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The critical case for a Berestycki-Lions theorem

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Abstract We consider the existence of the ground states solutions to the following Schrödinger equation:

$$-\Delta u + V(x)u = f(u), \quad u \in H^1(\mathbb{R}^N),$$

where $N \geqslant 3$ and f has critical growth. We generalize an earlier theorem due to Berestycki and Lions about the subcritical case to the current critical case.

Keywords meromorphic function, normal family, the sequence of omitted functions

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1 Introduction

In this paper, we are concerned with the following problem:

$$-\Delta u + V(x)u = f(u), \quad u \in H^1(\mathbb{R}^N), \tag{1.1}$$

where $N \geqslant 3$. Such equations arise in various branches of mathematical physics and they have been the subject of extensive study in recent years. We recall that u is said to be a ground state of (1.1) if and only if u solves (1.1) and minimizes the functional associated with (1.1) among all possible nontrivial solutions. The problem of finding such a type of solutions is a very classical problem. It has been introduced in [7]. Later, in the celebrated papers [5,6], almost necessary and sufficient conditions for the existence of ground states to the problem

$$-\Delta u = h(u), \quad u \in H^1(\mathbb{R}^N) \tag{1.2}$$

are given by Berestycki and Lions [6] when $N \geqslant 3$ and Berestycki et al. [5] when $N = 2$. In [6], Berestycki and Lions assumed that the following conditions hold for h:

(H_1) $h(s) \in C(\mathbb{R}, \mathbb{R})$ is odd;

(H_2) $-\infty < \liminf_{s\to 0} \frac{h(s)}{s} \leqslant \limsup_{s\to 0} \frac{h(s)}{s} = -a < 0$ for $N \geqslant 3$;

(H_3) When $N \geqslant 3$, $\limsup_{s\to\infty} \frac{h(s)}{|s|^{\frac{N+2}{N-2}}} \leqslant 0$;

(H_4) There exists $\xi_0 > 0$ such that $H(\xi_0) = \int_0^{\xi_0} h(s)ds > 0$.

Under the above assumptions, they showed that problem (1.2) has a radial ground state.

Theorem (See [6]). *Assume that* (H_1)–(H_4) *hold. Then the problem* (1.1) *admits a radial ground state.*

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When $N = 2$, a theorem due to Berestycki et al. can be seen in [5]. The nonlinearity $h(s)$ is independent of $x \in \mathbb{R}^N$ in both papers [5,6], which makes it possible to work in the space $H_r^1(\mathbb{R}^N)$ of radial symmetric functions. More important, the imbedding $H_r^1(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ is compact for $p \in (2, 2^*)$. Note that (H_3) characterizes the problem to be subcritical.

Now the problem arises: what happens when $h(s) = h(x, s)$ is of critical growth and depending on x non-radially? Essentially, this open problem has not been completely solved so far by variational methods. In this paper, and try to complete the study for this question, we consider the case

$$h = h(x, u) = -V(x)u + f(u).$$

Firstly, we assume that the potential $V(x)$ satisfies the following hypotheses:

(V_1) $V(x) \in C(\mathbb{R}^N, \mathbb{R})$;

(V_2) there exists $V_0 > 0$ such that $\inf_{x\in\mathbb{R}^N} V(x) \geqslant V_0$;

(V_3) $V(x) \leqslant V_\infty := \lim_{|x|\to\infty} V(x) < \infty$ for all $x \in \mathbb{R}^N$;

(V_4) there exists a function $\phi \in L^2(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$ such that $|x||\nabla V(x)| \leqslant \phi^2(x)$, $\forall\ x \in \mathbb{R}^N$.

The nonlinear term $f(t)$ verifies the following conditions:

(f_1) $f \in C(\mathbb{R}^+, \mathbb{R})$;

(f_2) $f(t) = o(t)$ as $t \to 0_+$;

(f_3) $\lim_{t\to+\infty} \frac{f(t)}{t^{\frac{N+2}{N-2}}} = K > 0$;

(f_4) there exist $D > 0$ and $2 < q < 2^*$ such that $f(t) \geqslant Kt^{\frac{N+2}{N-2}} + Dt^{q-1}$ for $t \geqslant 0$, where $2^* = \frac{2N}{N-2}$;

(f_5) $f \in C^1(\mathbb{R}^+, \mathbb{R})$, $|f'(t)| \leqslant C(1 + |t|^{\frac{4}{N-2}})$ for $t \geqslant 0$ and some $C > 0$.

The first result of the present paper is about the case of constant potential $V \equiv$ constant (which satisfies (V_1)–(V_4) automatically), which plays an important role for studying the case of critical-nonradial case. For this particular case, some hypotheses above may be dropped.

Theorem 1.1. *Assume $N = 3$ with $q > 4$, or $N \geqslant 4$. If $V(x) \equiv V > 0$ and (f_1)–(f_4) hold, then the problem (1.1) has a ground state.*

This theorem can be regarded as a form of generalization of the Berestycki-Lions theorem to the critical case. The ideas for proving Theorem 1.1 will play a key role for the following main theorem of this paper, which concerns with the non-radial potential V and critical nonlinear term f:

Theorem 1.2. *Assume $N = 3$ with $q > 4$, or $N \geqslant 4$. If (V_1)–(V_4) and (f_1)–(f_5) hold, then the problem (1.1) has a ground state.*

Theorem 1.2 can be regarded as a form of generalization of the Berestycki-Lions theorem to the critical and non-radial case. The conditions (f_3) and (f_4) characterize the equation (1.1) to be the critical growth case. The condition (f_4) plays an important role to ensure the existence of ground states. Without (f_4), Theorem 1.1 may not hold. For example, if we consider $f(t) = (t^+)^{2^*-1}$, where $t^+ = \max\{t, 0\}$, then $f(t)$ satisfies (f_1)–(f_3). The Pohozăev type identity implies that there exists no nontrivial solution to (1.1). However, (f_5) is a technical condition. Actually, (f_5) is not necessary when $V(x) \equiv V > 0$. In particular, we do not need the following (super) quadratic condition:

$$\mu \int_0^t f(s)ds \leqslant tf(t) \quad \text{for some } \mu \geqslant 2 \text{ and } \forall\, t \geqslant 0.$$

Recently, in [1], the authors made an attempt to complete the study made in [5,6] by considering a class of nonlinearities with critical growth in $\mathbb{R}^N (N \geqslant 2)$. More precisely, for $N \geqslant 3$, they studied the problem (1.1) with $V(x) \equiv V > 0$ and assumed the following hypotheses on f:

(G_1) $f \in C(\mathbb{R}, \mathbb{R})$;

(G_2) $f(t) = o(t)$ as $t \to 0_+$;

(G_3) $\lim_{t\to+\infty} \frac{f(t)}{t^{\frac{N+2}{N-2}}} \leqslant 1$;

(G_4) $2\int_0^t f(s)ds \leqslant tf(t)$ for $t \geqslant 0$;

(G_5) there exist $\lambda > 0$ and $2 < p < 2^*$ such that $f(t) \geqslant \lambda t^{p-1}$ for $t \geqslant 0$.

The existence of a ground state to (1.1) was obtained under the above conditions with λ sufficiently large.

We should mention that in our paper [17], we remove the condition (G_4). Moreover, the restriction for λ in (G_5) is weakened quite a lot. However, for small $\lambda > 0$, it is still unknown whether problem (1.1) has a ground state. In our Theorem 1.1, instead of (G_5), we use a different condition (f_4), imposing no restriction on $D > 0$. Observe that it is not sure whether (f_4) is a stronger condition than (G_5) with λ sufficiently large. The condition (f_4) is first introduced in [18], where the authors assumed $V(x) \equiv V > 0$, $f \in C(\mathbb{R}, \mathbb{R})$ is odd and satisfies the conditions (f_2)–(f_4). The existence of a radial ground state to (1.1) was established for $N = 3$ with $q > 4$, or $N \geqslant 4$. Compared with [18], in our Theorem 1.1, we use a different method, imposing no radial restrictions. In fact, a global compactness theorem in the critical case is established, which plays a crucial role in Theorems 1.1 and 1.2. We believe that the global compactness theorem is important and can be used in similar problems. It is interesting to know whether the ground state in Theorem 1.1 and the radial ground state obtained in [18] are the same. However, we cannot answer the question now.

