathrm{e}^{-1})}$, there exists a constant C large enough and independent of t such that

$$(\mathbb{E} H_t^{-4}) \vee (\mathbb{E} H_t^{-2}) \leqslant C\bigg(1 + \frac{1}{t - \frac{8}{(\theta\wedge 1)(1-\mathrm{e}^{-1})}}\bigg). \tag{3.23}$$

Combining this with the triangle inequality and Hölder's inequality, one can obtain

$$\begin{aligned}|\nabla_\xi \mathbb{E} f(X_t^x)| &\leqslant \|f\|_\infty \bigg\{\mathbb{E}\bigg\{H_t^{-1}\bigg|\int_0^t \int_{\mathbb{R}_0^n} \frac{\nabla(\rho(z)h(z))}{\rho(z)} \cdot (\sigma_s^{-1} J_s \xi)\widetilde{N^1}(dz, ds)\bigg|\bigg\} \\ &\quad + \mathbb{E}\bigg\{H_t^{-2}\bigg|\int_0^t \int_{\mathbb{R}_0^n} \nabla h(z) \cdot (\sigma_s^{-1} J_s \xi) N^1(dz, ds)\bigg|\bigg\}\bigg\} \\ &\leqslant \|f\|_\infty \bigg\{(\mathbb{E} H_t^{-2})^{\frac{1}{2}} \bigg(\mathbb{E}\int_0^t \int_{\mathbb{R}_0^n} \bigg(\frac{\nabla(\rho(z)h(z))}{\rho(z)} \cdot (\sigma_s^{-1} J_s \xi)\bigg)^2 \rho(z) dz ds\bigg)^{\frac{1}{2}} \\ &\quad + (\mathbb{E} H_t^{-4})^{\frac{1}{2}} \bigg(\mathbb{E}\bigg|\int_0^t \int_{\mathbb{R}_0^n} \nabla h(z) \cdot (\sigma_s^{-1} J_s \xi) N^1(dz, ds)\bigg|^2\bigg)^{\frac{1}{2}}\bigg\} \\ &\leqslant C\bigg(1 + \frac{1}{t - \frac{8}{(\theta\wedge 1)(1-\mathrm{e}^{-1})}}\bigg) \mathrm{e}^{\|\nabla b\|_\infty t} \beta \|f\|_\infty |\xi| \\ &\quad \times \bigg\{\bigg\|\frac{|\nabla(h\rho)|}{\rho}\bigg\|_{L^2} + 2\|\nabla h\|_\infty \sqrt{\|h\|_{L^2}^2 + \|h\|_{L^1}^2}\bigg\}. \end{aligned}$$

□

Corollary 3.7. *Let* (H1) *hold. If* $\rho(z) = \frac{C_\alpha}{|z|^{n+\alpha}}$ *with* $C_\alpha > 0$ *and* $0 < \alpha < 2$*, then for* $t \geqslant 1 + \frac{8}{1-\mathrm{e}^{-1}}$*,* $\xi \in \mathbb{R}^n$*,* $f \in C_b(\mathbb{R}^n)$ *and* $\gamma \geqslant \frac{\alpha}{2} + 2$*,* $\|\nabla_\xi P_t f\|_\infty \leqslant C(n,\gamma,\alpha)\|f\|_\infty |\xi| \beta \mathrm{e}^{\|\nabla b\|_\infty t}$*, where* $C(n,\gamma,\alpha)$ *denotes a constant only depending on* n*,* γ *and* α*.*

Proof. Let $\phi : \mathbb{R}^n \longrightarrow [0,\infty)$ be a smooth function satisfying $0 \leqslant \phi \leqslant 1$, $\phi(z) = 1$ for $|z| \leqslant \frac{1}{2}$ and $\phi(z) = 0$ for $|z| \geqslant 1$. Take $h(z) = |z|^\gamma \phi(z)$ with $\gamma \geqslant 2 + \frac{\alpha}{2}$. Then h is bounded and so is $Dh(z) = \gamma |z|^{\gamma-2} z\phi(z) + |z|^\gamma \nabla\phi(z)$.

