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Sharp constants in the doubly weighted Hardy-Littlewood-Sobolev inequality

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Abstract It is well known that the doubly weighted Hardy-Littlewood-Sobolev inequality is as follows,

$$\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{f(x)g(y)}{|x|^{\alpha}|x-y|^{\lambda}|y|^{\beta}}dxdy\right|\leqslant B(p,q,\alpha,\lambda,\beta,n)\|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^q(\mathbb{R}^n)}.$$

The main purpose of this paper is to give the sharp constants $B(p,q,\alpha,\lambda,\beta,n)$ for the above inequality for three cases: (i) $p=1$ and $q=1$; (ii) $p=1$ and $1<q\leqslant\infty$, or $1<p\leqslant\infty$ and $q=1$; (iii) $1<p,q<\infty$ and $\frac{1}{p}+\frac{1}{q}=1$. In addition, the explicit bounds can be obtained for the case $1<p,q<\infty$ and $\frac{1}{p}+\frac{1}{q}>1$.

Keywords best constants, Hardy-Littlewood-Sobolev inequality, Schur's lemma

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1 Introduction

A classical inequality that was obtained by Hardy and Littlewood [5] when $n=1$ and by Sobolev [8] for general n, called Hardy-Littlewood-Sobolev inequality, states that

$$\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{f(x_1)g(x_2)}{|x_1-x_2|^{\lambda}}dx_1dx_2\right|\leqslant B(p,q,\lambda,n)\|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^q(\mathbb{R}^n)},$$

with $1<p,q<\infty$, $0<\lambda<n$ and $\frac{1}{p}+\frac{1}{q}+\frac{\lambda}{n}=2$.

Stein and Weiss [9] obtained the doubly weighted Hardy-Littlewood-Sobolev inequality, i.e., the following inequality

$$\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{f(x)g(y)}{|x|^{\alpha}|x-y|^{\lambda}|y|^{\beta}}dxdy\right|\leqslant B(p,q,\alpha,\lambda,\beta,n)\|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^q(\mathbb{R}^n)}$$

holds, provided that the following three conditions,

$$\frac{1}{p}+\frac{1}{q}+\frac{\alpha+\lambda+\beta}{n}=2, \tag{1.1}$$

$$\alpha+\beta\geqslant 0,\quad \alpha<\frac{n}{p'},\quad \beta<\frac{n}{q'},\quad \lambda<n, \tag{1.2}$$

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and

$$\frac{1}{p}+\frac{1}{q}\geqslant 1 \tag{1.3}$$

hold simultaneously.

Recently, some related developments for the weighted Hardy-Littlewood-Sobolev inequalities in the Heisenberg group have been established by [4].

In order to further study Hardy-Littlewood-Sobolev inequality from the operator's point of view, we first give a definition.

Definition 1. The bilinear integral operator $T_{\alpha,\lambda,\beta}$ on $L^p(\mathbb{R}^n)\times L^q(\mathbb{R}^n)$ $(0<p,q\leqslant\infty)$ is defined as

$$T_{\alpha,\lambda,\beta}(f,g)(x,y):=\frac{f(x)g(y)}{|x|^{\alpha}|x-y|^{\lambda}|y|^{\beta}}. \tag{1.4}$$

The operator $\mathcal{T}_{\alpha,\lambda,\beta}$ is defined by

$$\mathcal{T}_{\alpha,\lambda,\beta}g(x):=\frac{1}{|x|^{\alpha}}\int_{\mathbb{R}^n}\frac{g(y)}{|x-y|^{\lambda}|y|^{\beta}}dy. \tag{1.5}$$

Let $\mathcal{T}^t_{\alpha,\lambda,\beta}$ denote the transpose operator of $\mathcal{T}_{\alpha,\lambda,\beta}$. Obviously, we have

$$\mathcal{T}^t_{\alpha,\lambda,\beta}f(y)=\frac{1}{|y|^{\beta}}\int_{\mathbb{R}^n}\frac{f(x)}{|x|^{\alpha}|x-y|^{\lambda}}dx. \tag{1.6}$$

Note that the boundedness of $T_{\alpha,\lambda,\beta}$ from $L^p(\mathbb{R}^n)\times L^q(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n\times\mathbb{R}^n)$ is equivalent to that of $\mathcal{T}_{\alpha,\lambda,\beta}$ from $L^q(\mathbb{R}^n)$ to $L^{p'}(\mathbb{R}^n)$, and also is equivalent to that of $\mathcal{T}^t_{\alpha,\lambda,\beta}$ from $L^p(\mathbb{R}^n)$ to $L^{q'}(\mathbb{R}^n)$ with $1\leqslant p,q\leqslant\infty$.

