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On a semilinear stochastic partial differential equation with double-parameter fractional noises

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Abstract We study the existence, uniqueness and Hölder regularity of the solution to a stochastic semilinear equation arising from 1-dimensional integro-differential scalar conservation laws. The equation is driven by double-parameter fractional noises. In addition, the existence and moment estimate are also obtained for the density of the law of such a solution.

Keywords stochastic partial differential equations, double-parameter fractional noises, Hölder regularity, density of the law, Malliavin calculus

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1 Introduction

This paper is concerned with the initial problem for the following stochastic semilinear equation,

$$\begin{cases} \left(\dfrac{\partial}{\partial t} - \mathcal{A}\right) u(t,x) = -\dfrac{\partial}{\partial x} q(t,x,u(t,x)) + \dot{W}^H(t,x), \quad \text{in } [0,T]\times\mathbb{R}, \\ u(0,\cdot) = u_0(\cdot), \quad x \in \mathbb{R}, \end{cases} \tag{1.1}$$

on the given domain $[0,T]\times\mathbb{R}$ with $L^p(\mathbb{R})$ initial condition with $p \geqslant 2$, where $\mathcal{A}$ is a symmetric integro-differential operator, the $L^2(\mathbb{R})$-generator of a symmetric, nonlocal, regular Dirichlet form which generates a strong Feller semigroup $\{e^{s\mathcal{A}}\}_{s>0}$ with transition density kernel G. Here, $q : [0,T]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is measurable and corresponds to the "nonlinearity", and $\dot{W}^H(t,x)$ is the so-called double-parameter fractional noises (see Section 2 for the definition).

Recently, there has been increasing interest in studying integro-differential scalar conservation laws of nonlocal type involving generators of Lévy type (see Biler et al. [4–6] and references therein) as well as in studying white noise perturbations of Burgers-type nonlinear partial differential equations with random initial data, see e.g., Bertoin [3], Giraud [13], Wehr et al. [34], Winkel [36] and references therein, or white noise driven stochastic Burgers and fractal Burgers equations, respectively in [30] and [31] where the mild solution is investigated in the initial problem for both the stochastic Burgers and stochastic fractal Burgers equation with Lévy time-space white noise.

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On the other hand, there also has been more and more studies on stochastic partial differential equation driven by fractional noises. Hu [15] proposed the multiple stochastic integral with respect to multiple-parameter fractional noises, then showed via chaos expansion the existence and uniqueness of the solution of a class of second-order stochastic heat equation, and further estimated the Lyapunov exponents of the solutions. Linear stochastic evolution equations in a Hilbert space driven by an additive cylindrical fractional Brownian motion with Hurst parameter H were studied by Duncan et al. [12] in the case $H \in (1/2, 1)$ and by Tindel et al. [29] in the general case, where they provided necessary and sufficient conditions for the existence and uniqueness of an evolution solution. Nualart and Ouknine [28] discussed the existence and uniqueness of the mild solutions to a class of second-order heat equations with additive fractional noises (fractional in time and white in space) with the Hurst parameter $H > 1/2$ under some restrictive conditions. Hu and Nualart [16] studied the d-dimensional stochastic heat equation with a multiplicative Gaussian noise which is white in space and has the covariance of a fractional Brownian motion with Hurst parameter $H \in (0, 1)$ in time. First they considered the equation in the Itô-Skorohod sense, and later in the Stratonovich sense. An explicit chaos expansion for the solution was obtained. Moreover, the moments of the solution are expressed in terms of the exponential moments of some weighted intersection local time of the Brownian motion. Bo et al. [7,8], Jiang et al. [17,18,20] and Wei [35] studied a class of four-order stochastic partial differential equations (including the Anderson models and the Cahn-Hilliard equations among others) with fractional noises, where the existence, uniqueness, regularity and the absolute continuity of the solutions were established. Liu et al. [25] studied a jump-type stochastic fractional partial differential equation with fractional noises and proved the existence and uniqueness of the global mild solution by the fixed point principle under some suitable assumptions. Jiang et al. [19] studied a class of stochastic heat equation with first order fractional noises and modeled the term structure for forward rate with such a solution.

Motivated by the above results, in this paper, we consider a semilinear stochastic partial differential equation driven by double-parameter fractional noises, i.e., (1.1) and we will prove that there exists a unique mild solution to (1.1). Moreover, we investigated the Hölder regularity and absolute continuity of the law of the solution.

Throughout this paper, we always consider (1.1) under the following assumptions on the coefficient q and the initial condition u_0 and then as "Assumption A" in the sequel.

(A1) For each $T > 0$, there exists a constant $C > 0$ such that for $(t, x) \in [0, T] \times \mathbb{R}$ and $u, v \in \mathbb{R}$,

$$|q(t, x, y)| \leqslant C(1 + |y|), \tag{1.2}$$

$$|q(t, x, u) - q(s, y, v)| \leqslant C(|t - s| + |x - y| + |u - v|). \tag{1.3}$$

(A2) For some $p \geqslant 2$,

$$\sup_{x\in\mathbb{R}} E(|u_0(x)|^p) < \infty. \tag{1.4}$$

(A3) For some $p \geqslant 2$, there exists some $\theta \in (0, 1)$ such that for $p\theta < 1$,

$$\sup_{x\in\mathbb{R}} E(|u_0(x + y) - u_0(x)|^p) < C_p|y|^{p\theta}. \tag{1.5}$$

The rest of the paper is organized as follows. In Section 2, we present some preliminaries on the integro-differential operator $\mathcal{A}$, the double-parameter fractional noises and Malliavin calculus with respect to the double-parameter fractional noise. Section 3 is devoted to proving the existence and uniqueness of the mild solution to (1.1). The Hölder regularity of the solution $u(t, x)$ to (1.1) is investigated in Section 4. In Section 5, we prove the existence of the density and established that the law of the solution to (1.1) is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$ by estimates of Malliavin derivative and the divergence operator.

Note. Most of the estimates in this paper contain unspecified constants. An unspecified positive and finite constant will be denoted by C, which may not be the same in each occurrence. Sometimes we shall emphasize the dependence of these constants upon parameters.

2 Preliminaries

In this section, we will present the symmetric integro-differential operators, double-parameter fractional noises and Malliavin calculus with respect to double-parameter fractional noises.

2.1 Symmetric integro-differential operators

According to, e.g., Komatsu [22] and Stroock [37], a Lévy type operator $\mathcal{A}$ is a second-order elliptic pseudo-differential operator having the following representation

$$(\mathcal{A}\varphi)(x) = \sigma^2(x)\varphi''(x) + b(x)\varphi'(x) + \int_{\mathbb{R}\setminus\{0\}} \left[\varphi(x+z) - \varphi(x) - \frac{z\varphi'(x)1_{\{|z|<1\}}(z)}{1+|z|^2}\right] \mu(x,dz),$$

for certain suitable function $\varphi : \mathbb{R} \to \mathbb{R}$ (for example, φ could be a Schwartz test function on $\mathbb{R}$), where $\sigma, b : \mathbb{R} \to \mathbb{R}$, and $\mu(x,dz)$ is the so-called Lévy kernel, i.e., $\forall\, x \in \mathbb{R}$, $\mu(x,dz)$ satisfies

$$\int_{\mathbb{R}\setminus\{0\}} (|z|^2 \wedge 1)\mu(x,dz) < \infty.$$

In this paper, we are interested in a special class of Lévy type operators. Namely, we consider only the integral part of the Lévy type operators

$$(\mathcal{A}\varphi)(x) = \int_{\mathbb{R}\setminus\{0\}} \left[\varphi(x+z) - \varphi(x) - \frac{z\varphi'(x)1_{\{|z|<1\}}(z)}{1+|z|^2}\right] \mu(x,dz), \tag{2.1}$$

and

$$\mu(x,dz) = \frac{c(x,x+z)dz}{|z|^{1+\alpha}},$$

for $\alpha \in (0,2)$, where $c : \mathbb{R}\times\mathbb{R} \to [d_1,d_2]$ is a symmetric, measurable function with certain given constants $d_2 \geqslant d_1 > 0$. In this case, the integro-differential operator $\mathcal{A}$ is symmetric with respect to $L^2(\mathbb{R})$. Moreover, from the theory of Dirichlet forms, there is a Feller semigroup, denoted by $\{\mathrm{e}^{s\mathcal{A}}\}_{s\geqslant 0}$. A typical example of a symmetric integro-differential operator $\mathcal{A}$ is the (one-dimensional) fractional Laplacian which can be defined as follows (see Albeverio et al. [1], Truman and Wu [32] and the references therein for more details). Let $c(x,y)=1$. Then

$$\mathcal{A} = \rho_\alpha(-\Delta)^{\frac{\alpha}{2}} = \rho_\alpha\left(-\frac{d^2}{dx^2}\right)^{\frac{\alpha}{2}},$$

with the symmetric stable semigroup as its Feller semigroup, where ρ_α is a negative constant determined by

$$\rho_\alpha = \int_{\mathbb{R}\setminus\{0\}} (\cos z - 1)\frac{1}{|z|^{1+\alpha}}dz.$$

The particular case when $\alpha = 1$ corresponds to the Cauchy semigroup. More examples and information of Lévy type operators in terms of pseudo-differential can be found in Komatsu [23], Truman and Wu [32] and etc.

