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Multidimensional stability of V-shaped traveling fronts in the Allen-Cahn equation

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Abstract This paper is concerned with the multidimensional asymptotic stability of V-shaped traveling fronts in the Allen-Cahn equation under spatial decaying initial values. We first show that V-shaped traveling fronts are asymptotically stable under the perturbations that decay at infinity. Then we further show that there exists a solution that oscillates permanently between two V-shaped traveling fronts, which indicates that V-shaped traveling fronts are not always asymptotically stable under general bounded perturbations. Our main technique is the supersolutions and subsolutions method coupled with the comparison principle.

Keywords Allen-Cahn equation, asymptotic stability, multidimensional, V-shaped, traveling front

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1 Introduction

In this paper, we consider the large time behavior of solutions to the following Allen-Cahn equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + f(u), \quad x \in \mathbb{R}^{n-2}, \quad y \in \mathbb{R}, \quad z \in \mathbb{R}, \quad t > 0, \\ u(x,y,z,0) = u_0(x,y,z), \quad x \in \mathbb{R}^{n-2}, \quad y \in \mathbb{R}, \quad z \in \mathbb{R}, \end{cases} \tag{1.1}$$

where the initial value u_0 is assumed to be smooth and bounded, and the function $f \in C^1(\mathbb{R})$ satisfies the following hypotheses:

(F1) $f(-1) = f(1) = 0$, $f'(-1) < 0$, $f'(1) < 0$.

(F2) $\int_{-1}^{1} f(s)ds > 0$.

(F3) $f(s) > 0$ and $f'(s) < 0$ for $s < -1$; $f(s) < 0$ and $f'(s) < 0$ for $s > 1$.

By the continuity of f and the hypotheses (F1), there exist some positive constants $\delta_\pm$ such that

$$f > 0 \ \text{ in } (-\infty,-1) \cup (1-\delta_-, 1) \quad \text{and} \quad f < 0 \ \text{ in } (-1,-1+\delta_+) \cup (1,\infty). \tag{1.2}$$

A typical example of such f is

$$f(u) = (u+1)(u-a)(1-u),$$

where $|a| < 1$ is a given number.

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It is known from [37] that the one-dimensional Allen-Cahn equation has a traveling front under the bistable assumption on f. Namely, there exist a unique function $\phi : \mathbb{R} \to \mathbb{R}$ and a unique constant $l \in \mathbb{R}$ such that

$$\begin{cases} -\phi'' - l\phi' - f(\phi) = 0, \\ \phi(-\infty) = 1 \quad \text{and} \quad \phi(\infty) = -1. \end{cases} \tag{1.3}$$

This function ϕ is called the planar traveling front of (1.1) on $\mathbb{R}$, which has been extensively studied since the pioneer works of Fisher [6] and Kolmogrov et al. [18]. One can refer to [1, 2, 5, 21, 37, 40] and the references therein for the existence, uniqueness and stability of the planar traveling front in one or higher dimensions. Here, it should be emphasized that the stability of planar fronts in multidimensional spaces has recently been studied by many authors. For example, Matano et al. [25] first considered the asymptotic stability of planar traveling fronts in $\mathbb{R}^N$ under the spatial decaying and non-decaying (almost periodic perturbations) perturbations, then Matano and Nara [24] further studied how a planar front behaves when arbitrarily large (but bounded) perturbation is given near the front region. They showed that the behavior of the disturbed front can be approximated by that of the mean curvature flow with a drift term for all large time t up to $+\infty$. Roquejoffre et al. [30] studied the large time behavior of planar traveling fronts and showed that the dynamics of planar fronts are similar to that of the heat equation. For earlier related results, we refer to [17, 20, 41]. In addition, we also refer to [22, 23] for monostable reaction-diffusion equations.

Recently, the study on *nonplanar traveling fronts* has attracted much attention. For example, Ninomiya and Taniguchi [28, 29] studied the existence, uniqueness and asymptotic stability of V-shaped traveling fronts to (1.1) in $\mathbb{R}^2$; Taniguchi [35, 36] considered a pyramidal traveling front in $\mathbb{R}^3$ which is uniquely determined and asymptotically stable. For other related results on nonplanar traveling fronts, we refer to [3, 4, 8–15, 19, 30, 34, 38] for autonomous reaction-diffusion equations and [31, 39] for non-autonomous equations (i.e., $f = f(u, t)$).

In this paper, we are interested in the multidimensional stability of nonplanar traveling fronts in $\mathbb{R}^n$. In particular, we deal with the stability of V-shaped traveling fronts to the Cauchy problem (1.1), where a V-shaped traveling front is referred to $V(y, s) = V(y, z-ct)$ for some positive constant c. For simplicity, we still denote $V(y, s)$ by $V(y, z)$. We remark here that the profile equation for V is

$$V_{yy} + V_{zz} + cV_z + f(V) = 0. \tag{1.4}$$

Just as Volpert et al. [37] pointed out that the stability of nonplanar traveling fonts is an important problem, which has been already investigated by some authors. See [10] for the conical-shaped traveling fronts with combustion nonlinearity; [16] for the Fisher-KPP equation; [28, 29] and [35, 36] for the Allen-Cahn equation. It should be remarked that the exponential stability of V-shaped traveling fronts recently has been established by Sheng et al. [32, 33] by employing the squeezing technique combined with the comparison principle.

However, as far as we know, there are no results devoted to the multidimensional stability of V-shaped traveling fronts in multidimensional spaces. It is then natural to ask what about the stability of V-shaped traveling fronts under appropriate perturbations in multidimensional space. Inspired by [24, 25], we first show that the V-shaped traveling front is asymptotic stable with the algebraic convergence rate under spatially decaying initial perturbations, then we further prove that the V-shaped traveling front is not asymptotically stable under general bounded initial perturbations. The main technique we use is the supersolution and subsolution method coupled with the comparison principle. To the best of our knowledge this is probably the first time that the multidimensional asymptotic stability of V-shaped traveling fronts is considered.