Wu et al. [10] considered the doubly weighted Hardy-Littlewood-Sobolev inequality for the whole ranges of p and q, i.e., $0<p\leqslant\infty$ and $0<q\leqslant\infty$ and characterized the sufficient and necessary conditions, which ensure validity of the doubly weighted Hardy-Littlewood-Sobolev inequality.

The following results are obtained in [10].

Theorem A. *Let $1<p,q<\infty$. The operator $T_{\alpha,\lambda,\beta}$ defined by* (1.4) *is bounded from $L^p(\mathbb{R}^n)\times L^q(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n\times\mathbb{R}^n)$, if and only if the three conditions* (1.1)–(1.3) *hold simultaneously.*

Theorem B. *Let $1<p\leqslant\infty$ and $q=1$. The operator $T_{\alpha,\lambda,\beta}$ is bounded from $L^p(\mathbb{R}^n)\times L^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n\times\mathbb{R}^n)$, if and only if*

$$\frac{1}{p}+\frac{\alpha+\lambda+\beta}{n}=1, \tag{1.7}$$

and

$$\beta<0,\quad \alpha+\beta>0,\quad \alpha<\frac{n}{p'}. \tag{1.8}$$

Theorem C. *Let $p=q=1$. The operator $T_{\alpha,\lambda,\beta}$ is bounded from $L^1(\mathbb{R}^n)\times L^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n\times\mathbb{R}^n)$, if and only if $\alpha=\lambda=\beta=0$.*

Theorem D. *Suppose that the condition* I *is $0<p<1$ or $0<q<1$ and the condition* II *is $p=\infty$ and $1<q\leqslant\infty$. If one of the condition* I *and the condition* II *holds, then the operator $T_{\alpha,\lambda,\beta}$ is not bounded from $L^p(\mathbb{R}^n)\times L^q(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n\times\mathbb{R}^n)$ for every real numbers α,β and γ.*

In fact, we can use Figure 1 to indicate a domain where the operator $T_{\alpha,\lambda,\beta}$ may be bounded from $L^p(\mathbb{R}^n)\times L^q(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n\times\mathbb{R}^n)$. Theorems A, B, C and D clearly imply that the domain is just the closed triangle area denoted by $\triangle EFG$. Consequently, the point $(\frac{1}{p},\frac{1}{q})\in\triangle EFG$ is only necessary condition which makes the operator $T_{\alpha,\lambda,\beta}$ be bounded from $L^p(\mathbb{R}^n)\times L^q(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n\times\mathbb{R}^n)$. An important and interesting question is how to find the sharp bound of the operator $T_{\alpha,\lambda,\beta}$.

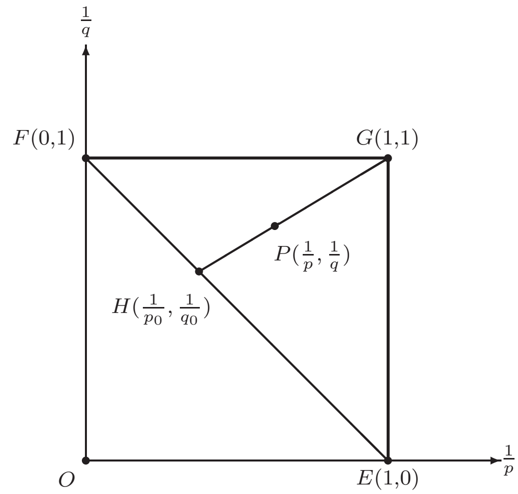

Figure 1 The set of points $(\frac{1}{p}, \frac{1}{q})$

In 1983, Lieb [6] obtained the sharp constants only when one of p and q equals 2 or $p = q$. When $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ hold, the sharp constant is given by Beckner in [1] and [2].

In this paper, we use the Selberg integral formula to give the sharp constants $B(p, q, \alpha, \lambda, \beta, n)$ as long as the point $(\frac{1}{p}, \frac{1}{q})$ is on the boundary of $\triangle EFG$. In addition, the explicit bounds can be obtained for the case $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} > 1$. It should be pointed out that for the case $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, we use a novel method to give the sharp constants.

For convenience, we denote p' as the dual number of p, the point $e_1 := (1, 0, \dots, 0) \in \mathbb{R}^n$ and define a constant $C_{d_1,d_2,n}$ related to d_1, d_2 and n as follows.

Definition 2. Let $d_1 < n$, $d_2 < n$ and $d_1 + d_2 > n$. Define

$$C_{d_1,d_2,n} := \pi^{\frac{n}{2}} \frac{\Gamma(\frac{n-d_1}{2})\Gamma(\frac{n-d_2}{2})\Gamma(\frac{d_1+d_2-n}{2})}{\Gamma(\frac{d_1}{2})\Gamma(\frac{d_2}{2})\Gamma(n - \frac{d_1+d_2}{2})}. \tag{1.9}$$

Obviously, we have $C_{d_1,d_2,n} = C_{d_2,d_1,n}$.