2.2 Double-parameter fractional noises

A one-dimensional fractional Brownian motion $B^H = \{B_t^H, t\in[0,T]\}$ with Hurst parameter $H\in(0,1)$ in $[0,T]$ is a centered Gaussian process on some probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geqslant 0}, P)$ with covariance

$$E[B_t^H B_s^H] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

Hu [15] (see also Wei [35] and Jiang et al. [20]) introduced a double-parameter fractional Brownian field

Definition 2.1. A one-dimensional double-parameter fractional Brownian field $W^H = \{W^H(t,x), (t,x) \in [0,T]\times\mathbb{R}\}$ with double-parameter $H = (H_1, H_2)$ for $H_i \in (0,1)$, $i = 1, 2$, is a centered Gaussian random field on some probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geqslant 0}, P)$ with covariance

$$\begin{aligned} R(t,s;x,y) &= E[W^H(t,x)W^H(s,y)] \\ &= \frac{1}{4}(|t|^{2H_1} + |s|^{2H_1} - |t-s|^{2H_1})(|x|^{2H_2} + |y|^{2H_2} - |x-y|^{2H_2}), \end{aligned}$$

for all $s,t \in [0,T]$ and $x,y \in \mathbb{R}$.

We denote by $\mathscr{E}$ the set of step functions on $[0,T]\times\mathbb{R}$. Let L^2_Ψ be the Hilbert space defined as the closure of $\mathscr{E}$ with respect to the scalar product

$$\langle 1_{[0,t]\times[0,x]}, 1_{[0,s]\times[0,y]}\rangle_{L^2_\Psi} = R(t,s;x,y).$$

Thus, the mapping $1_{[0,t]\times[0,x]} \mapsto W^H([0,t]\times[0,x])$ is an isometry between $\mathscr{E}$ and the linear space span of $\{W^H([0,t]\times[0,x]), (t,x) \in [0,T]\times\mathbb{R}\}$. Moreover, the mapping can be extended to an isometry from L^2_Ψ to a Gaussian space associated with W^H. This isometry will be denoted by $\varphi \mapsto W^H(\varphi)$ for $\varphi \in L^2_\Psi$. Therefore, we can regard $W^H(\varphi)$ as the stochastic integral with respect to W^H. In general, we use the notation

$$W^H(\varphi) = \int_0^T \int_{\mathbb{R}} \varphi(s,y)W^H(ds,dy), \quad \varphi \in L^2_\Psi.$$

The following embedding property from Bo et al. [7], Wei [35] and Jiang et al. [20] enables us to define the integral for $\psi \in L^2_\Psi$ with respect to W^H.

Proposition 2.2. *For $h > \frac{1}{2}$ we have*

$$L^2([0,T]\times\mathbb{R}) \subset L^{\frac{1}{h}}([0,T]\times\mathbb{R}) \subset L^2_\Psi.$$

For any $0 \leqslant s < t \leqslant T$ and $x,y \in \mathbb{R}$, let

$$\Psi_H(t,s;x,y) = 4H_1H_2(2H_1-1)(2H_2-1)|t-s|^{2H_1-2}|x-y|^{2H_2-2}. \tag{2.2}$$

Furthermore, the following properties hold.

Proposition 2.3. *For $f, g \in L^2_\Psi$, we have*

$$E\left[\int_0^t \int_{\mathbb{R}} f(s,y)W^H(dx,ds)\right] = 0,$$

and

$$\begin{aligned} &E\left[\int_0^t \int_{\mathbb{R}} f(s,x)W^H(dx,ds)\int_0^t \int_{\mathbb{R}} g(s,x)W^H(dx,ds)\right] \\ &\quad = \int_{[0,t]^2}\int_{\mathbb{R}^2} \Psi_h(u,v;x,y)f(u,x)g(v,y)dydxdvdu. \end{aligned}$$

Proposition 2.4. *If $h \in (1/2,1)$ and $f,g \in L^{\frac{1}{h}}([a,b])$, then*

$$\int_a^b\int_a^b f(u)g(v)|u-v|^{2h-2}dudv \leqslant C_h\|f\|_{L^{\frac{1}{h}}([a,b])}\|g\|_{L^{\frac{1}{h}}([a,b])},$$

where $C_h > 0$ is a constant depending only on h.

2.3 Malliavin calculus with respect to double-parameter fractional noises

Since $W^H(t,x),(t,x)\in[0,T]\times\mathbb{R}$ is Gaussian, we might develop the Malliavin calculus (refer to Nualart [27] for more details) with respect to double-parameter fractional noises in order to prove the existence of the density of the law of the solutions to stochastic partial differential equation driven by double-parameter fractional noises.

Let $W^H(\varphi)=\int_0^T\int_{\mathbb{R}}\varphi(t,x)W^H(dt,dx)$ for $\varphi\in L^2_\Psi$, and let $\mathcal{S}$ be the class of smooth and cylindrical random variables of the form

$$F=f(W^H(\varphi_1),\dots,W^H(\varphi_n)),$$

where $f\in C_b^\infty(\mathbb{R}^n)$ (the set of all functions with bounded derivatives of all orders) and $\varphi_i\in L^2_\Psi$ ($i=1,\dots,n$ and $n\in\mathbb{N}$). For each $F\in\mathcal{S}$, define the derivative $D_{t,x}F$ by

$$D_{t,x}F:=\sum_{i=1}^n\frac{\partial f}{\partial x}(W^H(\varphi_1),\dots,W^H(\varphi_n))\varphi_i(t,x).$$

Let $\mathbb{D}^{1,2}$ be the completion of $\mathcal{S}$ under the norm

$$\|F\|_{1,2}^2=E[|F|^2+\|DF\|^2_{L^2_\Psi}].$$

Then $\mathbb{D}^{1,2}$ is the domain of the closed operator D on $L^2(\Omega)$ with the domain $\mathbb{D}_h$ being the closure of $\mathcal{S}$ under the norm

$$\|F\|_h^2=E[|F|^2+|D_hF|^2].$$

Let $\{h_n,n\geqslant 1\}$ be an orthonormal basis of L^2_Ψ. Then $F\in\mathbb{D}^{1,2}$ if and only if $F\in\mathbb{D}_{h_n}$ for each $n\in\mathbb{N}$ and $\sum_{n=1}^\infty E|D_{h_n}F|^2<\infty$. In this case, $D_hF=\langle DF,h\rangle_{L^2_\Psi}$.

On the other hand, the divergence operator δ is the adjoint of the derivative operator D characterized by

$$E\langle DF,u\rangle_{L^2_\Psi}=E(F\delta(u)),\quad \text{for any } F\in\mathcal{S},$$

where $u\in L^2(\Omega;L^2_\Psi)$. Then $\mathrm{Dom}(\delta)$, the domain of δ, is the set of all functions $u\in L^2(\Omega,L^2_\Psi)$ such that

$$E|\langle DF,u\rangle_{L^2_\Psi}|\leqslant C(u)\|F\|_{L^2(\Omega)},$$

where $C(u)$ is some constant depending on u.

The following propositions that can be found in Wei [35] and Jiang et al. [20] ensure us to use Malliavin calculus with respect to fractional noises to deduce the laws for solutions to the corresponding stochastic partial differential equations.

Proposition 2.5. *Let $\mathscr{F}_A:=\sigma\{W^H(B),B\subset A\}$ for $A\in\mathcal{B}([0,T]\times\mathbb{R})$. If F is a square integrable random variable that is measurable with respect the σ-field $\mathscr{F}_{A^c}$, then*

$$DF1_A=0,\quad a.s.$$

Remark 2.6. Let $\{u(t,x),(t,x)\in[0,T]\times\mathbb{R}\}$ be an $\{\mathscr{F}_t,t\in[0,T]\}$-adapted random field. By Proposition 2.5, we have $D_{s,y}u(t,x)=0$, a.s. for any $0\leqslant s<t\leqslant T$ and $x,y\in\mathbb{R}$.

Proposition 2.7. *Given $F\in\mathbb{D}^{1,2}$, if $\|DF\|^2_{L^2_\Psi}>0$ a.s., then the distribution of the random variable F is absolutely continuous with respect to Lebesgue measure.*

Remark 2.8. Propositions 2.5 and 2.7 can be proved similarly as in Nualart [27] with Wiener white noise replaced by the fractional noise.

3 Existence and uniqueness of the solution

Let $(\Omega,\mathscr{F},\mathscr{F}_t,P)$ be given as in the previous section. In this section, we will study the Cauchy problem for the stochastic semilinear equation (1.1).

