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Acyclic improper colouring of graphs with maximum degree 4

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Abstract A k-colouring (not necessarily proper) of vertices of a graph is called *acyclic*, if for every pair of distinct colours i and j the subgraph induced by the edges whose endpoints have colours i and j is acyclic. We consider acyclic k-colourings such that each colour class induces a graph with a given (hereditary) property. In particular, we consider acyclic k-colourings in which each colour class induces a graph with maximum degree at most t, which are referred to as *acyclic t-improper k-colourings*. The *acyclic t-improper chromatic number* of a graph G is the smallest k for which there exists an acyclic t-improper k-colouring of G. We focus on acyclic colourings of graphs with maximum degree 4. We prove that 3 is an upper bound for the acyclic 3-improper chromatic number of this class of graphs. We also provide a non-trivial family of graphs with maximum degree 4 whose acyclic 3-improper chromatic number is at most 2, namely, the graphs with maximum average degree at most 3. Finally, we prove that any graph G with $\Delta(G) \leqslant 4$ can be acyclically coloured with 4 colours in such a way that each colour class induces an acyclic graph with maximum degree at most 3.

Keywords acyclic colouring, acyclic improper colouring, bounded degree graph, hereditary property

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1 Introduction

We consider only finite, simple graphs. We use standard notation. For a graph G, we denote its vertex set and edge set by $V(G)$ and $E(G)$, respectively. Let $v \in V(G)$. By $N_G(v)$ (or $N(v)$) we denote the set of the neighbours of v in G. The cardinality of $N_G(v)$ is called the *degree* of v, denoted by $d_G(v)$ (or $d(v)$). The maximum and minimum vertex degrees in G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. The notation $H \subseteq G$ means that H is a subgraph of G. For undefined concepts, we refer the reader to [22].

A *k-colouring* of a graph G is a mapping c from the set of vertices of G to the set $\{1, \dots, k\}$ of colours. We can also regard a k-colouring of G as a partition of the set $V(G)$ into *colour classes* $V_1, \dots, V_k$ such that each V_i is the set of vertices with colour i. In many situations, it is desired that the particular set V_i has some property. For example, if we require that each set V_i is independent, then we have a proper k-colouring. Assuming each V_i induces a graph with maximum degree at most t yields a t-improper k-colouring. One can also require that for any pair of distinct colours i and j, the subgraph induced by the edges whose endpoints have colours i and j satisfies a given property, for example, is acyclic. This

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yields to the concept of acyclic colouring. In this paper, we will mainly concentrate on colourings, which are both t-improper and acyclic. To precise this notion, we need to introduce or recall several definitions and notation.

Let $\mathcal{P}_1, \ldots, \mathcal{P}_k$ be nonempty classes of graphs closed with respect to isomorphism. A k-colouring of a graph G is called a *$(\mathcal{P}_1, \ldots, \mathcal{P}_k)$-colouring* of G if for $i \in \{1, \ldots, k\}$ the subgraph induced in G by the colour class V_i belongs to $\mathcal{P}_i$. Such a colouring is called an *acyclic $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$-colouring* if for every two distinct colours i and j the subgraph induced by the edges whose endpoints have colours i and j is acyclic. In other words, every bichromatic cycle in G contains at least one monochromatic edge. A bichromatic cycle (resp. path) having no monochromatic edge is called an *alternating cycle* (resp. *path*). By $\mathcal{P}_1 \circ \cdots \circ \mathcal{P}_k$ we denote the class of all graphs having a $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$-colouring. Similarly, $\mathcal{P}_1 \odot \cdots \odot \mathcal{P}_k$ stands for the class of all graphs having an acyclic $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$-colouring.

A graph property $\mathcal{P}$ is called *hereditary*, if $G \in \mathcal{P}$ and $H \subseteq G$ implies $H \in \mathcal{P}$. For a survey on hereditary graph properties, see [8]. Assume that $\mathcal{P}_1, \ldots, \mathcal{P}_k$ are hereditary graph properties. It is well known that $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$-colourings are monotone (see [8]). It is straightforward that acyclic $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$-colourings are also monotone. Below we state this fact formally.

Observation 1. *If G has an acyclic $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$-colouring and the properties $\mathcal{P}_1, \ldots, \mathcal{P}_k$ are hereditary, then each subgraph of G also has an acyclic $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$-colouring. It means that the property $\mathcal{P}_1 \odot \cdots \odot \mathcal{P}_k$ is hereditary.*

In this paper, we focus on two hereditary graph properties: The class of graphs with bounded maximum degree and the class of acyclic graphs. We use the following notation:

$$\mathcal{S}_d = \{G : \Delta(G) \leqslant d\},$$
$$\mathcal{D}_1 = \{G : G \text{ is an acyclic graph}\}.$$

An acyclic $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$-colouring of G is called an *acyclic k-colouring*, if for $i \in \{1, \ldots, k\}$ the class $\mathcal{P}_i$ is the set of all edgeless graphs. The minimum k such that G has an acyclic k-colouring is called the *acyclic chromatic number* of G, denoted by $\chi_a(G)$. An acyclic $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$-colouring such that for $i \in \{1, \ldots, k\}$ the class $\mathcal{P}_i$ is the set of graphs with maximum degree at most t, (i.e., $G[V_i] \in \mathcal{S}_t$, for $i \in \{1, \ldots, k\}$) is called an *acyclic t-improper k-colouring*. The *acyclic t-improper chromatic number* $\chi_a^t(G)$ is the smallest k for which there exists an acyclic t-improper k-colouring of G. Thus, $\chi_a^0(G)$ equals $\chi_a(G)$.