The main results are as follows.

Theorem 1.1. *Assume that* (F1)–(F3) *hold. Assume further that the initial value* $u_0(x, y, z)$ *of* (1.1) *satisfies*

$$\lim_{R\to\infty} \sup_{|x|+|y|+|z|\geqslant R} |u_0(x, y, z) - V(y, z)| = 0.$$

Then the solution $u(x,y,z,t)$ *to* (1.1) *satisfies*

$$\lim_{t\to\infty}\sup_{(x,y,z)\in\mathbb{R}^n}|u(x,y,z,t)-V(y,z-ct)|=0. \tag{1.5}$$

Theorem 1.1 shows that the V-shaped traveling front is asymptotically stable under the initial perturbations that decay at infinity. The following theorem gives the convergence rate for (1.5) when the initial perturbations belong to L^1 in a certain sense.

Theorem 1.2. *Assume that* (F1)–(F3) *hold. Assume further that the initial value* $u_0(x,y,z)$ *of* (1.1) *is given by*

$$u_0(x,y,z)=V(y,z-v_0(x)), \tag{1.6}$$

for some smooth function $v_0\in L^1(\mathbb{R}^{n-2})\cap L^\infty(\mathbb{R}^{n-2})$*. Then the solution* $u(x,y,z,t)$ *to* (1.1) *satisfies*

$$\sup_{(x,y,z)\in\mathbb{R}^n}|u(x,y,z,t)-V(y,z-ct)|\leqslant Ct^{-\frac{n-2}{2}},\quad t>0, \tag{1.7}$$

where $C>0$ *is a constant depending on* f, $\|v_0\|_{L^1(\mathbb{R}^{n-2})}$ *and* $\|v_0\|_{L^\infty(\mathbb{R}^{n-2})}$.

The following proposition shows that the convergence rate (1.7) is optimal in some sense.

Proposition 1.3. *Let* u_0 *be defined as in* (1.6) *and assume that either* $v_0\geqslant 0$, $v_0\not\equiv 0$ *or* $v_0\leqslant 0$, $v_0\not\equiv 0$*. Then there exist constants* $C_1>0$ *and* $C_2>0$ *such that*

$$\frac{C_1}{(1+t)^{\frac{n-2}{2}}}\leqslant\sup_{(x,y,z)\in\mathbb{R}^n}|u(x,y,z,t)-V(y,z-ct)|\leqslant\frac{C_2}{t^{\frac{n-2}{2}}},\quad t\geqslant 0. \tag{1.8}$$

Remark 1.4. It is known from [28, 29] that the V-shaped traveling front of (1.1) is asymptotically stable in $\mathbb{R}^2$. However, the large time behavior of V-shaped traveling fronts of (1.1) is unknown in $\mathbb{R}^n$ with $n\geqslant 3$. Our Theorems 1.1 and 1.2 show that the V-shaped traveling front is not only asymptotic stable, but also is algebraic stable under certain perturbations. In particular, our Proposition 1.3 further implies that this convergence rate is optimal in some sense. Comparing these results with those of [32] highlights the gap between the dynamics in dimension 2 and dimension n with $n\geqslant 3$.

Next, we present a result on the existence of a solution to (1.1) that oscillates permanently between two V-shaped traveling fronts.

Theorem 1.5. *Let* $n=3$ *and* (F1)–(F3) *hold. Then there exists a bounded function* $v_0^*(x)$ *on* $\mathbb{R}$ *with* $\|v_0^*\|_{L^\infty(\mathbb{R})}=\delta$ *such that the solution* $u(x,y,z,t)$ *to* (1.1) *with* $u(x,y,z,0)=V(y,z-v_0^*(x))$ *satisfies*

$$\lim_{m\to\infty}\sup_{|x|\leqslant m!-1,(y,z)\in\mathbb{R}^2}|u(x,y,z,t_m)-V(y,z-ct_m+(-1)^m\delta)|=0,$$

where $t_m=m(m!)^2/4$.

Remark 1.6. In fact, Roquejoffre and Roussier-Michon [30] showed that the asymptotic stability of conical traveling fronts breaks down as soon as the assumptions are relaxed as low as the initial value of (1.1) lies between two conical fronts. Our result further showed that the ω-limit set of (1.1) is nontrivial in general. In particular, we find two ω-limit points of (1.1).

This paper is organized as follows. We summarize some preliminaries including some known results of the curvature flow problem in Section 2. Section 3 is due to prove Theorems 1.1 and 1.2 by constructing various type of supersolutions and subsolutions. In the proof, we will express the solution $u(x,y,z,t)$ in a moving frame with speed c, so that the V-shaped traveling front can be viewed as stationary state. Let

$$u(x,y,z,t)=w(x,y,s,t),\quad s=z-ct.$$

Then the equation (1.1) is rewritten as

$$w_t=\Delta w+cw_s+f(w),\quad x\in\mathbb{R}^{n-2},\quad y\in\mathbb{R},\quad z\in\mathbb{R},\quad t>0,$$

where $\Delta := \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_{n-2}^2 + \partial^2/\partial y^2 + \partial^2/\partial s^2$. For simplicity, we denote $w(x,y,s,t)$ as $u(x,y,z,t)$ and consider the problem of the form

$$u_t = \Delta u + cu_z + f(u), \quad x \in \mathbb{R}^{n-2}, \quad y \in \mathbb{R}, \quad z \in \mathbb{R}, \quad t > 0, \tag{1.9}$$

$$u(x,y,z,0) = u_0(x,y,z), \quad x \in \mathbb{R}^{n-2}, \quad y \in \mathbb{R}, \quad z \in \mathbb{R}. \tag{1.10}$$

In Section 4, by using the supersolutions and subsolutions constructed in Section 3, we prove that the V-shaped traveling front is not always asymptotically stable, i.e., we prove Theorem 1.5. We give some discussions in the last section.