Definition 3. Suppose that the operator $T_{\alpha,\lambda,\beta}$ is bounded from $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n \times \mathbb{R}^n)$. The norm of the operator $T_{\alpha,\lambda,\beta}$ is defined by

$$\|T_{\alpha,\lambda,\beta}\|_{L^p(\mathbb{R}^n)\times L^q(\mathbb{R}^n)\to L^1(\mathbb{R}^n\times\mathbb{R}^n)} = \sup_{\|f\|_{L^p(\mathbb{R}^n)}\neq 0, \|g\|_{L^p(\mathbb{R}^n)}\neq 0} \frac{\|T_{\alpha,\lambda,\beta}(f,g)\|_{L^1(\mathbb{R}^n\times\mathbb{R}^n)}}{\|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^q(\mathbb{R}^n)}}.$$

Now we formulate our main results as follows.

Theorem 1.1. *If $1 < p \leqslant \infty$, $q = 1$, and the two conditions* (1.7) *and* (1.8) *hold, then we have*

$$\|T_{\alpha,\lambda,\beta}\|_{L^p(\mathbb{R}^n)\times L^1(\mathbb{R}^n)\to L^1(\mathbb{R}^n\times\mathbb{R}^n)} = (C_{p'\alpha,p'\lambda,n})^{\frac{1}{p'}}.$$

Theorem 1.2. *Let $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p < \infty$. If the two conditions* (1.1) *and* (1.2) *are satisfied, then we have*

$$\|T_{\alpha,\lambda,\beta}\|_{L^p(\mathbb{R}^n)\times L^q(\mathbb{R}^n)\to L^1(\mathbb{R}^n\times\mathbb{R}^n)} = C_{\alpha+\frac{n}{p},\lambda,n} = C_{\beta+\frac{n}{q},\lambda,n}.$$

Theorem 1.3. *If $p = q = 1$, then the norm of operator $T_{\alpha,\lambda,\beta}$ is satisfied as follows,*

$$\|T_{\alpha,\lambda,\beta}\|_{L^1(\mathbb{R}^n)\times L^1(\mathbb{R}^n)\to L^1(\mathbb{R}^n\times\mathbb{R}^n)} = 1.$$

For the case $1 < p, q < \infty$, we obtain an explicit upper bound estimate of the operator $T_{\alpha,\lambda,\beta}$.

Theorem 1.4. *Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} > 1$ and $\alpha + \beta > 0$. If the conditions* (1.1) *and* (1.2) *hold, then*

$$\|T_{\alpha,\lambda,\beta}\|_{L^p(\mathbb{R}^n)\times L^q(\mathbb{R}^n)\to L^1(\mathbb{R}^n\times\mathbb{R}^n)} \leqslant (C_{\frac{\alpha}{\theta}+\frac{n}{p_0},\frac{\lambda}{\theta},n})^{\theta} = (C_{\frac{\beta}{\theta}+\frac{n}{q_0},\frac{\lambda}{\theta},n})^{\theta},$$

where $p_0 = 1 + \frac{q'}{p'}$, $q_0 = 1 + \frac{p'}{q'}$ and $\theta = \frac{1}{p'} + \frac{1}{q'}$.

2 Some lemmas

To prove our theorems, we first provide some lemmas which will be used in the following.

Lemma 2.1. *If $1 \leqslant p, q \leqslant \infty$, then we have*

$$\|T_{\alpha,\lambda,\beta}\|_{L^p(\mathbb{R}^n)\times L^q(\mathbb{R}^n)\to L^1(\mathbb{R}^n\times\mathbb{R}^n)} = \|\mathcal{T}_{\alpha,\lambda,\beta}\|_{L^q(\mathbb{R}^n)\to L^{p'}(\mathbb{R}^n)} = \|\mathcal{T}^t_{\alpha,\lambda,\beta}\|_{L^p(\mathbb{R}^n)\to L^{q'}(\mathbb{R}^n)}.$$

Lemma 2.2 (See [3]). *Let $f_d(x) := |x|^{-d}, x \in \mathbb{R}^n$. If $d_1 < n$, $d_2 < n$ and $d_1 + d_2 > n$, then we have*

$$f_{d_1} * f_{d_2}(x) = C_{d_1,d_2,n}|x|^{-d_1-d_2+n}. \tag{2.1}$$

Lemma 2.3 (See [3]). *Let (X,μ) and (Y,ν) be two σ-finite measure spaces, where μ and ν are positive measures. Let $1 < p < \infty$ and $0 < A < \infty$. Suppose that $\mathcal{T}$ is the linear operator defined by*

$$\mathcal{T}f(x) = \int_Y K(x,y)f(y)d\nu(y)$$

and T^t is transpose operator of $\mathcal{T}$,

$$\mathcal{T}^t g(y) = \int_X K(x,y)g(x)d\mu(x),$$

where $K(\cdot,\cdot)$ is a nonnegative measurable function on $X \times Y$.

