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Best constants for Hausdorff operators on n-dimensional product spaces

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Abstract In this paper, we study certain Hausdorff operators in the high-dimensional product spaces. We obtain their power weighted boundedness from L^p to L^q and characterize the necessary and sufficient conditions for their boundedness on the power weighted L^p spaces. Moreover, in the case $p = q$, we obtain the sharp bound constants.

Keywords Hausdorff operators, Hardy operators, product spaces, power weights

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1 Introduction

Hausdorff operators have been widely studied by many researchers. They were first introduced in 1917 with summability of number series. The general Hausdorff operator of a Fourier-Stieltjes transform was introduced by Georgakis in [8]. Since then, various boundedness of Hausdorff operators have been studied by Liflyand, Lerner, Móricz, Miyachi et al.; see, for example, [1, 2, 4, 7, 11, 12] and the references therein for details. The one-dimensional Hausdorff operator is defined by

$$h_\Phi f(x) = \int_0^\infty \frac{\Phi(\frac{x}{t})}{t} f(t)dt, \quad x > 0,$$

where $\Phi(t)$ is a locally integrable function in $(0, \infty)$. By the generalized Minkowski inequality and the scaling property, one can easily show that h_Φ is bounded on $L^p(\mathbb{R}^+)$ if

$$\int_0^\infty |\Phi(t)| t^{-1+\frac{1}{p}} dt < +\infty.$$

There are several high-dimensional extensions of h_Φ. Below is one of them introduced in [1, 2]:

$$\widetilde{H}_\Phi f(x) = \int_{\mathbb{R}^n} \frac{\Phi(x|y|^{-1})}{|y|^n} f(y)dy,$$

where $y = (y_1, y_2, \dots, y_n), |y| = \sqrt{y_1^2 + y_2^2 + \cdots + y_n^2}$. In [2], Chen et al. proved various boundednesses for such Hausdorff operators. Later, in [3], Chen et al. extended it to the one-parameter multilinear case.

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They obtained the sharp bounds of multilinear Hausdorff operators on Herz type spaces. Below is one of their theorems.

Theorem A. *Let $p \geqslant 1$. If the non-negative radial function Φ satisfies $\int_0^\infty \Phi(t)t^{-1+\frac{n}{p}}dt < \infty$, then $\widetilde{H}_\Phi$ is bounded on $L^p(\mathbb{R}^n)$; moreover*

$$\|\widetilde{H}_\Phi\|_{L^p\to L^p} = \omega_n \int_0^\infty |\Phi(t)|t^{-1+\frac{n}{p}}dt,$$

where ω_n is the area of the unit sphere.

In this paper, we consider a different kind of multi-parameter Hausdorff operators and obtain their boundedness. Denote $G = (0,+\infty)^n$. Let $\Phi(t_1, t_2, \ldots, t_n)$ be a locally integrable function on G. For any $x = (x_1, x_2, \ldots, x_n) \in G$, we define

$$H_\Phi f(x_1, x_2, \ldots, x_n) = \int_G \frac{\Phi(\frac{x_1}{t_1}, \frac{x_2}{t_2}, \ldots, \frac{x_n}{t_n})}{t_1 t_2 \cdots t_n} f(t_1, t_2, \ldots, t_n)dt_1 dt_2 \cdots dt_n,$$

and call it the Hausdorff operator on the one-dimensional product spaces G. It is easy to verify that if $\Phi(t_1, t_2, \ldots, t_n) = \chi_{(1,+\infty)}(t_1)t_1^{-1} \cdots \chi_{(1,+\infty)}(t_n)t_n^{-1}$, then

$$H_\Phi f(x) = H_n f(x) = \frac{1}{x_1 \cdots x_n} \int_0^{x_1} \cdots \int_0^{x_n} f(t_1, \ldots, t_n)dt_1 \cdots dt_n,$$

which is the Hardy operator on the one-dimensional product spaces. If $\Phi(t_1, t_2, \ldots, t_n) = \chi_{(0,1)}(t_1) \cdots \chi_{(0,1)}(t_n)$, then

$$H_\Phi f(x) = H_n^* f(x) = \int_{x_1}^\infty \cdots \int_{x_n}^\infty \frac{f(t_1 \cdots t_n)}{t_1 \cdots t_n} dt_1 \cdots dt_n,$$

which is the adjoint of H_n. When $n = 1$, H_1 becomes the famous Hardy operator introduced by Hardy in [10]. This operator and its variants attracted a lot of attention. Their boundedness estimates and sharp bounds were studied, for example, in [5, 6, 13–15]. In [13], Pachpatte studied the boundedness of H_n and H_n^* on the Lebesgue space $L^p(G)$. Recently, Wang et al. [14] obtained the boundedness of H_n and H_n^* with power weights on one-dimensional product spaces and found their explicit bounds.

Here, we will consider the operator H_Φ on the weighted Lebesgue spaces. We prove the power weighted L^p to L^q boundedness of H_Φ and obtain the sharp constant for $1 \leqslant p = q \leqslant \infty$. The proof of sufficiency here is different from that in [2, 3]. And the sharp bound is archived in a different way from [14]. In this paper, we will also study the Hausdorff operator on high-dimensional product spaces. Suppose $m \in \mathbb{N}, n_i \in \mathbb{N}, 1 \leqslant i \leqslant m$,

$$H_\Psi^m f(x) = \int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_2}} \cdots \int_{\mathbb{R}^{n_m}} \frac{\Psi(\frac{x_1}{|u_1|}, \frac{x_2}{|u_2|}, \ldots, \frac{x_m}{|u_m|})}{|u_1|^{n_1}|u_2|^{n_2} \cdots |u_m|^{n_m}} f(u_1, u_2, \ldots, u_m)du_1 du_2 \cdots du_m,$$

where $u_i \in \mathbb{R}^{n_i}$, $x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_m}$.