2 Preliminaries

In this section, we state some known results of the curvature flow problem [26,27] and give the definition of the supersolution and subsolution.

The mean curvature flow for a graphical surface $u(x,t)$ on $\mathbb{R}^{n-2}$ is given by the following Cauchy problem:

$$\frac{v_t}{\sqrt{1+|\nabla v|^2}} = \mathrm{div}\bigg(\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\bigg), \quad x \in \mathbb{R}^{n-2}, \quad t > 0, \tag{2.1}$$

$$v(x,0) = v_0(x), \quad x \in \mathbb{R}^{n-2}. \tag{2.2}$$

Taking some constant $k > 0$ large, we obtain

$$\begin{aligned} 0 &= v_t - \sqrt{1+|\nabla v|^2} \cdot \mathrm{div}\bigg(\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\bigg) \\ &= v_t - \Delta v - \sum_{i,j=1}^{n-2} \frac{v_{x_i} v_{x_j} v_{x_i x_j}}{\sqrt{1+|\nabla v|^2}} \\ &\geqslant v_t - \Delta v - k|\nabla v|^2, \end{aligned} \tag{2.3}$$

under the assumption that the first and the second derivatives of u with respect to x are all bounded on $\mathbb{R}^{n-2}$. It is obvious from (2.3) that $v(x,t)$ is a subsolution to the following Cauchy problem:

$$\begin{aligned} &v_t^+ = \Delta v^+ + k|\nabla v^+|^2, \quad x \in \mathbb{R}^{n-2}, \quad t > 0, \\ &v^+(x,0) = u_0(x), \quad x \in \mathbb{R}^{n-2}. \end{aligned}$$

Taking the Cole-Hopf transformation $w(x,t) = \exp(kv(x,t))$, we have

$$\begin{aligned} &w_t = \Delta w, \quad x \in \mathbb{R}^{n-2}, \quad t > 0, \\ &w(x,0) = \exp(ku_0(x)), \quad x \in \mathbb{R}^{n-2}. \end{aligned}$$

This is a standard heat equation and the solution is given by

$$w(x,t) = \int_{\mathbb{R}^{n-2}} \Gamma(x-\eta,t) \exp(ku_0(\eta)) d\eta, \tag{2.4}$$

where Γ is defined by

$$\Gamma(\xi,\tau) := \frac{1}{(4\pi\tau)^{\frac{n-2}{2}}} \exp\bigg(\frac{-|\xi|^2}{4\tau}\bigg).$$

Thus, the explicit expression for $v^+(x,t)$ is given by

$$v^+(x,t) = \frac{1}{k} \ln\bigg(\int_{\mathbb{R}^{n-2}} \Gamma(x-\eta,t) \exp(ku_0(\eta)) d\eta\bigg), \tag{2.5}$$

which gives an upper estimate for $u(x,t)$ to (2.1) and (2.2). The lower estimate can be obtained in a similar way by considering the equation

$$v_t^- = \Delta v^- - k|\nabla v^-|^2, \quad x \in \mathbb{R}^{n-2}, \quad t > 0$$

with initial value $v^-(x,0) = u_0(x)$.

Now, we introduce a lemma which gives the large time behavior of solutions to

$$v_t^\pm = \Delta v^\pm \pm k|\nabla v^\pm|^2, \quad x \in \mathbb{R}^{n-2}, \quad t > 0.$$

Lemma 2.1 (See [25, Lemmas 2.4 and 2.5])**.** *Let $k > 0$ be any constant and $v^\pm(x,t)$ be solutions to the following Cauchy problems*:

$$\begin{aligned} &v_t^\pm = \Delta v^\pm \pm k|\nabla v^\pm|^2, \quad x \in \mathbb{R}^{n-2}, \quad t > 0, \\ &v^\pm(x,0) = v_0(x). \end{aligned}$$

If the initial value $v_0(x)$ is bounded and continuous on $\mathbb{R}^{n-2}$ and satisfies $\lim_{|x|\to\infty} |v_0(x)| = 0$*, then the solutions $v^\pm(x,t)$ satisfy*

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}^{n-2}} |v^\pm(x,t)| = 0, \tag{2.6}$$

respectively. Moreover, if we further assume that $v_0 \in L^1(\mathbb{R}^{n-2})$, then we have

$$\sup_{x\in\mathbb{R}^{n-2}} |v^\pm(x,t)| \leqslant \frac{1}{k} \| \exp(kv_0) - 1\|_{L^1(\mathbb{R}^{n-2})} \cdot t^{-\frac{n-2}{2}}, \quad t > 0. \tag{2.7}$$

Next, we give the definition of the supersolution and subsolution.

Definition 2.2. A function $u^+(x,y,z,t) \in C^{2,1}(\mathbb{R}^n \times (0,\infty))$ is called a supersolution to (1.9), if it satisfies

$$\frac{\partial u^+}{\partial t} \geqslant \Delta u^+ + cu_z^+ + f(u^+), \quad (x,y,z) \in \mathbb{R}^n, \quad t > 0. \tag{2.8}$$

Similarly, we can define a subsolution $u^-(x,y,z,t)$ by reversing the inequality in (2.8).

3 Asymptotic stability under spatially decaying initial perturbations

In this section, we first show that the second derivative of the V-shaped traveling front with respect to the variable z can be dominated by its first derivative. Then we prove that the functions defined by

$$u^\pm(x,y,z,t) := V(y, z - v^\pm(x,t))$$

are a supersolution and a subsolution to (1.9) and (1.10), respectively. Finally, we give proofs of Theorems 1.1 and 1.2 and Proposition 1.3.