We would like to say a few words on the subcritical growth case. In the work of [9], the authors established the existence of a positive solution for an asymptotically linear elliptic problem on $\mathbb{R}^N$. We also mention the article [12] where the authors considered a more general Schrödinger equation

$$-\Delta u + V(x)u = f(x, u), \quad u \in H^1(\mathbb{R}^N). \tag{1.3}$$

By using the Nehari manifold method, a more natural super-quadratic condition was considered and the existence of a ground state solution was obtained. For other related results, we refer the reader to [2–4, 8, 14] and the references therein. In particular, the paper [11] considered the problem (1.1) for $N \geqslant 2$. In [11], they imposed the similar assumptions to (V_1)–(V_4) above, but moreover, $f'(0) < \inf \sigma(-\Delta + V(x))$, where $\sigma(-\Delta + V(x))$ denotes the spectrum of the self-adjoint operator $-\Delta + V(x) : H^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$. The nonlinear term f is of subcritical. They showed that under the above assumptions, problem (1.1) has a ground state. The main obstacle in treating this class of Schrödinger equations is the boundedness of the Palais-Smale sequence because no global condition is required on f. To overcome the difficulty, the authors introduced the condition (V_4). Then the desired result was obtained by applying an indirect approach developed in [10]. In this paper, we complete [11]'s work in the critical growth case for $N \geqslant 3$.

We note that the functional associated with (1.2) is

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} H(u) dx,$$

where $u \in H^1(\mathbb{R}^N)$ and $H(u) = \int_0^u h(s)ds$. Then a natural method to solve (1.2) is to look for critical points of the functional J on $H^1(\mathbb{R}^N)$ directly. However, general assumptions imposed on the nonlinearity bring on the obstacle in proving the boundedness of the Palais-Smale sequences. Moreover, the lack of compactness due to the unboundedness of the domain prevents us from checking the Palais-Smale condition. To avoid the difficulties mentioned above, in [6], the authors investigated the constraint minimization problem

$$\min\left\{\frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx : u \in H^1(\mathbb{R}^N), \int_{\mathbb{R}^N} H(u)dx = 1\right\}, \tag{1.4}$$

where $N \geqslant 3$. The Schwarz symmetrization allows to work in the space $H_r^1(\mathbb{R}^N)$, where the compact embedding holds. By a change of scale, the minimum of (1.4) gives rise to the existence of a radial ground state for the problem (1.2). The critical exponential growth makes the problem on $\mathbb{R}^N$ ($N \geqslant 3$) more complicated due to the loss of the compactness for the embedding $H_r^1(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$. To overcome the difficulties, the authors in [1, 18] obtained the existence of ground states by modifying the minimization methods with constraints used in [6]. It should be noted that in [1, 6, 18], the radial symmetry plays an essential role. Therefore, their methods are invalid for the non-radial case. In this paper, we use a different approach to deal with the critical problem, which has the advantages of treating radial and non-radial cases in a unified frame work.

The outline of this paper is as follows: in Section 2, we establish some important lemmas. In Section 3, we employ an indirect approach to prove Theorem 1.1. In Section 4, we prove Theorem 1.2.

Notation: • C denotes a universal positive constant.
• $B_r(x_0)$ denotes the open ball centered at x_0 with radius $r > 0$.
• S denotes the best Sobolev constant:

$$S := \inf_{u \in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{(\int_{\mathbb{R}^N} |u|^{2^*} dx)^{\frac{2}{2^*}}}.$$

2 Preliminary lemmas

Because we look for positive solutions, we may assume that $f(t) = 0$ for all $t \leqslant 0$. Let $H = \{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u|^2 dx < \infty\}$ be the Hilbert space equipped with the norm $\|u\|^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2)dx$. The functional associated with (1.1) is

$$I(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(u)dx, \tag{2.1}$$

where $u \in H$, $F(u) = \int_0^u f(t)dt$. The conditions (f_1)–(f_3) imply that the functional $I : H \mapsto \mathbb{R}$ is of class C^1. Moreover, the critical points of I are weak solutions to (1.1). For simplicity, we may assume that $K = 1$. Set $g(t) = f(t) - (t^+)^{\frac{N+2}{N-2}}$, where $t^+ = \max\{t, 0\}$. Then

$$I(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} G(u)dx - \frac{1}{2^*}\int_{\mathbb{R}^N} (u^+)^{2^*} dx,$$

where $u \in H$, $G(u) = \int_0^u g(t)dt$. For $\lambda \in [\frac{1}{2}, 1]$, we consider the family of functionals $I_\lambda : H \mapsto \mathbb{R}$ defined by

$$I_\lambda(u) = \frac{1}{2}\|u\|^2 - \lambda\int_{\mathbb{R}^N} F(u)dx. \tag{2.2}$$

The following abstract result established in [10] will be needed.

Theorem 2.1. *Let X be a Banach space equipped with a norm $\|\cdot\|_X$ and let $J \subset \mathbb{R}^+$ be an interval. We consider a family $(I_\lambda)_{\lambda\in J}$ of C^1-functionals on X of the form*

$$I_\lambda(u) = A(u) - \lambda B(u), \quad \forall\, \lambda \in J,$$

where $B(u) \geqslant 0$, $\forall u \in X$, and either $A(u) \to +\infty$ or $B(u) \to +\infty$ as $\|u\|_X \to \infty$. We assume there are two points v_1, v_2 in X such that

$$c_\lambda = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\lambda(\Gamma(t)) > \max\{I_\lambda(v_1), I_\lambda(v_2)\}, \quad \forall\ \lambda \in J,$$

where $\Gamma = \{\gamma \in C([0,1], X); \gamma(0) = v_1, \gamma(1) = v_2\}$. Then, for almost every $\lambda \in J$, there is a sequence $\{v_n\} \subset X$ such that
(i) *$\{v_n\}$ is bounded,*
(ii) *$I_\lambda(v_n) \to c_\lambda$,*
(iii) *$I'_\lambda(v_n) \to 0$ in X^{-1}.*
Moreover, the map $\lambda \to c_\lambda$ is continuous from the left-hand side.

Now, we give a lemma which will be used later.

Lemma 2.2. *Assume* (V_1)–(V_2) *and* (f_1)–(f_4). *Then the conclusions of Theorem* 2.1 *hold. Moreover, if $V(x) \in L^\infty(\mathbb{R}^N)$, then for $N = 3$ with $q > 4$, or $N \geqslant 4$,*

$$c_\lambda < \frac{1}{N}\frac{S^{\frac{N}{2}}}{\lambda^{\frac{N-2}{2}}}. \tag{2.3}$$

Proof. Set $X = H$, $\|\cdot\|_X = \|\cdot\|$, $J = [\frac{1}{2}, 1]$, $A(u) = \frac{1}{2}\|u\|^2$ and $B(u) = \int_{\mathbb{R}^N} F(u)dx$ in Theorem 2.1. It is easy to see that $B(u) \geqslant 0$, $\forall\ u \in H$ and $A(u) \to +\infty$ as $\|u\| \to \infty$. On the other hand, the conditions

(f_1)–(f_3) imply that $\forall\ \varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that $|F(u)| \leqslant \varepsilon|u|^2 + C(\varepsilon)|u|^{2^*}$. Hence, there exists $r > 0$ such that for $\|u\| = r$, $I_\lambda(u) \geqslant \alpha > 0$, where r, α are independent of λ. From (f_4),

$$I_\lambda(u) \leqslant \frac{1}{2}\|u\|^2 - \frac{1}{2}\frac{1}{2^*}\int_{\mathbb{R}^N}(u^+)^{2^*}dx - \frac{D}{2q}\int_{\mathbb{R}^N}(u^+)^q dx.$$

Set $\varphi \in H$ such that $\varphi \geqslant 0$, $\varphi \neq 0$. Then $\lim_{t\to+\infty} I_\lambda(t\varphi) = -\infty$. Thus, there exists $t_0 > 0$ such that $\|t_0\varphi\| > r$ and $I_\lambda(t_0\varphi) < 0$ for all $\lambda \in [\frac{1}{2}, 1]$. We also have $I_\lambda(0) = 0$. Set $v_1 = 0$, $v_2 = t_0\varphi$. Then the conclusions of Theorem 2.1 hold. For $\varepsilon,\ r > 0$, define

$$u_\varepsilon(x) = \frac{\psi(x)\varepsilon^{\frac{N-2}{4}}}{(\varepsilon + |x|^2)^{\frac{N-2}{2}}}, \tag{2.4}$$

where $\psi \in C_0^\infty(B_{2r}(0))$ such that $0 \leqslant \psi(x) \leqslant 1$ and $\psi(x) = 1$ on $B_r(0)$. It is well known that S is attained by the functions $\frac{\varepsilon^{\frac{N-2}{4}}}{(\varepsilon+|x|^2)^{\frac{N-2}{2}}}$. A direct calculation can derive that

$$\int_{\mathbb{R}^N}|\nabla u_\varepsilon|^2 dx = (N-2)^2\int_{\mathbb{R}^N}\frac{|x|^2}{(1+|x|^2)^N}dx + O(\varepsilon^{\frac{N-2}{2}}) =: K_1 + O(\varepsilon^{\frac{N-2}{2}}), \tag{2.5}$$

$$\int_{\mathbb{R}^N}|u_\varepsilon|^{2^*}dx = \int_{\mathbb{R}^N}\frac{1}{(1+|x|^2)^N}dx =: K_2 + O(\varepsilon^{\frac{N}{2}}), \tag{2.6}$$

$$\int_{\mathbb{R}^N}|u_\varepsilon|^t dx = \begin{cases} K\varepsilon^{\frac{2N-(N-2)t}{4}}, & t > \dfrac{N}{N-2},\\ K\varepsilon^{\frac{N}{4}}|\ln\epsilon|, & t = \dfrac{N}{N-2},\\ K\varepsilon^{\frac{t(N-2)}{4}}, & t < \dfrac{N}{N-2}, \end{cases} \tag{2.7}$$

where K_1, K_2, K are positive constants. Moreover, $S = \frac{K_1}{K_2^{\frac{2}{2^*}}}$. By (2.5) and (2.6),

$$\frac{\int_{\mathbb{R}^N}|\nabla u_\varepsilon|^2 dx}{(\int_{\mathbb{R}^N}u_\varepsilon^{2^*}dx)^{\frac{2}{2^*}}} = S + O(\varepsilon^{\frac{N-2}{2}}). \tag{2.8}$$