Noting that for $n_1 = n_2 = \cdots = n_m = 1$, the operator H_Ψ^m will be equal to H_Ψ defined above. If $\Psi(t_1, t_2, \ldots, t_m) = \chi_{(1,+\infty)}(t_1)t_1^{-n_1} \cdots \chi_{(1,+\infty)}(t_m)t_m^{-n_m}$, then we will get the product Hardy operator H_n^m, i.e.,

$$H_\Psi^m f = H_n^m f(x) = \frac{1}{|x_1|^{n_1} \cdots |x_m|^{n_m}} \int_{|u_1|<|x_1|} \cdots \int_{|u_m|<|x_m|} f(u_1, \ldots, u_m)du_1 \cdots du_m.$$

If $\Psi(t_1, t_2, \ldots, t_m) = \chi_{(0,1)}(t_1) \cdots \chi_{(0,1)}(t_m)$, then $H_\Psi^m f = H_n^{m*} f$, which is the adjoint operator of H_n^m, i.e.,

$$H_\Psi^m f = H_n^{m*} f(x) = \int_{|u_1|>|x_1|} \cdots \int_{|u_m|>|x_m|} \frac{f(u_1, \ldots, u_m)}{|u_1|^{n_1} \cdots |u_m|^{n_m}} du_1 \cdots du_m.$$

It is not hard to see that for $m = 1$,

$$H_n^m f(x) = \widetilde{H_n} f(x) = \frac{1}{|x|^n} \int_{|u|<|x|} f(u)du, \quad H_n^{m*} f(x) = \widetilde{H_n}^* f(x) = \int_{|u|>|x|} \frac{f(u)}{|u|^n} du.$$

They are the n-dimensional Hardy operator and its adjoint. In 1995, Christ and Grafakos [5] proved that the $L^p(\mathbb{R}^n)$ norm of $\widetilde{H_n}$ is $\frac{p}{p-1}$, which is independent of dimension n. But the $L^p(\mathbb{R}^n)$ norm of H_n obtained in [14] depends on n. See [14] for details. So there are some differences between these two extensions of the classical Hardy operator. In [6], Fu et al. studied the boundedness of $\widetilde{H_n}$ on L^p spaces with power weights, and got the explicit bound of $\widetilde{H_n}$. The above observation motives us to study the Hausdorff operators in the high dimensional product spaces as both the Hardy operator and the adjoint Hardy operator are its special models. Here, we will discuss the boundedness of H^m_Ψ on Lebesgue spaces with power weights. We will get the power weighted L^p to L^q boundedness by using the boundedness of H_Φ. Moreover, we will work out the best constants of H^m_Ψ on these spaces for $1 \leqslant p = q \leqslant \infty$ by constructing a special function. As a corollary, we can get the sharp bounds of H^m_n and H^{m*}_n with power weights.

Now we introduce some notations frequently used in this paper. For two vectors $\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_m) \in \mathbb{R}^m, \vec{\beta} = (\beta_1, \beta_2, \dots, \beta_m) \in \mathbb{R}^m$, $\vec{\alpha}\vec{\beta} = \alpha_1\beta_1 + \alpha_2\beta_2 + \cdots + \alpha_m\beta_m$. Throughout this paper, the notation $\vec{\alpha} < \vec{\beta}$ means $\alpha_i < \beta_i$ for each $i = 1, 2, \dots, m$. Let $x = (x_1, x_2, \dots, x_m) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_m}, x^{\vec{\alpha}}$ be $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_m^{\alpha_m}$, and $|x|^{\vec{\alpha}}$ stands for $|x_1|^{\alpha_1}|x_2|^{\alpha_2}\cdots|x_m|^{\alpha_m}$, where $|x_i| = (\sum_{j=1}^{n_i} x_{ij}^2)^{1/2}, x_i \in \mathbb{R}^{n_i}$. We write $|x|^{\vec{\alpha}} < |y|^{\vec{\beta}}$ if $|x_i|^{\alpha_i} < |y_i|^{\beta_i}$ for each $i = 1, 2, \dots, m$. Denote also $\boldsymbol{p} = (p, p, \dots, p)$. Then it is clear that $x^{\mathbf{1}} = x_1 x_2 \cdots x_m$ and $|x|^{\mathbf{1}} = |x_1||x_2|\cdots|x_m|$.

This paper is organized as follows. In Section 2, we will formulate our results on H_Φ and give the proofs of the theorems. In Section 3, the boundedness of H^m_Ψ on Lebesgue spaces with power weights will be studied.

2 The boundedness of H_Φ

We formulate our results as follows.

Theorem 2.1. *Suppose that $1 \leqslant p \leqslant q \leqslant \infty$ and r satisfies $\frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{r}$. If the vectors $\vec{\alpha}, \vec{\beta} \in \mathbb{R}^n$ satisfy $\frac{\vec{\alpha}+\mathbf{1}}{p} = \frac{\vec{\beta}+\mathbf{1}}{q}$ and*

$$\mathcal{A} := \left(\int_G |\Phi(x)|^r x^{\frac{r(\vec{\beta}+\mathbf{1})}{q} - \mathbf{1}} dx \right)^{\frac{1}{r}} < \infty,$$

then

$$\|H_\Phi f\|_{L^q(G, x^{\vec{\beta}})} \leqslant \mathcal{A} \|f\|_{L^p(G, x^{\vec{\alpha}})}.$$

Remark 2.2. When $\Phi(t_1, t_2, \dots, t_n) = \chi_{(1,+\infty)}(t_1) t_1^{-1} \cdots \chi_{(1,+\infty)}(t_n) t_n^{-1}$, then $H_\Phi f = H_n f$. By a simple computation, we get that if $\vec{\alpha} + \mathbf{1} < \boldsymbol{p}$, then

$$\mathcal{A} = \left(\prod_{i=1}^n \frac{q}{r(q - \beta_i - 1)} \right)^{\frac{1}{r}}.$$

When $\Phi(t_1, t_2, \dots, t_n) = \chi_{(0,1)}(t_1) \cdots \chi_{(0,1)}(t_n)$, then $H_\Phi f = H^*_n f$. If $\vec{\alpha} > -\mathbf{1}$, then

$$\mathcal{A} = \left(\prod_{i=1}^n \frac{q}{r(\beta_i + 1)} \right)^{\frac{1}{r}}.$$

This confirms the constants in [14].