The concept of acyclic colouring of graphs was introduced by Grűnbaum [16] and has been widely considered in the recent past. Even more attention has been paid to this problem since it was proved by Coleman et al. [12, 13] that acyclic colorings can be used in computing Hessian matrices via the substitution method, see also [14, 15].

However, determining $\chi_a(G)$ is quite difficult. Kostochka [18] proved that it is an NP-complete problem to decide for a given arbitrary graph G whether $\chi_a(G) \leqslant 3$. The acyclic chromatic number was studied for several classes of graphs, below we mention only some of the results. A lot of attention has been paid to acyclic colourings of planar graphs. A famous theorem of Borodin [5] states that the acyclic chromatic number of a planar graph is at most 5. The reader interested in other results in this direction is referred to, for instance, [2, 6, 7].

Focusing on the family of graphs with small maximum degree, it was shown [16] that $\chi_a(G) \leqslant 4$ for any graph with maximum degree 3 (see also [21]). Burstein [11] proved that $\chi_a(G) \leqslant 5$ for any graph with maximum degree 4. Recently, Kostochka and Stocker [19] proved that $\chi_a(G) \leqslant 7$ for any graph with maximum degree 5. Concerning graphs with maximum degree 6, Hocquard [17] proved that 11 colours are enough for an acyclic colouring.

In 1999, Boiron et al. [3, 4] began the study on the problem of acyclic $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$-colourings of outerplanar and planar graphs, and bounded degree graphs. In particular, they proved that any graph $G \in \mathcal{S}_3$ has an acyclic $(\mathcal{D}_1, \mathcal{S}_2)$-colouring as well as an acyclic $(\mathcal{S}_1, \mathcal{S}_1, \mathcal{S}_1)$-colouring [4]. Addario-Berry et al. [1] proved that each graph from $\mathcal{S}_3$ has an acyclic $(\mathcal{S}_2, \mathcal{S}_2)$-colouring, i.e., $\chi_a^2(G) \leqslant 2$ for any $G \in \mathcal{S}_3$. This theorem was also proved in [9], where a polynomial-time algorithm was presented. The above result cannot be generalised for graphs with arbitrary maximum degree, since it was shown in [9] that for all

$d \geqslant 4$ there exists a graph G with maximum degree d and with $\chi_a^{d-1}(G) \geqslant 3$. In [10], a polynomial-time algorithm that provides an acyclic $(\mathcal{D}_1, \mathcal{LF})$-colouring of any graph from $\mathcal{S}_3 \setminus \{K_4, K_{3,3}\}$ was given ($\mathcal{LF}$ is the set of acyclic graphs with maximum degree at most 2, i.e., *linear forests*).

In this paper, we continue the previous work and consider acyclic improper colourings of graphs with maximum degree at most 4. We prove that $\chi_a^3(G) \leqslant 3$ for every graph $G \in \mathcal{S}_4$ (Theorem 3.1) and we show an example of a graph with maximum degree 4, whose acyclic 3-improper chromatic number equals 3. Thus, Theorem 3.1 is optimal (with respect to the number of colours). This motivated us to ask the question, which graphs with maximum degree 4 admit acyclic 3-improper 2-colourings? We use the notion of the maximum average degree of a graph (denoted Mad(G)). We prove that if $G \in \mathcal{S}_4$ and $\mathrm{Mad}(G) \leqslant 3$, then $\chi_a^3(G) \leqslant 2$. Next, we focus on such acyclic colourings in which each colour class induces an acyclic graph. Observe first that, for $n \geqslant 2$, K_{n+1} needs n colours for any such colouring. Thus, we have the following.

Proposition 1.1. *For $d \geqslant 2$, $\mathcal{S}_d \nsubseteq \mathcal{P}_1 \odot \mathcal{P}_2 \odot \cdots \odot \mathcal{P}_{d-1}$, where $\mathcal{P}_i = \mathcal{D}_1$ $(i = 1, \ldots, d-1)$.*

We prove that each graph $G \in \mathcal{S}_4$ has an acyclic $(\mathcal{D}_1, \mathcal{D}_1, \mathcal{D}_1, \mathcal{D}_1)$-colouring (Theorem 4.1). By Proposition 1.1, the number of colours in this theorem cannot be reduced. But this result can be improved in another way. Namely, instead of $\mathcal{D}_1$, one can try to take a "smaller" property. Indeed, we prove that every graph with maximum degree at most 4 can be acyclically coloured with 4 colours in such a way that each colour class induces an acyclic graph with maximum degree at most 3.