We can check $h \in \mathcal{H}_\rho$. In fact, one can easily obtain the following three estimates:

$$\int_{\mathbb{R}_0^n} h^2(z)\rho(z)dz \leqslant \int_{\mathbb{R}_0^n} h(z)\rho(z)dz \leqslant \int_{[0<|z|\leqslant 1]} |z|^\gamma \frac{C_\alpha}{|z|^{n+\alpha}} dz \leqslant C_\alpha \omega_n \int_0^1 r^{\gamma-1-\alpha} dr = \frac{C_\alpha \omega_n}{\gamma - \alpha}, \tag{3.24}$$

$$\begin{aligned}\int_{\mathbb{R}_0^n} |\nabla h(z)|^2 \rho(z) dz &\leqslant C_\alpha \int_{0\leqslant|z|\leqslant 1} |z|^{2\gamma-2-n-\alpha} dz + C_\alpha \|\nabla\phi\|_\infty^2 \int_{0\leqslant|z|\leqslant 1} |z|^{2\gamma-n-\alpha} dz \\ &\leqslant \frac{C_\alpha \omega_n}{2\gamma - \alpha - 2} + \frac{C_\alpha \|\nabla\phi\|_\infty^2 \omega_n}{2\gamma - \alpha}, \end{aligned}\tag{3.25}$$

$$\int_{\mathbb{R}_0^n} h^2(z)\frac{|\nabla\rho(z)|^2}{\rho(z)}dz \leqslant \frac{C_\alpha}{(n+\alpha)^2}\int_{0\leqslant|z|\leqslant 1}|z|^{2\gamma-2-n-\alpha}dz \leqslant \frac{C_\alpha\omega_n}{(n+\alpha)^2(2\gamma-\alpha-2)}. \tag{3.26}$$

For each $k\in\mathbb{N}$, define

$$\phi_k(z)=\begin{cases}1, & |z|\geqslant \dfrac{1}{k},\\ \dfrac{1}{2}\{1+\cos(k(k+1)|z|-(k+1))\pi\}, & \dfrac{1}{k+1}\leqslant|z|\leqslant\dfrac{1}{k},\\ 0, & |z|\leqslant\dfrac{1}{k+1}.\end{cases} \tag{3.27}$$

Then ϕ_k is a smooth function. It satisfies $0\leqslant\phi_k\leqslant 1$, $|\nabla\phi_k|\leqslant\frac{k(k+1)\pi}{2}I_{[\frac{1}{k+1}<|z|<\frac{1}{k}]}$ and for each $z\in\mathbb{R}^n$, $\phi_k(z)\to 1$ as $k\to\infty$. Let $h_k=h\phi_k$, then $h_k\in\mathscr{C}$ for each $k\in\mathbb{N}$. Moreover,

$$\begin{aligned}
\|h_k-h\|_\rho &= \left\{\int_{\mathbb{R}_0^n}|\nabla h_k(z)-\nabla h(z)|^2\rho(z)dz\right\}^{\frac{1}{2}}+\left\{\int_{\mathbb{R}_0^n}|h_k(z)-h(z)|^2\frac{|\nabla\rho(z)|^2}{\rho(z)}dz\right\}^{\frac{1}{2}}\\
&= \left\{\int_{\mathbb{R}_0^n}|\nabla(h(z)\phi_k(z))-\nabla h(z)|^2\rho(z)dz\right\}^{\frac{1}{2}}+\left\{\int_{\mathbb{R}_0^n}h^2(z)\frac{|\nabla\rho(z)|^2}{\rho(z)}(1-\phi_k(z))^2dz\right\}^{\frac{1}{2}}\\
&\leqslant \sqrt{2}\left\{\int_{\mathbb{R}_0^n}|\nabla h(z)|^2(1-\phi_k(z))^2\rho(z)dz\right\}^{\frac{1}{2}}+\sqrt{2}\left\{\int_{\mathbb{R}_0^n}|h(z)|^2|\nabla\phi_k(z)|^2\rho(z)dz\right\}^{\frac{1}{2}}\\
&\quad+\left\{\int_{\mathbb{R}_0^n}h^2(z)\frac{|\nabla\rho(z)|^2}{\rho(z)}(1-\phi_k(z))^2dz\right\}^{\frac{1}{2}}\\
&\leqslant \sqrt{2}\left\{\int_{\mathbb{R}_0^n}|\nabla h(z)|^2(1-\phi_k(z))^2\rho(z)dz\right\}^{\frac{1}{2}}+\pi\left\{k^2(k+1)^2\int_{\frac{1}{k+1}<|z|<\frac{1}{k}}|z|^{2\gamma}\frac{C_\alpha}{|z|^{n+\alpha}}dz\right\}^{\frac{1}{2}}\\
&\quad+\left\{\int_{\mathbb{R}_0^n}h^2(z)\frac{|\nabla\rho(z)|^2}{\rho(z)}(1-\phi_k(z))^2dz\right\}^{\frac{1}{2}}\\
&\leqslant \sqrt{2}\left\{\int_{\mathbb{R}_0^n}|\nabla h(z)|^2(1-\phi_k(z))^2\rho(z)dz\right\}^{\frac{1}{2}}+\pi\left\{\omega_nC_\alpha k^2(k+1)^2\int_{\frac{1}{k+1}}^{\frac{1}{k}}r^{2\gamma-\alpha-1}dr\right\}^{\frac{1}{2}}\\
&\quad+\left\{\int_{\mathbb{R}_0^n}h^2(z)\frac{|\nabla\rho(z)|^2}{\rho(z)}(1-\phi_k(z))^2dz\right\}^{\frac{1}{2}}\\
&\leqslant \sqrt{2}\left\{\int_{\mathbb{R}_0^n}|\nabla h(z)|^2(1-\phi_k(z))^2\rho(z)dz\right\}^{\frac{1}{2}}+\pi\sqrt{\frac{\omega_nC_\alpha}{\gamma-\alpha}}\left\{\frac{k^2(k+1)^2}{k^{\gamma-\alpha}}-\frac{k^2(k+1)^2}{(k+1)^{\gamma-\alpha}}\right\}^{\frac{1}{2}}\\
&\quad+\left\{\int_{\mathbb{R}_0^n}h^2(z)\frac{|\nabla\rho(z)|^2}{\rho(z)}(1-\phi_k(z))^2dz\right\}^{\frac{1}{2}}.
\end{aligned}$$