We begin with a lemma which shows the monotonicity of V-shaped traveling fronts.

Lemma 3.1 (See [28, Lemmas 4.3 and 4.4] and [32, Lemma 2.2])**.** *Let $V(y,z)$ be a V-shaped traveling front to* (1.4)*. Then there exist a constant γ, $\delta_0 > 0$ and a constant $R^* > 0$ such that*

$$\begin{aligned} &-V_z \geqslant \gamma > 0 \quad \textit{if} -1 + \delta_0 \leqslant V(y,z) \leqslant 1 - \delta_0, \\ &-V_z > 0, \quad (y,z) \in \mathbb{R}^2, \end{aligned}$$

and

$$\lim_{R^*\to\infty} \sup_{|z-m_*|y||\geqslant R^*} |V_z| = 0, \quad (y,z) \in \mathbb{R}^2.$$

Now, we introduce a lemma which plays a key role in constructing supersolutions and subsolutions.

Lemma 3.2. *There exists a constant $k>0$ which depends only on n, α and f such that*

$$kV_z(y,z) \leqslant V_{zz}(y,z) \leqslant -kV_z(y,z). \tag{3.1}$$

Proof. We note that the function V satisfy

$$\Delta V + cV_z + f(V) = 0, \quad (y,z) \in \mathbb{R}^2.$$

Following this equation, we have

$$\Delta V_z + c(V_z)_z + f'(V)V_z = 0, \quad (y,z) \in \mathbb{R}^2. \tag{3.2}$$

It follows from [7, Theorem 6.2] that

$$|V_z|^*_{2,\alpha;B(x,2)} \leqslant C_1 |V_z|_{0;B(x,2)}, \tag{3.3}$$

where C_1 is a constant depending on n, α and the coefficients of (3.2). Here

$$|V_z|^*_{k,\alpha;B(x,2)} := |V_z|^*_{k;B(x,2)} + [V_z]^*_{k,\alpha;B(x,2)},$$
$$|V_z|^*_{k;B(x,2)} := |V_z|^*_{k,0;B(x,2)} = \sum_{j=0}^{k} [V_z]^*_{j,\alpha;B(x,2)},$$
$$[V_z]^*_{k,\alpha;B(x,2)} := \sup_{x,y\in B(x,2),\beta=k} d^{k+\alpha}_{x,y} \frac{|D^\beta V_z(x) - D^\beta V_z(y)|}{|x-y|^\alpha}, \quad 0<\alpha\leqslant 1,$$

where

$$d_{x,y} := \min\{d_x, d_y\}$$

and

$$d_x := \operatorname{dist}(x, \partial B(x,2)), \quad d_y := \operatorname{dist}(y, \partial B(x,2)).$$

In addition, if $B(x,1) \subset\subset B(x,2)$ and $\rho = \operatorname{dist}(B(x,1), \partial B(x,2))$, then it follows from these interior norms that

$$\min\left\{1, \rho^{k+\alpha}\right\} |V_z|_{k,\alpha;B(x,1)} \leqslant |V_z|^*_{k,\alpha;B(x,2)}.$$

Thus, we have

$$|V_z|_{2,\alpha;B(x,1)} \leqslant |V_z|^*_{2,\alpha;B(x,2)}. \tag{3.4}$$

Combining (3.3) and (3.4), we get

$$|V_z|_{2,\alpha;B(x,1)} \leqslant C_1 |V_z|_{0;B(x,2)}.$$

Then the Harnack inequality implies that there exists another constant C_1' such that

$$|V_z|_{2,\alpha;B(x,1)} \leqslant -C_1 C_1' V_z,$$

because $V_z < 0$. Taking

$$k := C_1 C_1',$$

we get

$$|V_{zz}| \leqslant -kV_z,$$

which yields the conclusion of the lemma. The proof is complete. □

Next, we show that the functions $V(y, z - v^{\pm}(x,t))$ are a supersolution and a subsolution to (1.9) and (1.10), respectively. In the sequel, the signal, Δ_x and ∇_x denote the $(n-2)$-dimensional Laplacian and the $(n-2)$-dimensional gradient operator, respectively.

Lemma 3.3. *Suppose that the functions $v^+(x,t)$ and $v^-(x,t)$ are the solutions to the following problem:*

$$\frac{\partial}{\partial t}v^+(x,t) = \Delta_x v^+ + k|\nabla_x v^+|^2, \quad x \in \mathbb{R}^{n-2}, \quad t > 0, \tag{3.5}$$

$$\frac{\partial}{\partial t}v^-(x,t) = \Delta_x v^- - k|\nabla_x v^-|^2, \quad x \in \mathbb{R}^{n-2}, \quad t > 0 \tag{3.6}$$

with initial value, $v^+(x,0)$ and $v^-(x,0)$, respectively, where $k > 0$ is the constant defined in Lemma 3.2. *Let $u(x,y,z,t)$ be a solution to* (1.9) *and* (1.10) *with the initial value $u_0(x,y,z)$ satisfying*

$$V(y, z - v^-(x,0)) \leqslant u_0(x,y,z) \leqslant V(y, z - v^+(x,0)), \quad (x,y,z) \in \mathbb{R}^n.$$

Then we have

$$V(y, z - v^-(x,t)) \leqslant u(x,y,z,t) \leqslant V(y, z - v^+(x,t)), \quad (x,y,z) \in \mathbb{R}^n, \quad t \geqslant 0. \tag{3.7}$$