From the definition of c_λ, it can be derived that $c_\lambda \leqslant \sup_{t\geqslant 0} I_\lambda(tu_\varepsilon)$. Define $y(t) := \frac{1}{2}t^2\|u_\varepsilon\|^2 - \frac{\lambda}{2^*}t^{2^*}\int_{\mathbb{R}^N}u_\varepsilon^{2^*}dx$. We note that $y(t)$ attains its maximum at $t_0 = (\frac{\|u_\varepsilon\|^2}{\lambda\int_{\mathbb{R}^N}|u_\varepsilon|^{2^*}dx})^{\frac{N-2}{4}}$. Thus, $y(t_0) = \frac{1}{N}[\frac{\|u_\varepsilon\|^2}{(\lambda\int_{\mathbb{R}^N}|u_\varepsilon|^{2^*}dx)^{\frac{2}{2^*}}}]^{\frac{N}{2}}$. Observe that there exists $t' \in (0,1)$ such that for $\varepsilon < 1$,

$$\sup_{0\leqslant t\leqslant t'} I_\lambda(tu_\varepsilon) \leqslant \sup_{0\leqslant t\leqslant t'}\frac{1}{2}t^2\|u_\epsilon\|^2 < \frac{1}{N}\frac{S^{\frac{N}{2}}}{\lambda^{\frac{N-2}{2}}}. \tag{2.9}$$

On the other hand, by (f_4),

$$\sup_{t\geqslant t'} I_\lambda(tu_\varepsilon) \leqslant \sup_{t\geqslant 0} y(t) - \lambda\frac{D}{q}(t')^q\int_{\mathbb{R}^N}u_\varepsilon^q dx. \tag{2.10}$$

For $N > 4$, we derive from (2.7), (2.8) and (2.10) that

$$\sup_{t\geqslant t'} I_\lambda(tu_\varepsilon) \leqslant \frac{1}{N}\frac{1}{\lambda^{\frac{N-2}{2}}}S^{\frac{N}{2}} + O(\varepsilon) - C\varepsilon^{\frac{2N-(N-2)q}{4}}.$$

Observe that $\frac{2N-(N-2)q}{4} < 1$. Then there exists $\varepsilon_0 < 1$ small enough such that for $\varepsilon \in (0, \varepsilon_0)$, $\sup_{t\geqslant t'} I_\lambda(tu_\varepsilon) < \frac{1}{N}\frac{1}{\lambda^{\frac{N-2}{2}}}S^{\frac{N}{2}}$. Together with (2.9), we get (2.3) holds for $N > 4$. Similar argument shows that (2.3) holds for $N = 3$ with $q > 4$ or $N = 4$. □

Remark 2.3. In the present paper, we assume that for $\lambda \in [\frac{1}{2}, 1]$, if $\{u_n\} \subset H$ is a sequence satisfying

$$\|u_n\| < \infty, \quad I_\lambda(u_n) \to c_\lambda, \quad I'_\lambda(u_n) \to 0, \tag{2.11}$$

then $u_n \geqslant 0$ in H. In fact, we have $(I'_\lambda(u_n), u_n^-) = o(1)$, where $u_n^- = \min\{u_n, 0\}$. Thus, $\|u_n^-\| = o(1)$, from which we can derive that $\|u_n^+\| < \infty$, $I_\lambda(u_n^+) \to c_\lambda$ and $I'_\lambda(u_n^+) \to 0$.

We shall make use of the following Pohozǎev type identity. The proof can be done similarly to that in [6] and details are omitted here.

Lemma 2.4. *For $\lambda \in [\frac{1}{2}, 1]$, if u_λ is a critical point of I_λ, then u_λ satisfies*

$$\frac{N-2}{2N}\int_{\mathbb{R}^N}|\nabla u_\lambda|^2dx + \frac{1}{2N}\int_{\mathbb{R}^N}\nabla V(x)\ x\ u_\lambda^2dx = \int_{\mathbb{R}^N}\left[\lambda F(u_\lambda) - \frac{1}{2}V(x)u_\lambda^2\right]dx. \tag{2.12}$$

Remark 2.5. Standard argument shows that if (V_1)–(V_2) and (f_1)–(f_3) hold, then there exists $\beta > 0$ independent of $\lambda \in [\frac{1}{2}, 1]$ such that any nontrivial critical point u_λ of I_λ satisfies $\|u_\lambda\| \geqslant \beta > 0$.

Motivated by the ideas in [11], we establish the following lemma, which plays a fundamental role in our proof. We emphasize that the argument in [11] cannot be applied directly. New techniques must be developed to overcome the difficulty caused by the critical growth.

Lemma 2.6. *Assume $V(x) \equiv V > 0$ and (f_1)–(f_4) hold. For $\lambda \in [\frac{1}{2}, 1]$, let $\{u_n\} \subset H$ be a sequence such that $u_n \geqslant 0$, $\|u_n\| < \infty$, $I_\lambda(u_n) \to c_\lambda$ and $I'_\lambda(u_n) \to 0$. Moreover, $c_\lambda < \frac{1}{N}\frac{1}{\lambda^{\frac{N-2}{2}}}S^{\frac{N}{2}}$. Then there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, an integer $k \in \mathbb{N}\cup\{0\}$ and $w_\lambda^j \in H$ for $1 \leqslant j \leqslant k$ such that*

(i) $u_n \to u_\lambda$ weakly in H with $I'_\lambda(u_\lambda) = 0$,
(ii) $w_\lambda^j \neq 0$, $w_\lambda^j \geqslant 0$ and $I'_\lambda(w_\lambda^j) = 0$ for $1 \leqslant j \leqslant k$,
(iii) $c_\lambda = I_\lambda(u_\lambda) + \sum_{j=1}^k I_\lambda(w_\lambda^j)$,

where we agree that in the case $k = 0$, the above hold without w_λ^j.

Proof. We will take five steps to finish the proof.

Step 1. $\|u_n\| < \infty$ implies that up to a subsequence, $u_n \to u_\lambda$ weakly in H. It is not difficult to check that $I'_\lambda(u_\lambda) = 0$. Thus, (i) holds.

Set $v_n^1 = u_n - u_\lambda$.

Step 2. If $\lim_{n\to\infty}\sup_{z\in\mathbb{R}^N}\int_{B_1(z)}|v_n^1|^2dx = 0$, then $u_n \to u_\lambda$ in H and Lemma 2.6 holds with $k = 0$.