Combining this with (3.21) and (3.25), by dominated convergence theorem, we can obtain $\|h_k-h\|_\rho\to 0$ as $k\to\infty$. Therefore, $h\in\mathcal{H}_\rho$.

Note that

$$\begin{aligned}
\theta :=\liminf_{x\to\infty}\frac{\int_{[h\geqslant x^{-1}]}\rho(z)dz}{\log x} &\geqslant \liminf_{x\to\infty}\frac{\int_{[\frac{1}{2^\gamma}\geqslant|z|^\gamma\geqslant x^{-1}]}\rho(z)dz}{\log x}\\
&=\lim_{x\to\infty}\frac{C_\alpha C_n\int_{x^{-1}}^{\frac{1}{2}}r^{-1-\alpha}dr}{\log x}=\lim_{x\to\infty}\frac{C_\alpha C_n(x^{\frac{\alpha}{\gamma}}-1)}{\alpha\log x}=\infty,
\end{aligned}$$

where C_n is a constant depending on n. Combining above estimates and Theorem 2.3, we have, for $t\geqslant 1+\frac{8}{1-e^{-1}}$, $\|\nabla_\xi P_t f\|_\infty\leqslant C(n,\gamma,\alpha)\|f\|_\infty|\xi|\beta e^{\|\nabla b\|_\infty t}$, where $C(n,\gamma,\alpha)$ denotes a constant only depending on n, γ and α. □

Proof of Theorem 2.4. From the proof of Theorem 2.2, we can arrive at $|J_t| \leqslant \mathrm{e}^{-Kt}|x-y|$. Combining this with Theorem 2.3, for $t \geqslant 1+\frac{8}{(\theta\wedge 1)(1-\mathrm{e}^{-1})}$, one can derive

$$\|\nabla P_t f\|_\infty \leqslant C\|f\|_\infty \int_0^t |J_s| ds \leqslant \frac{C}{K}\|f\|_\infty, \tag{3.28}$$

where C is a constant independent of t and the values may be different from line to line. For $t > \frac{1}{1-\alpha}+\frac{8}{(1-\alpha)(\theta\wedge 1)(1-\mathrm{e}^{-1})}$, it follows from (3.28) and Markov property that

$$|P_t f(x) - P_t f(y)| \leqslant \mathbb{E}|P_{(1-\alpha)t} f(X^x_{\alpha t}) - P_{(1-\alpha)t} f(X^y_{\alpha t})| \leqslant \frac{C}{K}\|f\|_\infty |X^x_{\alpha t} - X^y_{\alpha t}| \leqslant \frac{C}{K}\|f\|_\infty \mathrm{e}^{-\alpha K t}.$$

This implies $\|P_t(x,\cdot) - P_t(y,\cdot)\|_{\mathrm{Var}} \leqslant \frac{C(\alpha)}{K}\mathrm{e}^{-\alpha Kt}$ for a proper constant $C(\alpha)$ and any $t > 0$. □

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