2 Notation

The following definitions and notation, which will be used later in the proofs, deal with a *partial k-colouring* of a graph G, defined as an assignment c of colours from the set $\{1, \ldots, k\}$ to a subset C of $V(G)$. Given a partial k-colouring c of G, the set C is the set of coloured vertices. Notice that it may happen that C is empty or equals $V(G)$. For $W \subseteq V(G)$ let $c(W) = \bigcup_{v \in W \cap C} c(v)$. Let C_v denote the multiset of colours assigned by c to the coloured neighbors of v. A coloured vertex v is called *t-saturated*, if it has exactly t neighbours coloured with $c(v)$. If it is clear from the context, then we will skip the parameter t and simply call such a vertex *saturated*. A vertex v (coloured or not) is called *rainbow*, if all its coloured neighbours have distinct colours.

Let c be a partial k-colouring of G and i, j be distinct colours. An (i, j)-alternating cycle (resp. path) is an alternating cycle (resp. path) with each vertex coloured i or j. Let F be a cycle in G containing v. Cycle F is called *(i, j)-dangerous for v*, if colouring v with i results in an (i, j)-alternating cycle. A cycle is called *i-mono-dangerous* for v, if colouring v with i results in a monochromatic cycle containing v. When it is convenient, all (i, j)-dangerous cycles and l-mono-dangerous cycles for v will be called simply *dangerous cycles* for v.

A partial k-colouring of G such that the set of coloured vertices induces a graph with an acyclic $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$-colouring is called a *partial acyclic $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$-colouring*. A *partial (acyclic) t-improper k-colouring* of G is defined analogously.

Given a partial (acyclic) t-improper k-colouring c of G, a colour i is called *admissible* for a vertex v, if assigning i to v results in the colouring that has neither an alternating cycle containing v nor a monochromatic subgraph $K_{1,t+1}$ containing v.

3 Acyclic improper colourings

In this section, we consider acyclic 3-improper colourings of graphs from $\mathcal{S}_4$. First, we prove that each such graph has an acyclic 3-improper 3-colouring; using the terminology of hereditary properties, it means that $\mathcal{S}_4 \subseteq \mathcal{S}_3 \odot \mathcal{S}_3 \odot \mathcal{S}_3$.

Theorem 3.1. *If $G \in \mathcal{S}_4$, then $\chi_a^3(G) \leqslant 3$.*

Proof. Let $G \in \mathcal{S}_4$. First, we consider the case G is 4-regular. At the beginning, we observe that G has a 3-improper 3-colouring. (It follows from the well-known and more general theorem due to Lovász [20] which states that $\mathcal{S}_{p+q+1} \subseteq \mathcal{S}_p \circ \mathcal{S}_q$, for any $p, q \geqslant 0$.) We choose such a colouring c with the smallest possible number of alternating cycles. We will show that we can recolour some vertices of G in such a way that we obtain a 3-improper 3-colouring of G that has less alternating cycles than c. Let F be an $(1, 2)$-alternating cycle of G and w_1, v, w_2 be three consecutive vertices of F. Without loss of generality, we assume that $c(w_1) = c(w_2) = 1$ and $c(v) = 2$. For $t_i \in V(F)$ we denote $N'(t_i) = N(t_i) \setminus \{t_{i-1}, t_{i+1}\}$, where $(t_1, t_2, \ldots, t_p)$ are consecutive vertices of F (all indices are mod p). Let $N'(v) = \{u_1, u_2\}$. In the six cases we consider all possible colours of u_1, u_2.

Case 1. $c(u_1) = 2, c(u_2) = 3$.

Since w_1, w_2 are not saturated, the colour 1 is admissible for v. Thus, we recolour v with colour 1. In this way, we destroy the alternating cycle F, and no new alternating cycle appears, and the new colouring is still a 3-improper 3-colouring of G.

In all the remaining cases, we assume that there is no admissible colour for v, since otherwise we can recolour v. We also assume that we consider the next case only if none of the preceding cases can be applied to any vertex of F.

Case 2. $c(u_1) = 1, c(u_2) = 2$.

Since colour 1 is not admissible for v, the vertex u_1 must be saturated. Since u_1 is saturated and 3 is not admissible for v, there exists a $(1, 3)$-alternating path joining w_1 and w_2. Thus, w_1 has a neighbour coloured with 3 that is not saturated. Since we are not in Case 1, the second vertex of $N'(w_1)$, namely w, is coloured either with 2 or 3. If colour 3 is admissible for w_1, we recolour w_1. Otherwise, there must be a $(3, 2)$-dangerous cycle for w_1 or w_1 has a saturated neighbour coloured with 3. Thus, the vertex w is coloured with 2 and is not saturated or w is coloured with 3 and is saturated. In both cases, we can recolour w_1 with 2 and after such a move G is still 3-improperly 3-coloured and we have less alternating cycles.

Case 3. $c(u_1) = 2$, $c(u_2) = 2$.

