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q-Bernoulli polynomials and q-umbral calculus

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Abstract In this paper, we investigate some properties of q-Bernoulli polynomials arising from q-umbral calculus. We find a formula for expressing any polynomial as a linear combination of q-Bernoulli polynomials with explicit coefficients. Also, we establish some connections between q-Bernoulli polynomials and higher-order q-Bernoulli polynomials.

Keywords q-Bernoulli polynomial, q-umbral calculus

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1 Introduction and preliminaries

Throughout this paper we will assume q to be a fixed number between 0 and 1. We denote by D_q the q-derivative of a function (see [8, 10])

$$(D_q f)(x) = \frac{f(qx) - f(x)}{(q-1)x}. \tag{1.1}$$

The Jackson definite q-integral of the function f is defined by (see [8, 12, 13])

$$\int_0^x f(t) d_q t = (1-q) \sum_{a=0}^{\infty} f(q^a x) x q^a. \tag{1.2}$$

From (1.1) and (1.2), we note that

$$D_q \int_0^x f(t) d_q t = f(x), \quad \int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x.$$

In this paper, we use the following notation:

$$[x]_q = \frac{1-q^x}{1-q}, \quad (a+b)_q^n = \prod_{i=0}^{n-1} (a + q^i b) \quad (n \in \mathbb{Z}^+) \tag{1.3}$$

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and

$$(1+a)_q^\infty = \prod_{j=0}^{\infty}\left(1+q^j a\right), \quad [n]_q! = [n]_q[n-1]_q\cdots[2]_q[1]_q. \tag{1.4}$$

The q-analogue of exponential function is defined by (see [5, 6, 8, 10])

$$e_q(t) = \frac{1}{(1-(1-q)\,t)_q^\infty} = \sum_{n=0}^{\infty}\frac{t^n}{[n]_q!}. \tag{1.5}$$

In [10], the q-analogues of Bernoulli polynomials are defined by the generating function to be (see [8–14])

$$\frac{t}{e_q(t)-1}e_q(xt) = \sum_{n=0}^{\infty}B_{n,q}(x)\frac{t^n}{[n]_q!}. \tag{1.6}$$

In the special case, $x=0$, $B_{n,q}(0)=B_{n,q}$ is called the n-th q-Bernoulli number.

The reader is referred to [1–3, 17] for other types of q-Bernoulli polynomials and to [7] for some related work.

From (1.6), we can derive the following equation:

$$B_{n,q}(x) = \sum_{l=0}^{n}\binom{n}{l}_q x^{n-l}B_{l,q} = \sum_{l=0}^{n}B_{n-l,q}x^l\binom{n}{l}_q, \tag{1.7}$$

where $\binom{n}{l}_q = \frac{[n]_q!}{[l]_q![n-l]_q!} = \frac{[n]_q[n-1]_q\cdots[n-l+1]_q}{[l]_q!}$.

Let $\mathbb{C}$ be the complex number field and let $\mathcal{F}$ be the set of all formal power series in variable t over $\mathbb{C}$ with

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty}\frac{a_k}{[k]_q!}t^k \,\middle|\, a_k\in\mathbb{C}\right\}. \tag{1.8}$$

Let $\mathbb{P}=\mathbb{C}[t]$ and let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. Now we denote by $\langle L|p(x)\rangle$ the action of the linear functional L on the polynomial $p(x)$. We remind that the vector space operations on $\mathbb{P}^*$ are defined by $\langle L+M|p(x)\rangle = \langle L|p(x)\rangle + \langle M|p(x)\rangle, \langle cL|p(x)\rangle = c\langle L|p(x)\rangle$, where c is any constant in $\mathbb{C}$ (see [15, 16]).