Theorem 2.3. *Let $1 \leqslant p \leqslant \infty$. If $\Phi(x)$ is a non-negative function on G satisfying*

$$\mathcal{B} := \int_G \Phi(x) x^{\frac{\vec{\alpha}+\mathbf{1}}{p} - \mathbf{1}} dx < \infty,$$

then

$$\|H_\Phi f\|_{L^p(G, x^{\vec{\alpha}})} \leqslant \mathcal{B} \|f\|_{L^p(G, x^{\vec{\alpha}})}.$$

Moreover, we have

$$\|H_\Phi\|_{L^p(G,x^{\vec{\alpha}})\to L^p(G,x^{\vec{\alpha}})} = \mathcal{B}.$$

Remark 2.4. When $\vec{\alpha} = (0, 0, \dots, 0)$, we get the boundedness of H_Φ on Lebesgue space $L^p(G)$. That is, for $1 \leqslant p \leqslant \infty$, if $\int_G \Phi(x)x^{\frac{\mathbf{1}}{p}-\mathbf{1}}dx < \infty$, then

$$\|H_\Phi f\|_{L^p(G)} \leqslant \int_G \Phi(x)x^{\frac{\mathbf{1}}{p}-\mathbf{1}}dx\|f\|_{L^p(G)}.$$

Moreover, we have

$$\|H_\Phi\|_{L^p(G)\to L^p(G)} = \int_G \Phi(x)x^{\frac{\mathbf{1}}{p}-\mathbf{1}}dx.$$

Comparing with Theorem A, we see that these two extensions of the Hausdorff operators have different L^p-norms. It means that there are some differences between these two extensions.

In order to prove our theorems, we need the following lemma, which can be found in [9].

Lemma 2.5 (Young's inequality [9])**.** *Let $1 \leqslant p, q, r \leqslant \infty$ satisfy $\frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{r}$, and μ be a Haar measure on G. Then for any f in $L^p(G, \mu)$ and any g in $L^r(G, \mu)$, we have*

$$\|f * g\|_{L^q(G,\mu)} \leqslant \|g\|_{L^r(G,\mu)}\|f\|_{L^p(G,\mu)},$$

where $(f * g)(x) = \int_G f(y)g(y^{-1}x)d\mu(y)$,

$$L^p(G, \mu) = \left\{f : \|f\|_{L^p(G,\mu)} = \left(\int_G |f(x)|^p d\mu(x)\right)^{\frac{1}{p}} < \infty\right\}$$

for $1 \leqslant p < +\infty$, and

$$L^\infty(G, \mu) = \{f : \|f\|_{L^\infty} < \infty\}$$

for $p = +\infty$.

Now let us prove Theorem 2.1.

Proof of Theorem 2.1. The proof is based on an idea used in [5] for proving a result on Hardy operator; also see [14]. As we all know, the multiplicative group $\mathbb{R}^+$ is a locally compact group. So is $G = (0, +\infty)^n$. The Haar measure μ on G is $\frac{dx}{x_1x_2\cdots x_n}$; see [9]. Let $f_1(x) = f(x)x^{\frac{\vec{\alpha}+\mathbf{1}}{p}}$ and $f_2(x) = x^{\frac{\vec{\beta}+\mathbf{1}}{q}}\Phi(x)$. $\|\cdot\|_{L^p(G,\mu)}$ denotes the L^p-norm with respect to the Haar measure on G and $\|\cdot\|_{L^p(G,x^{\vec{\alpha}})}$ respects the L^p-norm with power weight $x^{\vec{\alpha}}$ on G. By a simple calculation, we have

$$\|f_1\|_{L^p(G,\mu)} = \left(\int_G |f(x)|^p x^{\vec{\alpha}+\mathbf{1}}\frac{dx}{x^{\mathbf{1}}}\right)^{\frac{1}{p}} = \left(\int_G |f(x)|^p x^{\vec{\alpha}}dx\right)^{\frac{1}{p}} = \|f\|_{L^p(G,x^{\vec{\alpha}})},$$

and

$$\|f_2\|_{L^r(G,\mu)} = \left(\int_G |\Phi(x)|^r x^{\frac{r(\vec{\beta}+\mathbf{1})}{q}}\frac{dx}{x^{\mathbf{1}}}\right)^{\frac{1}{r}} = \left(\int_G |\Phi(x)|^r x^{\frac{r(\vec{\beta}+\mathbf{1})}{q}-\mathbf{1}}dx\right)^{\frac{1}{r}} = \|\Phi\|_{L^r(G,x^{\frac{r(\vec{\beta}+\mathbf{1})}{q}-\mathbf{1}})}.$$

Noticing that $\frac{\vec{\alpha}+\mathbf{1}}{p} = \frac{\vec{\beta}+\mathbf{1}}{q}$, we have

$$\begin{aligned} f_1 * f_2(x) &= \int_G f_1(y)f_2(y^{-1}x)\frac{dy}{y^{\mathbf{1}}} \\ &= \int_G f(y)y^{\frac{\vec{\alpha}+\mathbf{1}}{p}}(y^{-1}x)^{\frac{\vec{\beta}+\mathbf{1}}{q}}\Phi(y^{-1}x)\frac{dy}{y^{\mathbf{1}}} \\ &= x^{\frac{\vec{\beta}+\mathbf{1}}{q}}\int_G f(y)\frac{\Phi(\frac{x_1}{y_1}, \dots, \frac{x_n}{y_n})}{y_1\cdots y_n}dy_1\cdots dy_n. \end{aligned}$$

So,

$$\begin{aligned}\|f_1 * f_2\|_{L^q(G,\mu)} &= \left(\int_G x^{\vec{\beta}+\mathbf{1}}\left|\int_G f(y)\frac{\Phi(\frac{x_1}{y_1},\ldots,\frac{x_n}{y_n})}{y_1\cdots y_n}dy_1\cdots dy_n\right|^q\frac{dx}{x^{\mathbf{1}}}\right)^{\frac{1}{q}}\\ &= \left(\int_G \left|\int_G f(y)\frac{\Phi(\frac{x_1}{y_1},\ldots,\frac{x_n}{y_n})}{y_1\cdots y_n}dy\right|^q x^{\vec{\beta}}dx\right)^{\frac{1}{q}}\\ &= \|H_\Phi f\|_{L^q(G,x^{\vec{\beta}})}.\end{aligned}$$

Then by Lemma 2.5, we have

$$\|H_\Phi f\|_{L^q(G,x^{\vec{\beta}})} \leqslant \left(\int_G |\Phi(x)|^r x^{\frac{r(\vec{\beta}+\mathbf{1})}{q}-\mathbf{1}}dx\right)^{\frac{1}{r}}\|f\|_{L^p(G,x^{\vec{\alpha}})}.$$

This finishes the proof. □

Now, we turn to the proof of Theorem 2.3.