To avoid trivialities, we assume that there is a compactly supported, bounded, and positive ν-a.e. function h_1 on Y such that $\mathcal{T}(h_1) > 0$ μ-a.e. Then the following three statements are equivalent:

(I) *$\mathcal{T}$ maps $L^p(Y)$ into $L^p(X)$ with norm at most A*;

(II) *For all $B > A$ there is a measurable function h on Y that satisfies $0 < h < \infty$ ν-a.e., $0 < \mathcal{T}(h) < \infty$ μ-a.e., and such that*

$$\mathcal{T}^t(\mathcal{T}(h)^{\frac{p}{p'}}) \leqslant B^p h^{\frac{p}{p'}};$$

(III) *For all $B > A$ there are measurable functions u on X and v on Y such that $0 < u < \infty$ μ-a.e., $0 < v < \infty$ ν-a.e., and such that*

$$\mathcal{T}(v^{p'}) \leqslant Bu^{p'} \quad \text{and} \quad \mathcal{T}^t(u^p) \leqslant Bv^p.$$

We remark that the proof of Lemma 2.1 immediately follows from the elementary properties of functional analysis. The proof of Lemma 2.2 can be found in [3]. Lemma 2.3 is also called Schur's lemma, and its proof can be found in [3].

Lemma 2.4. *If $d_1, d_2, d_3 < n$ and $d_1 + d_2 + d_3 = 2n$, then*

$$C_{d_1,d_2,n} = C_{d_1,d_3,n} = C_{d_2,d_3,n}. \tag{2.2}$$

Proof. The equality (2.2) immediately follows from Definition 2. □

3 The proofs of theorems

Proof of Theorem 1.1. Without loss of generality, we always let $f, g \geqslant 0$. We conclude that

$$\begin{aligned}
\|T_{\alpha,\lambda,\beta}(f,g)\|_{L^1(\mathbb{R}^n\times\mathbb{R}^n)} &= \left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{f(x)g(y)}{|x|^\alpha|x-y|^\lambda|y|^\beta}dxdy\right| \\
&\leqslant \|g\|_{L^1(\mathbb{R}^n)}\left\|\int_{\mathbb{R}^n}\frac{f(x)}{|x|^\alpha|x-\cdot|^\lambda|\cdot|^\beta}dx\right\|_{L^\infty(\mathbb{R}^n)} \\
&\leqslant \|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^1(\mathbb{R}^n)}\left\|\left(\int_{\mathbb{R}^n}\frac{1}{|x|^{p'\alpha}|x-\cdot|^{p'\lambda}|\cdot|^{p'\beta}}dx\right)^{\frac{1}{p'}}\right\|_{L^\infty(\mathbb{R}^n)}.
\end{aligned}$$

Since the conditions (1.7) and (1.8) are satisfied, we obtain that

$$p'\alpha < n, \quad p'\lambda = n - p'(\alpha+\beta) < n, \quad p'\alpha + p'\lambda = n - p'\beta > n$$

and

$$p'\alpha + p'\lambda + p'\beta = n.$$

Thus it follows from Lemma 2.2 that

$$\int_{\mathbb{R}^n} \frac{1}{|x|^{p'\alpha}|x-y|^{p'\lambda}|y|^{p'\beta}} dx = |y|^{-p'\beta} C_{p'\alpha,p'\lambda,n} |y|^{-p'\alpha - p'\beta + n} = C_{p'\alpha,p'\lambda,n}.$$

Therefore, we have

$$\|T_{\alpha,\lambda,\beta}(f,g)\|_{L^1(\mathbb{R}^n\times\mathbb{R}^n)} \leqslant (C_{p'\alpha,p'\lambda,n})^{\frac{1}{p'}} \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^1(\mathbb{R}^n)},$$

which implies that

$$\|T_{\alpha,\lambda,\beta}\|_{L^p(\mathbb{R}^n)\times L^1(\mathbb{R}^n)\to L^1(\mathbb{R}^n\times\mathbb{R}^n)} \leqslant (C_{p'\alpha,p'\lambda,n})^{\frac{1}{p'}}. \tag{3.1}$$

Now we will prove the inverse inequality.

We consider the question for two cases: $p=\infty$ and $1<p<\infty$, respectively.