For $f(t)=\sum_{k=0}^{\infty}\frac{a_k}{[k]_q!}t^k\in\mathcal{F}$, we define the linear functional on $\mathbb{P}$ by setting

$$\langle f(t)|x^n\rangle = a_n, \quad \text{for all } n\geqslant 0. \tag{1.9}$$

Thus, by (1.8) and (1.9), we note that

$$\langle t^k|x^n\rangle = [n]_q!\delta_{n,k}, \quad n,k\geqslant 0, \tag{1.10}$$

where $\delta_{n,k}$ is the Kronecker's symbol.

Let $f_L(t)=\sum_{k=0}^{\infty}\frac{\langle L|x^k\rangle}{[k]_q!}t^k$. Then, by (1.8) and (1.9), we see that $\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle$ and so as linear functionals $L=f_L(t)$. It is easy to show that the map $L\longmapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ denotes both the algebra of formal power series in t and the vector space of all linear functionals on $\mathbb{P}$, and so an element $f(t)$ of $\mathcal{F}$ is thought of as both a formal power series and a linear functional. We call $\mathcal{F}$ the q-umbral algebra. The q-umbral calculus is the study of q-umbral algebra. By (1.5) and (1.10), we easily see that $\langle e_q(yt)|x^n\rangle = y^n$ and so $\langle e_q(yt)|p(x)\rangle = p(y)$.

Notice that for all $f(t)$ in $\mathcal{F}$,

$$f(t) = \sum_{k=0}^{\infty}\frac{\langle f(t)|x^k\rangle}{[k]_q!}t^k, \tag{1.11}$$

and for all polynomials $p(x)$ (see [15, 16]),

$$p(x) = \sum_{k=0}^{\infty}\frac{\langle t^k|p(x)\rangle}{[k]_q!}x^k. \tag{1.12}$$

For $f_1(t), f_2(t), \ldots, f_n(t) \in \mathcal{F}$, we have

$$\langle f_1(t)\cdots f_m(t)|x^n\rangle = \sum_{i_1+\cdots+i_m=n} \binom{n}{i_1,\ldots,i_m}_q \langle f_1(t)|x^{i_1}\rangle\cdots\langle f_m(t)|x^{i_m}\rangle, \tag{1.13}$$

where $\binom{n}{i_1,\ldots,i_m}_q = \frac{[n]_q!}{[i]_q!\cdots[i_m]_q!}$.

The order $O(f(t))$ of the power series $f(t)\,(\neq 0)$ is the smallest integer k for which a_k does not vanish. If $O(f(t)) = 0$, then $f(t)$ is called an invertible series. If $O(f(t)) = 1$, then $f(t)$ is called a delta series.

Let $p^{(k)}(x) = D_q^k p(x)$. Then, by (1.12), we get

$$p^{(k)}(x) = \sum_{l=k}^{\infty} \frac{\langle t^l|p(x)\rangle}{[l]_q!}[l]_q[l-1]_q\cdots[l-k+1]_q x^{l-k}. \tag{1.14}$$

From (1.14), we have

$$p^{(k)}(0) = \langle t^k|p(x)\rangle \quad \text{and} \quad \langle 1|p^{(k)}(x)\rangle = p^{(k)}(0). \tag{1.15}$$

By (1.15), we get

$$t^k p(x) = p^{(k)}(x) = D_q^k p(x). \tag{1.16}$$

Let $f(t), g(t) \in \mathcal{F}$ with $O(f(t)) = 1$ and $O(g(t)) = 0$. Then there exists a unique sequence $s_n(x)$ ($\deg s_n(x) = n$) of polynomials such that $\langle g(t)f(t)^k|s_n(x)\rangle = [n]_q!\delta_{n,k}$ $(n,k \geqslant 0)$, which is denoted by $s_n(x) \sim (g(t), f(t))$. The sequence $s_n(x)$ is called the q-Sheffer sequence for $(g(t), f(t))$. For $h(t), f(t), g(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$, we have

$$h(t) = \sum_{k=0}^{\infty} \frac{\langle h(t)|s_k(x)\rangle}{[k]_q!} g(t)f(t)^k, \quad p(x) = \sum_{k=0}^{\infty} \frac{\langle g(t)f(t)^k|p(x)\rangle}{[k]_q!} s_k(x), \tag{1.17}$$

and

$$\frac{1}{g(\bar{f}(t))} e_q(y\bar{f}(t)) = \sum_{k=0}^{\infty} \frac{s_k(y)}{[k]_q!} t^k, \quad \text{for all } y \in \mathbb{C}, \tag{1.18}$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$ [15,16].

