SCIENCE CHINA
Mathematics

• REVIEWS •

December 2013 Vol. 56 No. 12: 2411–2438
doi: 10.1007/s11425-013-4660-7

Some progress in spectral methods

Dedicated to Professor Shi Zhong-Ci on the Occasion of his 80th Birthday

GUO BenYu[1,2,3]

[1]*Department of Mathematics, Shanghai Normal University, Shanghai 200234, China;*
[2]*Scientific Computing Key Laboratory of Shanghai Universities, Shanghai 200234, China;*
[3]*Division of Computational Science, E-institute of Shanghai Universities, Shanghai 200234, China*
Email: byguo@shnu.edu.cn

Received March 12, 2013; accepted May 19, 2013; published online May 29, 2013

Abstract In this paper, we review some results on the spectral methods. We first consider the Jacobi spectral method and the generalized Jacobi spectral method for various problems, including degenerated and singular differential equations. Then we present the generalized Jacobi quasi-orthogonal approximation and its applications to the spectral element methods for high order problems with mixed inhomogeneous boundary conditions. We also discuss the related spectral methods for non-rectangular domains and the irrational spectral methods for unbounded domains. Next, we consider the Hermite spectral method and the generalized Hermite spectral method with their applications. Finally, we consider the Laguerre spectral method and the generalized Laguerre spectral method for many problems defined on unbounded domains. We also present the generalized Laguerre quasi-orthogonal approximation and its applications to certain problems of non-standard type and exterior problems.

Keywords Jacobi, Hermite and Laguerre spectral approximations, Jacobi and Laguerre quasi-orthogonal approximations, spectral and spectral element methods, degenerated and singular problems, problems on non-rectangular and unbounded domains, problems of non-standard type, exterior problems

MSC(2010) 65M70, 65N35, 41A10, 41A20, 41A30, 35J25, 35J35, 35K20, 35M13

Citation: Guo B Y. Some progress in spectral methods. Sci China Math, 2013, 56: 2411–2438, doi: 10.1007/s11425-013-4660-7

1 Introduction

The spectral methods have been successfully used in scientific computations, see the books of Bernardi and Maday [8, 9], Bernardi et al. [10], Boyd [13], Canuto et al. [14–16], Funaro [25], Gottlieb and Orszag [27], Guo [30], Hesthaven et al. [82], Shen and Tang [104], Shen et al. [105], Karniadakis and Sherwin [87], the review papers of Guo [38], Guo et al. [76], and Shen and Wang [111], and the references therein.

The traditional spectral methods are available for periodic problems and many problems defined on rectangular domains. Their mathematical foundations are the Fourier, Legendre and Chebyshev approximations. Guo [34, 35], and Guo and Wang [54, 56] developed the Jacobi approximation, and proposed the Jacobi spectral method for degenerated problems. We also refer to the work of Babuška and Guo [3], Funaro [25], and Ma and Sun [95]. Later, Guo et al. [47, 48] provided the generalized Jacobi approximation, which leads to a class of new spectral methods for high order problems. It is also appreciated for singular problems. Recently, Guo et al. [53], and Guo and Wang [60] proposed the Jacobi quasi-orthogonal approximation, which plays an important role in the spectral element methods for mixed inhomogeneous boundary value problems of various differential equations.

The Jacobi approximation also serves as a powerful tool for other numerical algorithms. Bernardi and Maday [6], and Guo and Huang [39] used certain specific Jacobi approximations for axisymmetric and spherically symmetrical domains. Dubiner [21], Guo and Wang [58], Li et al. [91], Owens [100], and Shen et al. [113] considered the spectral methods for triangles. Guo and Jia [40], and Jia and Guo [84] developed the spectral element methods on polygons. Some authors studied pseudospectral element methods with their applications, see [10, 16, 87] and the references therein. We could also use the Jacobi approximation, coupled with variable transformation, to solve some problems defined on unbounded domains, see [31, 33, 36, 37].

The Jacobi approximation is related to several irrational spectral methods for unbounded domains. In the early work, the used basis functions were induced by the Legendre and Chebyshev polynomials, see the work of Boyd [11, 12], Christov [19], Guo and Shen [45], Guo et al. [49, 50], Guo and Wang [64], and Wang and Guo [135, 136]. Later, Guo and Shen [46], and Wang and Guo [137] used the basis functions induced by the Jacobi polynomials. Recently, Guo and Yi [72], and Yi and Guo [150] proposed the irrational spectral methods with the basis functions induced by the generalized Jacobi functions, which match various boundary conditions at infinity closely.

We could solve many problems on the whole space directly by the Hermite spectral method. Guo [32], and Xu and Guo [143] studied the approximation using the Hermite polynomials, and designed the spectral methods for the whole line. Meanwhile, Funaro and Kavin [26], Guo et al. [51], Ma et al. [96], Tang et al. [121] and Xiang and Wang [147] developed the spectral methods by using several kinds of Hermite functions. Recently, Guo and Zhang [78] and Zhang and Guo [153] proposed the new generalized Hermite approximations fitting the asymptotic behaviors of exact solutions properly.

On the other hand, Funaro [24], Guo and Shen [44], Guo et al. [59], Guo and Zhang [80], Maday et al. [98], and Xu and Guo [143] developed the approximations using the Laguerre and generalized Laguerre polynomials, and provided the spectral methods for the half line. Guo and Ma [43], Guo and Zhang [81], and Shen [102] studied the approximations by using the Laguerre and generalized Laguerre functions, and the related spectral methods. Recently, Guo and Zhang [77], and Zhang and Guo [152] considered the more general Laguerre approximations and the Laguerre quasi-orthogonal approximations, which are specially appropriate for problems of non-standard type, and exterior problems.

This paper is organized as follows. In the next section, we review the recent results on the Jacobi and generalized spectral methods, the spectral methods using the Jacobi quasi-orthogonal approximation, the spectral methods for non-rectangular domains, and the irrational spectral methods for unbounded domains. In Section 3, we review the recent results on the spectral methods using the Hermite polynomials and functions, and the generalized Hermite spectral methods. We also review the recent results on the Laguerre and generalized Laguerre spectral methods, the spectral methods using the Laguerre functions and the Laguerre quasi-orthogonal approximation, and the spectral methods for problems of non-standard type, and exterior problems.

2 Jacobi spectral method

2.1 Jacobi spectral approximation

The Jacobi spectral method and the related spectral methods have been used widely for numerical solutions of differential equations defined on various bounded domains.

Let $\Lambda = \{x \mid |x| < 1\}$ and $\chi(x)$ be a certain weight function. For any integer $r \geqslant 0$, we define the weighted Sobolev spaces $H^r_\chi(\Lambda)$ and $H^r_{0,\chi}(\Lambda)$ in the usual way, equipped with the semi-norm $|v|_{r,\chi}$ and the norm $\|v\|_{r,\chi}$. In particular, the inner product and the norm of the space $L^2_\chi(\Lambda)$ are denoted by $(u, v)_\chi$ and $\|v\|_\chi$, respectively.

Let $\alpha, \beta > -1$, and $J^{(\alpha,\beta)}_l(x)$ be the Jacobi polynomial of degree l. The Jacobi weight function $\chi^{(\alpha,\beta)}(x) = (1-x)^\alpha(1+x)^\beta$. The set of all $J^{(\alpha,\beta)}_l(x)$ is a complete $L^2_{\chi^{(\alpha,\beta)}}(\Lambda)$-orthogonal system.

For any integer $N \geqslant 0$, $\mathcal{P}_N(\Lambda)$ stands for the set of all algebraic polynomials of degree at most N,

$$ {}_0\mathcal{P}_N(\Lambda) = \{v \mid v \in \mathcal{P}_N(\Lambda), v(-1) = 0\}, \quad \mathcal{P}_N^0(\Lambda) = \{v \mid v \in \mathcal{P}_N(\Lambda), v(-1) = v(1) = 0\}. $$

We denote by c a generic positive constant independent of any function and N.

The $L^2_{\chi^{(\alpha,\beta)}}(\Lambda)$-orthogonal projection $P_{N,\alpha,\beta} : L^2_{\chi^{(\alpha,\beta)}}(\Lambda) \to \mathcal{P}_N(\Lambda)$ is defined by

$$ (P_{N,\alpha,\beta} v - v, \phi)_{\chi^{(\alpha,\beta)}} = 0, \quad \forall\, \phi \in \mathcal{P}_N(\Lambda). $$

For description of approximation errors, we introduce the Jacobi weighted space $H^r_{\chi^{(\alpha,\beta)},A}(\Lambda)$ for any integer $r \geqslant 0$, equipped with the norm $\|v\|_{r,\chi^{(\alpha,\beta)},A} = (\sum_{k=0}^{r} \|\partial_x^k v\|^2_{\chi^{(\alpha+k,\beta+k)}})^{\frac{1}{2}}$. It was proved by Guo and Wang [56] that for any $v \in H^r_{\chi^{(\alpha,\beta)},A}(\Lambda)$ and integers $r \geqslant 0$, $r \leqslant N+1$,

$$ \|P_{N,\alpha,\beta} v - v\|_{\chi^{(\alpha,\beta)}} \leqslant c_{\alpha,\beta} (N(N+\alpha+\beta))^{-\frac{r}{2}} \|\partial_x^r v\|_{\chi^{(r+\alpha,r+\beta)}}, $$

where $c_{\alpha,\beta}$ is an explicit function of α and β.

In many practical problems, the coefficients of derivatives of different orders may degenerate in different ways. In these cases, we should compare the numerical solutions with the exact solutions in certain non-uniformly weighted spaces. For this purpose, we let $\alpha, \beta, \gamma, \delta > -1$, and define the space $H^1_{\alpha,\beta,\gamma,\delta}(\Lambda)$, equipped with the norm $\|v\|_{1,\alpha,\beta,\gamma,\delta} = (|v|^2_{1,\chi^{(\alpha,\beta)}} + \|v\|^2_{\chi^{(\gamma,\delta)}})^{\frac{1}{2}}$. Let

$$ a_{\alpha,\beta,\gamma,\delta}(u, v) = (\partial_x u, \partial_x v)_{\chi^{(\alpha,\beta)}} + (u, v)_{\chi^{(\gamma,\delta)}}, \quad \forall\, u, v \in H^1_{\alpha,\beta,\gamma,\delta}(\Lambda). $$

The orthogonal projection $P^1_{N,\alpha,\beta,\gamma,\delta}(\Lambda) : H^1_{\alpha,\beta,\gamma,\delta}(\Lambda) \to \mathcal{P}_N(\Lambda)$ is defined by

$$ a_{\alpha,\beta,\gamma,\delta}(P^1_{N,\alpha,\beta,\gamma,\delta} v - v, \phi) = 0, \quad \forall\, \phi \in \mathcal{P}_N(\Lambda). $$

Let $\alpha \leqslant \gamma + 2$ and $\beta \leqslant \delta + 2$. If $\partial_x^r v \in L^2_{\chi^{(\alpha+r-1,\beta+r-1)}}(\Lambda)$ and integers $1 \leqslant r \leqslant N+1$, then

$$ \|P^1_{N,\alpha,\beta,\gamma,\delta} v - v\|_{1,\alpha,\beta,\gamma,\delta} \leqslant c_{\alpha,\beta} (N(N+\alpha+\beta))^{\frac{1-r}{2}} \|\partial_x^r v\|_{\chi^{(\alpha+r-1,\beta+r-1)}}. \tag{2.1} $$

If, in addition, $\alpha \leqslant \gamma + 1$ and $\beta \leqslant \delta + 1$, then

$$ \|P^1_{N,\alpha,\beta,\gamma,\delta} v - v\|_{\chi^{(\gamma,\delta)}} \leqslant c_{\alpha,\beta} (N(N+\alpha+\beta))^{-\frac{r}{2}} \|\partial_x^r v\|_{\chi^{(\alpha+r-1,\beta+r-1)}}. \tag{2.2} $$

In some problems, the approximated functions vanish at one of the extreme points, say $x = -1$. So we need another projection. Let

$$ {}_0H^1_{\alpha,\beta,\gamma,\delta}(\Lambda) = \{v \mid v \in H^1_{\alpha,\beta,\gamma,\delta}(\Lambda) \text{ and } v(-1) = 0\}. $$

The orthogonal projection ${}_0P^1_{N,\alpha,\beta,\gamma,\delta} :\ {}_0H^1_{\alpha,\beta,\gamma,\delta}(\Lambda) \to {}_0\mathcal{P}_N(\Lambda)$ is defined by

$$ a_{\alpha,\beta,\gamma,\delta}({}_0P^1_N v - v, \phi) = 0, \quad \forall\, \phi \in {}_0\mathcal{P}_N(\Lambda). $$

We have the error estimates similar to (2.1) and (2.2).