Proof of Theorem 2.3. Take $p=q$ in Theorem 2.1. Then $r=1$ and $\vec{\alpha}=\vec{\beta}$. So we already get the sufficient part of the theorem. For the necessary part, let

$$f_k(x) = x^{-\frac{\vec{\alpha}+\mathbf{1}+\frac{\mathbf{1}}{\boldsymbol{k}}}{p}}\chi_{\{x^{\mathbf{1}}>\mathbf{1}\}}(x),$$

where

$$\frac{\mathbf{1}}{\boldsymbol{k}} = \left(\frac{1}{k},\frac{1}{k},\ldots,\frac{1}{k}\right) \quad (k>0).$$

Then by a simple calculation, we have $\|f_k\|_{L^p(G,x^{\vec{\alpha}})} = k^{\frac{n}{p}}$. Using a changing of variables, we obtain that

$$\begin{aligned}\|H_\Phi f_k\|_{L^p(G,x^{\vec{\alpha}})} &\geqslant \left(\int_{x^{\mathbf{1}}\geqslant\boldsymbol{k}}\left|\int_{u^{\mathbf{1}}>\mathbf{1}}\frac{\Phi(\frac{x_1}{u_1},\ldots,\frac{x_n}{u_n})}{u_1\cdots u_n}u^{-\frac{\vec{\alpha}+\mathbf{1}+\frac{\mathbf{1}}{\boldsymbol{k}}}{p}}du\right|^p x^{\vec{\alpha}}dx\right)^{\frac{1}{p}}\\ &= \left(\int_{x^{\mathbf{1}}\geqslant\boldsymbol{k}} x^{-\mathbf{1}-\frac{\mathbf{1}}{\boldsymbol{k}}}\left|\int_{0<y^{\mathbf{1}}<x^{\mathbf{1}}}\Phi(y)y^{\frac{\vec{\alpha}+\mathbf{1}+\frac{\mathbf{1}}{\boldsymbol{k}}}{p}-\mathbf{1}}dy\right|^p dx\right)^{\frac{1}{p}}\\ &\geqslant (k^{1-\frac{1}{k}})^{\frac{n}{p}}\int_{0<y^{\mathbf{1}}<\boldsymbol{k}}\Phi(y)y^{\frac{\vec{\alpha}+\mathbf{1}+\frac{\mathbf{1}}{\boldsymbol{k}}}{p}-\mathbf{1}}dy\\ &= k^{-\frac{1}{k}\cdot\frac{n}{p}}\int_{0<y^{\mathbf{1}}<\boldsymbol{k}}\Phi(y)y^{\frac{\vec{\alpha}+\mathbf{1}+\frac{\mathbf{1}}{\boldsymbol{k}}}{p}-\mathbf{1}}dy\|f_k\|_{L^q(G,x^{\vec{\alpha}})}.\end{aligned}$$

On the other hand,

$$\|H_\Phi f_k\|_{L^p(G,x^{\vec{\alpha}})} \leqslant \|H_\Phi\|_{L^p(G,x^{\vec{\alpha}})\to L^p(G,x^{\vec{\alpha}})}\|f_k\|_{L^p(G,x^{\vec{\alpha}})}.$$

So,

$$\|H_\Phi\|_{L^p(G,x^{\vec{\alpha}})\to L^p(G,x^{\vec{\alpha}})} \geqslant k^{-\frac{1}{k}\cdot\frac{n}{p}}\int_{0<y^{\mathbf{1}}<\boldsymbol{k}}\Phi(y)y^{\frac{\vec{\alpha}+\mathbf{1}+\frac{\mathbf{1}}{\boldsymbol{k}}}{p}-\mathbf{1}}dy.$$

Let $\boldsymbol{k}\to\infty$ (using $k^{\frac{1}{k}}\to 1$). We obtain

$$\|H_\Phi\|_{L^p(G,x^{\vec{\alpha}})\to L^p(G,x^{\vec{\alpha}})} \geqslant \int_G \Phi(y)y^{\frac{\vec{\alpha}+\mathbf{1}}{p}-\mathbf{1}}dy.$$

Consequently,

$$\|H_\Phi\|_{L^p(G,x^{\vec{\alpha}})\to L^p(G,x^{\vec{\alpha}})} = \int_G \Phi(y)y^{\frac{\vec{\alpha}+\mathbf{1}}{p}-\mathbf{1}}dy.$$

This completes the proof. □

3 The boundedness of H_Ψ^m

In this section, we will discuss the Hausdorff operator on high-dimensional product space. Our results can be stated as follows. Denote $\boldsymbol{n}=(n_1,n_2,\ldots,n_m),\mathbb{R}^{\boldsymbol{n}}=\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\cdots\times\mathbb{R}^{n_m}$ and $\mathbb{S}^{\boldsymbol{n-1}}=\mathbb{S}^{n_1-1}\times\mathbb{S}^{n_2-1}\times\cdots\times\mathbb{S}^{n_m-1}$. $\Psi(t_1,t_2,\ldots,t_m)$ is a locally integrable function on $\mathbb{R}^+\times\mathbb{R}^+\times\cdots\times\mathbb{R}^+$. For convenience, we also use $\Psi(x)$ to denote $\Psi(|x_1|,|x_2|,\ldots,|x_m|)$.