Recently, several authors have studied q-Bernoulli and Euler polynomials. In this paper, we investigate some properties of q-Bernoulli polynomials arising from q-umbral calculus. Finally, we derive some interesting identities of q-Bernoulli polynomials from our results.

2 *q*-Bernoulli polynomials and *q*-umbral calculus

From (1.6), we note that

$$B_{n,q}(x) \sim \left(\frac{e_q(t)-1}{t}, t\right). \tag{2.1}$$

By (2.1), we get

$$B_{n,q}(x) = \left(\frac{t}{e_q(t)-1}\right) x^n, \quad n \geqslant 0. \tag{2.2}$$

From (1.7) and (1.16), we note that

$$tB_{n,q}(x) = D_q B_{n,q}(x) = [n]_q B_{n-1,q}(x). \tag{2.3}$$

By (1.1) and (1.10), we easily see that

$$\begin{aligned} \left\langle \frac{e_q(t)-1}{t}\middle| x^n \right\rangle &= \frac{1}{[n+1]_q}\left\langle \frac{e_q(t)-1}{t}\middle| tx^{n+1}\right\rangle = \frac{1}{[n+1]_q}\langle e_q(t)-1|x^{n+1}\rangle \\ &= \frac{1}{[n+1]_q} = \int_0^1 x^n d_q x. \end{aligned} \tag{2.4}$$

Thus, from (2.4), we have

$$\left\langle \frac{e_q(t)-1}{t} \middle| p(x) \right\rangle = \int_0^1 p(x) d_q x, \quad \text{for } p(x) \in \mathbb{P}. \tag{2.5}$$

In particular, if we take $p(x) = B_{n,q}(x)$, then

$$\int_0^1 B_{n,q}(x) d_q x = \left\langle \frac{e_q(t)-1}{t} \middle| B_{n,q}(x) \right\rangle = \left\langle 1 \middle| \frac{e_q(t)-1}{t} B_{n,q}(x) \right\rangle = \langle t^0 | x^n \rangle = [n]_q! \delta_{n,0}. \tag{2.6}$$

From (1.7), we can derive

$$\int_0^1 B_{n,q}(x) d_q x = \sum_{k=0}^{n} B_{n-k,q} \binom{n}{k}_q \int_0^1 x^k d_q x = \sum_{k=0}^{n} \frac{B_{n-k,q}}{[k+1]_q} \binom{n}{k}_q. \tag{2.7}$$

Therefore, by (2.6) and (2.7), we obtain the following proposition.

Proposition 1. *For $n \in \mathbb{Z}^+$, we have*

$$B_{0,q} = 1, \quad \sum_{k=1}^{n} \binom{n}{k}_q \frac{1}{[k+1]_q} B_{n-k,q} = -B_{n,q}, \quad n > 0.$$

By (1.17) and (2.1), we get

$$\begin{aligned} p(x) &= \sum_{k=0}^{\infty} \frac{1}{[k]_q!} \left\langle \frac{e_q(t)-1}{t} t^k \middle| p(x) \right\rangle B_{k,q}(x) = \sum_{k=0}^{\infty} \frac{1}{[k]_q!} \left\langle \frac{e_q(t)-1}{t} \middle| t^k p(x) \right\rangle B_{k,q}(x) \\ &= \sum_{k=0}^{\infty} \frac{1}{[k]_q!} B_{k,q}(x) \int_0^1 t^k p(x) d_q x. \end{aligned} \tag{2.8}$$