Since colour 1 is not admissible for v, there must be a $(1, 2)$-dangerous cycle for v. Thus, vertices u_1 and u_2 both have a neighbour coloured with 1. Since colour 3 is also not admissible for v, there is also a $(3, 1)$-dangerous or a $(3, 2)$-dangerous cycle for v. This cycle goes through w_1, w_2 or u_1, u_2. This leads us to two subcases:

Subcase 3.1. The $(3, 1)$-dangerous cycle for v goes through w_1, w_2.

Thus, the vertex w_1 has a neighbour $x \in N'(w_1)$ coloured with 3. Furthermore, x also has a neighbour coloured with 1. Let $y \in N'(w_1) \setminus \{x\}$. Suppose that 3 is not admissible for w_1. Hence, y has colour 3 and is saturated or there is a $(3, 2)$-dangerous cycle for w_1, i.e., $c(y) = 2$ and y has a neighbour colored with 3. In both cases, colour 2 is admissible for w_1.

Subcase 3.2. The $(3, 2)$-dangerous cycle for v goes through u_1, u_2.

Clearly, u_1 has a neighbour x coloured with 1 and a neighbour y coloured with 3. Let $z \in N(u_1) \setminus \{v, x, y\}$. First, we recolour v with 1. Then we consider several cases, depending on the colour of z. If z has colour 2, then we recolour vertex u_1 with 1. After that we obtain a 3-improper 3-colouring in which F is not an alternating cycle. Suppose now that $c(z) = 1$. If z is saturated, then we recolour u_1 with 3. If z is not saturated, then we recolour u_1 with 1. Assume now that z is coloured with 3. If z is saturated, then we recolour u_1 with 1. Otherwise, we recolour u_1 with 3. It is easy to observe that there is neither an alternating cycle containing v nor an alternating cycle containing u_1 and that the new colouring is still a 3-improper 3-colouring.

Case 4. $c(u_1) = 3$, $c(u_2) = 3$.

Since colour 1 is not admissible for v, each of u_1, u_2 has a neighbour coloured with 1 that is not saturated. Since colour 3 is not admissible for v, there is a $(3, 1)$-alternating cycle that goes through w_1 and w_2. Thus, w_1 has a neighbour in $N'(w_1)$ that is coloured with 3 and is not saturated. Let w be the other vertex in $N'(w_1)$. Since we are not in Case 1, $c(w) = 2$ or $c(w) = 3$.

Subcase 4.1. $c(w) = 2$.

If colour 2 is not admissible for w_1, then w is saturated. Suppose that also 3 is not admissible for w_1. Since w_1 has no saturated neighbour coloured with 3, there must be a $(3, 2)$-dangerous cycle for w_1 that goes through v. Thus, u_1 or u_2 belongs to this cycle. Assume that u_1 belongs to this cycle. Let $N(u_1) \setminus \{v\} = \{x, y, z\}$. Hence $c(x) = 1, c(y) = 2$. First, we recolour v with 1. Next, we consider all possible colours of z. If $c(z) = 3$, then we recolour u_1 with 1. Suppose that $c(z) = 2$. If z is saturated, then we also recolour u_1 with 1, otherwise we recolour u_1 with 2. Suppose that $c(z) = 1$. If z is saturated, then we recolour u_1 with 2, otherwise we recolour u_1 with 1.

Subcase 4.2. $c(w) = 3$.

Suppose that colour 3 is not admissible for w_1. Thus, w_1 has a saturated neighbour coloured with 3 or there is a dangerous cycle for w_1. If w_1 has a saturated neighbour, then we recolour w_1 with 2. Otherwise, there is a $(3, 2)$-dangerous cycle for w_1 that goes through v and either u_1 or u_2. Assume that u_1 belongs to this cycle. Let $N(u_1) \setminus \{v\} = \{x, y, z\}$. Hence, $c(x) = 1, c(y) = 2$. Then similarly as in Subcase 4.1 we consider all colourings of z to show that we can recolour v and u_1.

Case 5. $c(u_1) = 1$, $c(u_2) = 3$.

Since colour 1 is not admissible for v, the vertex u_1 is saturated. Since we cannot recolour v with 3, it follows that v has a saturated neighbour coloured with 3 or there is a dangerous cycle for v.

Subcase 5.1. u_2 is saturated.

If we can recolour u_1 with 2 or 3, then we obtain Case 1 or Case 4, respectively. Assume that colours 2 and 3 are both not admissible for u_1. Thus, there is a $(2, 1)$-dangerous cycle and a $(3, 1)$-dangerous cycle for u_1. This implies that there is a neighbour $x \in N(u_1) \setminus \{v\}$ that has the neighbours coloured with 2 and 3. Let us denote $N(x) \setminus \{u_1\} = \{x_1, x_2, x_3\}$, $c(x_1) = 2, c(x_2) = 3$ and x_1, x_2 are not saturated. We recolour v with 1 and then we consider some cases depending on the colour of x_3. If $c(x_3) = 1$, then we recolour x with 3. Assume that $c(x_3) = 2$. If x_3 is saturated, then we recolour x with 3. Otherwise, we recolour x with 2. Suppose now that $c(x_3) = 3$. If x_3 is saturated, then we recolour x with 2. Otherwise, we recolour x with 3.