In studying movements of fluid flows with non-slip walls, populations of bud worms with lethal boundary conditions, and some problems in other topics, the homogenous boundary conditions are imposed. In these cases, we set

$$ H^1_{0,\alpha,\beta,\gamma,\delta}(\Lambda) = \{v \mid v \in H_{\alpha,\beta,\gamma,\delta}(\Lambda) \text{ and } v(-1) = v(1) = 0\}. $$

The orthogonal projection $P^{1,0}_{\alpha,\beta,\gamma,\delta} : H^1_{0,\alpha,\beta,\gamma,\delta}(\Lambda) \to \mathcal{P}^0_N(\Lambda)$ is defined by

$$ a_{\alpha,\beta,\gamma,\delta}(P^{1,0}_{N,\alpha,\beta,\gamma,\delta} v - v, \phi) = 0, \quad \phi \in \mathcal{P}^0_N(\Lambda). $$

We also have the approximation results similar to (2.1) and (2.2).

As an example, we consider the simplest model problem

$$-\partial_x(k(x)\partial_x U(x)) + b(x)U(x) = f(x), \quad x \in \Lambda,$$

where $k(x) \geqslant 0, b(x) \geqslant 0$ and $f(x)$ are given functions. Assume that $k(x)$ and $b(x)$ degenerate in such a way that $k(x) \sim \chi^{(\alpha,\beta)}(x), b(x) \sim \chi^{(\gamma,\delta)}(x)$. Also suppose that $k(x)U(x)\partial_x U(x) \to 0$ as $|x| \to 1$. A weak formulation of the above problem is to find $U \in H^1_{\alpha,\beta,\gamma,\delta}(\Lambda)$, satisfying

$$(\partial_x U, \partial_x v)_k + (bU, v) = (f, v), \quad \forall\, v \in H^1_{\alpha,\beta,\gamma,\delta}(\Lambda).$$

If $f \in (H^1_{\alpha,\beta,\gamma,\delta}(\Lambda))'$, then it has a unique solution. Let $V_N(\Lambda) = H^1_{\alpha,\beta,\gamma,\delta}(\Lambda) \cap \mathcal{P}_N(\Lambda)$. The corresponding spectral method is to seek $u_N \in V_N(\Lambda)$ such that

$$(\partial_x u_N, \partial_x \phi)_k + (bu_N, \phi) = (f, \phi), \quad \forall\, \phi \in V_N(\Lambda).$$

In actual computation, we take the proper Jacobi polynomials as the basis functions fitting the singularity of the exact solution. Then we compare the exact solution with the numerical solution in the space $H^1_{\alpha,\beta,\gamma,\delta}(\Lambda)$. If $k(x)$ degenerates at several distinct points, then we can use the Jacobi spectral method coupled with domain decomposition.

Guo and Wang [55], and Wang and Guo [124] investigated the multiple-dimensional Jacobi spectral method for some problems, in which the coefficients of differential equations degenerate, or the source terms and the boundary values grow up somewhere. Recently, Shen and Wang [112] developed the sparse spectral scheme of multiple-dimensional problems, which keeps the same numerical accuracy as the usual spectral scheme for certain specific solutions, but saves much computational time. The key point is to use the hyperbolic cross approximation. To show this, let $\mathbb{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}, \mathbb{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)^{\mathrm{T}}, \mathbb{\beta} = (\beta_1, \beta_2, \ldots, \beta_n)^{\mathrm{T}}$ and $\mathbb{l} = (l_1, l_2, \ldots, l_n)^{\mathrm{T}}$. We set

$$\mathbb{J}_{\mathbb{l}}^{(\mathbb{\alpha},\mathbb{\beta})}(\mathbb{x}) = \prod_{i=1}^{n} J_{l_i}^{(\alpha_i,\beta_i)}(x_i), \quad \chi^{(\mathbb{\alpha},\mathbb{\beta})}(\mathbb{x}) = \prod_{i=1}^{n} \chi^{(\alpha_i,\beta_i)}(x_i).$$

Furthermore,

$$\bar{\mathcal{P}}_N(\Lambda^n) = \mathrm{span}\left\{ \mathbb{J}_{\mathbb{l}}^{(\mathbb{\alpha},\mathbb{\beta})}(\mathbb{x}) \,\middle|\, 1 \leqslant \prod_{i=1}^{n} \max(1, l_i) \leqslant N \right\}.$$

The orthogonal projection from $L^2_{\chi^{(\mathbb{\alpha},\mathbb{\beta})}}(\Lambda^n)$ onto $\bar{\mathcal{P}}_N(\Lambda^n)$ is defined by

$$(P_{N,\mathbb{\alpha},\mathbb{\beta}} v - v, \mathbb{\phi})_{L^2_{\chi^{(\mathbb{\alpha},\mathbb{\beta})}}(\Lambda^n)} = 0, \quad \forall\, \mathbb{\phi} \in \bar{\mathcal{P}}_N(\Lambda^n).$$

Since $\mathrm{Card}(\bar{\mathcal{P}}_N(\Lambda^n)) = \lambda_n N (\ln N)^{n-1}$, λ_n being a positive constant depending on n, such orthogonal projection reduces the $\lambda_n N^n$ operations for the standard multiple-dimensional orthogonal projection to the $\lambda_n N(\ln N)^{n-1}$ operations.

The second order Jacobi orthogonal approximation and the related interpolation were considered by Guo et al. [70], and Wan et al. [123], which are applicable to the spectral and pseudospectral methods of fourth order problems. For the mapped Jacobi spectral methods, we refer to the work of Shen and Wang [106], and Wang and Shen [129], which oftentimes provide better numerical results.

The Gegenbauer approximation and the related spectral methods were studied by Guo [34]. In particular, Babuška and Guo [2,3] considered the Gegenbauer approximation in the Jacobi weighted Besov space. The sharpest inverse estimate was obtained, which leads to the optimal error estimates of the p-version of finite element method for the Poisson equation. For other applications of Gegenbauer approximation to the analysis of finite element method, could be found in [85, 115] and the references therein. The Gegenbauer approximation was also used early for recovering the Jibbs phenomena for piecewise analytic functions, see the work of Gottlieb and Shu [28, 29].

2.2 Generalized Jacobi spectral method

Guo et al. [47,48] proposed the generalized Jacobi orthogonal approximation, with arbitrary real parameters α and β.

Let $[\alpha]$ be the largest integer $\leqslant \alpha$. Furthermore, $\hat{\alpha}=\bar{\alpha}=-\alpha$ for $\alpha\leqslant -1$, and $\hat{\alpha}=0,\bar{\alpha}=\alpha$ otherwise. The notation $\hat{\beta},\bar{\beta}$ and $[\beta]$ have the same meanings. Let $\bar{l}_{\alpha,\beta}=[\hat{\alpha}]+[\hat{\beta}]$. The generalized Jacobi functions are defined by

$$\bar{J}_l^{(\alpha,\beta)}(x)=\chi^{(\hat{\alpha},\hat{\beta})}(x)J_{l-\bar{l}_{\alpha,\beta}}^{(\bar{\alpha},\bar{\beta})}(x),\quad l\geqslant \bar{l}_{\alpha,\beta},$$

which form a complete $L^2_{\chi^{(\alpha,\beta)}}(\Lambda)$-orthogonal system.

Let

$$Q_{N,\alpha,\beta}(\Lambda)=\mathrm{span}\{\bar{J}_l^{(\alpha,\beta)}(x),\bar{l}_{\alpha,\beta}\leqslant l\leqslant N\}.$$

The orthogonal projection $P_{N,\alpha,\beta}:L^2_{\chi^{(\alpha,\beta)}}(\Lambda)\to Q_{N,\alpha,\beta}(\Lambda)$ is defined by

$$(P_{N,\alpha,\beta}v-v,\phi)_{\chi^{(\alpha,\beta)}}=0,\quad \forall\,\phi\in Q_{N,\alpha,\beta}(\Lambda).$$

If $\partial_x^r v\in L^2_{\chi^{(\alpha+r,\beta+r)}}(\Lambda)$ and one of the following conditions holds,

(i) α is a negative integer and $\beta>-1$,

(ii) β is a negative integer and $\alpha>-1$,

(iii) α and β are negative integers,

then for integers $r\geqslant 1, 0\leqslant k\leqslant r$,

$$\|\partial_x^k(P_{N,\alpha,\beta}v-v)\|_{\chi^{(\alpha+k,\beta+k)}}\leqslant c_{\alpha,\beta}N^{k-r}\|\partial_x^r v\|_{\chi^{(\alpha+r,\beta+r)}}. \tag{2.3}$$

The above approximation was used for the spectral methods of high order problems, see [47,48]. Based on this approximation, Shen [103], and Shen and Wang [107,108] developed the spectral method with its applications to KdV-like equations. Ma and Sun [94,95] used the similar trick for some problems of odd order, and obtained the better error estimates. On the other hand, this approximation with $\alpha,\beta\leqslant -1$ is suitable for certain problems with coefficients growing up somewhere.

We now focus on the specific case with $\alpha=-m$ and $\beta=-n$, m and n being positive integers. For notational convenience, we denote $Q_{N,-m,-n}(\Lambda)$ and $P_{N,-m,-n}v$ by $\bar{Q}_{N,m,n}(\Lambda)$ and $\bar{P}_{N,m,n}v$, respectively. We introduce the space $H^r_{m,n,A}(\Lambda)$, equipped with norm $\|v\|_{H^r_{m,n,A}}=(\sum_{k=0}^r\|\partial_x^k v\|^2_{\chi^{(-m+k,-n+k)}})^{\frac{1}{2}}$. For $r\geqslant\max(m,n)$, we define the space

$$H^r_{0,m,n,A}(\Lambda)=\{v\in H^r_{m,n,A}(\Lambda)\mid \partial_x^k v(-1)=0 \text{ for } 0\leqslant k\leqslant n-1, \text{ and } \partial_x^k v(1)=0 \text{ for } 0\leqslant k\leqslant m-1\}.$$

For any integer $\mu\geqslant\max(m,n)$, the operator $\bar{P}^{\mu,0}_{N,m,n}:H^\mu_{0,m,n,A}(\Lambda)\to\bar{Q}_{N,m,n}(\Lambda)$ is defined by

$$(\partial_x^\mu(v-\bar{P}^{\mu,0}_{N,m,n}v),\partial_x^\mu\phi)_{\chi^{(-m+\mu,-n+\mu)}}=0,\quad\forall\,\phi\in\bar{Q}_{N,m,n}(\Lambda).$$

In fact, $\bar{P}^{\mu,0}_{N,m,n}v=\bar{P}_{N,m,n}v$ for any $v\in H^\mu_{0,m,n,A}(\Lambda)$. Thus, if $v\in H^\mu_{0,m,n,A}(\Lambda)$, $\partial_x^r v\in L^2_{\chi^{(-m+r,-n+r)}}(\Lambda)$ and integers $m,n,r\geqslant 1, N\geqslant m+n, 0\leqslant k\leqslant r\leqslant N+1,\mu\geqslant\max(m,n,k)$, then

$$\|\partial_x^k(v-\bar{P}^{\mu,0}_{N,m,n}v)\|_{\chi^{(-m+k,-n+k)}}\leqslant cN^{k-r}\|\partial_x^r v\|_{\chi^{(-m+r,-n+r)}}. \tag{2.4}$$

Sun and Guo [116] also considered the generalized Jacobi orthogonal approximation in several dimensions and its applications. Guo and Jiao [42] designed the spectral method for the Navier-Stokes equations with slip boundary conditions by using some results on the generalized Jacobi orthogonal approximation. In this case, the numerical solution satisfies the incompressibility automatically. This trick is also available for some other problems with divergence-free solutions.