It is known that

$$(x-1)_q^n = (x-1)(x-q)\cdots(x-q^{n-1}) \sim (e_q(t), t) \tag{2.9}$$

From (1.17) and (2.9), we have

$$\begin{aligned} B_{n,q}(x) &= \sum_{k=0}^{n} \frac{1}{[k]_q!} \langle e_q(t) t^k | B_{n,q}(x) \rangle (x-1)_q^k = \sum_{k=0}^{n} \frac{1}{[k]_q!} \langle e_q(t) | t^k B_{n,q}(x) \rangle (x-1)_q^k \\ &= \sum_{k=0}^{n} \binom{n}{k}_q B_{n-k,q}(1)(x-1)_q^k. \end{aligned} \tag{2.10}$$

From (1.3), we can derive

$$(x-1)_q^n = \sum_{m=0}^{n} \binom{n}{m}_q (-1)^{n-m} q^{\binom{n-m}{2}} x^m. \tag{2.11}$$

Thus, by (2.11), we get

$$\begin{aligned} t^k (x-1)_q^n &= \sum_{m=k}^{n} \binom{n}{m}_q (-1)^{n-m} q^{\binom{n-m}{2}} \frac{[m]_q!}{[m-k]_q!} x^{m-k} \\ &= \frac{[n]_q!}{[n-k]_q!} \sum_{m=0}^{n-k} \binom{n-k}{m}_q (-1)^{n-k-m} q^{\binom{n-k-m}{2}} x^m \\ &= \frac{[n]_q!}{[n-k]_q!} (x-1)_q^{n-k}. \end{aligned} \tag{2.12}$$

By (1.17) and (2.12), we get

$$(x-1)_q^n = \sum_{k=0}^{n} \frac{1}{[k]_q!} \left\langle \frac{e_q(t)-1}{t} t^k \middle| (x-1)_q^n \right\rangle B_{k,q}(x)$$

$$
\begin{aligned}
&= \sum_{k=0}^{n} \binom{n}{k}_q B_{k,q}(x) \left\langle \frac{e_q(t)-1}{t} \middle| (x-1)_q^{n-k} \right\rangle \\
&= \sum_{k=0}^{n} \binom{n}{k}_q B_{k,q}(x) \int_0^1 (x-1)_q^{n-k} d_q x \\
&= \sum_{k=0}^{n} \sum_{m=0}^{n-k} \binom{n}{k}_q \binom{n-k}{m}_q B_{k,q}(x)(-1)^{n-k-m} q^{\binom{n-k-m}{2}} \frac{1}{[m+1]_q}. \qquad (2.13)
\end{aligned}
$$

From (1.6) and (1.10), we note that

$$
\left\langle \frac{t}{e_q(t)-1} \middle| x^n \right\rangle = \sum_{k=0}^{\infty} \frac{B_{k,q}}{[k]_q!} \langle t^k | x^n \rangle = B_{n,q}. \qquad (2.14)
$$

Let $\mathbb{P}_n = \{p(x) \in \mathbb{C}[x] \mid \deg p(x) \leqslant n\}$. For $p(x) \in \mathbb{P}_n$, let us assume that

$$
p(x) = \sum_{k=0}^{n} b_{k,q} B_{k,q}(x). \qquad (2.15)
$$

By (2.1), we see that

$$
\left\langle \left(\frac{e_q(t)-1}{t} \right) t^k \middle| B_{n,q}(x) \right\rangle = [n]_q! \delta_{n,k}, \quad n,k \geqslant 0. \qquad (2.16)
$$

Thus, from (2.15) and (2.16), we have

$$
\left\langle \left(\frac{e_q(t)-1}{t} \right) t^k \middle| p(x) \right\rangle = \sum_{l=0}^{n} b_{l,q} \left\langle \left(\frac{e_q(t)-1}{t} \right) t^k \middle| B_{l,q}(x) \right\rangle = \sum_{l=0}^{n} b_{l,q} [l]_q! \delta_{l,k} = [k]_q! b_{k,q}. \qquad (2.17)
$$