Applying the Lions lemma in [13], we obtain that

$$v_n^1 \to 0 \quad \text{in} \quad L^t(\mathbb{R}^N), \quad \forall\, t \in (2, 2^*). \tag{2.13}$$

A direct calculation shows that

$$\begin{aligned}\|v_n^1\|^2 &= (I'_\lambda(u_n), v_n^1) + \lambda\int_{\mathbb{R}^N}(f(u_n) - f(u_\lambda))v_n^1dx \\ &= \lambda\int_{\mathbb{R}^N}(g(u_n) - g(u_\lambda))v_n^1dx + \lambda\int_{\mathbb{R}^N}(|u_n|^{2^*-2}u_n - |u_\lambda|^{2^*-2}u_\lambda)v_n^1dx + o(1).\end{aligned}$$

Note that (f_1)–(f_3) imply that $\forall\,\varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that

$$|g(u)| \leqslant \varepsilon(|u| + |u|^{2^*-1}) + C(\varepsilon)|u|^{q-1}. \tag{2.14}$$

Combining (2.13) with (2.14), $\|v_n^1\|^2 = \lambda\int_{\mathbb{R}^N}(|u_n|^{2^*-2}u_n - |u_\lambda|^{2^*-2}u_\lambda)v_n^1dx + o(1)$. By elliptic estimates, $u_\lambda \in L^\infty(\mathbb{R}^N)$. From [16, Lemma 8.9],

$$\left|\int_{\mathbb{R}^N}[|u_n|^{2^*-2}u_n - |u_\lambda|^{2^*-2}u_\lambda - |u_n - u_\lambda|^{2^*-2}(u_n - u_\lambda)]\varphi dx\right| = o(1)\|\varphi\|, \quad \forall\,\varphi \in H. \tag{2.15}$$

Thus,

$$\|v_n^1\|^2 = \lambda\int_{\mathbb{R}^N}|v_n^1|^{2^*}dx + o(1). \tag{2.16}$$

On the other hand, by the Brezis-Lieb lemma, we obtain that

$$\|v_n^1\|^2 = \|u_n\|^2 - \|u_\lambda\|^2 + o(1) \tag{2.17}$$

and

$$\int_{\mathbb{R}^N} |v_n^1|^{2^*} dx = \int_{\mathbb{R}^N} |u_n|^{2^*} dx - \int_{\mathbb{R}^N} |u_\lambda|^{2^*} dx + o(1). \tag{2.18}$$

We claim that

$$\int_{\mathbb{R}^N} G(v_n^1) dx = \int_{\mathbb{R}^N} G(u_n) dx - \int_{\mathbb{R}^N} G(u_\lambda) dx + o(1). \tag{2.19}$$

In fact, by (2.14) and the mean value theorem,

$$|G(u_n) - G(v_n^1)| \leqslant C[(|v_n^1| + |u_\lambda|) + (|v_n^1| + |u_\lambda|)^{2^*-1}]|u_\lambda|.$$

For $R > 0$, by Hölder's inequality,

$$\begin{aligned} &\int_{|x|\geqslant R} |G(u_n) - G(v_n^1)| dx \\ &\quad \leqslant C\bigg(\int_{|x|\geqslant R} |v_n^1|^2 dx\bigg)^{\frac{1}{2}} \bigg(\int_{|x|\geqslant R} |u_\lambda|^2 dx\bigg)^{\frac{1}{2}} + C\int_{|x|\geqslant R} |u_\lambda|^2 dx \\ &\qquad + C\bigg(\int_{|x|\geqslant R} |v_n^1|^{2^*} dx\bigg)^{\frac{2^*-1}{2^*}} \bigg(\int_{|x|\geqslant R} |u_\lambda|^{2^*} dx\bigg)^{\frac{1}{2^*}} + C\int_{|x|\geqslant R} |u_\lambda|^{2^*} dx. \end{aligned} \tag{2.20}$$

From (f_1)–(f_3), we also have

$$\int_{|x|\geqslant R} |G(u_\lambda)| dx \leqslant C\int_{|x|\geqslant R} |u_\lambda|^2 dx + C\int_{|x|\geqslant R} |u_\lambda|^{2^*} dx. \tag{2.21}$$

By (2.20) and (2.21), $\forall\, \varepsilon > 0$, there exists $R > 0$ such that

$$\int_{|x|\geqslant R} |G(u_n) - G(v_n^1) - G(u_\lambda)| dx \leqslant \varepsilon. \tag{2.22}$$

On the other hand, observe that $\lim_{t\to\infty} \frac{G(t)}{|t|^{2^*}} = 0$. We also have $\int_{\mathbb{R}^N} |u_n|^{2^*} dx < \infty$. From the compactness lemma of Strass [15],

$$\lim_{n\to\infty} \int_{|x|\leqslant R} |G(u_n) - G(u_\lambda)| dx = 0. \tag{2.23}$$

Similarly,

$$\lim_{n\to\infty} \int_{|x|\leqslant R} |G(v_n^1)| dx = 0. \tag{2.24}$$

It follows from (2.22)–(2.24) that (2.19) holds. Combining (2.17)–(2.19), there holds

$$c_\lambda - I_\lambda(u_\lambda) = \frac{1}{2}\|v_n^1\|^2 - \lambda\int_{\mathbb{R}^N} G(v_n^1) dx - \frac{\lambda}{2^*}\int_{\mathbb{R}^N} |v_n^1|^{2^*} dx + o(1). \tag{2.25}$$

From (2.13), (2.14) and (2.25),

$$c_\lambda - I_\lambda(u_\lambda) = \frac{1}{2}\|v_n^1\|^2 - \frac{\lambda}{2^*}\int_{\mathbb{R}^N} |v_n^1|^{2^*} dx + o(1). \tag{2.26}$$

Since $I'_\lambda(u_\lambda) = 0$, by Lemma 2.4, $I_\lambda(u_\lambda) \geqslant 0$. Thus, $c_\lambda - I_\lambda(u_\lambda) < \frac{1}{N}\frac{1}{\lambda^{\frac{N-2}{2}}} S^{\frac{N}{2}}$. We may assume that $\|v_n^1\|^2 \to l \geqslant 0$. Then $\lambda\int_{\mathbb{R}^N} |v_n^1|^{2^*} dx \to l$. If $l > 0$, then Sobolev embedding theorem implies that $S \leqslant \frac{\|v_n^1\|^2}{(\int_{\mathbb{R}^N} |v_n^1|^{2^*} dx)^{\frac{2}{2^*}}}$, from which we conclude that $l \geqslant \frac{1}{\lambda^{\frac{N-2}{2}}} S^{\frac{N}{2}}$. Thus, by (2.26), $c_\lambda - I_\lambda(u_\lambda) \geqslant \frac{1}{N}\frac{1}{\lambda^{\frac{N-2}{2}}} S^{\frac{N}{2}}$, a contradiction. Therefore, $\|v_n^1\| \to 0$ and Step 2 is completed.

Step 3. If $\exists\, \{z_n\} \subset \mathbb{R}^N$ such that $\int_{B_1(z_n)} |v_n^1|^2 dx \to d > 0$, then, after extracting a subsequence if necessary, the following hold

(1) $|z_n| \to \infty$,

(2) $u_n(\cdot + z_n) \to w_\lambda \neq 0$ weakly in H,

(3) $w_\lambda \geqslant 0$ and $I'_\lambda(w_\lambda) = 0$.

Note that (1) and (2) are obvious. For (3), set $\tilde{u}_n(\cdot) = u_n(\cdot + z_n)$. We note that $\tilde{u}_n \geqslant 0$ in H and $I'_\lambda(\tilde{u}_n) = o(1)$. Then (3) holds.

Step 4. If there exist $m \geqslant 1$, $\{y_n^k\} \subset \mathbb{R}^N$, $w_\lambda^k \in H$ for $1 \leqslant k \leqslant m$ such that

(i) $|y_n^k| \to \infty$, $|y_n^k - y_n^{k'}| \to \infty$, if $k \neq k'$,

(ii) $u_n(\cdot + y_n^k) \to w_\lambda^k \neq 0$ weakly in H, $\forall\ 1 \leqslant k \leqslant m$,

(iii) $w_\lambda^k \geqslant 0$ and $I'_\lambda(w_\lambda^k) = 0$, $\forall\ 1 \leqslant k \leqslant m$,

then one of the following conclusions must hold:

(1) If $\sup_{z\in\mathbb{R}^N} \int_{B_1(z)} |u_n - u_0 - \sum_{k=1}^m w_\lambda^k(\cdot - y_n^k)|^2 dx \to 0$, then

$$\left\| u_n - u_0 - \sum_{k=1}^m w_\lambda^k(\cdot - y_n^k) \right\| \to 0.$$

(2) If $\exists\ (z_n) \subset \mathbb{R}^N$ such that

$$\int_{B_1(z_n)} \left| u_n - u_0 - \sum_{k=1}^m w_\lambda^k(\cdot - y_n^k) \right|^2 dx \to d > 0,$$

then, after extracting a subsequence if necessary, the following hold:

(i) $|z_n| \to \infty$, $|z_n - y_n^k| \to \infty$ for all $1 \leqslant k \leqslant m$,

(ii) $u_n(\cdot + z_n) \to w_\lambda^{m+1} \neq 0$ weakly in H,

(iii) $w_\lambda^{m+1} \geqslant 0$ and $I'_\lambda(w_\lambda^{m+1}) = 0$.