For the case $p=\infty$, set $f\equiv 1$. Then it follows from Lemma 2.2 that

$$\|\mathcal{T}^t_{\alpha,\lambda,\beta} 1\|_{L^\infty(\mathbb{R}^n)} = C_{\alpha,\lambda,n}.$$

This means that

$$\|\mathcal{T}^t_{\alpha,\lambda,\beta}\|_{L^\infty(\mathbb{R}^n)\to L^\infty(\mathbb{R}^n)} \geqslant C_{\alpha,\lambda,n}. \tag{3.2}$$

For the case $1<p<\infty$, let

$$h_y(x) := \frac{1}{|x|^\alpha |x-y|^\lambda |y|^\beta}.$$

A direct calculation leads to

$$\|h_{e_1}\|_{L^{p'}(\mathbb{R}^n)} = C^{\frac{1}{p'}}_{p'\alpha,p'\lambda,n}.$$

Let $f(x) = (h_{e_1}(x))^{p'-1}$. We easily have $\|f\|_{L^p(\mathbb{R}^n)} = (C_{p'\alpha,p'\lambda,n})^{\frac{1}{p}} < \infty$. It follows that $\mathcal{T}^t_{\alpha,\lambda,\beta} f(e_1) = C_{p'\alpha,p'\lambda,n}$. Now we consider the continuous property of

$$\mathcal{T}^t_{\alpha,\lambda,\beta} f(y) = \int_{\mathbb{R}^n} h_y(x) f(x) dx$$

on the point e_1.

We deduce from Hölder's inequality that

$$|\mathcal{T}^t_{\alpha,\lambda,\beta} f(y)| \leqslant \|f\|_{L^p(\mathbb{R}^n)} \|h_y\|_{L^{p'}(\mathbb{R}^n)} = \|f\|_{L^p(\mathbb{R}^n)} (C_{p'\alpha,p'\lambda,n})^{\frac{1}{p'}} = C_{p'\alpha,p'\lambda,n}. \tag{3.3}$$

Let y tend to e_1. Then it implies from Fatou's lemma that

$$\liminf_{y\to e_1} \mathcal{T}^t_{\alpha,\lambda,\beta} f(y) \geqslant \int_{\mathbb{R}^n} f(x) \liminf_{y\to e_1} h_y(x) dx = \int_{\mathbb{R}^n} f(x) h_{e_1}(x) dx = C_{p'\alpha,p'\lambda,n}. \tag{3.4}$$

By (3.3) and (3.4), we have $\mathcal{T}^t_{\alpha,\lambda,\beta} f(y)$ is continuous on e_1 and thus

$$\|\mathcal{T}^t_{\alpha,\lambda,\beta} f\|_{L^\infty(\mathbb{R}^n)} = C_{p'\alpha,p'\lambda,n}.$$

Consequently, we have that

$$\|\mathcal{T}^t_{\alpha,\lambda,\beta}\|_{L^p(\mathbb{R}^n)\to L^\infty(\mathbb{R}^n)} \geqslant \frac{\|\mathcal{T}^t_{\alpha,\lambda,\beta} f\|_{L^\infty(\mathbb{R}^n)}}{\|f\|_{L^p(\mathbb{R}^n)}} \geqslant (C_{p'\alpha,p'\lambda,n})^{\frac{1}{p'}}. \tag{3.5}$$

Thus, combining the inequality (3.2) with (3.5) yields that

$$\|\mathcal{T}^t_{\alpha,\lambda,\beta}\|_{L^p(\mathbb{R}^n)\to L^\infty(\mathbb{R}^n)} \geqslant (C_{p'\alpha,p'\lambda,n})^{\frac{1}{p'}}. \tag{3.6}$$

It immediately follows from Lemma 2.1 and the inequality (3.6) that

$$\|T_{\alpha,\lambda,\beta}\|_{L^p(\mathbb{R}^n)\times L^1(\mathbb{R}^n)\to L^1(\mathbb{R}^n\times\mathbb{R}^n)} \geqslant (C_{p'\alpha,p'\lambda,n})^{\frac{1}{p'}}. \tag{3.7}$$

Consequently, both the inequalities (3.1) and (3.7) evidently imply that

$$\|T_{\alpha,\lambda,\beta}\|_{L^p(\mathbb{R}^n)\times L^1(\mathbb{R}^n)\to L^1(\mathbb{R}^n\times\mathbb{R}^n)} = (C_{p'\alpha,p'\lambda,n})^{\frac{1}{p'}}. \qquad \square$$

Proof of Theorem 1.2. Since $\frac{1}{p}+\frac{1}{q}=1$, by Lemma 2.1, we merely show that

$$\|\mathcal{T}_{\alpha,\lambda,\beta}\|_{L^q(\mathbb{R}^n)\to L^q(\mathbb{R}^n)} = C_{\alpha+\frac{n}{p},\lambda,n} = C_{\beta+\frac{n}{q},\lambda,n}.$$