Theorem 3.1. *Soppose $1\leqslant p\leqslant q\leqslant\infty$, r satisfies $\frac{1}{q}+1=\frac{1}{p}+\frac{1}{r}$, the vectors $\vec{\gamma}$ and $\vec{\delta}\in\mathbb{R}^m$ satisfy $\frac{\vec{\gamma}+\boldsymbol{n}}{p}=\frac{\vec{\delta}+\boldsymbol{n}}{q}$ and $\Psi(x)=\Psi(|x_1|,|x_2|,\ldots,|x_m|)$ is a radial function in each variable. If*

$$\mathcal{C}:=\left(\int_{\mathbb{R}^{\boldsymbol{n}}}|\Psi(x)|^r|x|^{\frac{r(\vec{\delta}+\boldsymbol{n})}{q}-\boldsymbol{n}}dx\right)^{\frac{1}{r}}<\infty,$$

then

$$\|H_\Psi^m f\|_{L^q(\mathbb{R}^{\boldsymbol{n}},|x|^{\vec{\delta}})}\leqslant\mathcal{C}\|f\|_{L^p(\mathbb{R}^{\boldsymbol{n}},|x|^{\vec{\gamma}})}.$$

Theorem 3.2. *Let $1\leqslant p\leqslant\infty$. If $\Psi(x)=\Psi(|x_1|,|x_2|,\ldots,|x_m|)$ is a non-negative function satisfying*

$$\mathcal{D}:=\int_{\mathbb{R}^{\boldsymbol{n}}}\Psi(x)|x|^{\frac{\vec{\alpha}+\boldsymbol{n}}{p}-\boldsymbol{n}}dx<\infty,$$

then

$$\|H_\Psi^m f\|_{L^p(\mathbb{R}^{\boldsymbol{n}},|x|^{\vec{\alpha}})}\leqslant\mathcal{D}\|f\|_{L^p(\mathbb{R}^{\boldsymbol{n}},|x|^{\vec{\alpha}})}.$$

Moreover, we have

$$\|H_\Psi^m\|_{L^p(\mathbb{R}^{\boldsymbol{n}},x^{\vec{\alpha}})\to L^p(\mathbb{R}^{\boldsymbol{n}},x^{\vec{\alpha}})}=\mathcal{D}.$$

Remark 3.3. When $\Psi(t_1,t_2,\ldots,t_m)=\chi_{(1,+\infty)}(t_1)t_1^{-n_1}\cdots\chi_{(1,+\infty)}(t_m)t_m^{-n_m}$, H_Ψ^m is the product Hardy operator H_n^m. By a computation, we get that if $\alpha_i<(p-1)n_i$, then $\mathcal{D}=\prod_{i=1}^m\frac{p\omega_{n_i}}{n_ip-n_i-\alpha_i}$ is its sharp bound. When $\Psi(t_1,t_2,\ldots,t_m)=\chi_{(0,1)}(t_1)\cdots\chi_{(0,1)}(t_m)$, then $H_\Psi^m f=H_n^{m*}f$, which is the adjoint operator of H_n^m. If $\alpha_i>-n_i$, then $\mathcal{D}=\prod_{i=1}^m\frac{p\omega_{n_i}}{\alpha_i+n_i}$ is the sharp bound of H_n^{m*}. If $m=1$, we obtain that $\|\widetilde{H}_n\|_{L^p(\mathbb{R}^n,|x|^\alpha)\to L^p(\mathbb{R}^n,|x|^\alpha)}=\frac{p\omega_n}{pn-n-\alpha}$ and $\|\widetilde{H}_n^*\|_{L^p(\mathbb{R}^n,|x|^\alpha)\to L^p(\mathbb{R}^n,|x|^\alpha)}=\frac{p\omega_n}{n+\alpha}$, which is a main theorem in [6].

Remark 3.4. If $\vec{\alpha}=(0,0,\ldots,0)$, then we obtain the results on the Lebesgue space $L^p(\mathbb{R}^{\boldsymbol{n}})$. We omit the details. If $m=1$, then we obtain that the sharp bounds of $\widetilde{H}_\Psi$ on weighted Lebesgue space $L^p(\mathbb{R}^n,|x|^\alpha)$ is $\int_{\mathbb{R}^n}\Psi(x)|x|^{\frac{\alpha+n}{p}-n}dx$. It is exactly the sharp bound for $\widetilde{H}_\Psi$ in [2].

To prove Theorem 3.1, we need the following lemma.

Lemma 3.5. *Suppose $\vec{\delta}$ is a vector in $\mathbb{R}^m$. Let*

$$g(x)=\prod_{i=1}^m\frac{1}{\omega_{n_i}}\int_{\mathbb{S}^{n_1-1}}\int_{\mathbb{S}^{n_2-1}}\cdots\int_{\mathbb{S}^{n_m-1}}f(|x_1|x_1',|x_2|x_2'\cdots|x_m|x_m')d\sigma(x_1')d\sigma(x_2')\cdots d\sigma(x_m'),$$

where $\omega_{n_i}(i=1,\ldots,m)$ is the area of the unit sphere $\mathbb{S}^{n_i-1}$. Then it is easy to see that g is a radial function of f, and we have

(1)
$$H_\Psi^m f(x)=H_\Psi^m g(x);$$

(2)
$$\frac{\|H_\Psi^m f\|_{L^p(\mathbb{R}^{\boldsymbol{n}},|x|^{\vec{\delta}})}}{\|f\|_{L^p(\mathbb{R}^{\boldsymbol{n}},|x|^{\vec{\delta}})}}\leqslant\frac{\|H_\Psi^m g\|_{L^p(\mathbb{R}^{\boldsymbol{n}},|x|^{\vec{\delta}})}}{\|g\|_{L^p(\mathbb{R}^{\boldsymbol{n}},|x|^{\vec{\delta}})}}.$$