From (1.16), (2.5) and (2.17), we have

$$
b_{k,q} = \frac{1}{[k]_q!} \left\langle \left(\frac{e_q(t)-1}{t} \right) t^k \middle| p(x) \right\rangle = \frac{1}{[k]_q!} \left\langle \frac{e_q(t)-1}{t} \middle| D_q^k p(x) \right\rangle = \frac{1}{[k]_q!} \int_0^1 p^{(k)}(x) dx, \qquad (2.18)
$$

where $p^{(k)}(x) = D_q^k p(x)$. Therefore, by (2.15) and (2.18), we obtain the following theorem.

Theorem 2. *For $p(x) \in \mathbb{P}_n$, let $p(x) = \sum_{k=0}^{n} b_{k,q} B_{k,q}(x)$. Then we have*

$$
b_{k,q} = \frac{1}{[k]_q!} \left\langle \frac{e_q(t)-1}{t} \middle| p^{(k)}(x) \right\rangle = \frac{1}{[k]_q!} \int_0^1 p^{(k)}(x) d_q x,
$$

where $p^{(k)}(x) = D_q^k p(x)$.

Let us consider the q-Bernoulli polynomials of order r as follows:

$$
\left(\frac{t}{e_q(t)-1} \right)^r e_q(xt) = \underbrace{\left(\frac{t}{e_q(t)-1} \right) \times \cdots \times \left(\frac{t}{e_q(t)-1} \right)}_{r \text{ times}} e_q(xt) = \sum_{n=0}^{\infty} B_{n,q}^{(r)}(x) \frac{t^n}{[n]_q!}. \qquad (2.19)
$$

In the special case, $x=0$, $B_{n,q}^{(r)}(0) = B_{n,q}^{(r)}$ is called the n-th q-Bernoulli number of order r. It is easy to show that

$$
\left\langle \left(\frac{t}{e_q(t)-1} \right)^r \middle| x^n \right\rangle = \sum_{k=0}^{\infty} \frac{B_{k,q}^{(r)}}{[k]_q!} \langle t^k | x^n \rangle = B_{n,q}^{(r)}. \qquad (2.20)
$$

From (1.13), (2.14) and (2.20), we note that

$$
B_{n,q}^{(r)} = \left\langle \left(\frac{t}{e_q(t)-1} \right)^r \middle| x^n \right\rangle
$$

$$= \sum_{i_1+\cdots+i_r=n} \binom{n}{i_1,\ldots,i_r}_q \left\langle \frac{t}{e_q(t)-1} \middle| x^{i_1} \right\rangle \cdots \left\langle \frac{t}{e_q(t)-1} \middle| x^{i_r} \right\rangle$$

$$= \sum_{i_1+\ldots+i_r=n} \binom{n}{i_1,\cdots,i_r}_q B_{i_1,q} \cdots B_{i_r,q}. \tag{2.21}$$

Therefore, by (2.21), we have the following lemma.

Lemma 3. *For $n \geqslant 0$, we have*

$$B_{n,q}^{(r)} = \sum_{i_1+\cdots+i_r=n} \binom{n}{i_1,\ldots,i_r}_q B_{i_1,q} \cdots B_{i_r,q}.$$