Proof. We only show that the former inequality of (3.7) holds, since the latter case can be proven in a similar way. In order to prove it, it needs only to show that the function $u^-(x,y,z,t) := V(y, z-v^-(x,t))$ is a subsolution from the comparison principle. Namely, it satisfies

$$L[u^-] := u^-_t - \Delta u^- - cu^-_z - f(u^-) \leqslant 0. \tag{3.8}$$

Indeed, we have

$$\begin{aligned} L[u^-] &= -v^-_t V_z - \sum_{i=1}^{n-2}(-v^-_{x_i x_i} V_z + (v_{x_i})^2 V_{zz}) - V_{yy} - V_{zz} - cV_z - f(V) \\ &= -v^-_t V_z + \Delta_x v^- V_z - |\nabla_x v^-|^2 V_{zz} \\ &= |\nabla_x v^-|^2 (kV_z - V_{zz}) \leqslant 0. \end{aligned}$$

In the second equality, we use the fact $V_{yy} + V_{zz} + cV_z + f(V) = 0$, and the last inequality is obtained by using (3.6) and Lemma 3.2. The proof is complete. □

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Define the function $v^+(x,t)$ as in Lemma 3.3. It follows that

$$u(x,y,z,t) \leqslant V(y, z - v^+(x,t)) \leqslant V(y,z) + \|V_z\|_{L^\infty(\mathbb{R}^2)} \sup_{x\in\mathbb{R}^{n-2}} |v^+(x,t)|.$$

Thus by Lemma 2.1, we have

$$u(x,y,z,t) - V(y,z) \leqslant Ct^{-\frac{n-2}{2}}.$$

Similarly, we can use Lemma 2.1 to obtain

$$u(x,y,z,t) - V(y,z) \geqslant -Ct^{-\frac{n-2}{2}}.$$

Combining these two inequalities, we prove Theorem 1.2. The proof is complete. □

Now we prove Proposition 1.3.

Proof of Proposition 1.3. We only consider the case where $v_0 \geqslant 0$, $v_0 \not\equiv 0$, since the other case can be discussed in a similar way. By Theorem 1.2 and (3.7), it needs only to show that the solution $v(x,t)$ to the problem

$$v_t = \Delta v - k|\nabla_x v|^2, \quad x \in \mathbb{R}^{n-2}, \quad t > 0, \tag{3.9}$$

$$v(x,0) = v_0(x), \quad x \in \mathbb{R}^{n-2} \tag{3.10}$$

satisfies $v(0,t) \geqslant C(1+t)^{-(n-2)/2}$ for some constant $C>0$. Indeed, the first inequality of (3.7) yields that

$$\begin{aligned} u(0,0,0,t) &\geqslant V(0,-v(0,t)) \\ &\geqslant V(0,0)+\min_{z\in[-\|v\|_{L^\infty(\mathbb{R}^{n-2})},0]}|V_z(0,z)|\cdot v(0,t) \\ &\geqslant V(0,0)+C'(1+t)^{-\frac{n-2}{2}}, \quad t\geqslant 0. \end{aligned}$$

Similar to (2.5), we can obtain that the explicit expression of the solution to (3.9) and (3.10) is given by

$$v(x,t)=-\frac{1}{k}\ln\bigg(\int_{\mathbb{R}^{n-2}}\Gamma(x-\eta,t)\exp(-kv_0(\eta))d\eta\bigg).$$

Since $v_0\geqslant 0$, $v_0\not\equiv 0$, there exist a constant $\delta>0$ and a nonempty open set $D\subset\mathbb{R}^{n-2}$ such that $v_0\geqslant\delta$ for $x\in D$. Thus, we have

$$\begin{aligned} v(x,t) &\geqslant -\frac{1}{k}\ln\bigg(1-\int_D\Gamma(x-\eta,t)(1-\exp(-k\delta))d\eta\bigg) \\ &\geqslant -\frac{1}{k}\ln\Big(1-|D|(1-\exp(-k\delta))\cdot\min_{\eta\in D}\Gamma(x-\eta,t)\Big) \\ &\geqslant \frac{|D|}{k}(1-\exp(-k\delta))\cdot\min_{\eta\in D}\Gamma(x-\eta,t), \end{aligned}$$

which implies $v(0,t)\geqslant C'(1+t)^{-(n-2)/2}$. This completes the proof. □

In order to prove Theorem 1.1, we construct some new types of supersolutions and subsolutions.

Lemma 3.4. *Let $k>0$ be defined as in Lemma 3.2. Then there exist some constants $\delta_0>0$, $\beta>0$ and $\sigma\geqslant 1$ such that, for any $\delta\in(0,\delta_0]$ and any functions $v^\pm(x,t)$ satisfying*

$$v_t^\pm=\Delta_x v^\pm\pm k|\nabla_x v^\pm|^2,$$

the functions defined by

$$u^\pm(x,y,z,t):=V\left(y,z-v^\pm(x,t)\mp\sigma\delta\left(1-\mathrm{e}^{-\beta t}\right)\right)\pm\delta\mathrm{e}^{-\beta t} \tag{3.11}$$

are a supersolution and a subsolution to (1.9) *and* (1.10), *respectively.*

Proof. Indeed, we have

$$\begin{aligned} L[u^+] &= -v_t^+V_z-\sigma\delta\beta\mathrm{e}^{-\beta t}V_z-\delta\beta\mathrm{e}^{-\beta t}+(\Delta_x v^+)V_z \\ &\quad -|\nabla_x v^+|^2V_{zz}-V_{yy}-V_{zz}-cV_z-f(V+\delta\mathrm{e}^{-\beta t}) \\ &= (-v_t^++\Delta_x v^+)V_z-|\nabla_x v^+|^2V_{zz}-\sigma\delta\beta\mathrm{e}^{-\beta t}V_z \\ &\quad -\delta\beta\mathrm{e}^{-\beta t}-f(V+\delta\mathrm{e}^{-\beta t})+f(V) \\ &\geqslant (-kV_z-V_{zz})|\nabla_x v^+|^2-\sigma\delta\beta\mathrm{e}^{-\beta t}V_z-\delta\beta\mathrm{e}^{-\beta t} \\ &\quad -\delta\mathrm{e}^{-\beta t}\bigg(\int_0^1 f'(V+\tau\delta\mathrm{e}^{-\beta t})d\tau\bigg) \\ &\geqslant \delta\mathrm{e}^{-\beta t}\bigg(-\sigma\beta V_z-\beta-\int_0^1 f'(V+\tau\delta\mathrm{e}^{-\beta t})d\tau\bigg). \end{aligned}$$