Proof. The main idea of this proof comes from [6]. For (1), by the definition of g and Fubini's theorem, we have

$$H_\Psi^m g(x)=\int_{\mathbb{R}^{\boldsymbol{n}}}\frac{\Psi(\frac{x_1}{|u_1|},\frac{x_2}{|u_2|},\ldots,\frac{x_m}{|u_m|})}{|u_1|^{n_1}|u_2|^{n_2}\cdots|u_m|^{n_m}}g(u_1,u_2,\ldots,u_m)du_1du_2\cdots du_m$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^{\boldsymbol{n}}} \frac{\Psi(\frac{x_1}{|u_1|}, \frac{x_2}{|u_2|}, \ldots, \frac{x_m}{|u_m|})}{|u_1|^{n_1}|u_2|^{n_2}\cdots|u_m|^{n_m}} \prod_{i=1}^{m} \frac{1}{\omega_{n_i}} \\
&\quad \times \int_{\mathbb{S}^{\boldsymbol{n}-\boldsymbol{1}}} f(|u_1|u_1', |u_2|u_2', \ldots, |u_m|u_m') d\sigma(u_1')d\sigma(u_2')\cdots d\sigma(u_m') du_1 du_2 \cdots du_m \\
&= \prod_{i=1}^{m} \frac{1}{\omega_{n_i}} \int_{\mathbb{S}^{\boldsymbol{n}-\boldsymbol{1}}} \int_{\mathbb{R}^{\boldsymbol{n}}} \frac{\Psi(\frac{x_1}{|u_1|}, \frac{x_2}{|u_2|}, \ldots, \frac{x_m}{|u_m|})}{|u_1|^{n_1}|u_2|^{n_2}\cdots|u_m|^{n_m}} \\
&\quad \times f(u_1, u_2, \ldots, u_m) du_1 du_2 \cdots du_m d\sigma(u_1')d\sigma(u_2')\cdots d\sigma(u_m') \\
&= \int_{\mathbb{R}^{\boldsymbol{n}}} \frac{\Psi(\frac{x_1}{|u_1|}, \frac{x_2}{|u_2|}, \ldots, \frac{x_m}{|u_m|})}{|u_1|^{n_1}|u_2|^{n_2}\cdots|u_m|^{n_m}} f(u_1, u_2, \ldots, u_m) du_1 du_2 \cdots du_m \\
&= H_{\Psi}^{m} f(x).
\end{aligned}
$$

For (2), using the generalized Minkowski inequality, we have

$$
\begin{aligned}
\|g\|_{L^p(\mathbb{R}^{\boldsymbol{n}}, |x|^{\vec{\delta}})} &= \prod_{i=1}^{m} \frac{1}{\omega_{n_i}} \bigg(\int_{\mathbb{R}^{\boldsymbol{n}}} \int_{\mathbb{S}^{\boldsymbol{n}-\boldsymbol{1}}} |f(|x_1|x_1', |x_2|x_2', \ldots, |x_m|x_m') \\
&\quad \times d\sigma(x_1')d\sigma(x_2')\cdots d\sigma(x_m')|^p |x|^{\vec{\delta}} dx_1 dx_2 \cdots dx_m \bigg)^{\frac{1}{p}} \\
&\leqslant \prod_{i=1}^{m} \frac{1}{\omega_{n_i}} \int_{\mathbb{S}^{\boldsymbol{n}-\boldsymbol{1}}} \bigg(\int_{\mathbb{R}^{\boldsymbol{n}}} |f(x_1, x_2, \ldots, x_m)|^p \\
&\quad \times |x|^{\vec{\delta}} dx_1 dx_2 \cdots dx_m \bigg)^{\frac{1}{p}} d\sigma(x_1')d\sigma(x_2')\cdots d\sigma(x_m') \\
&= \bigg(\int_{\mathbb{R}^{\boldsymbol{n}}} |f(x_1, x_2, \ldots, x_m)|^p |x|^{\vec{\delta}} dx_1 dx_2 \cdots dx_m \bigg)^{\frac{1}{p}} \\
&= \|f\|_{L^p(\mathbb{R}^{\boldsymbol{n}}, |x|^{\vec{\delta}})}.
\end{aligned}
$$

So, we can conclude that

$$
\frac{\|H_{\Psi}^{m} f\|_{L^p(\mathbb{R}^{\boldsymbol{n}}, |x|^{\vec{\delta}})}}{\|f\|_{L^p(\mathbb{R}^{\boldsymbol{n}}, |x|^{\vec{\delta}})}} \leqslant \frac{\|H_{\Psi}^{m} g\|_{L^p(\mathbb{R}^{\boldsymbol{n}}, |x|^{\vec{\delta}})}}{\|g\|_{L^p(\mathbb{R}^{\boldsymbol{n}}, |x|^{\vec{\delta}})}}.
$$

The proof is complete. □

Now let us prove Theorem 3.1.

Proof of Theorem 3.1. For convenience, we suppose $m = 2$. For $m > 2$, we can prove the theorem similarly without any essential difficulty. Lemma 3.5 implies that the operator H_{Ψ}^{m} and its restriction to radial functions have the same $\|\cdot\|_{L^p(\mathbb{R}^{\boldsymbol{n}}, |x|^{\vec{\delta}})}$ norm. So, without loss of generality, we assume that f is a radial function. By the polar decomposition and Fubini's theorem, we have

$$
\begin{aligned}
\|H_{\Psi}^{2} f\|_{L^q(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}, |x|^{\vec{\delta}})} &= \bigg(\int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_2}} \bigg| \int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_2}} \frac{\Psi(\frac{|x_1|}{|u_1|}, \frac{|x_2|}{|u_2|})}{|u_1|^{n_1}|u_2|^{n_2}} f(u_1, u_2) du_1 du_2 \bigg|^q |x|^{\vec{\delta}} dx \bigg)^{\frac{1}{q}} \\
&= \bigg(\int_{\mathbb{S}^{n_1-1}} \int_{\mathbb{S}^{n_2-1}} \int_0^\infty \int_0^\infty \bigg| \int_{\mathbb{S}^{n_1-1}} \int_{\mathbb{S}^{n_2-1}} \int_0^\infty \int_0^\infty \frac{\Psi(\frac{t_1}{s_1}, \frac{t_2}{s_2})}{s_1^{n_1} s_2^{n_2}} \\
&\quad \times f(s_1, s_2) s_1^{n_1-1} s_2^{n_2-1} ds_1 ds_2 d\sigma(\theta_1) d\sigma(\theta_2) \bigg|^q t^{\vec{\delta}+\boldsymbol{n}-\boldsymbol{1}} dt d\sigma(\xi_1) d\sigma(\xi_2) \bigg)^{\frac{1}{q}} \\
&= \omega_{n_1}^{1+\frac{1}{q}} \omega_{n_2}^{1+\frac{1}{q}} \bigg(\int_0^\infty \int_0^\infty \bigg| \int_0^\infty \int_0^\infty \frac{\Psi(\frac{t_1}{s_1}, \frac{t_2}{s_2})}{s_1 s_2} f(s_1, s_2) ds \bigg|^q t^{\vec{\delta}+\boldsymbol{n}-\boldsymbol{1}} dt \bigg)^{\frac{1}{q}} \\
&= \omega_{n_1}^{1+\frac{1}{q}} \omega_{n_2}^{1+\frac{1}{q}} \|H_{\Psi} f\|_{L^q((0,\infty)\times(0,\infty), t^{\vec{\delta}+\boldsymbol{n}-\boldsymbol{1}})}.
\end{aligned}
$$