Subcase 5.2. There is a $(3, 1)$-dangerous cycle for v.

Thus, this cycle goes through w_1. Let us denote $N'(w_1) = \{x, y\}$. Hence $c(x) = 3$. Since we are neither in Case 1 nor in Case 4, $c(y) = 2$. From the fact that colour 2 is not admissible for w_1 it follows y is saturated. Since we are not in Subcase 5.1, the vertex u_2 is not saturated. We recolour v with 3 and w_1 with 3.

Case 6. $c(u_1) = 1$, $c(u_2) = 1$.

Since we are not in any of the Cases $1, \ldots, 5$, each vertex of $N'(w_1) \cup N'(w_2)$ is coloured with 2. Colour 3 is admissible neither for v nor for w_1, hence there is a $(3, 1)$-dangerous cycle for v and there is a $(3, 2)$-dangerous cycle for w_1. Hence vertices of $N'(v)$ and $N'(w_1)$ are not saturated. We recolour v with 1 and w_1 with 2.

To finish the proof it is enough to observe that for any graph $H \in \mathcal{S}_4$ which is not 4-regular, there exists a 4-regular graph G such that $H \subset G$. As we have shown, G has an acyclic 3-improper 3-colouring. Thus, H has such a colouring too. □

Theorem 3.1 implies that every graph with maximum degree at most 4 has an acyclic 3-improper 3-colouring. The graph presented in Figure 1 is an example of a graph with maximum degree 4 having no acyclic 3-improper 2-colouring. Thus the number of colours in Theorem 3.1 cannot be reduced. These motivated us to ask a question, which graphs with maximum degree 4 admit acyclic 3-improper 2-colourings? In the next theorem, we provide a non-trivial family of such graphs. We use the notion of the maximum average degree $\mathrm{Mad}(G)$ of a graph G, defined as follows:

$$\mathrm{Mad}(G) = \max\left\{\frac{2|E(H)|}{|V(H)|} : H \subseteq G\right\}.$$

Theorem 3.2. *Let $G \in \mathcal{S}_4$ be a graph such that $\mathrm{Mad}(G) \leqslant 3$. Then $\chi_a^3(G) \leqslant 2$.*

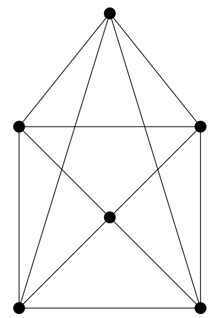

Figure 1 The graph with maximum degree 4 having no acyclic 3-improper 2-colouring

The proof of this theorem is based on a lemma, which provides some structural properties of graphs with maximum degree at most 4 and maximum average degree at most 3. Before we proceed, we introduce one more notation. For a vertex $v \in V(G)$, by $l_d(v)$ we denote the cardinality of the set of vertices of degree d in the neighbourhood of v.

Lemma 3.3. *Let $G \in \mathcal{S}_4$ satisfy $\delta(G) \geqslant 2$, $\Delta(G) = 4$ and $\mathrm{Mad}(G) \leqslant 3$. If G contains a vertex of degree 4 adjacent to at most one vertex of degree 2, then G contains at least one of the following configurations*:

(A1) *A vertex of degree 2 adjacent to at least one vertex of degree at most 3.*

(A2) *A vertex of degree 4 adjacent to at least three vertices of degree 2.*

Proof. Let $G = (V, E)$ be a graph satisfying the assumptions of the lemma. We use the discharging method. Initially, we define a mapping w on V as follows: For each $x \in V$, let $w(x) = d(x)$. Clearly, the fact that $Mad(G) \leqslant 3$ yields

$$\sum_{x \in V} w(x) \leqslant 3|V|. \tag{3.1}$$

In the discharging step, the values of w are redistributed between adjacent vertices according to the rule described below. In this way, we obtain a new mapping w'. After this procedure, each $x \in V$ has a new value $w'(x)$, but the sums of values of w' and w, counting over all the vertices, are the same. We show that if G contains neither A1 nor A2, then for each vertex x we have $w'(x) \geqslant 3$ and that there exists at least one vertex for which this value is strictly greater than 3, obtaining an obvious contradiction with inequality (3.1). We have only one rule for distributing the values between adjacent vertices:

(R) If x is a vertex of degree 4, then x gives $\frac{1}{2}$ to each neighbour of degree 2.

Now, we compute the values of vertices of G considering several cases, depending on the degree of $x \in V$.

If $d(x) = 2$, then $w'(x) = 2 + 2 \cdot \frac{1}{2} = 3$, because G does not contain Configuration (A1).

If $d(x) = 3$, then $w'(x) = w(x) = 3$.

If $d(x) = 4$, then x can be adjacent to at most two vertices of degree 2, since otherwise Configuration (A2) occurs. Hence, $w'(x) \geqslant 4 - 2 \cdot \frac{1}{2} = 3$.