From (F1) and (F3), there exist some constants $k_1>0$ and $\delta_0>0$ such that

$$-f'(s)\geqslant k_1>0, \quad s\in[-1-2\delta_0,-1+2\delta_0]\cup[1-2\delta_0,1+2\delta_0]. \tag{3.12}$$

For $-1+\delta_0\leqslant V(y,\xi(t))\leqslant 1-\delta_0$, where $\xi(t):=z-v^+(x,t)-\sigma\delta(1-\mathrm{e}^{-\beta t})$, by Lemma 3.1 we have

$$-\sigma\beta V_z-\beta-\int_0^1 f'(V+\tau\delta\mathrm{e}^{-\beta t})d\tau\geqslant\beta\bigg(\gamma\sigma-1-\frac{M}{\beta}\bigg),$$

where

$$M := \max_{-2\leqslant s\leqslant 2} f'(s). \tag{3.13}$$

For $V < -1 + \delta_0$ or $V > 1 - \delta_0$, it follows from Lemma 3.1 and (3.12) that

$$-\sigma\beta V_z - \beta - \int_0^1 f'\left(V + \tau\delta e^{-\beta t}\right) d\tau \geqslant k_1 - \beta.$$

Taking β small and σ large as

$$0 < \beta < k_1, \quad \sigma > \frac{\beta + M}{\beta\gamma},$$

we get $L[u^+] \geqslant 0$. Namely, $u^+(x,y,z,t)$ is a supersolution to (1.9). Similarly, we can prove that $u^-(x,y,z,t)$ is a subsolution of (1.9). This completes the proof. □

To prove Theorem 1.1, we need another auxiliary lemma.

Lemma 3.5. *Suppose that the initial value u_0 satisfies*

$$\lim_{R\to\infty} \sup_{|x|+|y|+|z|\geqslant R} |u_0(x,y,z) - V(y,z)| = 0. \tag{3.14}$$

Then, for any fixed $T > 0$, the solution to (1.9) *and* (1.10) *satisfies*

$$\lim_{R\to\infty} \sup_{|x|+|y|+|z|\geqslant R} |u(x,y,z,T) - V(y,z)| = 0.$$

Proof. Define a function $w(x,y,z,t)$ as

$$w(x,y,z,t) := u(x,y,z,t) - V(y,z).$$

Then $w(x,y,z,t)$ is the solution to the following Cauchy problem:

$$\begin{aligned} &w_t = \Delta w + cw_z + f'(V+\theta w)w, \quad x \in \mathbb{R}^{n-2}, \quad y \in \mathbb{R}, \quad z \in \mathbb{R}, \quad t > 0, \\ &w(x,y,z,0) = u_0(x,y,z) - V(y,z), \quad x \in \mathbb{R}^{n-2}, \quad y \in \mathbb{R}, \quad z \in \mathbb{R}, \end{aligned} \tag{3.15}$$

where $\theta(x,y,z,t)$ is a function that satisfies $0 \leqslant \theta(x,y,z,t) \leqslant 1$. In order to prove this lemma, it suffices to consider the case where $w(x,y,z,0) \geqslant 0$ and the case where $w(x,y,z,0) \leqslant 0$. In the sequel, we assume that $w(x,y,z,0) \geqslant 0$, since the other cases can be treated in the same way.

By the assumption (F3), it is easy to find that there exists a constant $K > 0$ such that $u(x,y,z,t) \leqslant K$ for all $(x,y,z) \in \mathbb{R}^n, t > 0$. Consequently, we have that there exists a positive constant N such that

$$|V + \theta(x,y,z,t)w(x,y,z,t)| \leqslant N, \quad x \in \mathbb{R}^{n-2}, \quad y \in \mathbb{R}, \quad z \in \mathbb{R}, \quad t > 0.$$

The maximum principle implies that $w(x,y,z,t) \geqslant 0$ since $w(x,y,z,0) \geqslant 0$. By the assumptions (F1) and (F3), and the boundedness of $V + \theta w$, there exists a constant $M_1 > 0$ satisfying

$$w_t = \Delta w + cw_z + f'(V+\theta w)w \leqslant \Delta w + cw_z + M_1 w.$$

Then we have the estimate

$$w(x,y,z,t) \leqslant e^{M_1 t} \int_{\mathbb{R}^{n-2}} \int_{\mathbb{R}} \int_{\mathbb{R}} Z(x,y,z,\xi,\eta,\zeta,t)(u_0(\xi,\eta,\zeta) - V(\eta,\zeta)) d\xi d\eta d\zeta,$$

where

$$Z(x,y,z,\xi,\eta,\zeta,t) := \frac{1}{(4\pi t)^{-\frac{n}{2}}} \exp\left(-\frac{|x-\xi|^2 + |y-\eta|^2 + |z-\zeta-ct|^2}{4t}\right).$$