Assume that (1) holds. Set $\xi_n = u_n - u_\lambda - \sum_{k=1}^m w_\lambda^k(\cdot - y_n^k)$. The Lions lemma implies that

$$\xi_n \to 0 \quad \text{in } L^t(\mathbb{R}^N), \quad \forall\ t \in (2, 2^*). \tag{2.27}$$

A direct calculation shows that

$$\|\xi_n\|^2 = (I'_\lambda(u_n), \xi_n) + \lambda \int_{\mathbb{R}^N} (f(u_n) - f(u_\lambda))\xi_n dx - \lambda \sum_{k=1}^m \int_{\mathbb{R}^N} f(w_\lambda^k)\xi_n(\cdot + y_n^k) dx.$$

Together with (2.14) and (2.27), there holds

$$\|\xi_n\|^2 = \lambda \int_{\mathbb{R}^N} (|u_n|^{2^*-2}u_n - |u_\lambda|^{2^*-2}u_\lambda)\xi_n dx - \lambda \sum_{k=1}^m \int_{\mathbb{R}^N} |w_\lambda^k|^{2^*-2} w_\lambda^k \xi_n(\cdot + y_n^k) dx + o(1).$$

Similar to (2.15), we obtain that

$$\begin{aligned}
\|\xi_n\|^2 &= \lambda \int_{\mathbb{R}^N} |u_n - u_\lambda|^{2^*-2}(u_n - u_\lambda)\xi_n dx - \lambda \int_{\mathbb{R}^N} |w_\lambda^1|^{2^*-2} w_\lambda^1 \xi_n(\cdot + y_n^1) dx \\
&\quad - \lambda \sum_{k=2}^m \int_{\mathbb{R}^N} |w_\lambda^k|^{2^*-2} w_\lambda^k \xi_n(\cdot + y_n^k) dx + o(1) \\
&= \lambda \int_{\mathbb{R}^N} |u_n(\cdot + y_n^1) - u_\lambda(\cdot + y_n^1)|^{2^*-2}(u_n(\cdot + y_n^1) - u_\lambda(\cdot + y_n^1))\xi_n(\cdot + y_n^1) dx \\
&\quad - \lambda \int_{\mathbb{R}^N} |w_\lambda^1|^{2^*-2} w_\lambda^1 \xi_n(\cdot + y_n^1) dx - \lambda \sum_{k=2}^m \int_{\mathbb{R}^N} |w_\lambda^k|^{2^*-2} w_\lambda^k \xi_n(\cdot + y_n^k) + o(1).
\end{aligned}$$

Since $|y_n^1| \to \infty$ and $u_n(\cdot + y_n^1) \to w_\lambda^1$ weakly in H, we have $u_n(\cdot + y_n^1) - u_\lambda(\cdot + y_n^1) \to w_\lambda^1$ weakly in H. Thus,

$$\|\xi_n\|^2 = \lambda \int_{\mathbb{R}^N} |u_n(\cdot + y_n^1) - u_\lambda(\cdot + y_n^1) - w_\lambda^1|^{2^*-2}(u_n(\cdot + y_n^1) - u_\lambda(\cdot + y_n^1) - w_\lambda^1)\xi_n(\cdot + y_n^1) dx$$

$$-\lambda\sum_{k=2}^{m}\int_{\mathbb{R}^N}|w_\lambda^k|^{2^*-2}w_\lambda^k\xi_n(\cdot+y_n^k)+o(1).$$

Continuing this process, we obtain that

$$\|\xi_n\|^2=\lambda\int_{\mathbb{R}^N}|\xi_n|^{2^*}dx+o(1). \tag{2.28}$$

On the other hand, since $u_n(\cdot+y_n^1)-u_\lambda(\cdot+y_n^1)\to w_\lambda^1$ weakly in H, arguing as Step 2, we obtain that

$$\begin{aligned}c_\lambda-I_\lambda(u_\lambda)&=\frac{1}{2}\|u_n-u_\lambda\|^2-\lambda\int_{\mathbb{R}^N}G(u_n-u_\lambda)dx-\frac{\lambda}{2^*}\int_{\mathbb{R}^N}|u_n-u_\lambda|^{2^*}dx+o(1)\\&=\frac{1}{2}\|u_n(\cdot+y_n^1)-u_\lambda(\cdot+y_n^1)-w_\lambda^1\|^2-\lambda\int_{\mathbb{R}^N}G(u_n(\cdot+y_n^1)-u_\lambda(\cdot+y_n^1)-w_\lambda^1)dx\\&\quad-\frac{\lambda}{2^*}\int_{\mathbb{R}^N}|u_n(\cdot+y_n^1)-u_\lambda(\cdot+y_n^1)-w_\lambda^1|^{2^*}dx+I_\lambda(w_\lambda^1)+o(1).\end{aligned}$$

Continuing this process, we obtain that

$$c_\lambda-I_\lambda(u_\lambda)-\sum_{k=1}^{m}I_\lambda(w_\lambda^k)=\frac{1}{2}\|\xi_n\|^2-\lambda\int_{\mathbb{R}^N}G(\xi_n)dx-\frac{\lambda}{2^*}\int_{\mathbb{R}^N}|\xi_n|^{2^*}dx+o(1). \tag{2.29}$$

Together with (2.27), there holds

$$c_\lambda-I_\lambda(u_\lambda)-\sum_{k=1}^{m}I_\lambda(w_\lambda^k)=\frac{1}{2}\|\xi_n\|^2-\frac{\lambda}{2^*}\int_{\mathbb{R}^N}|\xi_n|^{2^*}dx+o(1). \tag{2.30}$$

Because (2.28) and (2.30) hold, arguing as Step 2, we conclude that $\|\xi_n\|\to 0$.

Now we assume that (2) holds. The argument is standard, we omit it.

Step 5. Conclusion. By Step 1, Lemma 2.6(i) holds. If the assumption of Step 2 holds, then Lemma 2.6 holds with $k=0$. Otherwise, the assumption of Step 3 holds. Set $\{y_n^1\}=\{z_n\}$ and $w_\lambda^1=w_\lambda$ in Step 4. If (1) of Step 4 holds with $m=1$, from (2.34), we obtain the conclusions of Lemma 2.6. If not, (2) of Step 4 holds. Set $\{y_n^2\}=\{z_n\}$, $w_\lambda^2=w_\lambda^2$ and iterate Step 4. Observe that

$$\lim_{n\to\infty}\left(\|u_n\|^2-\|u_\lambda\|^2-\sum_{k=1}^{m}\|w_\lambda^k\|^2\right)=\lim_{n\to\infty}\left\|u_n-u_\lambda-\sum_{k=1}^{m}w_\lambda^k(\cdot-y_n^k)\right\|^2.$$

Remark 2.5 implies that $\|w_\lambda^k\|\geqslant\beta>0$ independent of λ. Thus, (1) in Step 4 must occur after a finite number of iterations. Together with (2.34), we conclude that Lemma 2.6 holds. □

3 Proof of Theorem 1.1

Lemma 3.1. *Assume the assumptions of Theorem* 1.1 *hold. Then for almost every* $\lambda\in[\frac{1}{2},1]$, I_λ *has a positive critical point.*

Proof. By Lemma 2.2, for almost every $\lambda\in[\frac{1}{2},1]$, there is a sequence $\{u_n\}\subset H$ such that (2.11) holds. Moreover, $c_\lambda\in(0,\frac{1}{N}\frac{S^{\frac{N}{2}}}{\lambda^{\frac{N-2}{2}}})$. From Remark 2.3, we may assume that $u_n\geqslant 0$ in H. Then there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, satisfying $u_n\to u_\lambda$ weakly in H. If $u_\lambda\neq 0$, then Lemma 3.1 holds obviously. Thus, we may assume that $u_n\to 0$ weakly in H. We claim that there exists $\delta>0$ such that

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_1(y)}|u_n|^2dx\geqslant\delta>0. \tag{3.1}$$

Otherwise, $\lim_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_1(y)}|u_n|^2dx=0$. Applying the Lions lemma, we obtain that

$$u_n\to 0\ \text{ in }\ L^t(\mathbb{R}^N),\quad\forall\, t\in(2,2^*). \tag{3.2}$$

From (2.14) and (3.2), there hold $\int_{\mathbb{R}^N} G(u_n)dx = o(1)$ and $\int_{\mathbb{R}^N} g(u_n)u_n dx = o(1)$. Together with (2.11), we obtain that

$$\frac{1}{2}\|u_n\|^2 - \frac{\lambda}{2^*}\int_{\mathbb{R}^N} u_n^{2^*} dx = c_\lambda + o(1) \tag{3.3}$$

and

$$\|u_n\|^2 - \lambda\int_{\mathbb{R}^N} u_n^{2^*} dx = o(1). \tag{3.4}$$

Note that $c_\lambda > 0$, we may assume that $\|u_n\|^2 \to l > 0$. Sobolev embedding theorem implies that $l \geqslant \frac{S^{\frac{N}{2}}}{\lambda^{\frac{N-2}{2}}}$. Thus, $c_\lambda \geqslant \frac{1}{N}\frac{S^{\frac{N}{2}}}{\lambda^{\frac{N-2}{2}}}$, a contradiction with $c_\lambda \in (0, \frac{1}{N}\frac{S^{\frac{N}{2}}}{\lambda^{\frac{N-2}{2}}})$. Therefore, we have (3.1), which implies that there exists $y_n \in \mathbb{R}^N$, $|y_n| \to \infty$ such that $\int_{B_1(y_n)} |u_n|^2 dx \geqslant \frac{\delta}{2} > 0$. Set $v_n = u_n(\cdot + y_n)$. In view of (2.11), we derive that $I_\lambda(v_n) \to c_\lambda$ and $I'_\lambda(v_n) \to 0$. Moreover, $v_n \to v_\lambda \neq 0$ weakly in H. Standard argument shows that $v_\lambda > 0$ in H. Thus, the proof is completed. □