Following Walsh [33], let us introduce a notation of mild solution to (1.1) in terms of the fundamental solution $G(s,z;t,x)$ for $\mathcal{A}$. An $L^p(\Omega)$ $\mathscr{F}_t$-adapted process $u:[0,T]\times\mathbb{R}\times\Omega\to\mathbb{R}$ is a solution to (1.1) if

$$\begin{aligned}u(t,x)=&\int_{\mathbb{R}}G(0,y;t,x)u_0(y)dy+\int_0^t\int_{\mathbb{R}}G(s,y;t,x)W^H(ds,dy)\\&+\int_0^t\int_{\mathbb{R}}[\partial_yG(s,y;t,x)]q(s,y,u(s,y))dyds,\end{aligned}\tag{3.1}$$

where $G(s,y;t,x)$ stands for the fundamental solution starting from $(s,y)\in[0,\infty)\times\mathbb{R}$, i.e., satisfying the following system

$$\begin{cases}\dfrac{\partial}{\partial t}v(t,y)=(\mathcal{A}v)(t,y), & (t,y)\in(s,\infty)\times\mathbb{R},\\ v(s,y)=\delta(y-x), & y\in\mathbb{R}.\end{cases}\tag{3.2}$$

We then summarize the following nice estimates for Green function G as follows (see Bass and Levin [2], Chen and Kumagai [10], Komatsu [23], Kolokoltsov [21] and Truman and Wu [32] for more details). There exists a constant C only depending on $\alpha\in(0,2)$ such that for any $0\leqslant s<t$ and $x,y\in\mathbb{R}$, the following estimates hold:

$$(t-s)^{1/\alpha}(1+(t-s)^{-1-1/\alpha}|x-y|^{1+\alpha})G(s,y;t,x)\leqslant C,\tag{3.3}$$

$$\left|\frac{\partial G(s,y;t,x)}{\partial x}\right|\leqslant C(t-s)^{-1-2/\alpha}|x-y|^{\alpha}(1+(t-s)^{-1-1/\alpha}|x-y|^{1+\alpha})^{-2},\tag{3.4}$$

$$\left|\frac{\partial G(s,y;t,x)}{\partial t}\right|\leqslant C\frac{(t-s)^{-1-\frac{1}{\alpha}}}{1+(t-s)^{-1-\frac{1}{\alpha}}|x-y|^{1+\alpha}}.\tag{3.5}$$

We have the following main result in this section.

Theorem 3.1. *Under the assumptions* (A1) *and* (A2)*, for $\alpha\in(1,2)$ and some $p\geqslant2$, then there exists a unique solution $u=\{u(t,x),(t,x)\in[0,T]\times\mathbb{R}\}$ to* (3.1) *satisfying*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}}E|u(t,x)|^p<\infty,$$

for all $T>0$.

Proof. We use the Picard iteration scheme to get a solution to (3.1). Define

$$u^{(0)}(t,x)=\int_{\mathbb{R}}G(0,y;t,x)u_0(y)dy,\tag{3.6}$$

$$\begin{aligned}u^{(n+1)}(t,x)=&\,u^{(0)}(t,x)+\int_0^t\int_{\mathbb{R}}\frac{\partial G}{\partial y}(s,y;t,x)q(s,y,u^{(n)}(s,y))dyds\\&+\int_0^t\int_{\mathbb{R}}G(s,y;t,x)W^H(dy,ds).\end{aligned}\tag{3.7}$$

Firstly, we will prove that

$$\sup_{n\in\mathbb{N}\cup\{0\}}\sup_{(t,x)\in[0,T]\times\mathbb{R}}E|u^{(n)}(t,x)|^p<\infty.$$

Apply Hölder's inequality on the measure $G(0,y;t,x)dy$, then

$$\begin{aligned}E|u^{(0)}(t,x)|^p&\leqslant\left(\int_{\mathbb{R}}|G(0,y;t,x)|dy\right)^{p-1}E\left(\int_{\mathbb{R}}|G(0,y;t,x)|\cdot|u_0(y)|^pdy\right)\\&\leqslant\sup_{x\in\mathbb{R}}E|u_0(x)|^p\left(\int_{\mathbb{R}}|G(0,y;t,x)|dy\right)^p\\&<\infty.\end{aligned}\tag{3.8}$$

This shows that $\sup_{(t,x)\in[0,T]\times\mathbb{R}} E|u^{(0)}(t,x)|^p < \infty$. On the other hand, it follows from (3.7) that for each $n \in \mathbb{N}$,

$$E|u^{(n+1)}(t,x)|^p \leqslant C_p(E|u^{(0)}(t,x)|^p + A_p^{(n)}(t,x) + B_p^{(n)}(t,x)), \tag{3.9}$$

where

$$A_p^{(n)}(t,x) = E\left|\int_0^t \int_{\mathbb{R}} \frac{\partial G}{\partial y}(s,y;t,x) q(s,y,u^{(n)}(s,y)) dyds\right|^p,$$

and

$$B_p^{(n)}(t,x) = E\left|\int_0^t \int_{\mathbb{R}} G(s,y;t,x) W^H(dy,ds)\right|^p.$$

Note that, by Hölder's inequality,

$$\begin{aligned}
A_p^{(n)}(t,x) &\leqslant C_p \left(\int_0^t \int_{\mathbb{R}} \left|\frac{\partial G}{\partial y}(s,y;t,x)\right| dyds\right)^{p-1} \\
&\quad \times E\left(\int_0^t \int_{\mathbb{R}} |q(s,y,u^{(n)}(s,y))|^p \cdot \left|\frac{\partial G}{\partial y}(s,y;t,x)\right| dyds\right) \\
&\leqslant C_p E\left[\int_0^t \int_{\mathbb{R}} (1+|u^{(n)}(s,y)|^p) \cdot \left|\frac{\partial G}{\partial y}(s,y;t,x)\right| dyds\right] \\
&\leqslant C_p \int_0^t \left(1+\sup_{y\in\mathbb{R}} E|u^{(n)}(s,y)|^p\right)\left(\int_{\mathbb{R}} \left|\frac{\partial G}{\partial y}(s,y;t,x)\right| dy\right) ds.
\end{aligned} \tag{3.10}$$

For the term $B_p^{(n)}(t,x)$, from Proposition 2.2 and (3.3), one can obtain

$$\begin{aligned}
B_p^{(n)}(t,x) &= E\left|\int_0^t \int_{\mathbb{R}} G(r,z;t,x) W^H(dr,dz)\right|^p \\
&\leqslant C\left(\int_0^t \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} G(r_1,z_1;t,x)\Psi(r_1,r_2,z_1,z_2) G(r_2,z_2;t,x) dz_1dz_2dr_1dr_2\right)^{\frac{p}{2}} \\
&= C\left(\int_{[0,t]^2} |r_1-r_2|^{2H_1-2} \int_{\mathbb{R}^2} |z_1-z_2|^{2H_2-2} G(r_1,z_1;t,x) G(r_2,z_2;t,x) dz_1dz_2dr_1dr_2\right)^{\frac{p}{2}} \\
&\leqslant C\left(\int_{[0,t]^2} |r_1-r_2|^{2H_1-2} \|G(r_1,\cdot;t,x)\|_{L^{\frac{1}{H_2}}(\mathbb{R})} \|G(r_2,\cdot;t,x)\|_{L^{\frac{1}{H_2}}(\mathbb{R})} dr_1dr_2\right)^{\frac{p}{2}} \\
&\leqslant C\left(\int_0^t (\|G(r,\cdot;t,x)\|_{L^{\frac{1}{H_2}}(\mathbb{R})})^{\frac{1}{H_1}} ds\right)^{pH_1} \\
&\leqslant C\left(\int_0^t |t-r|^{\frac{H_2-1}{\alpha H_1}} dr\right)^{pH_1} \\
&\leqslant Ct^{\frac{p}{\alpha}(\alpha H_1+H_2-1)} < \infty, \quad \text{if } \alpha \in (1,2),
\end{aligned} \tag{3.11}$$

where we have used the following fact,

$$\begin{aligned}
\|G(s,\cdot;t,x)\|_{L^{\frac{1}{H_2}}(\mathbb{R})} &= \left(\int_{\mathbb{R}} G(s,y;t,x)^{\frac{1}{H_2}} dy\right)^{H_2} \\
&\leqslant C\left(\int_{\mathbb{R}} \left(\frac{|t-s|^{-\frac{1}{\alpha}}}{1+|t-s|^{-1-\frac{1}{\alpha}}|x-y|^{1+\alpha}}\right)^{\frac{1}{H_2}} dy\right)^{H_2} \\
&= C|t-s|^{\frac{H_2}{\alpha}-\frac{1}{\alpha}} \left(\int_{\mathbb{R}} \left(\frac{1}{1+|y|^{1+\alpha}}\right)^{\frac{1}{H_2}} dy\right)^{H_2} \\
&= C|t-s|^{\frac{H_2}{\alpha}-\frac{1}{\alpha}} \left(2\int_0^1 \left(\frac{1}{1+y^{1+\alpha}}\right)^{\frac{1}{H_2}} dy + 2\int_1^{+\infty} \left(\frac{1}{1+y^{1+\alpha}}\right)^{\frac{1}{H_2}} dy\right)^{H_2} \\
&\leqslant C|t-s|^{\frac{H_2}{\alpha}-\frac{1}{\alpha}}.
\end{aligned} \tag{3.12}$$