By (2.19), we easily get

$$B_{n,q}^{(r)}(x) \sim \left(\left(\frac{t}{e_q(t)-1} \right)^r, t \right) \tag{2.22}$$

and

$$B_{n,q}^{(r)}(x) = \left(\frac{t}{e_q(t)-1} \right)^r x^n, \tag{2.23}$$

where $n, r \in \mathbb{Z}^+$. Let us take $p(x) = B_{n,q}^{(r)}(x) = \sum_{k=0}^n \binom{n}{k}_q B_{n-k,q}^{(r)} x^k \in \mathbb{P}_n$. Then we may write

$$p(x) = B_{n,q}^{(r)}(x) = \sum_{k=0}^n b_{k,q} B_{k,q}(x). \tag{2.24}$$

From (2.24), we have

$$p^{(k)}(x) = D_q^k B_{n,q}^{(r)}(x) = [n]_q [n-1]_q \cdots [n-k+1]_q B_{n-k,q}^{(r)}(x) = [k]_q! \binom{n}{k}_q B_{n-k,q}^{(r)}(x). \tag{2.25}$$

By (2.18) and (2.25), we get

$$b_{k,q} = \frac{1}{[k]_q!} \left\langle \left(\frac{e_q(t)-1}{t} \right) t^k \middle| p(x) \right\rangle = \frac{1}{[k]_q!} \left\langle \frac{e_q(t)-1}{t} \middle| D_q^k p(x) \right\rangle$$

$$= \binom{n}{k}_q \left\langle \frac{e_q(t)-1}{t} \middle| B_{n-k,q}^{(r)}(x) \right\rangle = \binom{n}{k}_q \left\langle t^0 \middle| \left(\frac{t}{e_q(t)-1} \right)^{r-1} x^{n-k} \right\rangle$$

$$= \binom{n}{k}_q B_{n-k,q}^{(r-1)}. \tag{2.26}$$

Therefore, by Theorem 2 and (2.24), we obtain the following theorem.

Theorem 4. *For $n \geqslant 0$, we have*

$$B_{n,q}^{(r)}(x) = \sum_{k=0}^n \binom{n}{k}_q \left\langle \frac{e_q(t)-1}{t} \middle| B_{n-k,q}^{(r)}(x) \right\rangle B_{k,q}(x) = \sum_{k=0}^n \binom{n}{k}_q B_{n-k,q}^{(r-1)} B_{k,q}(x).$$

For $p(x) \in \mathbb{P}_n$, let us assume that

$$p(x) = \sum_{k=0}^n b_{k,q}^{(r)} B_{k,q}^{(r)}(x). \tag{2.27}$$

By (2.22), we easily get

$$\left\langle \left(\frac{e_q(t)-1}{t} \right)^r t^k \middle| B_{n,q}^{(r)}(x) \right\rangle = [n]_q! \delta_{n,k}, \quad n, k \geqslant 0. \tag{2.28}$$

From (2.27) and (2.28), we have

$$\left\langle \left(\frac{e_q(t)-1}{t} \right)^r t^k \middle| p(x) \right\rangle = \sum_{l=0}^n b_{l,q}^{(r)} \left\langle \left(\frac{e_q(t)-1}{t} \right)^r t^k \middle| B_{l,q}^{(r)}(x) \right\rangle$$

$$= \sum_{l=0}^{n} b_{l,q}^{(r)} [l]_q! \delta_{l,k} = [k]_q! b_{k,q}^{(r)}. \tag{2.29}$$

By (2.29), we get

$$b_{k,q}^{(r)} = \frac{1}{[k]_q!} \left\langle \left(\frac{e_q(t)-1}{t} \right)^r t^k \middle| p(x) \right\rangle. \tag{2.30}$$

Therefore, by (2.27) and (2.30), we obtain the following theorem.

Theorem 5. *For $p(x) \in \mathbb{P}_n$, let $p(x) = \sum_{k=0}^{n} b_{k,q}^{(r)} B_{k,q}^{(r)}(x)$. Then we have*

$$b_{k,q}^{(r)} = \frac{1}{[k]_q!} \left\langle \left(\frac{e_q(t)-1}{t} \right)^r t^k \middle| p(x) \right\rangle.$$