Proof of Theorem 1.1. In view of the proof of Lemma 3.1, for almost every $\lambda \in [\frac{1}{2}, 1]$, there is a sequence $\{u_n\} \subset H$ such that $u_n \geqslant 0$ in H, $I_\lambda(u_n) \to c_\lambda$, $I'_\lambda(u_n) \to 0$ and $u_n \to u_\lambda > 0$ weakly in H. Moreover, $c_\lambda \in (0, \frac{1}{N}\frac{1}{\lambda^{\frac{N-2}{2}}}S^{\frac{N}{2}})$. From Lemma 2.6, $c_\lambda = I_\lambda(u_\lambda) + \sum_{j=1}^k I_\lambda(w_\lambda^j)$, $I'_\lambda(u_\lambda) = 0$ and $I'_\lambda(w_\lambda^j) = 0$, $j = 1, \dots, k$. By Lemma 2.4, we have $I_\lambda(u_\lambda) > 0$ and $I_\lambda(w_\lambda^j) \geqslant 0$, $j = 1, \dots, k$. Hence, there holds $c_\lambda \geqslant I_\lambda(u_\lambda) > 0$. Therefore, there exist $\lambda_n \in [\frac{1}{2}, 1]$, $c_{\lambda_n} \in (0, \frac{1}{N}\frac{1}{\lambda_n^{\frac{N-2}{2}}}S^{\frac{N}{2}})$ and $u_{\lambda_n} \in H$ satisfying $\lambda_n \to 1$, $u_{\lambda_n} > 0$, $I'_{\lambda_n}(u_{\lambda_n}) = 0$ and $0 < I_{\lambda_n}(u_{\lambda_n}) \leqslant c_{\lambda_n}$. By $I'_{\lambda_n}(u_{\lambda_n}) = 0$ and Lemma 2.4, we have $c_{\lambda_n} \geqslant I_{\lambda_n}(u_{\lambda_n}) = \frac{1}{N}\int_{\mathbb{R}^N} |\nabla u_{\lambda_n}|^2 dx > 0$. Thus, by the Sobolev embedding theorem, there holds $\int_{\mathbb{R}^N} |u_{\lambda_n}|^{2^*} dx < \infty$. From ($f_1$)–($f_3$) and Lemma 2.4, we obtain that $\forall\, \varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that

$$\frac{N-2}{2N}\int_{\mathbb{R}^N} |\nabla u_{\lambda_n}|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V|u_{\lambda_n}|^2 dx = \lambda_n \int_{\mathbb{R}^N} F(u_{\lambda_n})dx \leqslant \varepsilon\int_{\mathbb{R}^N} |u_{\lambda_n}|^2 dx + C(\varepsilon)\int_{\mathbb{R}^N} |u_{\lambda_n}|^{2^*} dx.$$

Hence, there holds $\|u_{\lambda_n}\| < \infty$. Without loss of generality, we may assume that $\lim_{n\to\infty} I_{\lambda_n}(u_{\lambda_n})$ exists. By Theorem 2.1, $\lambda \to c_\lambda$ is continuous from the left. Then there holds $0 \leqslant \lim_{n\to\infty} I_{\lambda_n}(u_{\lambda_n}) \leqslant c_1 < \frac{1}{N}S^{\frac{N}{2}}$. Observing that $I(u_{\lambda_n}) = I_{\lambda_n}(u_{\lambda_n}) + (\lambda_n - 1)\int_{\mathbb{R}^N} F(u_{\lambda_n})dx$, together with $\|u_{\lambda_n}\| < \infty$, there hold

$$0 \leqslant \lim_{n\to\infty} I(u_{\lambda_n}) \leqslant c_1 < \frac{1}{N}S^{\frac{N}{2}} \tag{3.5}$$

and

$$\lim_{n\to\infty} I'(u_{\lambda_n}) = 0. \tag{3.6}$$

By Remark 2.5, $\|u_{\lambda_n}\| \geqslant \beta > 0$, where $\beta > 0$ is independent of λ_n. Note that $\|u_{\lambda_n}\| < \infty$. Then following the same lines as in the proof of Lemma 3.1, we can obtain that (1.1) has a positive solution u_0. Moreover, by Lemma 2.6, $I(u_0) \leqslant \lim_{n\to\infty} I(u_{\lambda_n}) \leqslant c_1 < \frac{1}{N}S^{\frac{N}{2}}$. Let

$$m = \inf\{I(u) : u \in H, u \neq 0, I'(u) = 0\}.$$

Since $I'(u_0) = 0$, $m \leqslant I(u_0) < \frac{1}{N}S^{\frac{N}{2}}$. Lemma 2.4 implies that $m \geqslant 0$. Hence, $0 \leqslant m \leqslant I(u_0) < \frac{1}{N}S^{\frac{N}{2}}$. By the definition of m, there exists $\{u_n\} \subset H$ such that $u_n \neq 0$, $I(u_n) \to m$ and $I'(u_n) = 0$. Remark 2.5 implies that $\|u_n\| \geqslant \beta > 0$. It is easy to check that $\|u_n\| < \infty$. From Remark 2.3, we may assume that $u_n \geqslant 0$ in H. Then following the same lines as in the proof of Lemma 3.1, we know that there exists $\{v_n\} \subset H$ such that $v_n \geqslant 0$ in H, $v_n \to v_0 > 0$ weakly in H, $I(v_n) \to m$ and $I'(v_n) = 0$. From Lemma 2.6, $I'(v_0) = 0$ and $m \geqslant I(v_0)$. $I'(v_0) = 0$ implies that $I(v_0) \geqslant m$. Therefore, $v_0 > 0$ satisfies $I(v_0) = m$ and $I'(v_0) = 0$. □

4 Proof of Theorem 1.2

In view of Theorem 1.1, we may assume that $V(x)$ is not identical to V_∞ in this section.

For $\lambda \in [\frac{1}{2}, 1]$, consider the family of functionals $I_\lambda^\infty : H \mapsto \mathbb{R}$ defined by

$$I_\lambda^\infty(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V_\infty |u|^2)dx - \lambda \int_{\mathbb{R}^N} F(u)dx. \tag{4.1}$$

Similar argument as in [9] can derive the following result.

Lemma 4.1. *For $\lambda \in [\frac{1}{2}, 1]$, if $w_\lambda \in H$ is a nontrivial critical point of I_λ^∞, then there exists $\gamma_\lambda \in C([0,1], H)$ such that $\gamma_\lambda(0) = 0$, $I_{\frac{1}{2}}^\infty(\gamma_\lambda(1)) < 0$, $w_\lambda \in \gamma_\lambda[0,1]$ and $\max_{t\in[0,1]} I_\lambda^\infty(\gamma_\lambda(t)) = I_\lambda^\infty(w_\lambda)$. Moreover, $0 \notin \gamma_\lambda((0,1])$.*

Remark 4.2. From Theorem 1.1, we know that if (f_1)–(f_4) hold, then for $\lambda \in [\frac{1}{2}, 1]$, I_λ^∞ has a ground state.