Hence (3.9) implies that

$$\sup_{x\in\mathbb{R}} E|u^{(n+1)}(t,x)|^p \leqslant C_p + \int_0^t \Big(1+\sup_{y\in\mathbb{R}} E|u^{(n)}(s,y)|^p\Big) f_x(t-s)ds, \tag{3.13}$$

where $f_x(t-s)=\int_{\mathbb{R}}|\frac{\partial G}{\partial y}(s,y;t,x)|dy$ and

$$\begin{aligned}\int_{\mathbb{R}}\left|\frac{\partial G}{\partial y}(s,y;t,x)\right|dy &\leqslant C\int_{\mathbb{R}}\frac{|t-s|^{-1-\frac{2}{\alpha}}|x-y|^{\alpha}}{(1+|t-s|^{-\frac{1+\alpha}{\alpha}}|x-y|^{1+\alpha})^2}dy\\ &\leqslant C|t-s|^{-\frac{1}{\alpha}}\int_{\mathbb{R}}\frac{|z|^{\alpha}}{(1+|z|^{1+\alpha})^2}dz\\ &\leqslant C|t-s|^{-\frac{1}{\alpha}}. \end{aligned}\tag{3.14}$$

Moreover, $\int_0^T f_x(t-s)ds<\infty$ if $\alpha\in(1,2)$. By Lemma 15 in Dalang [11], one can obtain

$$\sup_{n\in\mathbb{N}\cup\{0\}}\ \sup_{(t,x)\in[0,T]\times\mathbb{R}} E|u^{(n)}(t,x)|^p<\infty. \tag{3.15}$$

Next let us prove that $\{u^{(n)}(t,x)\}_{n\geqslant 0}$ converges in $L^p(\Omega)$. As for $n\geqslant 2$,

$$\begin{aligned}&E(|u^{(n+1)}(t,x)-u^{(n)}(t,x)|^p)\\ &\quad= E\left(\left|\int_0^t\int_{\mathbb{R}}\frac{\partial G}{\partial y}(s,y;t,x)[q(s,y,u^{(n)}(s,y))-q(s,y,u^{(n-1)}(s,y))]dyds\right|^p\right)\\ &\quad\leqslant C_p\int_0^t E|u^{(n)}(s,y)-u^{(n-1)}(s,y)|^p\int_{\mathbb{R}}\left|\frac{\partial G}{\partial y}(s,y;t,x)\right|dyds\\ &\quad\leqslant C_p\int_0^t\sup_{y\in\mathbb{R}}E|u^{(n)}(s,y)-u^{(n-1)}(s,y)|^p f_x(t-s)ds,\end{aligned}\tag{3.16}$$

and

$$\sup_{x\in\mathbb{R}}E|u^{(1)}(s,y)-u^{(0)}(s,y)|^p\leqslant C_p(E|u^{(0)}(s,y)|^p+E|u^{(1)}(s,y)|^p)<\infty.$$

Then Gronwall's lemma yields that

$$\sum_{n\geqslant 0}\ \sup_{(t,x)\in[0,T]\times\mathbb{R}}E(|u^{(n+1)}(t,x)-u^{(n)}(t,x)|^p)<\infty. \tag{3.17}$$

Hence, $\{u^{(n)}(t,x)\}_{n\geqslant 0}$ is a Cauchy sequence in $L^p(\Omega)$. Let

$$u(t,x)=\lim_{n\to\infty}u^{(n)}(t,x).$$

Then for each $(t,x)\in[0,T]\times\mathbb{R}$,

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}}E|u(t,x)|^p<\infty. \tag{3.18}$$

Take $n\to\infty$ in $L^p(\Omega)$ at both sides of (3.7). Then, it shows that $u(t,x);(t,x)\in[0,T]\times\mathbb{R}$ satisfies (3.1).

Finally, we have to prove the uniqueness of the solution. Let u and $\widetilde{u}$ be the two solutions of (3.1), then

$$\begin{aligned}E(|u(t,x)-\widetilde{u}(t,x)|^p)&=E\left(\left|\int_0^t\int_{\mathbb{R}}\frac{\partial G}{\partial y}(s,y;t,x)[q(s,y,u(s,y))-q(s,y,\widetilde{u}(s,y))]dyds\right|^p\right)\\ &\leqslant C_p\int_0^t E\,|u(s,y)-\widetilde{u}(s,y)|^p\int_{\mathbb{R}}\left|\frac{\partial G}{\partial y}(s,y;t,x)\right|dyds\\ &\leqslant C_p\int_0^t\sup_{y\in\mathbb{R}}E\,|u(s,y)-\widetilde{u}(s,y)|^p f_x(t-s)ds.\end{aligned}\tag{3.19}$$

Then Gronwall's lemma shows that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} E\,|u(t,x)-\widetilde{u}(t,x)|^p = 0. \tag{3.20}$$

This completes the proof of this theorem. □

Remark 3.2. In the case $\alpha \in (0,1)$, our method used here does not work.

4 Hölder regularity of the solution

This section is devoted to studying the Hölder regularity of the solution $u = \{u(t,x),(t,x)\in[0,T]\times\mathbb{R}\}$ to (3.1). Actually, we will have the following theorem.

Theorem 4.1. *Let $\alpha \in (1,2)$ and $H_1, H_2 \in (\frac{1}{2},1)$ such that $\alpha H_1 < 2-H_2$. If Assumption* A *is satisfied, then there exists a continuous modification of $u(t,x)$ (for convenience, we still denote by $u(t,x)$) which is β-Hölder continuous in t with $\beta \in (0,\min\{\frac{\theta}{\alpha},\frac{\alpha H_1+H_2-1}{\alpha}\})$ and γ-Hölder continuous in x with $\gamma\in(0,\min\{\theta,\alpha H_1+H_2-1\})$.*

Proof. As for each $x\in\mathbb{R}$ and $0\leqslant s<t\leqslant T$, we have

$$\begin{aligned}
|u(t,x)-u(s,x)| &\leqslant \left|\int_{\mathbb{R}}[G(0,y;t,x)-G(0,y;s,x)]u_0(y)dy\right| \\
&\quad+\left|\int_0^t\int_{\mathbb{R}}\frac{\partial G}{\partial y}(r,y;t,x)q(r,y,u(r,y))dydr-\int_0^s\int_{\mathbb{R}}\frac{\partial G}{\partial y}G(r,y;s,x)q(r,y,u(r,y))dydr\right| \\
&\quad+\left|\int_0^t\int_{\mathbb{R}}G(r,y;t,x)W^H(dy,dr)-\int_0^s\int_{\mathbb{R}}G(r,y;s,x)W^H(dy,dr)\right| \\
&=:\sum_{j=0}^2|A_j(t,s,x)|.
\end{aligned}\tag{4.1}$$

By the semigroup property, (3.3) and Hölder's inequality,

$$\begin{aligned}
E|A_0(t,s,x)|^p &= E\left(\left|\int_{\mathbb{R}}\int_{\mathbb{R}}G(0,z;t-s,y)G(0,y;s,x)u_0(z)dydz-\int_{\mathbb{R}}G(0,y;s,x)u_0(y)dy\right|\right)^p \\
&= E\left|\int_{\mathbb{R}}G(0,z;t-s,0)\int_{\mathbb{R}}G(0,y;s,x)(u_0(y-z)-u_0(y))dydz\right|^p \\
&\leqslant C_p\int_{\mathbb{R}}|G(0,z;t-s,0)|\int_{\mathbb{R}}G(0,y;s,x)E|u_0(y-z)-u_0(y)|^pdydz \\
&\leqslant C_p\int_{\mathbb{R}}|G(0,y;s,x)|dy\int_{\mathbb{R}}\frac{|t-s|^{\frac{p\theta}{\alpha}}|z|^{p\theta}}{1+|z|^{1+\alpha}}dz \\
&\leqslant C_p|t-s|^{\frac{p\theta}{\alpha}}\left(\int_0^1\frac{z^{p\theta}}{1+z^{1+\alpha}}dz+\int_1^\infty\frac{z^{p\theta}}{1+z^{1+\alpha}}dz\right) \\
&\leqslant C_p|t-s|^{\frac{p\theta}{\alpha}},\quad \text{if } \alpha>p\theta.
\end{aligned}\tag{4.2}$$

Next, we consider $A_1(t,s,x)$. Let $r'=r-(t-s)$. Then

$$\begin{aligned}
|A_1(t,s,x)| &\leqslant \int_0^s\int_{\mathbb{R}}\partial_yG(r,y;s.x)[q(r+t-s,y,u(r+t-s,y))-q(r,y,u(r,y))]dydr \\
&\quad+\int_0^{t-s}\int_{\mathbb{R}}\partial_yG(r,y;t,x)q(r,y,u(r,y))dydr \\
&=:|A_{1,1}(t,s,x)|+|A_{1,2}(t,s,x)|.
\end{aligned}\tag{4.3}$$

Then, by Hölder's inequality and (A1),

$$\begin{aligned} E\left(|A_{1,1}(t,s,x)|^p\right) \leqslant C_p\bigg| \int_0^s \int_{\mathbb{R}} \partial_y G(r,y;s,x) E|q(r+t-s,y,u(r+t-s,y)) - q(r,y,u(r,y))|^p dydr \\ \times \left[\int_0^s \int_{\mathbb{R}} \partial_y G(r,y;s,x) dydr\right]^{p-1} \bigg| \\ \leqslant C_p\left[|t-s|^p + \int_0^s \sup_{y\in\mathbb{R}} E|u(r+t-s,y) - u(r,y)|^p dr\right]. \end{aligned} \tag{4.4}$$