By Theorem 2.1, the equalities $\frac{1}{q}+1=\frac{1}{p}+\frac{1}{r}$, $\frac{\vec{\gamma}+\boldsymbol{n}}{p}=\frac{\vec{\delta}+\boldsymbol{n}}{q}$, by substituting variables and polar decomposition again, we get that

$$\begin{aligned}\|H_\Psi^2 f\|_{L^q(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2},|x|^{\vec{\delta}})} &\leqslant \omega_{n_1}^{1+\frac{1}{q}}\omega_{n_2}^{1+\frac{1}{q}}\left(\int_0^\infty\int_0^\infty|\Psi(t)|^r t^{\frac{r(\vec{\delta}+\boldsymbol{n})}{q}-\boldsymbol{1}}dt\right)^{\frac{1}{r}}\left(\int_0^\infty\int_0^\infty|f(s)|^p s^{(\vec{\gamma}+\boldsymbol{n}-\boldsymbol{1})}ds\right)^{\frac{1}{p}}\\ &=\left(\int_{\mathbb{S}^{n_1-1}}\int_{\mathbb{S}^{n_2-1}}\int_0^\infty\int_0^\infty|\Psi(t_1,t_2)|^r t^{\frac{r(\vec{\delta}+\boldsymbol{n})}{q}-\boldsymbol{1}}dtd\sigma(\xi_1)d\sigma(\xi_2)\right)^{\frac{1}{r}}\\ &\quad\times\left(\int_{\mathbb{S}^{n_1-1}}\int_{\mathbb{S}^{n_2-1}}\int_0^\infty\int_0^\infty|f(s_1,s_2)|^p s^{(\vec{\gamma}+\boldsymbol{n}-\boldsymbol{1})}dsd\sigma(\zeta_1)d\sigma(\zeta_2)\right)^{\frac{1}{p}}\\ &=\left(\int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}|f(x_1,x_2)|^p|x|^{\vec{\gamma}}dx\right)^{\frac{1}{p}}\left(\int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}|\Psi(x)|^r|x|^{\frac{r(\vec{\delta}+\boldsymbol{n})}{q}-\boldsymbol{n}}dx\right)^{\frac{1}{r}}.\end{aligned}$$

Consequently,

$$\|H_\Psi^2 f\|_{L^q(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2},|x|^{\vec{\delta}})}\leqslant\mathcal{C}\|f\|_{L^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2},|x|^{\vec{\gamma}})}.$$

This finishes the proof. □

Now we are in a position to prove Theorem 3.2.

Proof of Theorem 3.2. The first part of the theorem is a special case of Theorem 3.1. So, we only need to prove the second part of the theorem. As the proof of Theorem 3.1, we also assume $m=2$ and f is a radial function. Let

$$f_{\boldsymbol{k}}(x)=|x|^{-\frac{\vec{\alpha}+\boldsymbol{n}+\frac{\boldsymbol{1}}{\boldsymbol{k}}}{\boldsymbol{p}}}\chi_{\{|x|^{\boldsymbol{1}}>\boldsymbol{1}\}}(x).$$

A normal computation by polar transformation shows that

$$\|f_{\boldsymbol{k}}\|_{L^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2},|x|^{\vec{\alpha}})}=\omega_{n_1}^{\frac{1}{p}}\omega_{n_2}^{\frac{1}{p}}k^{\frac{2}{p}}.$$

By polar decomposition again and a changing of variables, we get

$$\begin{aligned}H_\Psi^2 f_{\boldsymbol{k}}(x)&=\int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}}\frac{\Psi(\frac{|x_1|}{|u_1|},\frac{|x_2|}{|u_2|})}{|u_1|^{n_1}|u_2|^{n_2}}f_{\boldsymbol{k}}(u_1,u_2)du_1du_2\\ &=\int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}}\frac{\Psi(\frac{|x_1|}{|u_1|},\frac{|x_2|}{|u_2|})}{|u_1|^{n_1}|u_2|^{n_2}}|u|^{-\frac{\vec{\alpha}+\boldsymbol{n}+\frac{\boldsymbol{1}}{\boldsymbol{k}}}{\boldsymbol{p}}}\chi_{\{|u|^{\boldsymbol{1}}>\boldsymbol{1}\}}(u)du_1du_2\\ &=\int_{|u_1|>1}\int_{|u_2|>1}\frac{\Psi(\frac{|x_1|}{|u_1|},\frac{|x_2|}{|u_2|})}{|u_1|^{n_1}|u_2|^{n_2}}|u_1|^{-\frac{\alpha_1+n_1+\frac{1}{k}}{p}}|u_2|^{-\frac{\alpha_2+n_2+\frac{1}{k}}{p}}du_1du_2\\ &=\int_1^\infty\int_1^\infty\int_{\mathbb{S}^{n_1-1}}\int_{\mathbb{S}^{n_2-1}}\frac{\Psi(\frac{|x_1|}{t_1},\frac{|x_2|}{t_2})}{t_1t_2}t_1^{-\frac{\alpha_1+n_1+\frac{1}{k}}{p}}t_2^{-\frac{\alpha_2+n_2+\frac{1}{k}}{p}}d\sigma(u_1')d\sigma(u_2')dt_1dt_2\\ &=\omega_{n_1}\omega_{n_2}\int_0^{|x_2|}\int_0^{|x_1|}\Psi(s_1,s_2)s_1^{\frac{\alpha_1+n_1+\frac{1}{k}}{p}-1}s_2^{\frac{\alpha_2+n_2+\frac{1}{k}}{p}-1}|x|^{-\frac{\vec{\alpha}+\boldsymbol{n}+\frac{\boldsymbol{1}}{\boldsymbol{k}}}{\boldsymbol{p}}}ds_1ds_2.\end{aligned}$$