Lemma 4.3. *Assume the assumptions of Theorem 1.2, we obtain that for almost every $\mu \in [\frac{1}{2}, 1]$, I_μ has a positive critical point.*

Proof. From Lemma 2.2 and Remark 2.3, we may assume that for almost every $\mu \in [\frac{1}{2}, 1]$, there exists a sequence $\{u_n\} \subset H$, $u_n \geqslant 0$ such that $u_n \to u_\mu$ weakly in H, $I_\mu(u_n) \to c_\mu$ and $I_\mu'(u_n) \to 0$. Moreover, $c_\mu \in (0, \frac{1}{N}\frac{S^{\frac{N}{2}}}{\mu^{\frac{N-2}{2}}})$. We claim that $u_\mu \neq 0$. Otherwise, $u_\mu = 0$. Similar to the proof of Lemma 3.1, there exists $y_n \in \mathbb{R}^N$ such that $|y_n| \to \infty$ and $v_n = u_n(\cdot + y_n) \to v_\mu \neq 0$ weakly in H. On the other hand, since $u_n \to 0$ weakly in H, there hold $I_\mu^\infty(u_n) \to c_\mu$ and $I_\mu^{\infty'}(u_n) \to 0$. Thus, $I_\mu^\infty(v_n) \to c_\mu$ and $I_\mu^{\infty'}(v_n) \to 0$. Since $v_n \to v_\mu \neq 0$ weakly in H, there holds $I_\mu^{\infty'}(v_\mu) = 0$. From Lemma 2.6, $c_\mu \geqslant I_\mu^\infty(v_\mu)$. Remark 4.2 implies that I_μ^∞ has a ground state w_μ. Thus, $c_\mu \geqslant I_\mu^\infty(w_\mu)$. By Lemma 4.1, $c_\mu \geqslant \max_{t\in[0,1]} I_\mu^\infty(\gamma_\mu(t))$, where $\gamma_\mu \in C([0,1], H)$ such that $\gamma_\mu(0) = 0$, $I_{\frac{1}{2}}^\infty(\gamma_\mu(1)) < 0$. Moreover, $0 \notin \gamma_\mu((0,1])$. From (V_3), $I_\mu(\gamma_\mu(t)) < I_\mu^\infty(\gamma_\mu(t))$ for all $t \in (0,1]$. Set $v_1 = 0$ and $v_2 = \gamma_\mu(1)$ in Theorem 2.1. Thus, the definition of c_μ implies that $c_\mu \leqslant \max_{t\in[0,1]} I_\mu(\gamma_\mu(t)) < \max_{t\in[0,1]} I_\mu^\infty(\gamma_\mu(t)) \leqslant c_\mu$, a contradiction. Then we obtain that $u_\mu \neq 0$. Standard argument shows that $u_\mu > 0$. Thus, Lemma 4.3 holds. □

We need the following lemma which is the key to our proof of Theorem 1.2.

Lemma 4.4. *Assume (V_1)–(V_3) and (f_1)–(f_5) hold. For $\lambda \in [\frac{1}{2}, 1]$, let $\{u_n\} \subset H$ be a sequence such that $u_n \geqslant 0$, $\|u_n\| < \infty$, $I_\lambda(u_n) \to c_\lambda$ and $I_\lambda'(u_n) \to 0$. Moreover, $c_\lambda < \frac{1}{N}\frac{1}{\lambda^{\frac{N-2}{2}}}S^{\frac{N}{2}}$. Then there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that*

(i) *$u_n \to u_\lambda$ weakly in H with $I_\lambda'(u_\lambda) = 0$,*

(ii) *$c_\lambda \geqslant I_\lambda(u_\lambda)$.*

Proof. Since $\|u_n\| < \infty$, we may assume that $u_n \to u_\lambda$ weakly in H. It is easy to see that $I_\lambda'(u_\lambda) = 0$. Thus, (i) holds. Set $w_n^1 = u_n - u_\lambda$. Arguing as Lemma 2.6, there holds

$$c_\lambda - I_\lambda(u_\lambda) = \frac{1}{2}\|w_n^1\|^2 - \lambda\int_{\mathbb{R}^N} G(w_n^1)dx - \frac{\lambda}{2^*}\int_{\mathbb{R}^N} |w_n^1|^{2^*}dx + o(1). \tag{4.2}$$

We claim that

$$\left|\int_{\mathbb{R}^N} (g(u_n) - g(u_\lambda) - g(w_n^1))\varphi dx\right| = o(1)\|\varphi\|, \quad \forall\, \varphi \in H. \tag{4.3}$$

In fact, by (f_5) and the mean value theorem, we have

$$|g(u_n) - g(w_n^1)| \leqslant C[1 + (|w_n^1| + |u_\lambda|)^{\frac{4}{N-2}}]|u_\lambda|.$$

For $R > 0$, by Hölder's inequality,

$$\begin{aligned}\int_{|x|\geqslant R} |g(u_n) - g(w_n^1)||\varphi|dx &\leqslant C\left(\int_{|x|\geqslant R} |u_\lambda|^2 dx\right)^{\frac{1}{2}}\|\varphi\| + C\left(\int_{|x|\geqslant R} |u_\lambda|^{2^*}dx\right)^{\frac{N+2}{2N}}\|\varphi\| \\ &\quad + C\left(\int_{|x|\geqslant R} |w_n^1|^{2^*}dx\right)^{\frac{2}{N}}\left(\int_{|x|\geqslant R} |u_\lambda|^{2^*}dx\right)^{\frac{1}{2^*}}\|\varphi\|. \end{aligned}\tag{4.4}$$

We also have

$$\begin{aligned}\int_{|x|\geqslant R}|g(u_\lambda)||\varphi|dx &\leqslant C\int_{|x|\geqslant R}|u_\lambda||\varphi|dx + C\int_{|x|\geqslant R}|u_\lambda|^{2^*-1}|\varphi|dx\\ &\leqslant C\bigg(\int_{|x|\geqslant R}|u_\lambda|^2dx\bigg)^{\frac{1}{2}}\|\varphi\| + \bigg(\int_{|x|\geqslant R}|u_\lambda|^{2^*}dx\bigg)^{\frac{2^*-1}{2^*}}\|\varphi\|. \end{aligned}\tag{4.5}$$

By (4.4) and (4.5), we obtain that $\forall\ \varepsilon > 0$, there exists $R > 0$ such that

$$\bigg|\int_{|x|\geqslant R}(g(u_n) - g(u_\lambda) - g(w_n^1))\varphi dx\bigg| \leqslant \varepsilon\|\varphi\|.\tag{4.6}$$

On the other hand,

$$\int_{|x|\leqslant R}|g(u_n) - g(u_\lambda)||\varphi|dx \leqslant \bigg(\int_{|x|\leqslant R}|g(u_n) - g(u_\lambda)|^{\frac{2^*}{2^*-1}}dx\bigg)^{\frac{2^*-1}{2^*}}\bigg(\int_{|x|\leqslant R}|\varphi|^{2^*}\bigg)^{\frac{1}{2^*}}.$$

Observe that $\lim_{t\to\infty}\frac{g(t)^{\frac{2^*}{2^*-1}}}{t^{2^*}} = 0$. We also have $\int_{\mathbb{R}^N}|u_n|^{2^*}dx < \infty$. From the compactness lemma of Strass [15],

$$\lim_{n\to\infty}\int_{|x|\leqslant R}|g(u_n)|^{\frac{2^*}{2^*-1}}dx = \int_{|x|\leqslant R}|g(u_\lambda)|^{\frac{2^*}{2^*-1}}dx.$$

Thus, by Lebesgue's dominated convergence theorem,

$$\int_{|x|\leqslant R}|g(u_n) - g(u_\lambda)||\varphi|dx = o(1)\|\varphi\|.\tag{4.7}$$

Similarly,

$$\int_{|x|\leqslant R}|g(w_n^1)||\varphi|dx = o(1)\|\varphi\|.\tag{4.8}$$

It follows from (4.6)–(4.8) that (4.3) holds. For $\lambda\in[\frac{1}{2},1]$, define

$$\begin{aligned}H_\lambda(u) &= \frac{1}{2}\|u\|^2 - \lambda\int_{\mathbb{R}^N}G(u)dx - \frac{\lambda}{2^*}\int_{\mathbb{R}^N}|u|^{2^*}dx,\quad u\in H,\\ H_\lambda^\infty(u) &= \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + V_\infty u^2)dx - \lambda\int_{\mathbb{R}^N}G(u)dx - \frac{\lambda}{2^*}\int_{\mathbb{R}^N}|u|^{2^*}dx,\quad u\in H,\\ J_\lambda(u) &= \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + V_\infty u^2)dx - \frac{\lambda}{2^*}\int_{\mathbb{R}^N}|u|^{2^*}dx,\quad u\in H.\end{aligned}$$

From (4.2),

$$c_\lambda - I_\lambda(u_\lambda) = H_\lambda(w_n^1) + o(1).\tag{4.9}$$

By (2.15) and (4.3),

$$|(I'_\lambda(u_n) - I'_\lambda(u_\lambda),\varphi) - (H'_\lambda(w_n^1),\varphi)| = o(1)\|\varphi\|,\quad \forall\ \varphi\in H.\tag{4.10}$$

That is,

$$H'_\lambda(w_n^1) = o(1).\tag{4.11}$$

Since $w_n^1\to 0$ weakly in H, we obtain that

$$c_\lambda - I_\lambda(u_\lambda) = H_\lambda^\infty(w_n^1) + o(1)\tag{4.12}$$

and

$$H_\lambda^{\infty'}(w_n^1) = o(1).\tag{4.13}$$

We will consider two cases.