Recently, Guo et al. [53] proposed the generalized Jacobi quasi-orthogonal approximation (also see [76]). To do this, we introduce the following two families of polynomials of degree $m+n-1$,

$$q^-_{m,n,j}(x)=\frac{1}{2^m j!}(1-x)^m\sum_{l=0}^{n-1-j}\frac{(m+l-1)!}{2^l l!(m-1)!}(1+x)^{l+j},\quad m,n\geqslant 1,$$

$$q^+_{m,n,j}(x)=\frac{(-1)^j}{2^n j!}(1+x)^n\sum_{l=0}^{m-1-j}\frac{(n+l-1)!}{2^l l!(n-1)!}(1-x)^{l+j},\quad m,n\geqslant 1.$$

Let $\delta_{k,j}$ be the Kronecker symbol. It can be checked that

$$\begin{aligned}&\partial_x^{\mu} q_{m,n,j}^{-}(-1)=\delta_{\mu,j}, \quad \partial_x^{\nu} q_{m,n,j}^{-}(1)=0, \quad 0 \leqslant j,\mu \leqslant n-1, \quad 0 \leqslant \nu \leqslant m-1,\\ &\partial_x^{\nu} q_{m,n,j}^{+}(-1)=0, \quad \partial_x^{\mu} q_{m,n,j}^{+}(1)=\delta_{\mu,j}, \quad 0 \leqslant j,\mu \leqslant m-1, \quad 0 \leqslant \nu \leqslant n-1.\end{aligned}$$

Now, for any $v \in H_{m,n,A}^{\mu}(\Lambda)$ and $\mu \geqslant \max(m,n)$, we set

$$v_{m,n,b}(x)=\sum_{j=0}^{n-1} \partial_x^j v(-1) q_{m,n,j}^{-}(x)+\sum_{j=0}^{m-1} \partial_x^j v(1) q_{m,n,j}^{+}(x).$$

Furthermore, we let $\bar{v}(x)=v(x)-v_{m,n,b}(x)$. Clearly, $\bar{v} \in H_{0,m,n,A}^{\mu}(\Lambda)$. Thereby, we define the Jacobi quasi-orthogonal projection as

$$\bar{P}_{*,N,m,n}^{\mu} v(x)=\bar{P}_{N,m,n}^{\mu,0} \bar{v}(x)+v_{m,n,b}(x).$$

We have

$$\begin{aligned}&\partial_x^k \bar{P}_{*,N,m,n}^{\mu} v(-1)=\partial_x^k v(-1), \quad \text{for } 0 \leqslant k \leqslant n-1,\\ &\partial_x^k \bar{P}_{*,N,m,n}^{\mu} v(1)=\partial_x^k v(1), \quad \text{for } 0 \leqslant k \leqslant m-1.\end{aligned}$$

If $v \in H_{m,n,A}^{\max(m,n)}(\Lambda)$, $\partial_x^r v \in L^2_{\chi^{(-m+r,-n+r)}}(\Lambda)$ and integers $m,n,r \geqslant 1$, $N \geqslant m+n$, $0 \leqslant k \leqslant r \leqslant N+1$, $\max(m,n,k) \leqslant \mu \leqslant m+n$, then

$$\|\partial_x^k(\bar{P}_{*,N,m,n}^{\mu} v-v)\|_{\chi^{(-m+k,-n+k)}} \leqslant cN^{k-r}(\|\partial_x^r v\|_{\chi^{(-m+r,-n+r)}}+\|v\|_{H^{\max(m,n)}(\Lambda)}). \tag{2.5}$$

If, in addition, $r \geqslant m+n$ or $m,n \leqslant 4$, then

$$\|\partial_x^k(\bar{P}_{*,N,m,n}^{\mu} v-v)\|_{\chi^{(-m+k,-n+k)}} \leqslant cN^{k-r}\|\partial_x^r v\|_{\chi^{(-m+r,-n+r)}}. \tag{2.6}$$

Since the Jacobi quasi-orthogonal approximation fits certain derivatives of approximated functions at the endpoints of Λ, it is very helpful for the spectral element methods of high order problems, and spectral methods with essential imposition of various boundary conditions, see the work of Guo and Jia [40], Guo and Wang [60–62], Jia and Guo [84], Wang and Guo [132, 133], Wang and Wang [134], and Wang and Wang [142]. Recently, Yu and Guo [151] proposed the spectral element method for mixed inhomogeneous boundary value problems of fourth order.

2.3 Jacobi pseudospectral method

We now turn to the Jacobi pseudospectral method, with which we only need to evaluate the unknown functions on the nodes of the Jacobi interpolation, and could deal with nonlinear problems conveniently.

For any integer $r \geqslant 0$, we denote by $C^r(\bar{\Lambda})$ the space consisting of all r-times differentiable functions. Let $\zeta_{G,N,j}^{(\alpha,\beta)}, \zeta_{R,N,j}^{(\alpha,\beta)}$ and $\zeta_{L,N,j}^{(\alpha,\beta)}$ be the zeros of polynomials $J_{N+1}^{(\alpha,\beta)}(x), (1+x)J_N^{(\alpha,\beta+1)}(x)$ and $(1-x^2)\partial_x J_N^{(\alpha,\beta)}(x)$, respectively, $0 \leqslant j \leqslant N$. They are arranged in decreasing orders. The Gauss-type interpolations $\mathcal{I}_{Z,N,\alpha,\beta} v$ are defined by

$$\mathcal{I}_{Z,N,\alpha,\beta} v(\zeta_{Z,N,j}^{(\alpha,\beta)})=v(\zeta_{Z,N,j}^{(\alpha,\beta)}), \quad Z=G,R,L, \quad 0 \leqslant j \leqslant N,$$

where $Z=G,R,L$ correspond to the Jacobi-Gauss interpolation, the Jacobi-Gauss-Radau interpolation and the Jacobi-Gauss-Lobatto interpolation, respectively. Guo and Wang [56] estimated the errors of the previous interpolations, stated below.

- If $\partial_x^r v \in L^2_{\chi^{(\alpha+r,\beta+r)}}(\Lambda)$ and integers $1 \leqslant r \leqslant N+1$, then

$$\|\mathcal{I}_{Z,N,\alpha,\beta} v-v\|_{\chi^{(\alpha,\beta)}} \leqslant c_{\alpha,\beta} N^{-r}\|\partial_x^r v\|_{\chi^{(\alpha+r,\beta+r)}}, \quad Z=G,R. \tag{2.7}$$

If, in addition, $\partial_x^r v \in L^2_{\chi^{(\alpha+r-1,\beta+r-1)}}(\Lambda)$ and one of the following conditions holds,

$$\text{(i)}\ \ \alpha = \beta \geqslant -\frac{1}{2}, \quad \text{(ii)}\ \ \alpha \geqslant \beta + 1, \quad \text{(iii)}\ \ \frac{1}{2} \leqslant \alpha \leqslant \beta + 1, \tag{2.8}$$

then

$$\|\partial_x(\mathcal{I}_{Z,N,\alpha,\beta}v - v)\|_{\chi^{(\alpha,\beta)}} \leqslant c_{\alpha,\beta}N^{2-r}\|\partial_x^r v\|_{\chi^{(\alpha+r-1,\beta+r-1)}}, \quad Z = G, R. \tag{2.9}$$

- If $\partial_x^r v \in L^2_{\chi^{(\alpha+r-1,\beta+r-1)}}(\Lambda)$, $-1 < \alpha, \beta \leqslant 0$ or $0 < \alpha, \beta < 1$, and integers $1 \leqslant r \leqslant N + 1$, then

$$\|\mathcal{I}_{L,N,\alpha,\beta}v - v\|_{\chi^{(\alpha,\beta)}} \leqslant c_{\alpha,\beta}N^{-r}\|\partial_x^r v\|_{\chi^{(\alpha+r-1,\beta+r-1)}}. \tag{2.10}$$

If, in addition, (2.8) is fulfilled, then

$$\|\partial_x(\mathcal{I}_{L,N,\alpha,\beta}v - v)\|_{\chi^{(\alpha,\beta)}} \leqslant c_{\alpha,\beta}N^{2-r}\|\partial_x^r v\|_{\chi^{(\alpha+r-1,\beta+r-1)}}. \tag{2.11}$$

More general and precise results were given by Guo and Wang [54,56], which are useful for pseudospectral methods of differential equations of second order. Wang and Guo [128] studied the interpolation based on the Gauss-Lobatto-Legendre-Birkhoff quadrature, which leads to a new pseudospectral method with exact imposition of the Neumann boundary condition. Wang and Wang [142] also considered a collocation method with exact imposition of mixed boundary condition. Recently, Wang et al. [130] discovered the exponential convergence of the Gegenbauer-Gauss and Gegenbauer-Gauss-Lobatto interpolations for analytic functions. Zhang [157] obtained the exponential convergence of the Chebyshev-Gauss-Lobatto interpolation using the values of derivatives of approximated functions. We also refer to the work of Zhang [158,159] for the superconvergence of the Legendre-Gauss-Lobatto and Chebyshev-Gauss-Lobatto interpolations for certain smooth functions, and their applications to p-version of finite element and collocation methods.

The second order Jacobi-Gauss type interpolations were also considered, which are applicable to pseudospectral methods of fourth order problems, see [123]. On the other hand, Guo and Zhang [79] renewed the early results on the Jacobi-Gauss type interpolations in multiple dimensions with its applications, which also leads to the concept of the Legendre quasi-orthogonal approximation.

Guo and Wang [66] considered the initial value problem of first order ordinary differential equation

$$\begin{cases} \dfrac{d}{dt}U(t) = f(U(t), t), & 0 < t \leqslant T, \\ U(0) = U_0. \end{cases}$$

Let $\Lambda_T = \{t \mid 0 < t \leqslant T\}$. $L_l(t)$ stands for the Legendre polynomials and the shifted Legendre polynomials $L_{T,l}(t) = L_l(\frac{2t}{T} - 1)$. The zeros of $L_{T,N+1}(t)$ are denoted by $t^N_{T,j}, 0 \leqslant j \leqslant N$. The corresponding collocation method is to seek $u^N(t) \in \mathcal{P}_{N+1}(\Lambda_T)$ such that

$$\begin{cases} \dfrac{d}{dt}u^N(t^N_{T,k}) = f(u^N(t^N_{T,k}), t^N_{T,k}), & 0 \leqslant k \leqslant N, \\ u^N(0) = U_0. \end{cases}$$

If $f(z, t)$ satisfies certain conditions, then it has a unique solution. Usually, one expands the unknown $\frac{d}{dt}u^N(t)$ by the Lagrange interpolation and derives a system with its coefficients. But, the Lagrange interpolation is not stable for large N. We now expand $u^N(t)$ directly by the shifted Legendre polynomials, and derive a system with its coefficients, which leads to a stable algorithm even for large N, possessing the spectral accuracy. The multi-step procedure was also given by Guo and Wang [66]. The sharp error estimate is obtained by using the approximation results in this subsection. The numerical results demonstrate that this new approach provides more accurate numerical results and costs less computational time than other commonly used algorithms. Guo and Wang [68] applied this numerical process to collocation method for nonlinear Klein-Gordon equation. Guo and Wang [67], and Wang and Guo [138] designed the new Radau and Lobatto collocation methods for first order problems. We also refer to the work of Kanyamee and Zhang [86] for the Hamiltonian systems. Moreover, Guo and Yan [71] proposed the Gauss collocation method for second order problems. Recently, Wang and Wang [141] provided the new numerical algorithm for the delay ordinary differential equation

$$\begin{cases} \frac{d}{dt}U(t) = f(U(t), U(t-\theta(t)), t), & 0 < t \leqslant T, \\ U(t) = V(t), & t \leqslant 0, \end{cases}$$

where $\theta(t) > 0$ and $V(t)$ are given functions. Babuška and Janik [4], Shen and Wang [109], and Tang and Ma [117–119] also considered spectral approximation in time.