And by (A1), we also have

$$\begin{aligned} E\left(|A_{1,2}(t,s,x)|^p\right) &\leqslant C_p \int_0^{t-s} \int_{\mathbb{R}} \partial_y G(r,y;t,x) E|q(r,y,u(r,y))|^p dydr \\ &\quad \times \left[\int_0^{t-s}\int_{\mathbb{R}} \frac{\partial G}{\partial y}(r,y;t,x) dydr\right]^{p-1} \\ &\leqslant C_p|t-s|^p\left[1 + \sup_{(t,x)\in[0,T]\times\mathbb{R}} E|u(t,x)|^p\right] \\ &\leqslant C_p|t-s|^p. \end{aligned} \tag{4.5}$$

Then by (4.3)–(4.5), we have

$$E|A_1(t,s,x)|^p \leqslant C_p\left(|t-s|^p + \int_0^s \sup_{y\in\mathbb{R}} E|u(r+t-s,y) - u(r,y)|^p dr\right). \tag{4.6}$$

Next, we want to estimate $A_2(t,s,x)$. Note that

$$\begin{aligned} |A_2(t,s,x)| &\leqslant \left|\int_0^s \int_{\mathbb{R}} (G(r,z;t,x) - G(r,z;s,x)) W^H(dz,dr)\right| \\ &\quad + \left|\int_s^t \int_{\mathbb{R}} G(r,z;t,x) W^H(dz,dr)\right| \\ &= |A_{2,1}(t,s,x)| + |A_{2,2}(t,s,x)|. \end{aligned} \tag{4.7}$$

Let $\beta \in (0, \frac{\alpha H_1 + H_2 - 1}{\alpha}) \subset (0,1)$. For the first term $|A_{2,1}(t,s,x)|$, we have

$$\begin{aligned} E|A_{2,1}(t,s,x)|^p &\leqslant C_p\|G(\cdot,\cdot;t,x) - G(\cdot,\cdot;s,x)\|^p_{L^2_\Psi} \\ &\leqslant C_{p,\beta}(\||G(\cdot,\cdot;t,x) - G(\cdot,\cdot;s,x)|^\beta \cdot |G(\cdot,\cdot;t,x) - G(\cdot,\cdot;s,x)|^{1-\beta}\|^2_{L^2_\Psi})^{\frac{p}{2}} \\ &\leqslant C_{p,\beta}(\||G(\cdot,\cdot;t,x) - G(\cdot,\cdot;s,x)|^\beta \cdot |G(\cdot,\cdot;t,x)|^{1-\beta}\|^2_{L^2_\Psi} \\ &\quad + \||G(\cdot,\cdot;t,x) - G(\cdot,\cdot;s,x)|^\beta \cdot |G(\cdot,\cdot;s,x)|^{1-\beta}\|^2_{L^2_\Psi})^{\frac{p}{2}} \\ &\equiv C_{p,\beta}(|A_{2,1,1}(t,s,x)| + |A_{2,1,2}(t,s,x)|). \end{aligned}$$

Then using (3.3), (3.5), Proposition 2.4 and the mean-value theorem, for $\xi \in (s,t)$, one can get

$$\begin{aligned} |A_{2,1,1}(t,s,x)| &= \left(\left\|\left|\frac{\partial G}{\partial t}(\cdot,\cdot;\xi,x)\right|^\beta |t-s|^\beta |G(\cdot,\cdot;t,x)|^{1-\beta}\right\|^2_{L^2_\Psi}\right)^{\frac{p}{2}} \\ &= |t-s|^{p\beta}\bigg(\int_{[0,t]^2}\int_{\mathbb{R}^2}\left|\frac{\partial G}{\partial t}(r_1,z_1;\xi,x)\right|^\beta |G(r_1,z_1;t,x)|^{1-\beta} \\ &\quad \times \Psi_H(r_1,r_2,z_1,z_2)\left|\frac{\partial G}{\partial t}(r_2,z_2;\xi,x)\right|^\beta |G(r_2,z_2;t,x)|^{1-\beta} dz_1dz_2dr_1dr_2\bigg)^{\frac{p}{2}} \\ &\leqslant C|t-s|^{p\beta}\left(\int_0^T\left(\int_{\mathbb{R}}\left(\left|\frac{\partial G(r,z;t,x)}{\partial t}\right|^\beta \cdot |G(r,z;t,x)|^{1-\beta}\right)^{\frac{1}{H_2}} dz\right)^{\frac{H_2}{H_1}} dr\right)^{pH_1} \end{aligned}$$

$$\leqslant C|t-s|^{p\beta}. \tag{4.8}$$

In fact

$$\begin{aligned}
&\int_{\mathbb{R}}\left(\left|\frac{\partial G(r,z;t,x)}{\partial t}\right|^{\beta}\cdot|G(r,z;t,x)|^{1-\beta}\right)^{\frac{1}{H_2}}dz\\
&\quad\leqslant C\int_{\mathbb{R}}\left(\left|\frac{(t-r)^{-1-\frac{1}{\alpha}}}{1+(t-r)^{-1-\frac{1}{\alpha}}|x-z|^{1+\alpha}}\right|^{\beta}\cdot\left|\frac{|t-r|^{-\frac{1}{\alpha}}}{1+(t-r)^{-1-\frac{1}{\alpha}}|x-z|^{1+\alpha}}\right|^{1-\beta}\right)^{\frac{1}{H_2}}dz\\
&\quad=C|t-r|^{\frac{H_2-1-\alpha\beta}{\alpha H_2}}\int_{\mathbb{R}}\frac{1}{(1+|y|^{1+\alpha})^{\frac{1}{H_2}}}dy\\
&\quad=C|t-r|^{\frac{H_2-1-\alpha\beta}{\alpha H_2}}.
\end{aligned}$$

Then

$$\begin{aligned}
&\left(\int_0^T\left(\int_{\mathbb{R}}\left(\left|\frac{\partial G(r,z;t,x)}{\partial t}\right|^{\beta}\cdot|G(r,z;t,x)|^{1-\beta}\right)^{\frac{1}{H_2}}dz\right)^{\frac{H_2}{H_1}}dr\right)^{pH_1}\\
&\quad\leqslant C\left(\int_0^T|t-r|^{\frac{H_2-1-\alpha\beta}{\alpha H_1}}dr\right)^{pH_1}\leqslant C,
\end{aligned}$$

if $1+\frac{H_2-1-\alpha\beta}{\alpha H_1}>0$, i.e., $\beta<\frac{\alpha H_1+H_2-1}{\alpha}$.

Similarly, one can prove that

$$|A_{2,1,2}(t,s,x)|\leqslant C|t-s|^{p\beta}.$$

Then, it follows that

$$E|A_{2,1}(t,s,x)|^p\leqslant C|t-s|^{p\beta},\quad\text{with }\beta\in\left(0,\frac{\alpha H_1+H_2-1}{\alpha}\right). \tag{4.9}$$

On the other hand, we have

$$\begin{aligned}
E|A_{2,2}(t,s,x)|^p&=E\left|\int_s^t\int_{\mathbb{R}}G(r,z;t,x)W^H(dr,dz)\right|^p\\
&\leqslant C\left(\int_s^t\int_s^t\int_{\mathbb{R}}\int_{\mathbb{R}}G(r_1,z_1;t,x)\Psi(r_1,r_2,z_1,z_2)G(r_2,z_2;t,x)dz_1dz_2dr_1dr_2\right)^{\frac{p}{2}}\\
&=C\left(\int_{[s,t]^2}|r_1-r_2|^{2H_1-2}\int_{\mathbb{R}^2}|z_1-z_2|^{2H_2-2}G(r_1,z_1;t,x)G(r_2,z_2;t,x)dz_1dz_2dr_1dr_2\right)^{\frac{p}{2}}\\
&\leqslant C\left(\int_{[s,t]^2}|r_1-r_2|^{2H_1-2}\|G(r_1,\cdot;t,x)\|_{L^{\frac{1}{H_2}}(\mathbb{R})}\|G(r_2,\cdot;t,x)\|_{L^{\frac{1}{H_2}}(\mathbb{R})}dr_1dr_2\right)^{\frac{p}{2}}\\
&\leqslant C\left(\int_s^t(\|G(r,\cdot;t,x)\|_{L^{\frac{1}{H_2}}(\mathbb{R})})^{\frac{1}{H_1}}dr\right)^{pH_1}\\
&\leqslant C\left(\int_s^t|t-r|^{\frac{H_2-1}{\alpha H_1}}dr\right)^{pH_1}\\
&\leqslant C|t-s|^{\frac{p}{\alpha}(\alpha H_1+H_2-1)},
\end{aligned} \tag{4.10}$$

where we have used the following fact

$$\begin{aligned}
\|G(s,\cdot;t,x)\|_{L^{\frac{1}{H_2}}(\mathbb{R})}&=\left(\int_{\mathbb{R}}G(s,y;t,x)^{\frac{1}{H_2}}dy\right)^{H_2}\\
&\leqslant C\left(\int_{\mathbb{R}}\left(\frac{|t-s|^{-\frac{1}{\alpha}}}{1+|t-s|^{-1-\frac{1}{\alpha}}|x-y|^{1+\alpha}}\right)^{\frac{1}{H_2}}dy\right)^{H_2}
\end{aligned}$$

$$
\begin{aligned}
&= C|t-s|^{\frac{H_2}{\alpha}-\frac{1}{\alpha}}\left(\int_{\mathbb{R}}\left(\frac{1}{1+|y|^{1+\alpha}}\right)^{\frac{1}{H_2}}dy\right)^{H_2}\\
&= C|t-s|^{\frac{H_2}{\alpha}-\frac{1}{\alpha}}\left(2\int_0^1\left(\frac{1}{1+y^{1+\alpha}}\right)^{\frac{1}{H_2}}dy+2\int_1^{+\infty}\left(\frac{1}{1+y^{1+\alpha}}\right)^{\frac{1}{H_2}}dy\right)^{H_2}\\
&\leqslant C|t-s|^{\frac{H_2}{\alpha}-\frac{1}{\alpha}}. \qquad (4.11)
\end{aligned}
$$