As we have shown, for each vertex x of G the value $w'(x)$ is greater than or equal to 3. To obtain a contradiction with (3.1), it remains to prove that there exists at least one vertex, say x, such that $w'(x) > 3$. Observe that a vertex of degree 4 may have the final charge 3 if and only if it has exactly two neighbours of degree 2. From the assumption there exists a vertex of degree 4 which has at most one vertex of degree 2 in its neighbourhood and hence with the final charge greater than 3. □

Proof of Theorem 3.2. Let $G = (V, E)$ be a minimal, with respect to the number of edges, counterexample to the theorem. There is no loss of generality in assuming G is connected. Observe at the beginning, that G does not contain vertices of degree 1. Indeed, if v would be such a vertex, then by removing v we obtain a graph that has an acyclic 3-improper 2-colouring. Clearly, this colouring can be extended to an acyclic 3-improper 2-colouring of G. Now we provide some additional properties of G.

Claim 1. *G contains no vertex of degree 2 adjacent to a vertex of degree at most 3.*

Proof. Assume to the contrary that there is a vertex v of degree 2 adjacent to a vertex u of degree at most 3. Let $G' = G - vu$. From the fact that G is a minimal counterexample it follows G' has an acyclic 3-improper 2-colouring c. We claim that we can extend this colouring. Observe at the beginning, that if $c(v) = c(u)$, then the colouring c can be extended to G (because $d_G(u) \leqslant 3$). Therefore, we may assume $c(v) \neq c(u)$, w.l.o.g., $c(v) = 1, c(u) = 2$. If we cannot extend the colouring c to an acyclic 3-improper 2-colouring of G, then there is an alternating $(1, 2)$-path from v to u. Let w be the second neighbour of v. It follows $c(w) = 2$ and w is not saturated. Hence we can recolour v with 2. Clearly, the obtained colouring can be extended. □

Claim 2. *G contains no vertex of degree 4 adjacent to at least three vertices of degree 2.*

Proof. Assume to the contrary that v is a vertex of degree 4 with $l_2(v) \geqslant 3$. Let u_1, u_2 and u_3 be the vertices of degree 2 adjacent to v and let u_4 be the remaining neighbour of v. We may assume $d(u_4) = 4$. We consider a graph $G' = G - vu_1$. G' has an acyclic 3-improper 2-colouring c, because G is a minimal counterexample. We show that the colouring c can be extended. To this aim, we consider two cases. Let w_1 be the neighbour of u_1.

Case 1. Let $c(v) = c(u_1) = 1$.

Observe that we cannot extend the colouring c only if v is saturated, i.e., $c(u_2) = c(u_3) = c(u_4) = 1$. If we can recolour u_1 with 2, then we are done, because the obtained colouring can be extended. It follows $c(w_1) = 2$, $d_G(w_1) = 4$ and w_1 is saturated. If we can recolour v with 2, then we are also done. Observe that v cannot be recoloured only if there exists a $(2, 1)$-dangerous cycle for v. Obviously, any such cycle is passing through at least one of the vertices u_2, u_3. W.l.o.g., we may assume that such a cycle is passing through u_2. Let w_2 be the neighbour of u_2, $w_2 \neq v$. It follows $c(w_2) = 2$ and w_2 is not saturated. We recolour u_2 with 2. We obtain an acyclic 3-improper 2-colouring which can be extended (because v is not saturated).

Case 2. Assume $c(v) \neq c(u_1)$, w.l.o.g., $c(v) = 1, c(u_1) = 2$.

If the colouring c cannot be extended, then there is an alternating $(1, 2)$-path from v to u_1. Hence, $c(w_1) = 1$ and w_1 is not saturated. We recolour u_1 with 1 and obtain a colouring considered in Case 1. □

Claim 3. *G contains a vertex v of degree 4 adjacent to at most one vertex of degree 2.*

Proof. Assume to the contrary that each vertex of degree 4 has at least two neighbours of degree 2. By Claim 2, we may assume that each vertex of degree 4 has exactly two neighbours of degree 2. Let v be such a vertex and $w_1, w_2 \in N(v)$ be of degree 2. Assume w_3, w_4 are the remaining neighbours of v. Let $u_1 \in N(w_1) \setminus \{v\}, u_2 \in N(w_2) \setminus \{v\}$. Let $G' = G - vw_1$. G being a minimal counterexample implies G' has an acyclic 3-improper 2-colouring c. At the beginning we assume $c(v) \neq c(w_1)$, w.l.o.g., $c(v) = 1, c(w_1) = 2$. If we cannot extend c, then there is an alternating $(1, 2)$-path from v to w_1. Hence $c(u_1) = 1$ and neither v nor u_1 is saturated. We recolour w_1 with 1, obtaining an acyclic 3-improper 2-colouring which can be extended. Therefore, we may assume $c(v) = c(w_1) = 1$. The fact we cannot extend the colouring c implies v is saturated. If we can recolour w_1 with 2, then we are done, because such a colouring can be extended. Hence $c(u_1) = 2$, $d_G(u_1) = 4$, and u_1 is saturated. Similarly, if we can recolour w_2 with 2, then we are also done. Thus, $c(u_2) = 2$, $d_G(u_2) = 4$, and u_2 is saturated. We focus on v. If v can be recoloured with 2, then the obtained colouring can be extended. Otherwise, there is an alternating $(1, 2)$-path P from w_3 to w_4. Up to symmetry, there are two cases to consider.