Let us take $p(x) = B_{n,q}(x)$. Then, by Theorem 1.5, we get

$$B_{n,q}(x) = p(x) = \sum_{k=0}^{n} b_{k,q}^{(r)} B_{k,q}^{(r)}(x), \tag{2.31}$$

where

$$b_{k,q}^{(r)} = \frac{1}{[k]_q!} \left\langle \left(\frac{e_q(t)-1}{t} \right)^r t^k \middle| p(x) \right\rangle = \frac{1}{[k]_q!} \left\langle \left(\frac{e_q(t)-1}{t} \right)^r t^k \middle| B_{n,q}(x) \right\rangle. \tag{2.32}$$

For $k < r$, by (2.32), we have

$$\begin{aligned} b_{k,q}^{(r)} &= \frac{1}{[k]_q!} \left\langle (e_q(t)-1)^r \frac{1}{t^{r-k}} \middle| B_{n,q}(x) \right\rangle \\ &= \frac{1}{[k]_q!} \left(\frac{1}{[n+r-k]_q \cdots [n+1]_q} \right) \left\langle (e_q(t)-1)^r \left(\frac{1}{t}\right)^{r-k} \middle| t^{r-k} B_{n+r-k,q}(x) \right\rangle \\ &= \left(\frac{1}{[k]_q! [r-k]_q!} \right) \left(\frac{[r-k]_q!}{[n+r-k]_q \cdots [n+1]_q} \right) \langle (e_q(t)-1)^r | B_{n+r-k,q}(x) \rangle \\ &= \frac{1}{[r]_q!} \frac{\binom{r}{k}_q}{\binom{n+r-k}{r-k}_q} \sum_{j=0}^{r} \binom{r}{j} (-1)^{r-j} \langle (e_q(t))^j | B_{n+r-k,q}(x) \rangle \\ &= \frac{1}{[r]_q!} \frac{\binom{r}{k}_q}{\binom{n+r-k}{r-k}_q} \sum_{j=0}^{r} \binom{r}{j} (-1)^{r-j} \sum_{m=0}^{n+r-k} \sum_{m_1+\cdots+m_j=m} \binom{m}{m_1, \ldots, m_j}_q \\ &\quad \times \binom{n+r-k}{m}_q B_{n+r-k-m,q}. \end{aligned} \tag{2.33}$$

Let us assume that $k \geqslant r$. Then, by (2.32), we get

$$\begin{aligned} b_{k,q}^{(r)} &= \frac{1}{[k]_q!} \langle (e_q(t)-1)^r | t^{k-r} B_{n,q}(x) \rangle \\ &= \frac{1}{[k]_q!} [n]_q [n-1]_q \cdots [n-k+r+1]_q \langle (e_q(t)-1)^r | B_{n-k+r,q}(x) \rangle \\ &= \frac{[k-r]_q!}{[k]_q!} \binom{n}{k-r}_q \sum_{j=0}^{r} \binom{r}{j} (-1)^{r-j} \langle (e_q(t))^j | B_{n-k+r,q}(x) \rangle \\ &= \frac{1}{[r]_q!} \frac{\binom{n}{k-r}_q}{\binom{k}{r}_q} \sum_{j=0}^{r} \binom{r}{j} (-1)^{r-j} \sum_{m=0}^{n-k+r} \sum_{m_1+\cdots+m_j=m} \binom{m}{m_1, \ldots, m_j}_q \\ &\quad \times \frac{\langle t^m | B_{n-k+r,q}(x) \rangle}{[m]_q!} \\ &= \frac{1}{[r]_q!} \frac{\binom{n}{k-r}_q}{\binom{k}{r}_q} \sum_{j=0}^{r} \binom{r}{j} (-1)^{r-j} \sum_{m=0}^{n-k+r} \sum_{m_1+\cdots+m_j=m} \binom{m}{m_1, \ldots, m_j}_q \end{aligned}$$

$$\times \binom{n-k+r}{m}_q B_{n-k+r-m,q}. \tag{2.34}$$

Therefore, by (2.31), (2.33) and (2.34), we obtain the following theorem.