It follows from (3.14) that, for any fixed $\varepsilon > 0$, there exists $\overline{R}$ such that

$$\sup_{|x|+|y|+|z|\geqslant R} |u_0(x,y,z) - V(y,z)| < \frac{\varepsilon}{2} e^{-M_1 T} \tag{3.16}$$

for $R > \overline{R}$. Direct calculations show that

$$\begin{aligned}
&\int_{\mathbb{R}^{n-2}}\int_{\mathbb{R}}\int_{\mathbb{R}} Z(x,y,z,\xi,\eta,\zeta,t)(u_0(\xi,\eta,\zeta)-V(\eta,\zeta))d\xi d\eta d\zeta\\
&\quad=\int_{|\xi|+|\eta|+|\zeta|\geqslant\overline{R}} Z(x,y,z,\xi,\eta,\zeta,t)(u_0(\xi,\eta,\zeta)-V(\eta,\zeta))d\xi d\eta d\zeta\\
&\qquad+\int_{|\xi|+|\eta|+|\zeta|\leqslant\overline{R}} Z(x,y,z,\xi,\eta,\zeta,t)(u_0(\xi,\eta,\zeta)-V(\eta,\zeta))d\xi d\eta d\zeta\\
&\quad:= I_1+I_2.
\end{aligned}$$

Clearly, $I_1 \leqslant \frac{1}{2}\mathrm{e}^{-M_1T}\varepsilon$ because of (3.16). On the other hand, since

$$\lim_{R\to\infty}\sup_{|x|+|y|+|z|\geqslant R}\exp\left(-\frac{|x-\xi|^2+|y-\eta|^2+|z-\zeta-cT|^2}{4T}\right)=0$$

for any $|\xi|,|\eta|,|\zeta|\leqslant\overline{R}$, there exists a constant $\hat{R}>0$ such that $I_2\leqslant\frac{1}{2}\mathrm{e}^{-M_1T}\varepsilon$ for $|x|+|y|+|z|\geqslant\hat{R}$. Therefore, for $R\geqslant\max\{\hat{R},\overline{R}\}$, we have

$$\sup_{|x|+|y|+|z|\geqslant R}|w(x,y,z,T)|\leqslant \mathrm{e}^{M_1T}(I_1+I_2)\leqslant\varepsilon,$$

which implies that

$$\lim_{R\to\infty}\sup_{|x|+|y|+|z|\geqslant R}|w(x,y,z,T)|=0 \quad \text{for any fixed} \quad T>0.$$

The proof is complete. □

Proof of Theorem 1.1. We only show the upper estimate, since the lower estimate can be proven similarly. Take constants $k>0$ as in Lemma 3.2 and $\sigma\geqslant1$ as in Lemma 3.4. Set constants $\varepsilon>0$ and $\hat{\varepsilon}=\varepsilon/(2\|V_z\|_{L^\infty(\mathbb{R}^2)}+1)$. Since $f(s)<0$ for $s>1$ by (F3), there exists a constant $T_1\geqslant0$ such that

$$u(x,y,z,T_1)\leqslant 1+\frac{\hat{\varepsilon}}{2\sigma},\quad (x,y,z)\in\mathbb{R}^n.$$

Furthermore, Lemma 3.5 implies that there exists a constant $R>0$ such that

$$\sup_{|x|+|y|+|z|\geqslant R}|u(x,y,z,T_1)-V(y,z)|\leqslant\frac{\hat{\varepsilon}}{\sigma},\quad (x,y,z)\in\mathbb{R}^n.$$

Then we can choose a function $v_0(x)\geqslant0$ satisfying $\lim_{|x|\to\infty}v_0(x)=0$ and

$$u(x,y,z,T_1)\leqslant V(y,z-v_0(x))+\frac{\hat{\varepsilon}}{\sigma},\quad (x,y,z)\in\mathbb{R}^n.$$

Let $v(x,t)$ be the solution to the following equation:

$$\begin{aligned}
&v_t=\Delta_x v+k|\nabla_x v|^2,\quad x\in\mathbb{R}^{n-2},\quad t>0,\\
&v(x,0)=v_0(x),\quad x\in\mathbb{R}^{n-2}.
\end{aligned}$$

Then Lemma 2.1 implies that there exists a constant $T_2>0$ satisfying $v(x,t)\leqslant\hat{\varepsilon}$ for $x\in\mathbb{R}^{n-2}$ and $t\geqslant T_2$. Consequently, by using the comparison principle and the supersolution constructed in Lemma 3.4, we have

$$\begin{aligned}
u(x,y,z,t)&\leqslant V(y,z-v(x,t-T_1)-\hat{\varepsilon}(1-\mathrm{e}^{-\beta(t-T_1)}))+\frac{\hat{\varepsilon}}{\sigma}\mathrm{e}^{-\beta(t-T_1)}\\
&\leqslant V(y,z-2\hat{\varepsilon})+\hat{\varepsilon}\\
&\leqslant V(y,z)+(2\|V_z\|_{L^\infty(\mathbb{R}^2)}+1)\hat{\varepsilon}\\
&\leqslant V(y,z)+\varepsilon
\end{aligned}$$

for $t\geqslant T_1+T_2$. The proof is complete. □

4 Permanent oscillating solutions

In this section, we show that the V-shaped traveling fronts are not asymptotically stable under general bounded perturbations. In fact, we prove that even very small perturbations to the V-shaped traveling fronts can lead to permanent oscillation.

By combining Lemma 3.3 and the following Lemma 4.1, we construct a sequence of supersolutions and a sequence of subsolutions that push the solution back and forth in the z-direction, thus forcing the solution to oscillate permanently with non-decaying amplitude.