The Jacobi interpolation was used for numerical solutions of the Volterra integral equations of second kind,

$$U(t) = g(t) + \int_0^t (t-s)^{-\mu} K(t,s) U(s) ds, \quad \mu < 1, \quad 0 \leqslant t \leqslant T.$$

The related results could be found in [90], and [17,18]. We also refer to the early work of Tang et al. [122]. Recently, Lin et al. [92] studied the numerical solutions of the fractional cable equation,

$$\partial_t U(x,t) = -d^{\mu-1} {}_0D_t^{\mu} U(x,t) + {}_0D_t^{\nu} \partial_x^2 U(x,t), \quad 0 < \mu, \nu < 1,$$

where d is a constant, and ${}_0D_t^{\mu} v(x,t)$ is the Riemann-Louville fractional derivative operator, i.e.,

$${}_0D_t^{\mu} v(x,t) = \frac{1}{\Gamma(1-\mu)} \partial_t \int_0^t \frac{v(x,\tau)}{(t-\tau)^{\mu}} d\tau.$$

Recently, Guo et al. [53] proposed the generalized Jacobi-Gauss-Lobatto interpolation. For any integer $r \geqslant \max(m-1, n-1)$, we define

$$C_{0,m,n}^r(\bar{\Lambda}) = \{v \in C^r(\bar{\Lambda}) \mid \partial_x^k v(-1) = 0 \text{ for } 0 \leqslant k \leqslant n-1, \partial_x^k v(1) = 0 \text{ for } 0 \leqslant k \leqslant m-1\}.$$

Let

$$\zeta_{N,j}^{(m,n)} = \zeta_{G,N-m-n,j}^{(m,n)}, \quad m, n \geqslant 0, \quad 0 \leqslant j \leqslant N-m-n.$$

For any $v \in C_{0,m,n}^{\max(m-1,n-1)}(\bar{\Lambda})$ and $m, n \geqslant 1$, we introduce the auxiliary interpolation $\bar{\mathcal{I}}_{N,m,n} v \in Q_{N,m,n}(\Lambda)$, determined uniquely by

$$\bar{\mathcal{I}}_{N,m,n} v(\zeta_{N,j}^{(m,n)}) = v(\zeta_{N,j}^{(m,n)}), \quad 0 \leqslant j \leqslant N-m-n.$$

Next, let $v_{m,n,b}(x)$ be the same as in the last subsection. For any $v \in C_{m,n}^{\max(m-1,n-1)}(\bar{\Lambda})$, we set $\bar{v}(x) = v(x) - v_{m,n,b}(x)$. In fact, $\bar{v} \in C_{0,m,n}^{\max(m-1,n-1)}(\bar{\Lambda})$. Thus, we define the generalized Jacobi-Gauss-Lobatto interpolation as

$$\bar{\mathcal{I}}_{L,N,m,n} v(x) = \bar{\mathcal{I}}_{N,m,n} \bar{v}(x) + v_{m,n,b}(x).$$

Obviously,

$$\begin{aligned} &\bar{\mathcal{I}}_{L,N,m,n} v(\zeta_{N,j}^{(m,n)}) = v(\zeta_{N,j}^{(m,n)}), \quad \text{for } 0 \leqslant j \leqslant N-m-n, \\ &\partial_x^k \bar{\mathcal{I}}_{L,N,m,n} v(-1) = \partial_x^k v(-1), \quad \text{for } 0 \leqslant k \leqslant n-1, \\ &\partial_x^k \bar{\mathcal{I}}_{L,N,m,n} v(1) = \partial_x^k v(1), \quad \text{for } 0 \leqslant k \leqslant m-1. \end{aligned}$$

If $v \in H_{m,n,A}^{\max(m,n)}(\Lambda)$, $\partial_x^r v \in L^2_{\chi^{(-m+r,-n+r)}}(\Lambda)$ and integers $m, n, r \geqslant 1, N \geqslant m+n, 0 \leqslant k \leqslant r \leqslant N+1$, then

$$\|\partial_x^k(\bar{\mathcal{I}}_{L,N,m,n} v - v)\|_{\chi^{(-m+k,-n+k)}} \leqslant cN^{k-r}(\|\partial_x^r v\|_{\chi^{(-m+r,-n+r)}} + \|v\|_{H^{\max(m,n)}(\Lambda)}). \tag{2.12}$$

If, in addition, $1 \leqslant m, n \leqslant 4$ or $r \leqslant m+n$, then

$$\|\partial_x^k(\bar{\mathcal{I}}_{L,N,m,n} v - v)\|_{\chi^{(-m+k,-n+k)}} \leqslant cN^{k-r}\|\partial_x^r v\|_{\chi^{(-m+r,-n+r)}}. \tag{2.13}$$

The above result with $m = n = 1$ and $k = 0, 1$ was first obtained by Guo and Zhang [79], which is much better than (2.10) and (2.11).

If $m = n \geqslant 1$, then $\bar{\mathcal{I}}_{L,N,m,m}v$ is equivalent to the generalized Legendre-Gauss-Labatto polynomial interpolation $k_N^m v$ given by Bernardi and Maday [9]. By (13.26) of [9], we have $\|k_N^1 v - v\|_{\chi^{(-1,-1)}} \leqslant cN^{-r}\|v\|_{H^r(\Lambda)}$. Thus, the results (2.12) and (2.13) improve and generalize the earlier results. The result (2.13) also implies $\|\partial_x^m(\bar{\mathcal{I}}_{L,N,m,m}v)\| \leqslant c\|\partial_x^m v\|$. Thereby, the interpolation $\bar{\mathcal{I}}_{L,N,m,m}v$ is stable in the space $H^m_{m,m,A}(\Lambda)$. Bernardi and Maday [9] first proved that $\|k_N^1 v\|_{H^1(\Lambda)} \leqslant c\|v\|_{H^1(\Lambda)}$. Clearly, the estimate (2.13) generalizes the existing results. The above results are very applicable to psudospectral element and collocation methods for high order problems.

2.4 Spectral methods on non-rectangular domains

The Jacobi approximation was also used for the spectral methods of many problems defined on non-rectangular domains.

For example, we consider the Laplace equation on the cylinder Ω,

$$-\frac{1}{\rho}\partial_\rho(\rho\partial_\rho U(\rho,\theta,z)) - \frac{1}{\rho^2}\partial_\theta^2 U(\rho,\theta,z) - \partial_z^2 U(\rho,\theta,z) = f(\rho,\theta,z), \quad \text{in } \Omega. \tag{2.14}$$

We can solve (2.14) numerically, by using the Jacobi, Fourier and Legendre approximations in the ρ, θ and z directions, respectively. Bernardi et al. [6] treated with this problem early, by using some combinations of the Legendre polynomials in the ρ-direction. This is equivalent to the specific Jacobi approximation with $\alpha = 1$ and $\beta = 0$. In this case, we do not need any artificial boundary condition at $\rho = 0$, which is imposed usually in the finite difference methods.

Guo and Wang [127] considered the mixed Jacobi-Fourier spectral method on a disc. Guo and Huang [39] proposed the mixed Jacobi-spherical harmonic spectral method for the Navier-Stokes equations in a ball, in which the components of velocity are presented by using the Descartes coordinates. This treatment simplifies the computation essentially. Recently, Shen and Wang [110] considered the Helmholtz equation in an ellipse Ω with the focal distance $2d$,

$$\partial_{x_1}^2 U(x_1,x_2) + \partial_{x_2}^2 U(x_1,x_2) + k^2 U(x_1,x_2) = 0, \quad \text{in } \Omega.$$

Under the transformation $x = d\cosh\rho\cos\theta, y = d\sinh\rho\sin\theta$ and $U(x_1,x_2) = V(\rho,\theta)$, we obtain

$$\frac{2}{d^2(\cosh(2\rho) - \cos(2\theta))}(\partial_\rho^2 V(\rho,\theta) + \partial_\theta^2 V(\rho,\theta)) + k^2 V(\rho,\theta) = 0.$$

Then we could solve the Helmholtz equation by using the Legendre polynomials in the ρ-direction, and the Mathieu functions in the θ-direction, which are the eigenfunctions of the Sturm-Liouville equation

$$\frac{d^2\Phi}{d\theta^2} + \left(\lambda - \frac{1}{2}d^2k^2\right)\Phi = 0.$$

We next consider the orthogonal approximation on triangles. Let $\mathcal{T}$ be the reference triangle,

$$\mathcal{T} = \{(x,y) \mid 0 < x, y < 1, 0 < x + y < 1\}.$$

Dubiner [21] considered the polynomials

$$g_{l,m}(x,y) = 2^{l+\frac{3}{2}}(1-y)^l J_l^{(0,0)}\left(\frac{2x+y-1}{1-y}\right) J_m^{(2l+1,0)}(2y-1),$$
$$0 \leqslant l \leqslant N \leqslant M, \quad 0 \leqslant m \leqslant M, \quad 0 \leqslant l+m \leqslant M,$$

which form the normalized $L^2(\mathcal{T})$-orthogonal system. Let

$$\mathcal{P}_{N,M}(\mathcal{T}) = \{g_{l,m}(x,y) \mid 0 \leqslant l \leqslant N, 0 \leqslant m \leqslant M\}.$$

The orthogonal projection $P_{N,M} : L^2(\mathcal{T}) \to \mathcal{P}_{N,M}(\mathcal{T})$ is defined by

$$(P_{N,M}v - v, \phi)_{L^2(\mathcal{T})} = 0, \quad \forall\, \phi \in \mathcal{P}_{N,M}(\mathcal{T}).$$

We introduce the non-isotropic weighted space $H^{r,s}(\mathcal{T})$, equipped with the norm

$$\|v\|_{H^{r,s}(\mathcal{T})} = \bigg(\sum_{k=0}^{r} \sum_{j=0}^{k} \|x^j y^{\frac{r}{2}} (1-y)^{k-j-\frac{r}{2}} \partial_x^j \partial_y^{k-j} v\|^2_{L^2(\mathcal{T})} + \|x^{\frac{s}{2}} (1-x-y)^{\frac{s}{2}} \partial_x^s v\|^2_{L^2(\mathcal{T})} \bigg)^{\frac{1}{2}}.$$

Guo and Wang [58] proved that for any $v \in H^{r,s}(\mathcal{T})$ and integers $0 \leqslant r \leqslant N+1, 0 \leqslant s \leqslant M+1$,

$$\|P_{N,M}v - v\|_{L^2(\mathcal{T})} \leqslant c\bigg(\bigg(\frac{M(M+N)}{N^2} \bigg)^{-r} + N^{-s} \bigg) \|v\|_{H^{r,s}(\mathcal{T})}.$$

The $H_0^1(\mathcal{T})$-orthogonal approximation and the related spectral method were also investigated by Guo and Wang [58]. Recently, Li and Shen [88] established the new error estimates in certain weighted semi-norms for both of the $L^2(\mathcal{T})$-orthogonal approximation and the $H_0^1(\mathcal{T})$-orthogonal approximation, by using the generalized Koornwinder polynomials and the properties of the Sturm-Liouville operator on the triangle $\mathcal{T}$. These results were applied to the spectral method for partial differential equations of second and fourth orders. We also refer to the work of Owens [100] for spectral approximation on triangles, and the work of Sherwin and Karniadakis [114] for triangular and tetrahedral basis of high-order finite element methods.