Theorem 6. *For $n \in \mathbb{Z}^+$ and $r \in \mathbb{N}$, we have*

$$\begin{aligned} B_{n,q}(x) = & \sum_{k=0}^{r-1} \frac{1}{[r]_q!} \frac{\binom{r}{k}_q}{\binom{n+r-k}{r-k}_q} \Bigg\{ \sum_{j=0}^{r} \binom{r}{j} (-1)^{r-j} \sum_{m=0}^{n-k+r} \sum_{m_1+\cdots+m_j=m} \binom{m}{m_1,\ldots,m_j}_q \\ & \times \binom{n-k+r}{m}_q B_{n+r-k-m,q} \Bigg\} B_{k,q}^{(r)}(x) + \sum_{k=r}^{n} \frac{\binom{n}{k-r}_q}{[r]_q! \binom{r}{k}_q} \\ & \times \Bigg\{ \sum_{j=0}^{r} \binom{r}{j} (-1)^{r-j} \sum_{m=0}^{n-k+r} \sum_{m_1+\cdots+m_j+m} \binom{m}{m_1,\ldots,m_j}_q \binom{n-k+r}{m}_q \\ & \times B_{n-k+r-m,q} \Bigg\} B_{k,q}^{(r)}(x). \end{aligned}$$

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References

1 Açikgöz M, Erdal D, Araci S. A new approach to q-Bernoulli numbers and q-Bernoulli polynomials related to q-Bernstein polynomials. Adv Difference Equ, 2010, Article ID: 951764

2 Araci S, Acikgoz M, Kilicman A. Extended p-adic q-invariant integrals on $\mathbf{Z}_p$ associated with applications of Umbral calculus. Adv Difference Equ, 2013, 2013: 96

3 Bayad A, Kim T. Identities involving values of Bernstein, q-Bernstein, q-Bernoulli, and q-Euler polynomials. Russ J Math Phys, 2011, 18: 133–143

4 Carlitz L. q-Bernoulli numbers and polynomials. Duke Math J, 1948, 15: 987–1000

5 Hegazi A S, Mansour M. A note on q-Bernoulli numbers and polynomials. J Nonlinear Math Phys, 2006, 13: 9–18

6 Ismail M E H, Rahman M. Inverse operators, q-fractional integrals, and q-Bernoulli polynomials. J Approx Theory, 2002, 114: 269–307

7 Kim D S, Kim T, Lee S H, et al. Some identities of Bernoulli, Euler and Abel polynomials arising from umbral calculus. Adv Difference Equ, 2013, 2013: 15

8 Kim T. q-Generalized Euler numbers and polynomials. Russ J Math Phys, 2006, 13: 293–298

9 Kim T. q-Bernoulli numbers and polynomials associated with Gaussian binomial coefficients. Russ J Math Phys, 2008, 15: 51–57

10 Kupershmidt B O. Reflection symmetries of q-Bernoulli polynomials. J Nonlinear Math Phys, 2005, 12: 412–422

11 Kurt V, Cenkci M. A new approach to q-Genocchi numbers and polynomials. Bull Korean Math Soc, 2010, 47: 575–583

12 Mahmudov N I. On a class of q-Bernoulli and q-Euler polynomials. Adv Difference Equ, 2013, 2013: 108

13 Mahmudov N I, Keleshteri M E. On a class of generalized q-Bernoulli and q-Euler polynomials. Adv Difference Equ, 2013, 2013: 115

14 Rim S H, Bayad A, Moon E J, et al. A new construction on the q-Bernoulli polynomials. Adv Difference Equ, 2011, 2011: 34

15 Roman S. More on the umbral calculus, with emphasis on the q-umbral calculus. J Math Anal Appl, 1985, 107: 222–254

16 Roman S. The Umbral Calculus. New York: Dover, 2005

17 Sharma A. q-Bernoulli and Euler numbers of higher order. Duke Math J, 1958, 25: 343–353