Case 1. Assume w_3 is of degree 4.

Thus, $l_2(w_3) = 2$. Let $N(w_3) = \{v, t_1, t_2, t_3\}$ and $d(t_1) = d(t_2) = 2$. At the beginning, we consider the situation when P is passing through t_1 or t_2, say t_1. Let p_1 be the neighbour of t_1, different from w_3. It follows $c(t_1) = 2, c(p_1) = 1$ and p_1 is not saturated. We recolour t_1 with 1. In this way, we destroy the path P. Observe, that P was the only one alternating path from w_3 to w_4, because c was acyclic. Now we can recolour v with 2. We obtain an acyclic 3-improper 2-colouring, which can be extended. Thus, we may assume P is passing through t_3. Clearly, $c(t_3) = 2$ and t_3 is not saturated. We try to recolour w_3 with 2. If this is possible, then we are done, since v is no longer saturated and the obtained colouring can be extended. On the other hand, such a recolouring cannot be done only if there is an alternating

$(1,2)$-path from t_1 to t_2. It follows $c(t_1)=c(t_2)=1$ and p_1 is coloured with 2 and is not saturated. Hence we can recolour t_1 with 2, w_3 with 2 and extend the obtained colouring.

Case 2. Assume w_3 and w_4 are both of degree 3.

Observe that none of w_3, w_4 can be recoloured with 2 (if this was possible, then we would be done). Let t_1, t_2 be the neighbours of w_3 different from v. Assume that P is passing through t_1. Hence $c(t_1)=2$ and t_1 is not saturated. We cannot recolour w_3, thus $c(t_2)=2$, $d(t_2)=4$ and t_2 is saturated. From the assumptions we have $l_2(t_2)=2$. Let $s_1, s_2 \in N(t_2)$ be of degree 2. Assume for a moment that we can recolour t_2 with 1. In this case we are done, because we recolour w_3 with 2 and the obtained colouring can be extended. Hence t_2 cannot be recoloured. It follows there is an alternating $(1,2)$-path P' joining two neighbours of t_2. W.l.o.g., we may assume P' is passing through s_1. We recolour s_1 with 1, w_3 with 2. The obtained colouring can be extended, since v is not saturated. □

To finish the proof it is enough to observe that Claim 3 implies G satisfies the assumptions of Lemma 3.3. Hence G should contain (A1) or (A2), but this is impossible, because of Claims 1 and 2. □

4 Acyclic colourings in which each colour class induces an acyclic graph

In Section 4, we consider acyclic colourings of graphs from $\mathcal{S}_4$ such that each colour class induces an acyclic graph, i.e., a graph from $\mathcal{D}_1$. We prove that $\mathcal{S}_4 \subseteq \mathcal{D}_1 \odot \mathcal{D}_1 \odot \mathcal{D}_1 \odot \mathcal{D}_1$. Since by Proposition 1.1 the graph K_5 needs 4 colours in every such colouring, we cannot reduce the number of colours in Theorem 4.1.

Theorem 4.1. $\mathcal{S}_4 \subseteq \mathcal{D}_1 \odot \mathcal{D}_1 \odot \mathcal{D}_1 \odot \mathcal{D}_1$.

Before we can proceed with the proof, we need to observe the following easy fact.

Proposition 4.2. *For every partial acyclic $(\mathcal{D}_1,\mathcal{D}_1,\mathcal{D}_1,\mathcal{D}_1)$-colouring of a graph $G \in \mathcal{S}_4$ and a rainbow vertex u of G, we can colour or recolour u with each of the 4 colours.*

Proof of Theorem 4.1. We claim that it is enough to prove that the theorem holds for 4-regular graphs. Indeed, for any $H \in \mathcal{S}_4$ there exists a 4-regular graph G such that $H \subseteq G$. Furthermore, the existence of an acyclic $(\mathcal{D}_1,\mathcal{D}_1,\mathcal{D}_1,\mathcal{D}_1)$-colouring of G implies that H has such a colouring too, what follows from Observation 1. Therefore, we assume G is 4-regular. Let c be a partial acyclic $(\mathcal{D}_1,\mathcal{D}_1,\mathcal{D}_1,\mathcal{D}_1)$-colouring of G and v be an uncoloured vertex. We show that we can colour v. Let $N(v)=\{x,y,z,w\}$. The vertex v cannot be coloured only if for each colour i $(i=1,\ldots,4)$ there is an (i,j)-dangerous cycle or an i-mono-dangerous cycle for v.

By Proposition 4.2, it is enough to consider the following three cases:

Case 1. Two neighbours of v have the same colour (say $c(x)=c(y)=1$), others have distinct colours, and different from 1, or are uncoloured.

Since every $(i,1)$-dangerous cycle for v and every 1-mono-dangerous cycle for v must go through x, we can colour v with a colour j such that $j \notin c(N(x))$.