Lemma 4.1 (See [24, Lemmas 3.1 and 3.2]). *Let $k>0$ be defined as in Lemma 3.2 and $v^{\pm}(x,t)$ be solutions to the following Cauchy problem*:

$$\begin{aligned} &v_t^{\pm} = v_{xx}^{\pm} \pm k v^{\pm}, \quad x\in\mathbb{R}, \quad t>0,\\ &v^{\pm}(x,0) = v_0^{\pm}(x), \quad x\in\mathbb{R}, \end{aligned}$$

respectively. Suppose that initial values $v_0^{\pm}(x)$ are all bounded functions on $\mathbb{R}$ and satisfy

$$\begin{aligned} &v_0^{+}(x) \leqslant \delta, \quad x\in\mathbb{R},\\ &v_0^{+}(x) \leqslant -\delta, \quad |x|\in[m!+1,(m+1)!-1] \end{aligned}$$

and

$$\begin{aligned} &v_0^{-}(x) \geqslant -\delta, \quad x\in\mathbb{R},\\ &v_0^{-}(x) \geqslant \delta, \quad |x|\in[m!+1,(m+1)!-1] \end{aligned}$$

for some constant $\delta>0$ and some integer $m\geqslant 2$, respectively. Then there exists a constant $C>0$ which only depends on δ and k such that solutions $v^{\pm}(x,t)$ satisfying

$$\sup_{|x|\leqslant m!-1} v^{+}(x,T) \leqslant -\delta + C\int_{|\zeta|\in[0,2/\sqrt{m}]\cup[\sqrt{m},\infty]} \mathrm{e}^{-\zeta^2} d\zeta$$

and

$$\sup_{|x|\leqslant m!-1} v^{-}(x,T) \geqslant \delta - C\int_{|\zeta|\in[0,2/\sqrt{m}]\cup[\sqrt{m},\infty]} \mathrm{e}^{-\zeta^2} d\zeta,$$

respectively, where $T=m(m!)^2/4$.

Proof of Theorem 1.5. Let

$$I_m = [m!+1,(m+1)!-1], \quad \tilde{I}_m = [0,m!]\cup[(m+1)!,\infty].$$

Define two sequences of smooth functions $\{v_{0,i}^{\pm}(x)\}_{i=1,2,\ldots}$ such that

$$|v_{0,i}^{+}(x)| \leqslant \delta, \quad x\in\mathbb{R} \quad \text{and} \quad v_{0,i}^{+}(x) = \begin{cases} -\delta, & |x|\in I_{2i},\\ \delta, & |x|\in\tilde{I}_{2i} \end{cases}$$

and

$$|v_{0,i}^{-}(x)| \leqslant \delta, \quad x\in\mathbb{R} \quad \text{and} \quad v_{0,i}^{-}(x) = \begin{cases} \delta, & |x|\in I_{2i+1},\\ -\delta, & |x|\in\tilde{I}_{2i+1}, \end{cases}$$

respectively. Now we can choose a function $v_0^*(x)\in C^{\infty}(\mathbb{R})$ satisfying

$$v_{0,i}^{-}(x) \leqslant v_0^*(x) \leqslant v_{0,i}^{+}(x) \quad \text{for all} \quad i\geqslant 1.$$

Let $u^*(x,y,z,t)$ be the solution to (1.9) and (1.10) with $u^*(x,y,z,0)=V(y,z-v_0^*(x))$ and $v_i^{+}(x,t)$ be the solution to the following problem:

$$v_{i,t}^{+} = v_{i,xx}^{+} + k(v_{i,x}^{+})^2, \quad x\in\mathbb{R}, \quad t>0,$$

$$v_i^+(x,0) = v_{0,i}^+(x), \quad x \in \mathbb{R}.$$

By the definition of the function $v_0^*(x)$, we have

$$V(y, z+\delta) \leqslant V(y, z - v_0^*(x)) \leqslant V(y, z - v_{0,i}^+(x)) \leqslant V(y, z-\delta).$$

Thus Lemma 3.3 yields that

$$V(y, z+\delta) \leqslant u^*(x,y,z,t) \leqslant V(y, z - v_i^+(x,t)) \leqslant V(y, z-\delta).$$

It follows from Lemma 4.1 that

$$\begin{aligned} V(y,z+\delta) &\leqslant \sup_{|x|\leqslant (2i)!-1} u^*(x,y,z,t_{2i}) \\ &\leqslant \sup_{|x|\leqslant (2i)!-1} V(y, z - v_i^+(x, t_{2i})) \\ &\leqslant V(y,z+\delta) + \|V_z\|_{L^\infty(\mathbb{R}^2)} \cdot C \int_{|\zeta|\in[0,2/\sqrt{2i}]\cup[\sqrt{2i},\infty]} \mathrm{e}^{-\zeta^2} d\zeta, \end{aligned}$$

where $t_{2i} = (2i)((2i)!)^2/4$. This implies that

$$\lim_{i\to\infty} \sup_{(y,z)\in\mathbb{R}^2} \sup_{|x|\leqslant (2i)!-1} |u^*(x,y,z,t) - V(y,z+\delta)| = 0. \tag{4.1}$$

On the other hand, by using Lemma 4.1 again for v^- and the inequality $v_0^*(x) \geqslant v_{0,i}^-(x)$ for $i = 1, 2, \ldots,$ we have

$$\begin{aligned} V(y,z-\delta) &\geqslant \sup_{|x|\leqslant (2i+1)!-1} u^*(x,y,z,t_{2i+1}) \\ &\geqslant \sup_{|x|\leqslant (2i+1)!-1} V(y, z - v_i^-(x, t_{2i+1})) \\ &\geqslant V(y,z-\delta) - \|V_z\|_{L^\infty(\mathbb{R}^2)} \cdot C \int_{|\zeta|\in[0,2/\sqrt{2i+1}]\cup[\sqrt{2i+1},\infty]} \mathrm{e}^{-\zeta^2} d\zeta, \end{aligned}$$

where $t_{2i+1} = (2i+1)((2i+1)!)^2/4$. This yields that

$$\lim_{i\to\infty} \sup_{z\in\mathbb{R}} \sup_{|x|\leqslant (2i+1)!-1} |u^*(x,y,z,t) - V(y,z-\delta)| = 0. \tag{4.2}$$

The conclusion of Theorem 1.5 then obviously follows from (4.1) and (4.2). The proof is complete. □

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