Thus from the above estimates (4.2), (4.6), (4.9) and (4.10), we have

$$
E|u(t,x)-u(s,x)|^p \leqslant C\left[|t-s|^{p\beta}+\int_0^s \sup_{y\in\mathbb{R}} E|u(r+t-s,y)-u(r,y)|^p dr\right].
$$

Hence Gronwall's lemma yields that

$$
E|u(t,x)-u(s,x)|^p \leqslant C|t-s|^{p\beta}, \qquad (4.12)
$$

with $\beta\in(0,\min\{\frac{\theta}{\alpha},\frac{\alpha H_1+H_2-1}{\alpha}\})$.

Next, we consider the space variable. For each $t\in[0,T]$ and $x,y\in\mathbb{R}$,

$$
\begin{aligned}
|u(t,x)-u(t,y)| &\leqslant \left|\int_{\mathbb{R}}[G(0,z;t,x)-G(0,z;t,y)]u_0(z)dz\right|\\
&\quad+\left|\int_0^t\int_{\mathbb{R}}[\partial_z G(r,z;t,x)-\partial_z G(r,z;t,y)]q(r,z,u(r,z))dzdr\right|\\
&\quad+\left|\int_0^t\int_{\mathbb{R}}[G(r,z;t,x)-G(r,z;t,y)]W^H(dz,dr)\right|\\
&=: \sum_{k=0}^{2}|B_k(t,x,y)|.
\end{aligned}
$$

First, set $z'=z-(x-y)$. Then

$$
\begin{aligned}
E|B_0(t,x,y)|^p &= E\left(\left|\int_{\mathbb{R}}G(0,z;t,x)[u_0(z+x-y)-u_0(z)]dz\right|^p\right)\\
&\leqslant \sup_{z\in\mathbb{R}}E|u_0(z+x-y)-u_0(z)|^p\cdot\left|\int_{\mathbb{R}}G(0,z;t,x)dz\right|^p\\
&\leqslant C|x-y|^{p\theta}. \qquad (4.13)
\end{aligned}
$$

Now we turn to $|B_1(t,x,y)|$. By Hölder's inequality and (A1), we have

$$
\begin{aligned}
E|B_1(t,x,y)|^p &\leqslant C_pE\left(\left|\int_0^t\int_{\mathbb{R}}\left|\frac{\partial G}{\partial z}(r,z;t,x)\right|[q(r,z+x-y,u(r,z+x-y))-q(r,z,u(r,z))]dzdr\right|^p\right)\\
&\leqslant C_p\left[\int_0^t\int_{\mathbb{R}}\left|\frac{\partial G}{\partial z}(r,z;t,x)\right|E|q(r,z+x-y,u(r,z+x-y))-q(r,z,u(r,z))|^p dzdr\right.\\
&\quad\left.\times\left|\int_0^t\int_{\mathbb{R}}\left|\frac{\partial G}{\partial z}(r,z;t,x)\right|dzdr\right|^{p-1}\right]\\
&\leqslant C_p\left|\int_0^t\left(|x-y|^p+\sup_{z\in\mathbb{R}}E|u(r,z+x-y)-u(r,z)|^p\right)\left(\int_{\mathbb{R}}\left|\frac{\partial G}{\partial z}(r,z;t,x)\right|dz\right)dr\right|\\
&\leqslant C_{p,T}\left[|x-y|^p+\int_0^t\sup_{z\in\mathbb{R}}E|u(r,z+x-y)-u(r,z)|^p dr\right]. \qquad (4.14)
\end{aligned}
$$

Finally, let us consider the term $B_2(t,x,y)$. Let $\gamma\in(0,\alpha H_1+H_2-1)\subset(0,1)$. Then

$$
E|B_2(t,x,y)|^p \leqslant C_p\left(E\left|\int_0^t\int_{\mathbb{R}}[G(r,z;t,x)-G(r,z;t,y)]W^H(dr,dz)\right|^2\right)^{\frac{p}{2}}
$$

$$
\begin{aligned}
&= C_p\bigg(\int_{[0,t]^2}\int_{\mathbb{R}^2}|G(r_1,z_1;t,x)-G(r_1,z_1;t,y)|\Psi_H(r_1,r_2;z_1,z_2)\\
&\quad\times|G(r_2,z_2;t,x)-G(r_2,z_2;t,y)|dz_1dz_2dr_1dr_2\bigg)^{\frac{p}{2}}\\
&= C_p\|G(\cdot,\cdot;t,x)-G(\cdot,\cdot;t,y)\|^p_{L^2_\Psi}\\
&= C_p\||G(\cdot,\cdot;t,x)-G(\cdot,\cdot;t,y)|^\gamma\cdot|G(\cdot,\cdot;t,x)-G(\cdot,\cdot;t,y)|^{1-\gamma}\|^p_{L^2_\Psi}\\
&\leqslant C(p,\gamma)\||G(\cdot,\cdot;t,x)-G(\cdot,\cdot;t,y)|^\gamma\cdot|G(\cdot,\cdot;t,x)|^{1-\gamma}\|^p_{L^2_\Psi}\\
&\quad+C(p,\gamma)\||G(\cdot,\cdot;t,x)-G(\cdot,\cdot;t,y)|^\gamma\cdot|G(\cdot,\cdot;t,y)|^{1-\gamma}\|^p_{L^2_\Psi}\\
&\equiv C(p,\gamma)(B_{2,1}(t,x,y)+B_{2,2}(t,x,y)).
\end{aligned}
$$

Using (3.3), (3.4), Proposition 2.4 and the mean-value theorem, one can get

$$
\begin{aligned}
B_{2,1}(t,x,y) &= \bigg\|\bigg|\frac{\partial G(\cdot,\cdot;t,\xi)}{\partial x}\bigg|^\gamma\cdot|x-y|^\gamma\cdot|G(\cdot,\cdot;t,x)|^{1-\gamma}\bigg\|^p_{L^2_\Psi}\\
&\leqslant C_p|x-y|^{p\gamma}\bigg(\int_{[0,T]^2}\int_{\mathbb{R}^2}\bigg|\frac{\partial G(r_1,z_1;t,\xi)}{\partial x}\bigg|^\gamma\cdot|G(r_1,z_1;t,x)|^{1-\gamma}\Psi_H(r_1,r_2,z_1,z_2)\\
&\quad\times\bigg|\frac{\partial G(r_2,z_2;t,\xi)}{\partial x}\bigg|^\gamma\cdot|G(r_2,z_2;t,x)|^{1-\gamma}dz_1dz_2dr_1dr_2\bigg)^{\frac{p}{2}}\\
&\leqslant C|x-y|^{p\gamma}\bigg(\int_0^T\bigg(\int_{\mathbb{R}}\bigg(\bigg|\frac{\partial G(r,z;t,x)}{\partial x}\bigg|^\gamma\cdot|G(r,z;t,x)|^{1-\gamma}\bigg)^{\frac{1}{H_2}}dz\bigg)^{\frac{H_2}{H_1}}dr\bigg)^{pH_1}\\
&\leqslant C|x-y|^{p\gamma}.
\end{aligned}
$$