We choose two functions

$$u(x)=|x|^{-\frac{n}{pq}} \quad \text{and} \quad v(y)=|y|^{-\frac{n}{pq}}.$$

We easily check that u and v satisfy (III) of Lemma 2.3. In fact, we have

$$\mathcal{T}_{\alpha,\lambda,\beta}(v^{q'}) = C_{\beta+\frac{n}{q},\lambda,n}u^{q'}, \tag{3.8}$$

and

$$\mathcal{T}^t_{\alpha,\lambda,\beta}(u^{q}) = C_{\alpha+\frac{n}{p},\lambda,n}v^{q}. \tag{3.9}$$

Since the two conditions (1.1) and (1.2) are satisfied, and $1<p=q'<\infty$, we can obtain that

$$0<\alpha+\frac{n}{p}, \quad \text{and} \quad \lambda<n, \quad \beta+\frac{n}{q}<n.$$

Since

$$\left(\frac{n}{p}+\alpha\right)+\lambda+\left(\frac{n}{q}+\beta\right)=2n,$$

it follows from Lemma 2.4 that $C_{\beta+\frac{n}{q},\lambda,n}=C_{\alpha+\frac{n}{p},\lambda,n}$.

Thus Lemma 2.3 implies that $\mathcal{T}_{\alpha,\lambda,\beta}$ is bounded from $L^q(\mathbb{R}^n)$ to itself and

$$\|\mathcal{T}_{\alpha,\lambda,\beta}\|_{L^q(\mathbb{R}^n)\to L^q(\mathbb{R}^n)} \leqslant C_{\beta+\frac{n}{q},\lambda,n} = C_{\alpha+\frac{n}{p},\lambda,n}. \tag{3.10}$$

To complete the proof of Theorem 1.2, we have to show the inverse inequality.

Set

$$g_\varepsilon(y)=\chi_{|y|\leqslant 1}(y)|y|^{-\frac{n}{q}+\varepsilon}$$

with $\varepsilon>0$. For any fixed $x\in\mathbb{R}^n\backslash\{\mathbf{0}\}$, there must exist a rotation transformation denoted by A_x such that

$$A_xe_1=\frac{x}{|x|}.$$

By means of variable substitution, we can get that

$$\begin{aligned}
\mathcal{T}_{\alpha,\lambda,\beta}g_\varepsilon(x) &= |x|^{-\alpha}\int_{|y|\leqslant 1}|x-y|^{-\lambda}|y|^{-\frac{n}{q}-\beta+\varepsilon}dy\\
&= |x|^{-\alpha}\int_{||x|A_xy|\leqslant 1}||x|A_xe_1-|x|A_xy|^{-\lambda}||x|A_xy|^{-\frac{n}{q}-\beta+\varepsilon}d(|x|A_xy)\\
&= |x|^{-\alpha-\lambda-\beta-\frac{n}{q}+n+\varepsilon}\int_{|y|\leqslant\frac{1}{|x|}}|y-e_1|^{-\lambda}|y|^{-\frac{n}{q}-\beta+\varepsilon}dy\\
&= |x|^{-\frac{n}{q}+\varepsilon}\int_{|y|\leqslant\frac{1}{|x|}}|y-e_1|^{-\lambda}|y|^{-\frac{n}{q}-\beta+\varepsilon}dy.
\end{aligned}$$

Now fix a δ with $0<\delta<1$. We conclude that

$$\begin{aligned}\frac{\|\mathcal{T}_{\alpha,\lambda,\beta}g_\varepsilon\|^q_{L^q(\mathbb{R}^n)}}{\|g_\varepsilon\|^q_{L^q(\mathbb{R}^n)}}&\geqslant\frac{\|\mathcal{T}_{\alpha,\lambda,\beta}g_\varepsilon\|^q_{L^q(B_\delta(0))}}{\|g_\varepsilon\|^q_{L^q(\mathbb{R}^n)}}\\&\geqslant\frac{\int_{|x|\leqslant\delta}(|x|^{-\frac{n}{q}+\varepsilon}\int_{|y|\leqslant\frac{1}{\delta}}|y-e_1|^{-\lambda}|y|^{-\frac{n}{q}-\beta+\varepsilon}dy)^qdx}{\|g_\varepsilon\|^q_{L^q(\mathbb{R}^n)}}\\&=\frac{\int_{|x|\leqslant\delta}|x|^{-n+\varepsilon q}dx}{\|g_\varepsilon\|^q_{L^q(\mathbb{R}^n)}}\bigg(\int_{|y|\leqslant\frac{1}{\delta}}|y-e_1|^{-\lambda}|y|^{-\frac{n}{q}-\beta+\varepsilon}dy\bigg)^q\\&=\delta^{\varepsilon q}\bigg(\int_{|y|\leqslant\frac{1}{\delta}}|y-e_1|^{-\lambda}|y|^{-\frac{n}{q}-\beta+\varepsilon}dy\bigg)^q.\end{aligned}\tag{3.11}$$