Case 2. Three neighbours of v have the same colour (say $c(x)=c(y)=c(z)=1$), the remaining one has a colour different from 1 or is uncoloured.

If v cannot be coloured with any colour, then each of colours must be contained in at least two different multisets among C_x, C_y, C_z. We have four colours, hence at least one of x,y,z is rainbow. We recolour this vertex with a colour $i \neq c(w)$ or with 2, if w is uncoloured, and we are in Case 1.

Case 3. Four neighbours of v have the same colour (say, $c(x)=c(y)=c(z)=c(w)=1$) or two pairs of neighbours have the same colour (say, $c(x)=c(y)=1, c(z)=c(w)=2$).

If v cannot be coloured, then for every colour i there is an (i,j)-dangerous cycle or an i-mono-dangerous cycle for v. Thus, each of colours must be contained in at least two different multisets among C_x, C_y, C_z, C_w. If at least one vertex of x,y,z,w is rainbow, then we recolour this vertex (with 3) and we are either in Case 1 or in Case 2. Otherwise, each vertex of x,y,z,w belongs to at least two dangerous (or mono-dangerous) cycles for v and has two neighbours coloured with distinct colours. We focus on x. Let $N(x) \setminus \{v\} = \{x_1,x_2,x_3\}$. W.l.o.g. we may assume that both x_1 and x_2 belong to some dangerous (or mono-dangerous) cycles for v and $c(x_1) = c_1, c(x_2) = c_2, c(x_3) = c_2$, $c_1 \neq c_2$. First, we assume $c_1, c_2 \neq 1$. The vertex x is in both the $(c_1, 1)$-dangerous cycle and the $(c_2, 1)$-dangerous cycle for v, hence x_1 has two neighbours coloured with 1 and x_2 has two neighbours coloured with 1. Now, we claim that we can recolour x. If we can recolour x with 2, then we are in Case 2. Otherwise, x_2 must have a neighbour coloured with 2. If we can recolour x neither with 3 nor with 4, then x_2 must have a neighbour coloured with 3 and a neighbour coloured with 4. Since $d(x_2) \leqslant 4$, this is impossible. Thus, we can recolour x and obtain either Case 1 or Case 2. The case $c_1 = 1$ or $c_2 = 1$ can be done similarly. □

In the next theorem, we improve the previous result.

Theorem 4.3. *$\mathcal{S}_4 \subseteq \mathcal{P}_1 \odot \mathcal{P}_2 \odot \mathcal{P}_3 \odot \mathcal{P}_4$, where $\mathcal{P}_i = \mathcal{S}_3 \cap \mathcal{D}_1$ $(i = 1, \dots, 4)$.*

Proof. Let $G \in \mathcal{S}_4$. As above, we may assume G is 4-regular. Theorem 4.1 implies that there exists an acyclic $(\mathcal{D}_1, \mathcal{D}_1, \mathcal{D}_1, \mathcal{D}_1)$-colouring of G. Let c be such a colouring with the smallest possible number of vertices that have four neighbours coloured with its colour. Let v be such a vertex. We will show that we can recolour v or a neighbour of v in such a way that the new colouring is an acyclic $(\mathcal{D}_1, \mathcal{D}_1, \mathcal{D}_1, \mathcal{D}_1)$-colouring of G with smaller number of vertices having four neighbours coloured with its colour. Assume $c(v) = 1$ and $C_v = \{1, 1, 1, 1\}$.

If we cannot recolour v, then for every colour i, $i \in \{2, 3, 4\}$, there is an $(i, 1)$-dangerous cycle for v. Thus, there is a neighbour x of v that is in at least two dangerous cycles for v. Let $N(x) = \{x_1, x_2, x_3, v\}$. Without loss of generality, we may assume that x is in both a $(2, 1)$-dangerous and a $(3, 1)$-dangerous cycle for v and vertices x_1, x_2 also belong to these cycles. Hence, $c(x_1) = 2, c(x_2) = 3$ and x_1, x_2 are not saturated. If $c(x_3) = 4$, then we can recolour x with 2, since x is rainbow and x_1 is not saturated. If $c(x_3) = 1$, then also we can recolour x with 2. Observe that such a colouring does not create any alternating cycle, since an alternating cycle cannot go through v (v has three neighbours in its colour). Suppose now that $c(x_3) = 2$ (the case $c(x_3) = 3$ is similar). If we cannot recolour x with 3, then there must be a $(3, 2)$-dangerous cycle for x. Thus, x_1 and x_3 both have a neighbour coloured with 3. If we cannot recolour x with 2, then there must be a 2-mono-dangerous cycle for x, hence x_1 must have a neighbour coloured with 2. Since x_1 has two neighbour coloured with 1 (x_1 is in the $(2, 1)$-dangerous cycle for v) and $d(x_1) = 4$, we can recolour x with 4. In all cases, we obtain an acyclic $(\mathcal{D}_1, \mathcal{D}_1, \mathcal{D}_1, \mathcal{D}_1)$-colouring of G with smaller number of vertices having four neighbours coloured with its colour, a contradiction with the choice of c. □

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