Li et al. [91] introduced a new rectangle-to-triangle mapping, which pulls one edge of the triangle to two adjacent edges of the reference rectangle. For example, we make the variable transformation $x = \frac{1}{8}(1+\xi)(3-\eta)$, $y = \frac{1}{8}(3-\xi)(1+\eta)$. Then the triangle $\mathcal{T}$ becomes the square in the $\xi - \eta$ plan. Meanwhile, the corner with $\xi = \eta = 1$ becomes the midpoint of one edge with $x = y = \frac{1}{2}$. In contrast with the collapsed mapping, such a mapping is one-to-one, and allows an efficient implementation of spectral approximation on the triangle $\mathcal{T}$, by a direct using of nodal Lagrange polynomial basis on the reference rectangle with a slight modification. This technique was generalized to the three-dimensional problems defined on a tetrahedra, by using the variable transformation $x = \frac{1}{24}(1+\xi)(7-2\eta-2\zeta+\eta\zeta)$, $y = \frac{1}{24}(1+\eta)(7-2\xi-2\zeta+\xi\zeta)$, $z = \frac{1}{24}(1+\zeta)(7-2\xi-2\eta+\xi\eta)$. Recently, Samson et al. [101] designed a new spectral method with better interpolation points in triangles.

On the other hand, Shen et al. [113] provided the irrational basis functions on the triangle $\mathcal{T}$, which are induced by the polynomials in the reference square through a collapsed transformation. The $L^2(\mathcal{T})$-orthogonal projection and the $H_0^1(\mathcal{T})$-orthogonal projection were investigated, which lead to a new spectral method on triangles. Li and Wang [89] also proposed a spectral method on tetrahedra by using certain irrational basis functions.

We now turn to spectral method on a convex quadrilateral Ω in the $\xi - \eta$ plan. By a bilinear variable transformation, the quadrilateral is transformed to a reference square in the $x - y$ plan. The determinant of Jacobi matrix of the transformation is denoted by $J_\Omega(x, y)$. Let $L_l(x)$ be the Legendre polynomials as before. Guo and Jia [40], and Jia and Guo [84] proposed the spectral methods for the quadrilateral Ω, by using the following two kinds of basis functions,

$$\psi_{l,m}(\xi,\eta) = L_l(x(\xi,\eta)) L_m(y(\xi,\eta)) J_\Omega^{-\frac{1}{2}}(x(\xi,\eta), y(\xi,\eta)), \quad l, m \geqslant 0,$$
$$\psi_{l,m}(\xi,\eta) = L_l(x(\xi,\eta)) L_m(y(\xi,\eta)), \quad l, m \geqslant 0.$$

For the spectral element methods for arbitrary polygons, the key points are how to match the numerical solutions on the common boundaries of adjacent subdomains, and how to keep the spectral accuracy on the whole polygons. Guo and Jia [40], and Jia and Guo [84] investigated the two-dimensional Legendre quasi-orthogonal approximation and its applications to the spectral element methods for some problems defined on polygons, with the mixed inhomogeneous Dirichlet-Neumann-Robin boundary conditions.

Bernardi et al. [10], Canuto et al. [16], and Karniadakis and Sherwin [87] developed the pseudospectral element methods for non-rectangular domains, which are also called as spectral methods in many literatures, and succeed in scientific computing. We also refer to the nodal discontinuous Galerkin method, see [83].

The Jacobi approximation could be applied to some problems on unbounded domains. Let $\mathbb{R} = (-\infty,\infty)$ and $\mathbb{R}^+ = (0,\infty)$. As an example, we consider the logistic equation governing the population of budworm on an unbounded forest. Suppose that the boundary condition at $y=-1$ is lethal, and at least $\mathrm{e}^{-y}V(y,t)\partial_y V(y,t) \to 0$, as $y \to \infty$. This problem is of the form

$$\begin{cases} \partial_t V(y,t) - \partial_y^2(y,t) = V(y,t)(1-V(y,t)), & 0<y<\infty, \quad 0<t\leqslant T, \\ V(0,t) = \lim\limits_{y\to\infty} \mathrm{e}^{-y}V(y,t)\partial_y V(y,t) = 0, & 0\leqslant t\leqslant T, \\ V(y,0) = V_0(y), & 0<y<\infty. \end{cases} \tag{2.15}$$

We make the variable transformation $y(x) = -2\ln(1-x) + 2\ln 2$. Accordingly, $U(x,t) = V(y(x),t)$ and $U_0(x) = V_0(y(x))$. Then (2.15) becomes

$$\begin{cases} \partial_t U(x,t) - \dfrac{1}{4}(1-x)\partial_x((1-x)\partial_x U(x,t)) = U(x,t)(1-U(x,t)), & x\in\mathbb{R}^+, \quad 0\leqslant t\leqslant T, \\ U(-1,t) = \lim\limits_{x\to 1}(1-x)^2 U(x,t)\partial_x U(x,t) = 0, & 0\leqslant t\leqslant T, \\ U(x,0) = U_0(x), & x\in\mathbb{R}^+. \end{cases}$$

If $U_0 \in L^2(\mathbb{R}^+)$, then it admits a unique solution $U \in L^\infty(0,T;L^2(\mathbb{R}^+)) \cap L^2(0,T;{}_0H^1_{\chi^{(2,0)}}(\mathbb{R}^+))$. Obviously, we could solve this problem by the Jocobi spectral method with $\alpha=2$ and $\beta=\gamma=\delta=0$, see [37].

With the aid of variable transformation $y = \ln\frac{1+x}{1-x}$, we can use the Jacobi spectral method with $\alpha=\beta=2$ and $\gamma=\delta=0$ to solve some problems on the whole line $\mathbb{R}$, see [36].

If the solutions decay to zero at infinity, then we may solve the differential equations on the half line, by using the Jacobi spectral method with $\alpha=1, \beta=0, \gamma=-1$ and $\delta=0$. For the differential equations on the whole line, we could use the Jacobi spectral method with $\alpha=\beta=1$ and $\gamma=\delta=-1$, see [31,33]. For the multiple-dimensional problems, we use the Jacobi approximations with different parameters in different directions, see [125].

2.5 Jacobi irrational spectral methods for unbounded domains

The Jacobi approximation is related to various irrational spectral methods for unbounded domain closely, which are also called rational spectral methods usually.

In the early work of Boyd [12] and Christov [19], the irrational basis functions are induced by the Legendre or Chebyshev polynomials. Guo et al. [49,50], Guo and Wang [64], and Wang and Guo [135], proposed several new irrational orthogonal approximations and interpolations. By using some results on the Jacobi approximation, they established a series of approximation results. The related spectral and pseudospectral methods were provided.

The solutions of different problems have different asymptotic behaviors at infinity. Also, the same solution might behave differently at different endpoints of infinite intervals, such as the kink solitons in quantum physics and the heteroclinic waves in logistic models. In the previous irrational approximations, the basis functions are mutually orthogonal with the fixed weights. This feature limits their applications. For overcoming this deficiency, Wang and Guo [137] used the irrational basis functions on the half line, induced by the standard Jacobi polynomials. Recently, Guo and Yi [72] developed the generalized Jacobi irrational spectral method and so enlarged the applications of irrational spectral methods.

Let $\bar{J}_l^{(\alpha,\beta)}(x)$ be the generalized Jacobi functions. The generalized Jacobi irrational functions are defined by

$$R_l^{(\alpha,\beta)}(x) = \bar{J}_l^{(\alpha,\beta)}\left(\frac{x}{\sqrt{x^2+1}}\right), \quad l \geqslant \bar{l}_{\alpha,\beta}.$$

Let the weight function

$$\omega_R^{(\alpha,\beta)}(x) = (\sqrt{x^2+1}+x)^{\beta-\alpha}(x^2+1)^{\frac{-\alpha-\beta-3}{2}}.$$

The set of all $R_l^{(\alpha,\beta)}(x)$ is a complete $L^2_{\omega_R^{(\alpha,\beta)}}(\mathbb{R})$-orthogonal system. We set

$$\mathcal{V}_{N,\alpha,\beta}(\mathbb{R}) = \text{span}\{R_l^{(\alpha,\beta)}(x), \bar{l}_{\alpha,\beta} \leqslant l \leqslant N\}.$$

The orthogonal projection $P_{N,\alpha,\beta} : L^2_{\omega_R^{(\alpha,\beta)}}(\mathbb{R}) \to \mathcal{V}_{N,\alpha,\beta}(\mathbb{R})$ is defined by

$$(P_{N,\alpha,\beta}v - v, \phi)_{\omega_R^{(\alpha,\beta)}} = 0, \quad \forall\, \phi \in \mathcal{V}_{N,\alpha,\beta}(\mathbb{R}).$$

We introduce the space $H^r_{\omega_R^{(\alpha,\beta)},A}(\mathbb{R})$ equipped with the following semi-norm and norm,

$$\|v\|_{0,\omega_R^{(\alpha,\beta)},A} = \|v\|_{\omega_R^{(\alpha,\beta)}}, \quad |v|_{1,\omega_R^{(\alpha,\beta)},A} = \|(x^2+1)^{\frac{3}{2}}\partial_x v\|_{\omega_R^{(\alpha+1,\beta+1)}},$$
$$|v|_{k,\omega_R^{(\alpha,\beta)},A} = |(x^2+1)^{\frac{3}{2}}\partial_x v|_{k-1,\omega_R^{(\alpha+1,\beta+1)},A}, \quad k \geqslant 2, \quad \|v\|_{r,\omega_R^{(\alpha,\beta)},A} = \left(\sum_{k=0}^{r} |v|^2_{k,\omega_R^{(\alpha,\beta)},A}\right)^{\frac{1}{2}}.$$

Moreover,

$$H^r_{0,\omega_R^{(\alpha,\beta)},A}(\mathbb{R}) = \{v \mid v \in H^r_{\omega_R^{(\alpha,\beta)},A}(\mathbb{R}), v(\pm\infty) = 0\}.$$

Let $\mathbb{N}^-$ be the set of all negative integers. If one of the following conditions holds,

(i) $\alpha,\ \beta > -1$,

(ii) $\alpha > -1,\ \beta \leqslant -r-1$ or $\beta \in \mathbb{N}^-$,

(iii) $\alpha \leqslant -r-1$ or $\alpha \in \mathbb{N}^-,\ \beta > -1$,

(iv) $\alpha,\ \beta \leqslant -r-1$ or $\alpha,\ \beta \in \mathbb{N}^-$,

then for any $v \in H^r_{\omega_R^{(\alpha,\beta)},A}(\mathbb{R})$ and integers $r \geqslant 0$, $0 \leqslant k \leqslant r \leqslant N+1$,

$$\|P_{N,\alpha,\beta}v - v\|_{k,\omega_R^{(\alpha,\beta)},A} \leqslant cN^{k-r}|v|_{r,\omega_R^{(\alpha,\beta)},A}. \tag{2.16}$$

For numerical solutions of differential equations, we need other projections. To do this, we introduce the space $H^1_{\alpha,\beta,\gamma,\delta}(\mathbb{R})$, equipped with the norm $\|v\|_{1,\alpha,\beta,\gamma,\delta} = (|v|^2_{1,\omega_R^{(\alpha,\beta)}} + \|v\|^2_{\omega_R^{(\gamma,\delta)}})^{\frac{1}{2}}$. There are several projections depending on various underlying problems. For example, if $\alpha,\beta > -4$ and $\gamma,\delta > -1$, then we define the orthogonal projection $P^1_{N,\alpha,\beta,\gamma,\delta} : H^1_{\alpha,\beta,\gamma,\delta}(\mathbb{R}) \to \mathcal{V}_{N,\alpha,\beta}(\mathbb{R})$ by

$$(\partial_x(P^1_{N,\alpha,\beta,\gamma,\delta}v - v), \partial_x\phi)_{\omega_R^{(\alpha,\beta)}} + (P^1_{N,\alpha,\beta,\gamma,\delta}v - v, \phi)_{\omega_R^{(\gamma,\delta)}} = 0, \quad \forall\, \phi \in \mathcal{V}_{N,\alpha,\beta}(\mathbb{R}).$$

Let $\sigma,\theta \leqslant 3$. If $0 < \alpha+\sigma \leqslant \gamma+2$, $0 < \beta+\theta \leqslant \delta+2$ and $\gamma,\delta > -1$, then for any $v \in H^1_{\alpha,\beta,\gamma,\delta}(\mathbb{R}) \cap H^r_{\omega_R^{(\alpha+\sigma-1,\beta+\theta-1)},A}(\mathbb{R})$ and integers $1 \leqslant r \leqslant N+1$,

$$\|P^1_{N,\alpha,\beta,\gamma,\delta}v - v\|_{1,\alpha,\beta,\gamma,\delta} \leqslant cN^{1-r}|v|_{r,\omega_R^{(\alpha+\sigma-1,\beta+\theta-1)},A}. \tag{2.17}$$

If, in addition, $\alpha \leqslant \gamma+\sigma-5$ and $\beta \leqslant \delta+\theta-5$, then

$$\|P^1_{N,\alpha,\beta,\gamma,\delta}v - v\|_{\omega_R^{(\gamma,\delta)}} \leqslant cN^{-r}|v|_{r,\omega_R^{(\alpha+\sigma-1,\beta+\theta-1)},A}. \tag{2.18}$$

Guo and Yi [72] provided the irrational spectral schemes for the sine-Gordon, Klein-Gordon and Fisher equations defined on the whole line.