In fact,

$$
\begin{aligned}
&\int_{\mathbb{R}}\bigg(\bigg|\frac{\partial G(r,z;t,x)}{\partial x}\bigg|^\gamma\cdot|G(r,z;t,x)|^{1-\gamma}\bigg)^{\frac{1}{H_2}}dz\\
&\quad\leqslant\int_{\mathbb{R}}\bigg(\frac{|t-r|^{(-1-\frac{2}{\alpha})\gamma}|x-z|^{\alpha\gamma}}{(1+|t-r|^{-\frac{1+\alpha}{\alpha}}|x-z|^{1+\alpha})^{2\gamma}}\cdot\frac{|t-r|^{-\frac{(1-\gamma)}{\alpha}}}{(1+|t-r|^{-\frac{1+\alpha}{\alpha}}|x-z|^{1+\alpha})^{1-\gamma}}\bigg)^{\frac{1}{H_2}}dz\\
&\quad=|t-r|^{\frac{H_2-\gamma-1}{\alpha H_2}}\int_{\mathbb{R}}\frac{|y|^{\frac{\alpha\gamma}{H_2}}}{(1+|y|^{1+\alpha})^{\frac{\gamma+1}{H_2}}}dy\\
&\quad\leqslant C|t-r|^{\frac{H_2-\gamma-1}{\alpha H_2}}\bigg(\int_0^1 y^{\frac{\gamma\alpha}{H_2}}dy+\int_1^{+\infty}y^{\frac{-1-\gamma-\alpha}{H_2}}dy\bigg)\\
&\quad\equiv: C(\alpha,H_2,\gamma)|t-r|^{\frac{-\gamma-1+H_2}{\alpha H_2}}.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\bigg(\int_0^T\bigg(\int_{\mathbb{R}}\bigg(\bigg|\frac{\partial G(r,z;t,x)}{\partial x}\bigg|^\gamma\cdot|G(r,z;t,x)|^{1-\gamma}\bigg)^{\frac{1}{H_2}}dz\bigg)^{\frac{H_2}{H_1}}dr\bigg)^{pH_1}\\
&\quad\leqslant C\bigg(\int_0^T|t-r|^{\frac{-\gamma-1+H_2}{\alpha H_1}}dr\bigg)^{pH_1}\leqslant C(H_1,H_2,\gamma,T),
\end{aligned}
$$

if $\gamma<\alpha H_1+H_2-1$. So

$$E|B_{2,1}(t,x,y)|^p\leqslant C(H_1,H_2,\gamma,T)|x-y|^{p\gamma}.$$

Similarly,

$$E|B_{2,2}(t,x,y)|^p\leqslant C(H_1,H_2,\gamma,T)|x-y|^{p\gamma}.$$

From the above estimates, it follows that

$$E|B_2(t,x,y)|^p\leqslant C(H_1,H_2,\gamma,T)|x-y|^{p\gamma},\quad \gamma\in(0,\alpha H_1+H_2-1)\subset(0,1). \tag{4.15}$$

Together with (4.13)–(4.15), for $\gamma \in (0, \min\{\theta, \alpha H_1 + H_2 - 1\})$, we have

$$E|u(t,x) - u(t,y)|^p \leqslant C\left[|x-y|^{p\theta} + |x-y|^p + \int_0^t \sup_{z\in\mathbb{R}} E|u(r, z+x-y) - u(r,z)|^p dr + |x-y|^{p\gamma}\right]$$
$$\leqslant C\left[|x-y|^{p\gamma} + \int_0^t \sup_{z\in\mathbb{R}} E|u(r, z+x-y) - u(r,z)|^p dr\right].$$

Then Gronwall's lemma yields

$$E|u(t,x) - u(t,y)|^p \leqslant C|x-y|^{p\gamma}, \quad \text{with } \gamma \in (0, \min\{\theta, \alpha H_1 + H_2 - 1\}). \tag{4.16}$$

This completes the proof. □

5 Existence of the density of the law of the solution

In this section, we shall prove the absolute continuity of the law of the solution $\{u(t,x) : (t,x) \in [0,T]\times\mathbb{R}\}$ given in Section 3. We first prove $u(t,x) \in \mathbb{D}^{1,2}$ and then the derivative $Du(t,x)$.

Proposition 5.1. *Under the assumptions in Theorem* 3.1*, if we further assume that* $q(s,y,\cdot) \in C_b^1([0,T]\times\mathbb{R}\times\mathbb{R})$*, then for* $(t,x) \in [0,T]\times\mathbb{R}$*, the solution* $u(t,x) \in \mathbb{D}^{1,2}$ *and*

$$D_{v,z}u(t,x) = \int_v^t \int_{\mathbb{R}} \frac{\partial G}{\partial y}(s,y;t,x) q'(s,y,u(s,y)) D_{v,z}u(s,y)dyds + G(v,z;t,x), \tag{5.1}$$

for all $v \leqslant t$ *and* $z \in \mathbb{R}$*.*

Proof. Let $u^{(n)}$ be the solution of the following stochastic partial differential equation

$$\begin{cases} u_0(t,x) = \displaystyle\int_{\mathbb{R}} G(0,y;t,x)u_0(y)dy, \\ u^{(n+1)}(t,x) = u_0(t,x) + \displaystyle\int_0^t \int_{\mathbb{R}} \frac{\partial G}{\partial y}(s,y;t,x) q(s,y,u^{(n)}(s,y))dyds \\ \qquad\qquad + \displaystyle\int_0^t \int_{\mathbb{R}} G(s,y;t,x)W^H(s,y). \end{cases} \tag{5.2}$$

Then a similar argument to that in Zhang and Zheng [38] shows that for each $n \in \mathbb{N}$ and $h \in L^2_\Psi$, $u^{(n)}(t,x) \in \mathbb{D}_h$ and it satisfies that

$$D_h u^{(n)}(t,x) = \int_0^t \int_{\mathbb{R}} \frac{\partial G}{\partial y}(s,y;t,x) q'(s,y,u^{(n-1)}(s,y)) D_h u^{(n-1)}(s,y)dyds + \langle G(\cdot,\cdot;t,x), h\rangle_{L^2_\Psi}. \tag{5.3}$$

Since $u^{(n)}(t,x) \to u(t,x)$ as $n \to \infty$ in $L^p(\Omega)$ sense, there exists a random field $u_h(t,x)$ such that $D_h u^{(n)}(t,x) \to u_h(t,x)$ as $n \to \infty$ uniformly on $(t,x) \in [0,T]\times\mathbb{R}$, and the latter satisfies that

$$u_h(t,x) = \int_0^t \int_{\mathbb{R}} \frac{\partial G}{\partial y}(s,y;t,x) q'(s,y,u(s,y)) u_h(s,y)dyds + \langle G(\cdot,\cdot;t,x), h\rangle_{L^2_\Psi}. \tag{5.4}$$

Hence, from the closeness of the operator D_h, it follows that $u(t,x) \in \mathbb{D}_h$, $D_h u(t,x) = u_h(t,x)$ and

$$D_h u(t,x) = \int_0^t \int_{\mathbb{R}} \frac{\partial G}{\partial y}(s,y;t,x) q'(s,y,u(s,y)) D_h u(s,y)dyds + \langle G(\cdot,\cdot;t,x), h\rangle_{L^2_\Psi}. \tag{5.5}$$

Next, we proceed to prove that $u(t,x) \in \mathbb{D}^{1,2}$. Recall $\{h_n, n \geqslant 1\}$ in Section 2. By (5.5),

$$E|D_{h_n}u(t,x)|^2 = E\left|\int_0^t \int_{\mathbb{R}} \frac{\partial G}{\partial y}(s,y;t,x) q'(s,y,u(s,y)) D_{h_n}u(s,y)dyds + \langle G(\cdot,\cdot;t,x), h_n\rangle_{L^2_\Psi}\right|^2$$

$$\leqslant CE\bigg[\int_0^t\int_{\mathbb{R}}\left(\frac{\partial G}{\partial y}(s,y;t,x)\right)^2(D_{h_n}u(s,y))^2dyds\bigg]+C\langle G(\cdot,\cdot;t,x),h_n\rangle^2_{L^2_\Psi}, \tag{5.6}$$

where $C>0$ is a constant that may change from line to line in this section. Set

$$U_m(t)=\sup_{x\in\mathbb{R}}E\sum_{n=1}^m|D_{h_n}u(t,x)|^2.$$

Then by (5.6) and Hölder's inequality with $p=q=2$, we have

$$\begin{aligned}U_m(t)&\leqslant CE\bigg[\int_0^t\int_{\mathbb{R}}\left(\frac{\partial G}{\partial y}(s,y;t,x)\right)^2U_m(s)dyds\bigg]+C\|G(\cdot,\cdot;t,x)\|^2_{L^2_\Psi}\\&\leqslant C\int_0^t\int_{\mathbb{R}}\frac{(t-s)^{-2-\frac{4}{\alpha}}|x-y|^{2\alpha}}{(1+(t-s)^{\frac{1+\alpha}{\alpha}}|x-y|^{1+\alpha})^4}U_m(s)dyds+C\|G(\cdot,\cdot;t,x)\|^2_{L^{\frac{1}{H}}([0,t]\times\mathbb{R})}\\&\leqslant C+C\int_0^t(t-s)^{-\frac{3}{\alpha}}U_m(s)ds.\end{aligned}\tag{5.7}$$

Then the Gronwall's lemma yields that

$$U_m(t)\leqslant C\mathrm{e}^{CT^{1-\frac{3}{\alpha}}},$$

where C is independent of m. Let $m\to\infty$, to get

$$\sup_{x\in\mathbb{R}}E\sum_{n=1}^\infty|D_{h_n}u(t,x)|^2<\infty.$$

That means that $u(t,x)\in\mathbb{D}^{1,2}$.