Case 1. $\lim_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_1(y)}|w_n^1|^2dx=0$.

The Lions lemma implies that

$$w_n^1\to 0 \ \text{ in } \ L^t(\mathbb{R}^N), \quad \forall\, t\in(2,2^*). \tag{4.14}$$

Combining (2.13), (2.14) and (4.12)–(4.14), there hold $c_\lambda-I_\lambda(u_\lambda)=J_\lambda(w_n^1)+o(1)$ and $J'_\lambda(w_n^1)=o(1)$. Then $c_\lambda-I_\lambda(u_\lambda)=\frac{\lambda}{N}\int_{\mathbb{R}^N}|w_n^1|^{2^*}dx+o(1)$. Hence, $c_\lambda-I_\lambda(u_\lambda)\geqslant 0$.

Case 2. There exists $\gamma_1>0$ such that

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_1(y)}|w_n^1|^2dx\geqslant\gamma_1>0.$$

Then there exists $y_n^1\in\mathbb{R}^N$, $|y_n^1|\to\infty$ such that $\int_{B_1(y_n^1)}|w_n^1|^2dx\geqslant\frac{\gamma_1}{2}>0$, from which we derive that $w_n^1(\cdot+y_n^1)\to w_\lambda^1\neq 0$ weakly in H,

$$c_\lambda-I_\lambda(u_\lambda)=H_\lambda^\infty(w_n^1(\cdot+y_n^1))+o(1) \tag{4.15}$$

and

$$H_\lambda^{\infty'}(w_n^1(\cdot+y_n^1))=o(1). \tag{4.16}$$

Thus, $H_\lambda^{\infty'}(w_\lambda^1)=0$. If $c_\lambda-I_\lambda(u_\lambda)<\frac{1}{N}\frac{1}{\lambda^{\frac{N-2}{2}}}S^{\frac{N}{2}}$, then Lemma 4.4 follows from the argument of Lemma 2.6. Otherwise, set $w_n^2=w_n^1(\cdot+y_n^1)-w_\lambda^1$. Similar to the argument of (4.9) and (4.11), we have

$$c_\lambda-I_\lambda(u_\lambda)-H_\lambda^\infty(w_\lambda^1)+o(1)=H_\lambda^\infty(w_n^2) \tag{4.17}$$

and

$$H_\lambda^{\infty'}(w_n^2)=o(1). \tag{4.18}$$

Then either

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_1(y)}|w_n^2|^2dx=0, \tag{4.19}$$

or there exists $\gamma_2>0$ such that

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_1(y)}|w_n^2|^2dx\geqslant\gamma_2>0. \tag{4.20}$$

If (4.19) holds, by Case 1, we have $c_\lambda-I_\lambda(u_\lambda)-H_\lambda^\infty(w_\lambda^1)\geqslant 0$. By Lemma 2.4, $H_\lambda^\infty(w_\lambda^1)\geqslant 0$. Thus, $c_\lambda-I_\lambda(u_\lambda)\geqslant 0$. So we may assume (4.20) holds. Continuing this process, we obtain $w_n^i\in H$, $y_n^i\in\mathbb{R}^N$, $|y_n^i|\to\infty$, $i\in N$ such that $w_n^i(\cdot+y_n^i)\to w_\lambda^i\neq 0$ weakly in H, $H_\lambda^{\infty'}(w_\lambda^i)=0$,

$$c_\lambda-I_\lambda(u_\lambda)-\sum_{i=1}^{j}H_\lambda^\infty(w_\lambda^i)+o(1)=H_\lambda^\infty(w_n^{j+1}) \tag{4.21}$$

and

$$H_\lambda^{\infty'}(w_n^{j+1})=o(1), \tag{4.22}$$

where $w_n^{j+1}=w_n^j(\cdot+y_n^j)-w_\lambda^j$, $j\in N$. Since $H_\lambda^{\infty'}(w_\lambda^i)=0$, from Lemma 2.4, we have

$$H_\lambda^\infty(w_\lambda^i)=\frac{1}{N}\int_{\mathbb{R}^N}|\nabla w_\lambda^i|^2dx. \tag{4.23}$$

We claim that there exists $\gamma>0$ independent of i such that

$$\int_{\mathbb{R}^N}|\nabla w_\lambda^i|^2dx\geqslant\gamma>0. \tag{4.24}$$

From (f_1)–(f_3), we obtain that $\forall\,\varepsilon>0$, there exists $C(\varepsilon)>0$ such that

$$\int_{\mathbb{R}^N}(|\nabla w_\lambda^i|^2+V_\infty|w_\lambda^i|^2)dx\leqslant\varepsilon\int_{\mathbb{R}^N}|w_\lambda^i|^2dx+C(\varepsilon)\int_{\mathbb{R}^N}|w_\lambda^i|^{2^*}dx.$$

Thus, $\int_{\mathbb{R}^N} |\nabla w_\lambda^i|^2 dx \leqslant C \int_{\mathbb{R}^N} |w_\lambda^i|^{2^*} dx$. Sobolev embedding theorem implies that (4.24) holds. By (4.23) and (4.24), we have $c_\lambda - I_\lambda(u_\lambda) - \sum_{i=1}^j H_\lambda^\infty(w_\lambda^i) < \frac{1}{N}\frac{1}{\lambda^{\frac{N-2}{2}}} S^{\frac{N}{2}}$ at some $j = k$. Then Lemma 4.4 follows from Lemma 2.6. □

Proof of Theorem 1.2. In view of the proof of Lemma 4.3, for almost every $\lambda \in [\frac{1}{2}, 1]$, there is a sequence $\{u_n\} \subset H$ such that $I_\lambda(u_n) \to c_\lambda$, $I'_\lambda(u_n) \to 0$ and $u_n \to u_\lambda \neq 0$ weakly in H. Moreover, $c_\lambda \in (0, \frac{1}{N}\frac{1}{\lambda^{\frac{N-2}{2}}} S^{\frac{N}{2}})$. Then Lemma 4.4 holds. Thus, $I'_\lambda(u_\lambda) = 0$, $c_\lambda \geqslant I_\lambda(u_\lambda)$. Therefore, there exist $\lambda_n \in [\frac{1}{2}, 1]$, $c_{\lambda_n} \in (0, \frac{1}{N}\frac{1}{\lambda_n^{\frac{N-2}{2}}} S^{\frac{N}{2}})$ and $u_{\lambda_n} \in H$ satisfying $\lambda_n \to 1$, $u_{\lambda_n} \neq 0$, $I'_{\lambda_n}(u_{\lambda_n}) = 0$ and $I_{\lambda_n}(u_{\lambda_n}) \leqslant c_{\lambda_n}$. We claim that $\|u_{\lambda_n}\| < \infty$. The proof can be done similarly to Proposition 4.2 in [11] and details are omitted here. Set $I(u_{\lambda_n}) = I_{\lambda_n}(u_{\lambda_n}) + (\lambda_n - 1)\int_{\mathbb{R}^N} F(u_{\lambda_n}) dx$. Then we have $\lim_{n\to\infty} I(u_{\lambda_n}) \leqslant c_1 < \frac{1}{N} S^{\frac{N}{2}}$ and $\lim_{n\to\infty} I'(u_{\lambda_n}) = 0$. By Remark 2.5, $\|u_{\lambda_n}\| \geqslant \beta > 0$, where $\beta > 0$ is independent of λ_n. Note that $\|u_{\lambda_n}\| < \infty$. Then following the same lines as in the proof of Lemma 4.3, we can obtain that $u_n \to u_0 \neq 0$ weakly in H. By Lemma 4.4, $I'(u_0) = 0$, $I(u_0) \leqslant \lim_{n\to\infty} I(u_{\lambda_n}) \leqslant c_1 < \frac{1}{N} S^{\frac{N}{2}}$. Let

$$m = \inf\{I(u) : u \in H, u \neq 0, I'(u) = 0\}.$$

Since $I'(u_0) = 0$, $m \leqslant I(u_0) < \frac{1}{N} S^{\frac{N}{2}}$. By the definition of m, there exists $\{v_n\} \subset H$ such that $v_n \neq 0$, $I(v_n) \to m$ and $I'(v_n) = 0$. Remark 2.5 implies that $\|v_n\| \geqslant \beta > 0$. Similar to [11, Proposition 4.2], we have $\|v_n\| < \infty$. Thus, $m > -\infty$. Following the same lines as in the proof of Lemma 4.3, we have $v_n \to v_0 \neq 0$ weakly in H. From Lemma 4.4, $I'(v_0) = 0$, $I(v_0) \leqslant m$. $I'(v_0) = 0$ implies that $I(v_0) \geqslant m$. Therefore, $v_0 \neq 0$ satisfies $I(v_0) = m$ and $I'(v_0) = 0$. □

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