Yi and Guo [150] also introduced the generalized Jacobi irrational functions defined on the half line, as

$$R_l^{(\alpha,\beta)}(x) = \bar{J}_l^{(\alpha,\beta)}\left(\frac{x-1}{x+1}\right), \quad l \geqslant \bar{l}_{\alpha,\beta}.$$

Let the weight function $\omega_R^{(\alpha,\beta)}(x) = x^\beta(x+1)^{-\alpha-\beta-2}$. The set of all $R_l^{(\alpha,\beta)}(x)$ is a complete $L^2_{\omega_R^{(\alpha,\beta)}}(\mathbb{R}^+)$-orthogonal system. We set

$$\mathcal{V}_{N,\alpha,\beta}(\mathbb{R}^+) = \text{span}\{R_l^{(\alpha,\beta)}(x),\ \bar{l}_{\alpha,\beta} \leqslant l \leqslant N\}.$$

The orthogonal projection $P_{N,\alpha,\beta}: L^2_{\omega_R^{(\alpha,\beta)}}(\mathbb{R}) \to \mathcal{V}_{N,\alpha,\beta}(\mathbb{R})$ is defined by

$$(P_{N,\alpha,\beta}v - v, \phi)_{\omega_R^{(\alpha,\beta)}} = 0, \quad \forall\, \phi \in \mathcal{V}_{N,\alpha,\beta}(\mathbb{R}^+).$$

We introduce the space $H^r_{\omega_R^{(\alpha,\beta)},A}(\mathbb{R}^+)$, equipped with the following semi-norm and norm,

$$\|v\|_{0,\omega_R^{(\alpha,\beta)},A} = \|v\|_{\omega_R^{(\alpha,\beta)}}, \quad |v|_{1,\omega_R^{(\alpha,\beta)},A} = \|(x+1)^2\partial_x v\|_{\omega_R^{(\alpha+1,\beta+1)}},$$

$$|v|_{k,\omega_R^{(\alpha,\beta)},A} = |(x+1)^2\partial_x v|_{k-1,\omega_R^{(\alpha+1,\beta+1)},A}, \quad k \geqslant 2, \quad \|v\|_{r,\omega_R^{(\alpha,\beta)},A} = \left(\sum_{k=0}^{r} |v|^2_{k,\omega_R^{(\alpha,\beta)},A}\right)^{\frac{1}{2}}.$$

Moreover,

$$H^r_{0,\omega_R^{(\alpha,\beta)},A}(\mathbb{R}^+) = \{v \mid v \in H^r_{\omega_R^{(\alpha,\beta)},A}(\mathbb{R}^+), v(0) = v(\infty) = 0\}.$$

If one of the following conditions holds,

(i) $\alpha,\ \beta > -1$,

(ii) $\alpha > -1,\ \beta \leqslant -r-1$ or $\beta \in \mathbb{N}^-$,

(iii) $\alpha \leqslant -r-1$ or $\alpha \in \mathbb{N}^-,\ \beta > -1$,

(iv) $\alpha,\ \beta \leqslant -r-1$ or $\alpha,\ \beta \in \mathbb{N}^-$,

then for any $v \in H^r_{\omega_R^{(\alpha,\beta)},A}(\mathbb{R}^+)$ and integers $0 \leqslant \mu \leqslant r \leqslant N$,

$$\|P_{N,\alpha,\beta}v - v\|_{\mu,\omega_R^{(\alpha,\beta)},A} \leqslant cN^{\mu-r}|v|_{r,\omega_R^{(\alpha,\beta)},A}. \tag{2.19}$$

For numerical solutions of differential equations, we need other projections. For this purpose, we introduce the space $H^1_{\alpha,\beta,\gamma,\delta}(\mathbb{R}^+)$, equipped with the norm $\|v\|_{1,\alpha,\beta,\gamma,\delta} = (|v|^2_{1,\omega_R^{(\alpha,\beta)}} + \|v\|^2_{\omega_R^{(\gamma,\delta)}})^{\frac{1}{2}}$. There are also several projections. For example, if $\beta,\delta > -1, \alpha - 2\gamma > -3$ and $\gamma \leqslant -1$, then we define the orthogonal projection $P^1_{N,\alpha,\beta,\gamma,\delta}: H^1_{\alpha,\beta,\gamma,\delta}(\mathbb{R}^+) \to \mathbb{R}_{N,\alpha,\beta}(\mathbb{R}^+)$ by

$$(\partial_x(P^1_{N,\alpha,\beta,\gamma,\delta}v - v), \partial_x\phi)_{\omega_R^{(\alpha,\beta)}} + (P^1_{N,\alpha,\beta,\gamma,\delta}v - v, \phi)_{\omega_R^{(\gamma,\delta)}} = 0, \quad \forall\, \phi \in \mathcal{V}_{N,\alpha,\beta}(\mathbb{R}^+).$$

Denote the set of all positive integers by $\mathbb{N}^+$. Let $\gamma \leqslant -1$, $\delta > -1$, $\alpha - 2\gamma > -3$, $\gamma - \alpha - \sigma + 2 \in \mathbb{N}^+$, and $\sigma \leqslant 4$, $\theta \leqslant 0$. If one of the following conditions holds,

(i) $\alpha + \sigma \leqslant -r-1$ or $\alpha + \sigma - 1 \in \mathbb{N}^-$, $0 \leqslant \beta + \theta \leqslant \delta + 2$,

(ii) $\alpha + \sigma - 1 \in \mathbb{N}^-$, $\beta + \theta - 1 \in \mathbb{N}^-$, $\beta > -1$,

then for any $v \in H^1_{\alpha,\beta,\gamma,\delta}(\mathbb{R}^+) \cap H^r_{w_R^{\alpha+\sigma-1,\beta+\theta-1},A}(\mathbb{R}^+)$ and integers $1 \leqslant r \leqslant N+1$,

$$\|P^1_{N,\alpha,\beta,\gamma,\delta}v - v\|_{1,\alpha,\beta,\gamma,\delta} \leqslant cN^{1-r}|v|_{r,\omega_R^{(\alpha+\sigma-1,\beta+\theta-1)},A}. \tag{2.20}$$

If, in addition, $\alpha \leqslant \gamma + \sigma - 7$, $\beta \leqslant \delta + \theta + 1$ and $\alpha < \sigma - 8$, then

$$\|P^1_{N,\alpha,\beta,\gamma,\delta}v - v\|_{\omega^{(\gamma,\delta)}} \leqslant cN^{-r}|v|_{r,\omega_R^{(\alpha+\sigma-1,\beta+\theta-1)},A}. \tag{2.21}$$

The irrational spectral schemes for the Black-Scholes type equation were provided by Yi and Guo [150].

The nonlinear wave equations usually possess some conservations, which play important role in theoretical analysis and actual computation. However, the used weights might destroy this property. Some authors adopted the basis functions $\sqrt{\omega_R^{(\alpha,\beta)}(x)}R_l^{(\alpha,\beta)}(x)$. Then, the corresponding numerical solutions keep the same conservations as the exact solutions. We refer to the work of Guo and Shen [45], Guo and and Wang [63], and Wang and Guo [136], with the parameters $\alpha = \beta = 0$ or $\alpha = \beta = -\frac{1}{2}$. They derived the errors of the related orthogonal approximations and interpolations, and designed the spectral and psudospectral methods for many important nonlinear equations.

Guo and Shen [46] also considered the irrational spectral methods for exterior problems, by using the base functions

$$I_l^{(\alpha,\gamma,\delta)}(x) = \frac{1}{x^\gamma}J_l^{(\alpha,0)}\left(1 - \frac{2}{x^\delta}\right), \quad x > 0, \quad l \geqslant 0.$$

In actual computation, we first choose γ properly to fit the asymptotic behaviors of exact solutions closely. Next, we take α in such a way that $I_l^{(\gamma,\delta)}(x)$ are mutually orthogonal in the weighted space to which the exact ones belong. Finally, the suitable choice of $\delta > 0$ leads to better numerical results.

3 Hermite and Laguerre spectral methods

3.1 Hermite spectral and pseudospectral methods

The Hermite spectral and pseudospectral methods are widely used for problems defined on the whole line $\mathbb{R}$ and the related unbounded domains.

Let $\omega(x) = \mathrm{e}^{-x^2}$ and $H^r_\omega(\mathbb{R})$ be the weighted Sobolev space with the norm $\|v\|_{r,\omega}$. We denote the inner product and the norm of the space $L^2_\omega(\mathbb{R})$ by $(u,v)_\omega$ and $\|v\|_\omega$, respectively. In the standard Hermite spectral method and pseudospectral method, we take the Hermite polynomials $H_l(x), l \geqslant 0$, as the basis functions, which form a complete $L^2_\omega(\mathbb{R})$-orthogonal system.

Let $\mathcal{P}_N(\mathbb{R})$ be the set of all algebraic polynomials of degree at most N. The $L^2_\omega(\mathbb{R})$-orthogonal projection $P_N : L^2_\omega(\mathbb{R}) \to \mathcal{P}_N(\mathbb{R})$ is defined by

$$(P_N v - v, \phi)_\omega = 0, \quad \forall\, \phi \in \mathcal{P}_N(\mathbb{R}).$$

According to a slight modification of the result by Guo [32], we have that for any $v \in H^r_\omega(\mathbb{R})$ and integers $0 \leqslant k \leqslant r$,

$$\|\partial^k_x (P_N v - v)\|_\omega \leqslant c N^{\frac{k-r}{2}} \|\partial^r_x v\|_\omega. \tag{3.1}$$

The $H^m_\omega(\mathbb{R})$-orthogonal projection is exactly the same as P_N.