Since $u(t,x)$ is $\mathscr{F}_t$-adapted, there exists a measurable function $D_{v,z}u(t,x)\in L^2_\Psi$ such that $D_{v,z}u(t,x)=0$ if $v>t$ and for any $h\in L^2_\Psi$,

$$D_hu(t,x)=\langle Du(t,x),h\rangle_{L^2_\Psi}. \tag{5.8}$$

From (5.5), (5.8) and Fubini theorem, it follows that

$$\begin{aligned}\langle Du(t,x),h\rangle_{L^2_\Psi}&=\int_0^t\int_{\mathbb{R}}\frac{\partial G}{\partial y}(s,y;t,x)q'(s,y,u(s,y))D_hu(s,y)dyds+\langle G(\cdot,\cdot;t,x),h\rangle_{L^2_\Psi}\\&=\int_0^t\int_{\mathbb{R}}\frac{\partial G}{\partial y}(s,y;t,x)q'(s,y,u(s,y))\langle Du(s,y),h\rangle_{L^2_\Psi}dyds+\langle G(\cdot,\cdot;t,x),h\rangle_{L^2_\Psi}\\&=\int_0^t\int_{\mathbb{R}}\frac{\partial G}{\partial y}(s,y;t,x)q'(s,y,u(s,y))dyds\\&\quad\times\int_{[0,s]^2}\int_{\mathbb{R}^2}D_{v,z}u(s,y)h(v',z')\Psi_H(v,v';z,z')dzdz'dvdv'+\langle G(\cdot,\cdot;t,x),h\rangle_{L^2_\Psi}\\&=\int_{[0,t]^2}\int_{\mathbb{R}^2}h(v',z')\Psi_H(v,v';z,z')dzdz'dvdv'\\&\quad\times\int_v^t\int_{\mathbb{R}}\frac{\partial G}{\partial y}(s,y;t,x)q'(s,y,u(s,y))dyds+\langle G(\cdot,\cdot;t,x),h\rangle_{L^2_\Psi}.\end{aligned}$$

Therefore

$$D_{v,z}u(t,x)=\int_v^t\int_{\mathbb{R}}\frac{\partial G}{\partial y}(s,y;t,x)q'(s,y,u(s,y))D_{v,z}u(s,y)dyds+G(v,z;t,x).$$

Thus the proof of the proposition is completed. □

Theorem 5.2. *Under the conditions in Theorem* 3.1 *and furthermore assume that* $q(s,y,\cdot)\in C^1_b([0,T]\times\mathbb{R}\times\mathbb{R})$. *Then for* $(t,x)\in[0,T]\times\mathbb{R}$, *the law of the solution* $u(t,x)$ *of* (3.1) *is absolutely continuous with respect to Lebesgue measure.*

In order to prove Theorem 5.2, we firstly give a useful lemma.

Lemma 5.3. *For $\varepsilon \in (0,t)$, then there exists some constant $C>0$ such that*

$$\sup_{(s,y)\in[t-\varepsilon,t]\times\mathbb{R}} E\left(\int_{t-\varepsilon}^{t}\int_{\mathbb{R}} |D_{v,z}u(s,y)|^2 dzdv\right) < C\varepsilon. \tag{5.9}$$

Proof. For $s\in[t-\varepsilon,t]$, set

$$L_\varepsilon(s,y) = E\left(\int_{t-\varepsilon}^{s}\int_{\mathbb{R}} |D_{v,z}u(s,y)|^2 dzdv\right).$$

Then from the proof of Proposition 5.1, we get that

$$\sup_{(s,y)\in[0,T]\times\mathbb{R}} L_\varepsilon(s,y) < \infty.$$

Recall (5.1), and then

$$\begin{aligned} L_\varepsilon(s,y) &\leqslant 2\bigg[\int_{t-\varepsilon}^{s}\int_{\mathbb{R}} |G(v,z;s,y)|^2 dzdv \\ &\quad + E\int_{t-\varepsilon}^{s}\int_{\mathbb{R}}\left|\int_{v}^{s}\int_{\mathbb{R}} \frac{\partial G}{\partial y}(r_1,z_1;s,y)q(r_1,z_1,u(r_1,z_1))D_{v,z}u(r_1,z_1)dz_1dr_1\right|^2 dzdv\bigg] \\ &=: 2(L_{\varepsilon,1}(s,y)+L_{\varepsilon,2}(s,y)). \end{aligned} \tag{5.10}$$

With (3.3), it is easy to check that

$$\begin{aligned} L_{\varepsilon,1}(s,y) &\leqslant C\int_{t-\varepsilon}^{s}\int_{\mathbb{R}} \frac{(s-v)^{-\frac{2}{\alpha}}}{(1+(s-v)^{-\frac{1+\alpha}{\alpha}}|y-z|^{1+\alpha})^2}dzdv \\ &\leqslant C\int_{t-\varepsilon}^{s}(s-v)^{-\frac{1}{\alpha}}dv\int_{\mathbb{R}}\frac{1}{(1+|z|^{1+\alpha})^2}dz \\ &\leqslant C\epsilon^{1-\frac{1}{\alpha}}, \end{aligned} \tag{5.11}$$

and

$$L_{\varepsilon,2}(s,y) \leqslant C\int_{t-\varepsilon}^{s}\sup_{z_1\in\mathbb{R}} L_\varepsilon(r_1,z_1)dr_1 \leqslant C\varepsilon^{1-\frac{1}{\alpha}} + C\int_{t-\varepsilon}^{s}\sup_{z_1\in\mathbb{R}} L_{\varepsilon,2}(r_1,z_1)dr_1. \tag{5.12}$$

Then by Gronwall's lemma, we get (5.9). □

Proof of Theorem 5.2. To prove Theorem 5.2, we will adopt a technical argument, which was proposed by Cardon-Weber [9].

By Proposition 2.7, it suffices to prove that

$$\|Du(t,x)\|_{L^2_\Psi} > 0, \quad \text{a.s.}$$

Notice that

$$\|Du(t,x)\|_{L^2_\Psi} > 0 \Leftrightarrow \|Du(t,x)\|_{L^2([0,T]\times\mathbb{R})} > 0.$$

Hence, we only need to prove that $\|Du(t,x)\|_{L^2([0,T]\times\mathbb{R})} > 0$ a.s. For $0<\varepsilon<t$, recall (5.1), and we have

$$\int_0^t\int_{\mathbb{R}} |D_{r,z}u(t,x)|^2 dzdr \geqslant \int_{t-\varepsilon}^{t}\int_{\mathbb{R}} |D_{r,z}u(t,x)|^2 dzdr \geqslant C(I_1(t,x,\varepsilon)-I_2(t,x,\varepsilon)), \tag{5.13}$$

where

$$I_1(t,x,\varepsilon) = \int_{t-\varepsilon}^{t}\int_{\mathbb{R}} |G(r,z;t,x)|^2 dzdr$$

and

$$I_2(t,x,\varepsilon) = \int_{t-\varepsilon}^{t}\int_{\mathbb{R}}\left|\int_{r}^{t}\int_{\mathbb{R}} \frac{\partial G}{\partial z_1}(r_1,z_1;t,x)q'(r_1,z_1,u(r_1,z_1))D_{r,z}u(r_1,z_1)dz_1dr_1\right| dzdr.$$

Similar to the proof of (5.11), there exists a constant $K > 0$ such that

$$I_1(t,x,\varepsilon) = K\varepsilon^{1-\frac{1}{\alpha}}. \tag{5.14}$$

By (3.14) and Lemma 5.3, one gets

$$\begin{aligned} E|I_2(t,x,\varepsilon)| &\leqslant \int_{t-\varepsilon}^{t}\int_{\mathbb{R}} \frac{\partial G}{\partial z_1}(r_1,z_1;t,x) E\left(\int_{t-\varepsilon}^{r_1}\int_{\mathbb{R}} |D_{r,z}u(r_1,z_1)|^2 dzdr\right) dz_1 dr_1 \\ &\leqslant C\varepsilon \int_{t-\varepsilon}^{t}\int_{\mathbb{R}} \frac{\partial G}{\partial z_1}(r_1,z_1;t,x) dz_1 dr_1 \\ &\leqslant C\varepsilon^{2-\frac{1}{\alpha}}. \end{aligned} \tag{5.15}$$

Then for each $\varepsilon_0 > 0$, according to (5.13)–(5.15),

$$\begin{aligned} P\left(\int_0^t \int_{\mathbb{R}} |D_{r,z}u(t,x)|^2 dzdr > 0\right) &\geqslant \sup_{\varepsilon\in(0,\varepsilon_0]} P(C(I_1(t,x,\varepsilon) - I_2(t,x,\varepsilon)) > 0) \\ &\geqslant \sup_{\varepsilon\in(0,\varepsilon_0]} P(I_2(t,x,\varepsilon) \leqslant C I_1(t,x,\varepsilon)) \\ &\geqslant 1 - \inf_{\varepsilon\in(0,\varepsilon_0]} \left\{\frac{1}{C\varepsilon^{1-\frac{1}{\alpha}}} E|I_2(t,x,\varepsilon)|\right\} \\ &\geqslant 1 - \inf_{\varepsilon\in(0,\varepsilon_0]} C\varepsilon = 1. \end{aligned} \tag{5.16}$$

Thus the proof of the theorem is complete. □

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