It follows from Fadou's lemma and the inequality (3.11) that

$$\liminf_{\varepsilon\to0^+}\frac{\|\mathcal{T}_{\alpha,\lambda,\beta}g_\varepsilon\|^q_{L^q(\mathbb{R}^n)}}{\|g_\varepsilon\|^q_{L^q(\mathbb{R}^n)}}\geqslant\bigg(\int_{|y|\leqslant\frac{1}{\delta}}|y-e_1|^{-\lambda}|y|^{-\frac{n}{q}-\beta}dy\bigg)^q.$$

Letting $\delta\to0$ and using Lemma 2.2, we conclude that

$$\|\mathcal{T}_{\alpha,\lambda,\beta}\|_{L^q(\mathbb{R}^n)\to L^q(\mathbb{R}^n)}\geqslant\liminf_{\varepsilon\to0^+}\frac{\|\mathcal{T}_{\alpha,\lambda,\beta}g_\varepsilon\|_{L^q(\mathbb{R}^n)}}{\|g_\varepsilon\|_{L^q(\mathbb{R}^n)}}\geqslant\int_{\mathbb{R}^n}|y-e_1|^{-\lambda}|y|^{-\frac{n}{q}-\beta}dy=C_{\beta+\frac{n}{q},\lambda,n}.\tag{3.12}$$

Consequently, combining the inequality (3.10) with the inequality (3.12) immediately yields that

$$\|T_{\alpha,\lambda,\beta}\|_{L^p(\mathbb{R}^n)\times L^q(\mathbb{R}^n)\to L^1(\mathbb{R}^n\times\mathbb{R}^n)}=\|\mathcal{T}_{\alpha,\lambda,\beta}\|_{L^q(\mathbb{R}^n)\to L^q(\mathbb{R}^n)}=C_{\alpha+\frac{n}{p},\lambda,n}=C_{\beta+\frac{n}{q},\lambda,n}.$$

This finishes the proof of Theorem 1.2. □

Proof of Theorem 1.3. According to Theorem C and the boundedness of the operator $T_{\alpha,\lambda,\beta}$ from $L^1(\mathbb{R}^n)\times L^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n\times\mathbb{R}^n)$, we have that $\alpha=\lambda=\beta=0$. Evidently, we have that

$$\|T_{\alpha,\lambda,\beta}\|_{L^1(\mathbb{R}^n)\times L^1(\mathbb{R}^n)\to L^1(\mathbb{R}^n\times\mathbb{R}^n)}=\|T_{0,0,0}\|_{L^1(\mathbb{R}^n)\times L^1(\mathbb{R}^n)\to L^1(\mathbb{R}^n\times\mathbb{R}^n)}=1.$$ □

Proof of Theorem 1.4. Without loss of generality, we let $f,g\geqslant0$. Let $p_0=1+\frac{q'}{p'}$, $q_0=1+\frac{p'}{q'}$ and $\theta=\frac{1}{p'}+\frac{1}{q'}$. Obviously we have that

$$\frac{1}{p_0}+\frac{1}{q_0}=1,\quad(1-\theta)+\frac{\theta}{p_0}=\frac{1}{p}\quad\text{and}\quad(1-\theta)+\frac{\theta}{q_0}=\frac{1}{q}.$$

Rewrite $T_{\alpha,\lambda,\beta}(f,g)$ as

$$T_{\alpha,\lambda,\beta}(f,g)(x,y)=|f(x)|^{p(1-\theta)}|g(y)|^{q(1-\theta)}\frac{|f(x)|^{\frac{p\theta}{p_0}}|g(y)|^{\frac{q\theta}{q_0}}}{|x|^\alpha|x-y|^\lambda|y|^\beta}.$$

Notice that

$$0<\theta=\frac{1}{p'}+\frac{1}{q'}=2-\frac{1}{p}-\frac{1}{q}<1.$$

It follows from Hölder's inequality that

$$\begin{aligned}\|T_{\alpha,\lambda,\beta}(f,g)\|_{L^1(\mathbb{R}^n\times\mathbb{R}^n)}&\leqslant\|f^pg^q\|^{1-\theta}_{L^1(\mathbb{R}^n\times\mathbb{R}^n)}\bigg(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f(x)|^{\frac{p}{p_0}}|g(y)|^{\frac{q}{q_0}}}{|x|^{\frac{\alpha}{\theta}}|x-y|^{\frac{\lambda}{\theta}}|y|^{\frac{\beta}{\theta}}}dxdy\bigg)^\theta\\&=\|f\|^{p(1-\theta)}_{L^p(\mathbb{R}^n)}\|g\|^{q(1-\theta)}_{L^q(\mathbb{R}^n)}\bigg(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f(x)|^{\frac{p}{p_0}}|g(y)|^{\frac{q}{q_0}}}{|x|^{\frac{\alpha}{\theta}}|x-y|^{\frac{\lambda}{\theta}}|y|^{\frac{\beta}{\theta}}}dxdy\bigg)^\theta.\end{aligned}\tag{3.13}$$