In actual computation, the Hermite pseudospectral method is more preferable. For any integer $r \geqslant 0$, we denote by $C^r(\mathbb{R})$ the space consisting of all r-times differential functions. Let $\xi_{G,N,j}, 0 \leqslant j \leqslant N$, be the zeros of $H_{N+1}(x)$. For any $v \in C(\mathbb{R})$, the Hermite-Gauss interpolation $\mathcal{I}_{G,N} v \in \mathcal{P}_N(\mathbb{R})$ is determined uniquely by

$$\mathcal{I}_{G,N} v(\xi_{G,N,j}) = v(\xi_{G,N,j}), \quad 0 \leqslant j \leqslant N.$$

Guo and Xu [73] proved that for any $v \in H^r_\omega(\mathbb{R})$ and integers $r \geqslant 1$, $0 \leqslant k \leqslant r$,

$$\|\partial^k_x (\mathcal{I}_{G,N} v - v)\|_\omega \leqslant c N^{\frac{1}{3}+\frac{k-r}{2}} \|\partial^r_x v\|_\omega.$$

As an example, we consider the Burgers equation with the kinetic viscosity $\mu > 0$,

$$\begin{cases} \partial_s V(y,t) + \dfrac{1}{2} V(y,t)\partial_y V(y,t) - \mu \partial^2_y V(y,t) = g(y,t), & -\infty < y < \infty, \quad 0 < s \leqslant T, \\ V(y,0) = V_0(y), & -\infty < y < \infty. \end{cases} \tag{3.2}$$

In addition, $V(y,t)$ satisfies certain conditions at the infinity. The above problem is not well-posed in the Sobolev space with the weight function $\omega(x)$. To remedy this deficiency, we may take the similarity transformation $x = \frac{y}{2\sqrt{\mu(1+s)}}$ and $t = \ln(1+s)$. Accordingly, $U(x,t) = \mathrm{e}^{x^2} V(2\sqrt{\mu} x \mathrm{e}^{\frac{t}{2}}, \mathrm{e}^t - 1)$ and $f(x,t) = \mathrm{e}^{x^2+t} g(2\sqrt{\mu} x \mathrm{e}^{\frac{t}{2}}, \mathrm{e}^t - 1)$. Then (3.2) is changed to

$$\begin{cases} \partial_t U(x,t) + \dfrac{1}{2} U(x,t) + \dfrac{1}{2} x \partial_x U(x,t) + \dfrac{1}{4\sqrt{\mu}} \mathrm{e}^{x^2+\frac{t}{2}} \partial_x (\mathrm{e}^{-2x^2} U^2(x,t)) \\ \quad -\dfrac{1}{4} \partial^2_x U(x,t) = f(x,t), \quad x \in \mathbb{R}, \quad 0 < t \leqslant \ln(1+T), \\ U(x,0) = \mathrm{e}^{x^2} V_0(2\sqrt{\mu} x), \quad x \in \mathbb{R}. \end{cases}$$

This reformed problem is well-posed in the weighted Sobolev space. Therefore, we could design the proper Hermite spectral and pseudospectral schemes. Xu and Guo [145] also proposed the multiple-dimensional Hermite spectral and pseudospectral methods with their applications to the logistic equation.

The standard Hermite spectral method might not be the most appropriate for some problems. Guo et al. [51], and Tang [120] took the Hermite functions $\mathrm{e}^{-\frac{1}{2}x^2} H_l(x), l \geqslant 0$, as the basis functions, which are mutually orthogonal in the space $L^2(\mathbb{R})$. Thereby, the numerical schemes have the same weight functions as in the weak formulations of original problems. As a result, the numerical solutions keep the same conservations as the exact ones. This trick has been applied to the numerical simulations of Gaussian type functions, the Dirac equation an so on.

In some practical problems, the solutions decay exponentially as $|x| \to \infty$. In this case, it is reasonable to take the basis functions like $\mathrm{e}^{-\alpha^2 x^2} H_l(\alpha x), \alpha > 0, l \geqslant 0$, which are mutually orthogonal in the space with the weight function $\mathrm{e}^{\alpha^2 x^2}$. Funaro and Kavian [26] first used such functions with $\alpha = \frac{1}{2}$. Tang et al. [121], and Xiang and Wang [147] provided numerical algorithms with $\alpha > 0$, for simulating the dispersion of small particles in a turbulent flows and the growth and dispersal of population. Meanwhile, Fox et al. [23] proposed the composite Hermite spectral-finite difference method for the two-dimensional Fokker-Planck equation. Guo and Wang [60], and Wang and Guo [131] investigated the mixed Hermite-Legendre spectral and pseudospectral methods with $\alpha = 1$, for heat transfer in an infinite stripe. Ma et al. [96], and Ma and Zhao [97] developed the Hermite spectral method with time-dependent basis functions $\mathrm{e}^{-\alpha(t)x^2}\phi(x), \phi \in \mathcal{P}_N(\mathbb{R})$. The reasonable choice of $\alpha(t)$ depends on some coefficients of underlying evolutionary problems, which in turn affect the asymptotic behaviors of solutions as t increases.

Zhang and Guo [153] introduced a family of new generalized Hermite functions, which are mutually orthogonal with the weight function $(1+x^2)^{-\gamma}$, where γ is any real number. By adjusting the parameter γ suitably, they simulate the asymptotic behaviors of approximated functions at infinity reasonably. The corresponding orthogonal approximation and interpolation were used for numerical solutions of the sine-Gordon equation. Recently, Guo and Zhang [78] studied the more general orthogonal approximation and interpolation using the basis functions, which are mutually orthogonal with the weight function $(1+\frac{2}{\pi}\arctan x)^{-2\alpha}(1-\frac{2}{\pi}\arctan x)^{-2\gamma}$. By adjusting the arbitrary real parameters α and γ suitably, they simulate different asymptotic behaviors of approximated functions at the different endpoints of the whole line properly.

3.2 Laguerre spectral and pseudospectral methods

The Laguerre spectral and pseudospectral methods play important roles in numerical solutions of differential equations defined on various unbounded domains, as well as certain exterior problems.

Let $\omega^{(\alpha,\beta)}(x) = x^\alpha \mathrm{e}^{-\beta x}, \alpha > -1, \beta > 0$. We define the weighted space $H^r_{\omega^{(\alpha,\beta)}}(\mathbb{R}^+)$ and its norm $\|v\|_{r,\omega^{(\alpha,\beta)}}$ as usual. The inner product and the norm of the space $L^2_{\omega^{(\alpha,\beta)}}(\mathbb{R}^+)$ are denoted by $(u,v)_{\omega^{(\alpha,\beta)}}$ and $\|v\|_{\omega^{(\alpha,\beta)}}$, respectively.

Guo and Zhang [80] considered the scaled generalized Laguerre polynomials

$$\mathcal{L}_l^{(\alpha,\beta)}(x) = \frac{1}{l!}x^{-\alpha}\mathrm{e}^{\beta x}\partial_x^l(x^{l+\alpha}\mathrm{e}^{-\beta x}), \quad l \geqslant 0,$$

which form a complete $L^2_{\omega^{(\alpha,\beta)}}(\mathbb{R}^+)$-orthogonal system.

The orthogonal projection $P_{N,\alpha,\beta} : L^2_{\omega^{(\alpha,\beta)}}(\mathbb{R}^+) \to \mathcal{P}_N(\mathbb{R}^+)$ is defined by

$$(P_{N,\alpha,\beta}v - v, \phi)_{\omega^{(\alpha,\beta)}} = 0, \quad \forall\, \phi \in \mathcal{P}_N(\mathbb{R}^+).$$

If $\partial_x^r v \in L^2_{\omega^{(\alpha+r,\beta)}}(\mathbb{R}^+)$ and integers $r \geqslant 0$, $0 \leqslant k \leqslant r \leqslant N+1$, then

$$\|\partial_x^k(P_{N,\alpha,\beta}v - v)\|_{\omega^{(\alpha+k,\beta)}} \leqslant c_{\alpha,\beta}(\beta N)^{\frac{k-r}{2}}\|\partial_x^r v\|_{\omega^{(\alpha+r,\beta)}}, \tag{3.3}$$

where the positive constant $c_{\alpha,\beta}$ is given explicitly.

In many differential equations, the coefficients of derivatives of different orders of unknown functions grow or degenerate in different ways. Thus, we have to consider the orthogonal approximation in certain non-uniformly weighted Sobolev spaces. Now, we let $\alpha, \gamma > -1, \beta, \delta > 0$, and introduce the space $H^1_{\alpha,\beta,\gamma,\delta}(\mathbb{R}^+)$, equipped with the norm $\|v\|_{1,\alpha,\beta,\gamma,\delta} = (\|\partial_x v\|^2_{\omega^{(\alpha,\beta)}} + \|v\|^2_{\omega^{(\gamma,\delta)}})^{\frac{1}{2}}$. The orthogonal projection $P^1_{N,\alpha,\beta,\gamma,\delta} : H^1_{\alpha,\beta,\gamma,\delta}(\mathbb{R}^+) \to \mathcal{P}_N(\mathbb{R}^+)$ is defined by

$$(\partial_x(P^1_{N,\alpha,\beta,\gamma,\delta}v - v), \partial_x\phi)_{\omega^{(\alpha,\beta)}} + (P^1_{N,\alpha,\beta,\gamma,\delta}v - v, \phi)_{\omega^{(\gamma,\delta)}} = 0, \quad \forall\, \phi \in \mathcal{P}_N(\mathbb{R}^+).$$

As a special and important case, we assume that $-1 < \gamma \leqslant \alpha \leqslant \gamma+2$, $\beta = \delta$, or $\gamma > -1$, $-1 < \alpha \leqslant \gamma + 2$, $\beta < \delta$. If $v \in H^1_{\alpha,\beta,\gamma,\delta}(\mathbb{R}^+)$, $\partial_x^r v \in L^2_{\omega^{(\alpha+r-1,\beta)}}(\mathbb{R}^+)$ and integers $1 \leqslant r \leqslant N+1$, then

$$\|P^1_{N,\alpha,\beta,\gamma,\delta}v - v\|_{1,\alpha,\beta,\gamma,\delta} \leqslant c_{\alpha,\beta,\gamma,\delta}(\beta N)^{\frac{1-r}{2}}\|\partial_x^r v\|_{\omega^{(\alpha+r-1,\beta)}}, \tag{3.4}$$

where the positive constant $c_{\alpha,\beta,\gamma,\delta}$ is given explicitly.

Taking the boundary conditions into consideration, we denote by ${}_0H^1_{\omega_{\alpha,\beta}}(\mathbb{R}^+)$ and ${}_0\mathcal{P}_N(\mathbb{R}^+)$ the subsets of $H^1_{\omega_{\alpha,\beta}}(\mathbb{R}^+)$ and $\mathcal{P}_N(\mathbb{R}^+)$, respectively, in which all functions vanish at $x = 0$. We define the orthogonal projection ${}_0P^1_{N,\alpha,\beta}(\mathbb{R}^+) : {}_0H^1_{\omega^{(\alpha,\beta)}}(\mathbb{R}^+) \to {}_0\mathcal{P}_N(\mathbb{R}^+)$ by

$$(\partial_x({}_0P^1_{N,\alpha,\beta}v - v), \partial_x\phi)_{\omega^{(\alpha,\beta)}} = 0, \quad \forall\, \phi \in {}_0\mathcal{P}_N(\mathbb{R}^+).$$

If $\partial_x v \in L^2_{\omega^{(\alpha,\beta)}}(\mathbb{R}^+)$, $\partial^r_x v \in L^2_{\omega^{(\alpha+r-1,\beta)}}(\mathbb{R}^+)$ and $v(0) = 0$, then for integers $1 \leqslant r \leqslant N+1$,

$$\|\partial_x({}_0P^1_{N,\alpha,\beta}v - v)\|_{\omega^{(\alpha,\beta)}} \leqslant c_{\alpha,\beta}(\beta N)^{\frac{1-r}{2}}\|\partial^r_x v\|_{\omega^{(\alpha+r-1,\beta)}}. \tag{3.5}$$

If, in addition, $v \in L^2_{\omega^{(\alpha,\beta)}}(\mathbb{R}^+)$ and $|\alpha| < 1$, then

$$\|{}_0P^1_{N,\alpha,\beta}v - v\|_{\omega^{(\alpha,\beta)}} \leqslant c_{\alpha,\beta}(\beta N)^{\frac{1-r}{2}}\|\partial^r_x v\|_{\omega^{(\alpha+r-1,\beta)}}. \tag{3.6}$$

We now turn to the generalized Laguerre interpolation. For any integer $r \geqslant 0$, we denote by $C^r(\bar{\mathbb{R}}^+)$ the space consisting of all r-times differential functions. Let $\xi^{(\alpha,\beta)}_{G,N,j}$ and $\xi^{(\alpha,\beta)}_{R,N,j}$, $0 \leqslant j \leqslant N$, be the zeros of $\mathcal{L}^{(\alpha,\beta)}_{N+1}(x)$ and $x\mathcal{L}^{(\alpha+1,\beta)}_N(x)$, respectively. They are arranged in ascending order. For $v \in \bar{C}(\mathbb{R}^+)$, the generalized Laguerre-Gauss type interpolation $\mathcal{I}_{Z,N,\alpha,\beta}v \in \mathcal{P}_N(\mathbb{R}^+)$ is determined uniquely by

$$\mathcal{I}_{Z,N,\alpha,\beta}v(\xi^{(\alpha,\beta)}_{G,N,j}) = v(\xi^{(\alpha,\beta)}_{G,N,j}), \quad Z = G, R, \quad 0 \leqslant j \leqslant N,$$

where $Z = G, R$ correspond to the Laguerre-Gauss interpolation and the Laguerre-Radau interpolation, respectively.