Thus, we obtain that

$$\begin{aligned}&\|H_\Psi^2 f_{\boldsymbol{k}}\|_{L^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2},|x|^{\vec{\alpha}})}\\ &\quad=\left(\int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}}|H_\Psi^2 f_{\boldsymbol{k}}(x)|^p|x|^{\vec{\alpha}}dx\right)^{\frac{1}{p}}\\ &\quad=\omega_{n_1}\omega_{n_2}\left(\int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}}\left|\int_0^{|x_2|}\int_0^{|x_1|}\Psi(s_1,s_2)\cdot s_1^{\frac{\alpha_1+n_1+\frac{1}{k}}{p}-1}s_2^{\frac{\alpha_2+n_2+\frac{1}{k}}{p}-1}|x|^{-\frac{\vec{\alpha}+\boldsymbol{n}+\frac{\boldsymbol{1}}{\boldsymbol{k}}}{\boldsymbol{p}}}ds_1ds_2\right|^p|x|^{\vec{\alpha}}dx\right)^{\frac{1}{p}}\\ &\quad\geqslant\omega_{n_1}\omega_{n_2}\left(\int_{|x_1|\geqslant k}\int_{|x_2|\geqslant k}\left|\int_0^k\int_0^k\Psi(s_1,s_2)\cdot s_1^{\frac{\alpha_1+n_1+\frac{1}{k}}{p}-1}s_2^{\frac{\alpha_2+n_2+\frac{1}{k}}{p}-1}ds_1ds_2\right|^p|x|^{-\boldsymbol{n}-\frac{\boldsymbol{1}}{\boldsymbol{k}}}dx\right)^{\frac{1}{p}}\end{aligned}$$

$$= \omega_{n_1}\omega_{n_2}\left(\int_{|x_1|\geqslant k}\int_{|x_2|\geqslant k}|x|^{-\boldsymbol{n}-\frac{1}{\boldsymbol{k}}}dx\right)^{\frac{1}{p}}\cdot\int_0^k\int_0^k\Psi(s_1,s_2)s_1^{\frac{\alpha_1+n_1+\frac{1}{k}}{p}-1}s_2^{\frac{\alpha_2+n_2+\frac{1}{k}}{p}-1}ds_1ds_2.$$

By polar decomposition, we can calculate that

$$\left(\int_{|x_1|\geqslant k}\int_{|x_2|\geqslant k}|x|^{-\boldsymbol{n}-\frac{1}{\boldsymbol{k}}}dx\right)^{\frac{1}{p}} = \omega_{n_1}^{\frac{1}{p}}\omega_{n_2}^{\frac{1}{p}}k^{\frac{2}{p}}k^{-\frac{1}{k}\cdot\frac{2}{p}} = k^{-\frac{1}{k}\cdot\frac{2}{p}}\|f_{\boldsymbol{k}}\|_{L^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2},|x|^{\vec{\alpha}})}.$$

Thus,

$$\begin{aligned}\|H_\Psi^2 f_{\boldsymbol{k}}\|_{L^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2},|x|^{\vec{\alpha}})} \geqslant\ & k^{-\frac{1}{k}\cdot\frac{2}{p}}\|f_{\boldsymbol{k}}\|_{L^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2},|x|^{\vec{\alpha}})}\\ &\times\omega_{n_1}\omega_{n_2}\int_0^k\int_0^k\Psi(s_1,s_2)s_1^{\frac{\alpha_1+n_1+\frac{1}{k}}{p}-1}s_2^{\frac{\alpha_2+n_2+\frac{1}{k}}{p}-1}ds_1ds_2\\ =\ & k^{-\frac{1}{k}\cdot\frac{2}{p}}\|f_{\boldsymbol{k}}\|_{L^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2},|x|^{\vec{\alpha}})}\int_{\mathbb{S}^{n_1-1}}\int_{\mathbb{S}^{n_2-1}}\int_0^k\int_0^k\Psi(s_1,s_2)\\ &\times s_1^{\frac{\alpha_1+n_1+\frac{1}{k}}{p}-n_1}s_2^{\frac{\alpha_2+n_2+\frac{1}{k}}{p}-n_2}s^{\boldsymbol{n}-\boldsymbol{1}}ds_1ds_2d\sigma(\xi_1)d\sigma(\xi_2)\\ =\ & k^{-\frac{1}{k}\cdot\frac{2}{p}}\|f_{\boldsymbol{k}}\|_{L^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2},|x|^{\vec{\alpha}})}\int_{|x|<\boldsymbol{k}}\Psi(x)|x|^{\frac{\vec{\alpha}+\boldsymbol{n}+\frac{1}{\boldsymbol{k}}}{\boldsymbol{p}}-\boldsymbol{n}}dx.\end{aligned}$$

Consequently,

$$\|H_\Psi^2\|_{L^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2},|x|^{\vec{\alpha}})\to L^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2},|x|^{\vec{\alpha}})} \geqslant k^{-\frac{1}{k}\cdot\frac{2}{p}}\int_{|x|<\boldsymbol{k}}\Psi(x)|x|^{\frac{\vec{\alpha}+\boldsymbol{n}+\frac{1}{\boldsymbol{k}}}{\boldsymbol{p}}-\boldsymbol{n}}dx.$$

Letting $k\to\infty$, we get

$$\|H_\Psi^2\|_{L^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2},|x|^{\vec{\alpha}})\to L^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2},|x|^{\vec{\alpha}})} \geqslant \int_{\mathbb{R}^{\boldsymbol{n}}}\Psi(x)|x|^{\frac{\vec{\alpha}+\boldsymbol{n}}{\boldsymbol{p}}-\boldsymbol{n}}dx.$$

So,

$$\|H_\Psi^2\|_{L^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2},|x|^{\vec{\alpha}})\to L^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2},|x|^{\vec{\alpha}})} = \int_{\mathbb{R}^{\boldsymbol{n}}}\Psi(x)|x|^{\frac{\vec{\alpha}+\boldsymbol{n}}{\boldsymbol{p}}-\boldsymbol{n}}dx.$$

This completes the proof. □

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