Setting $F=f^{\frac{p}{p_0}}$ and $G=g^{\frac{q}{q_0}}$, we have

$$\|F\|_{L^{p_0}(\mathbb{R}^n)}=\|f\|^{\frac{p}{p_0}}_{L^p(\mathbb{R}^n)}\quad\text{and}\quad\|G\|_{L^{q_0}(\mathbb{R}^n)}=\|g\|^{\frac{q}{q_0}}_{L^q(\mathbb{R}^n)}.$$

Since $p_0 = 1+\frac{q'}{p'}$, $q_0 = 1+\frac{p'}{q'}$ and $\theta = \frac{1}{p'}+\frac{1}{q'}$ and the conditions (1.1) and (1.2) hold, a straightforward calculation shows that

$$\frac{\alpha}{\theta}+\frac{\lambda}{\theta}+\frac{\beta}{\theta}=n,$$
$$\frac{\alpha}{\theta}=\frac{\alpha p'q'}{p'+q'}<n\frac{q'}{p'+q'}=\frac{n}{q_0}=\frac{n}{p_0'}$$

and

$$\frac{\beta}{\theta}=\frac{\beta p'q'}{p'+q'}<n\frac{p'}{p'+q'}=\frac{n}{p_0}=\frac{n}{q_0'}.$$

Set $\alpha' = \frac{\alpha}{\theta}$, $\lambda' = \frac{\lambda}{\theta}$ and $\beta' = \frac{\beta}{\theta}$. We clearly have that

$$\frac{1}{p_0}+\frac{1}{q_0}=1, \tag{3.14}$$
$$\frac{1}{p_0}+\frac{1}{q_0}+\frac{\alpha'+\lambda'+\beta'}{n}=2, \tag{3.15}$$
$$\alpha'+\beta'>0, \quad \alpha'<\frac{n}{p_0'}, \quad \beta'<\frac{n}{q_0'}, \quad \lambda'<n. \tag{3.16}$$

Clearly by (3.14)–(3.16), we can easily verify that the functions F, G and the indexes $p_0, q_0, \alpha', \lambda'$ and β' satisfy all the conditions in Theorem 1.2, so we conclude from the inequality (3.13) that

$$\begin{aligned}\|T_{\alpha,\lambda,\beta}(f,g)\|_{L^1(\mathbb{R}^n\times\mathbb{R}^n)} &\leqslant (C_{\frac{\alpha}{\theta}+\frac{n}{p_0},\frac{\lambda}{\theta},n})^{\theta}\|f\|^{p(1-\theta)}_{L^p(\mathbb{R}^n)}\|g\|^{q(1-\theta)}_{L^q(\mathbb{R}^n)}\|F\|^{\theta}_{L^{p_0}(\mathbb{R}^n)}\|G\|^{\theta}_{L^{q_0}(\mathbb{R}^n)}\\ &= (C_{\frac{\alpha}{\theta}+\frac{n}{p_0},\frac{\lambda}{\theta},n})^{\theta}\|f\|^{p(1-\theta)}_{L^p(\mathbb{R}^n)}\|g\|^{q(1-\theta)}_{L^q(\mathbb{R}^n)}\|f\|^{\frac{p\theta}{p_0}}_{L^p(\mathbb{R}^n)}\|g\|^{\frac{q\theta}{q_0}}_{L^q(\mathbb{R}^n)}\\ &= (C_{\frac{\alpha}{\theta}+\frac{n}{p_0},\frac{\lambda}{\theta},n})^{\theta}\|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^q(\mathbb{R}^n)}.\end{aligned}$$

This means that

$$\|T_{\alpha,\lambda,\beta}\|_{L^p(\mathbb{R}^n)\times L^q(\mathbb{R}^n)\to L^1(\mathbb{R}^n\times\mathbb{R}^n)} \leqslant (C_{\frac{\alpha}{\theta}+\frac{n}{p_0},\frac{\lambda}{\theta},n})^{\theta} = (C_{\frac{\beta}{\theta}+\frac{n}{q_0},\frac{\lambda}{\theta},n})^{\theta}.$$

This completes the proof of Theorem 1.4. □

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