Guo et al. [59] proved that if $\partial^k_x v \in L^2_{\omega^{(\alpha,\beta)}}(\mathbb{R}^+)$, $\partial^r_x v \in L^2_{\omega^{(\alpha+r,\beta)}}(\mathbb{R}^+) \cap L^2_{\omega^{(\alpha+r-1,\beta)}}(\mathbb{R}^+) \cap L^2_{\omega^{(r+\alpha-k,\beta)}}(\mathbb{R}^+)$ and integers $r \geqslant 1$, $0 \leqslant k \leqslant r \leqslant N+1$, then

$$\begin{aligned}\|\partial^k_x(\mathcal{I}_{G,N,\alpha,\beta}v - v)\|_{\omega^{(\alpha,\beta)}} \leqslant{}& c_{\alpha,\beta}(\beta N)^{\frac{2k+1-r}{2}}(\beta^{-1}(\|\partial^r_x v\|_{\omega^{(\alpha+r-1,\beta)}} + N^{-\frac{1}{2}}\|\partial^r_x v\|_{\omega^{(\alpha+r-k,\beta)}})\\ &+ (1+\beta^{-\frac{1}{2}})(\ln N)^{\frac{1}{2}}\|\partial^r_x v\|_{\omega^{(\alpha+r,\beta)}}).\end{aligned} \tag{3.7}$$

If, in addition, $r > \alpha + 1$ or $|\alpha| < 1$, then

$$\begin{aligned}\|\partial^k_x(\mathcal{I}_{R,N,\alpha,\beta}v - v)\|_{\omega^{(\alpha,\beta)}} \leqslant{}& c_{\alpha,\beta}(\beta N)^{\frac{2k+1-r}{2}}(\beta^{-1}(\|\partial^r_x v\|_{\omega^{(\alpha+r-1,\beta)}} + N^{-\frac{1}{2}}\|\partial^r_x v\|_{\omega^{(\alpha+r-k,\beta)}})\\ &+ (1+\beta^{-\frac{1}{2}})(\ln N)^{\frac{1}{2}}\|\partial^r_x v\|_{\omega^{(\alpha+r,\beta)}}).\end{aligned} \tag{3.8}$$

Mastroianni and Monegato [99] also studied the generalized Laguerre interpolation with its application to integral equations. Xu and Guo [146] investigated the generalized Laguerre approximation in multiple dimensions.

In the early work, we used the Laguerre polynomials mostly, i.e., $\alpha = 0$ and $\beta = 1$. Coulaud et al. [20], Funaro [24,25], Guo and Shen [44], Maday et al. [98], and Xu and Guo [143] proposed the related spectral and pseudospectral methods for various steady and unsteady problems arising in fluid dynamics. For example, we consider the Burgers equation with the homogeneous boundary condition at $x = 0$. We make the variable transformation $y = x, U(x,t) = \mathrm{e}^{\frac{x}{2}}V(x,t)$ and $f(x,t) = \mathrm{e}^{\frac{x}{2}}g(x,t)$. Then, the Burgers equation on the half line becomes

$$\begin{cases} \partial_t U(x,t) - \mu(\partial^2_x U(x,t) - \partial_x U + \dfrac{1}{4}U(x,t)) + \dfrac{1}{2}\mathrm{e}^{\frac{x}{2}}\partial_x(\mathrm{e}^{-x}U^2(x,t)) = f(x,t), & x \in \mathbb{R}^+, \quad 0 < t \leqslant T,\\ U(0,t) = 0, & 0 \leqslant t \leqslant T,\\ U(x,0) = \mathrm{e}^{\frac{x}{2}}V_0(x), & x \in \mathbb{R}^+. \end{cases}$$

This problem is well-posed in the weighted Sobolev space. Hence, we could solve it by the Laguerre spectral and pseudospectral methods numerically.

Wang and Guo [126] proposed the stair Laguerre pseudospectral method. In other words, we first use the Laguerre interpolation with moderate mode N, and then use the shifted Laguerre interpolation to extend the numerical solutions step by step. This approach simplifies calculation and raises the numerical accuracy. On the other hand, Bernardi et al. [5], and Bernardi and Maday [7] developed the $H^2_{\omega_{0,1}}(\mathbb{R}^+)$-orthogonal approximation and the corresponding interpolations, with their applications to fourth order problems. Guo and Xu [74], and Xu and Guo [144] proposed the mixed Legendre-Laguerre spectral and pseudospectral methods for the stream function form of the incompressible fluid flows in an infinite strip.

The Laguerre approximation is available essentially for rectangular domains. In opposite, the generalized Laguerre approximation is applicable to a large class of other problems. As an example, let ρ, λ and θ be the radius, the longitude and the latitude, respectively. We consider the following problem with the spherical geometry,

$$\begin{cases} -\dfrac{1}{\rho^2}\partial_\rho(\rho^2\partial_\rho U(\rho,\lambda,\theta)) - \dfrac{1}{\rho^2\cos\theta}\partial_\theta(\cos\theta\partial_\theta U(\rho,\lambda,\theta)) \\ \quad -\dfrac{1}{\rho^2\cos^2\theta}\partial^2_\lambda U(\rho,\lambda,\theta) = f(\rho,\lambda,\theta), \quad \rho>\rho_0, \quad 0\leqslant\lambda<2\pi, \quad |\theta|<\dfrac{\pi}{2}, \\ \lim\limits_{\rho\to\infty} U(\rho,\lambda,\theta)=0, \quad 0\leqslant\lambda<2\pi, \quad -\dfrac{\pi}{2}\leqslant\theta<\dfrac{\pi}{2}, \\ U(\rho,\lambda+2\pi,\theta)=U(\rho,\lambda,\theta), \quad \rho>\rho_0, \quad 0\leqslant\lambda<2\pi, \quad -\dfrac{\pi}{2}\leqslant\theta<\dfrac{\pi}{2}. \end{cases} \tag{3.9}$$

In addition, $\partial_\lambda U(\rho,\lambda,\theta)=0$ for $|\theta|=\frac{1}{2}\pi$. If $\rho_0>0$, then we impose the boundary condition as $U(\rho_0,\lambda,\theta)=g(\lambda,\theta)$. This case corresponds to an exterior problem. We could adopt the spherical harmonic approximation in the (λ,θ)-directions, and benefit from the orthogonality of spherical harmonic functions. Furthermore, in the weak formulation of (3.9), the coefficients of terms $\partial_\rho U, \partial_\theta U$ and $\frac{1}{\cos\theta}\partial_\lambda U$ are $\rho^2, 1$ and 1, respectively. Thus, it is natural to use the generalized Laguerre approximation with $\alpha=2$, in the ρ-direction. Moreover, we may adjust the parameter β suitably so that the numerical solutions match the exact ones properly.

The generalized Laguerre approximation has been used successfully for a lot of problems. Guo and Zhang [80], and Zhang and Guo [155] used the above approach with $\alpha=2$, coupled with the spherical harmonic approximation, for problems in the whole three-dimensional space and outside a ball, respectively. Guo et al. [52] applied this trick with $\alpha=\beta=1$, coupled with the Fourier approximation, to two-dimensional exterior problems. Guo and Jiao [41] proposed the mixed spectral method for exterior problem of the Navier-Stokes equations, with a disc obstacle. Besides, Guo et al. [59] provided the pseudospectral method for spherically symmetrical solutions of (3.9) with $\rho_0=0$ and $\rho_0>0$, respectively.

3.3 Laguerre approximation using generalized Laguerre functions

Many practical problems are not well-posed in the Laguerre weighted Sobolev spaces. Thus, we often need certain variable transformations for deriving reasonable alternative forms, which are not convenient in multiple dimensions usually. On the other hand, the weight functions might destroy some properties of numerical solutions, which the exact solutions possess. They also bring difficulties for matching the numerical solutions of domain decomposition methods, on the common boundaries of adjacent subdomains. Indeed, for the functions belong to certain Sobolev spaces with the weight function $\omega^{(\alpha)}(x)=x^\alpha$, we prefer to the spectral methods using the generalized Laguerre functions.

Guo and Zhang [81] considered the scaled generalized Laguerre functions

$$\tilde{\mathcal{L}}^{(\alpha,\beta)}_l(x)=\mathrm{e}^{-\frac{1}{2}\beta x}\mathcal{L}^{(\alpha,\beta)}_l(x), \quad l\geqslant 0,$$

which form a complete $L^2_{\omega^{(\alpha)}}(\mathbb{R}^+)$-orthogonal system. Let

$$Q_{N,\alpha,\beta}(\mathbb{R}^+)=\{\mathrm{e}^{-\frac{1}{2}\beta x}\phi \mid \phi\in\mathcal{P}_N(\mathbb{R}^+)\}.$$

The orthogonal projection $\tilde{P}_{N,\alpha,\beta}: L^2_{\omega^{(\alpha)}}(\mathbb{R}^+)\to Q_{N,\alpha,\beta}(\mathbb{R}^+)$ is defined by

$$(\tilde{P}_{N,\alpha,\beta}v-v,\phi)_{\omega^{(\alpha)}}=0, \quad \forall\,\phi\in Q_{N,\alpha,\beta}(\mathbb{R}^+).$$

If $v \in L^2_{\omega^{(\alpha)}}(\mathbb{R}^+)$ and integers $r \geqslant 0$, $r \leqslant N+1$, then

$$\|\tilde{P}_{N,\alpha,\beta}v - v\|_{\omega^{(\alpha)}} \leqslant c(\beta N)^{-\frac{r}{2}}\|\partial_x^r(\mathrm{e}^{\frac{1}{2}\beta x}v)\|_{\omega^{(\alpha+r,\beta)}}, \tag{3.10}$$

provided that $\|\partial_x^r(\mathrm{e}^{\frac{1}{2}\beta x}v)\|_{\omega^{(\alpha+r,\beta)}}$ is finite. The special result with $\alpha = 0$ and $\beta = 1$ was given earlier by Shen [102].

In applications, we need several specific orthogonal projections in non-uniformly weighted spaces, which correspond to different underlying problems. For example, we define the space $H^1_{2,0}(\mathbb{R}^+)$ with the norm $\|v\|_{1,2,0} = (\|\partial_x v\|^2_{\omega^{(2)}} + \|v\|^2_{\omega^{(0)}})^{\frac{1}{2}}$. Let $\eta \geqslant 0$. The orthogonal projection $\tilde{P}^1_{N,\alpha,\beta} : H^1_{2,0}(\mathbb{R}^+) \to Q_{N,\alpha,\beta}(\mathbb{R}^+)$ is defined by

$$(\partial_x(\tilde{P}^1_{N,\alpha,\beta}v - v), \partial_x\phi)_{\omega^{(2)}} + \eta(\tilde{P}^1_{N,\alpha,\beta}v - v, \phi)_{\omega^{(0)}} = 0, \quad \forall\, \phi \in Q_{N,\alpha,\beta}(\mathbb{R}^+).$$

If $v \in H^1_{2,0}(\mathbb{R}^+)$ and integers $1 \leqslant r \leqslant N+1$, then

$$\|\tilde{P}^1_{N,\alpha,\beta}v - v\|_{1,2,0} \leqslant c(1+\eta)(\beta N)^{\frac{1-r}{2}}\|\partial_x^{r-1}(\mathrm{e}^{\frac{1}{2}\beta x}\partial_x v)\|_{\omega^{(r+1,\beta)}}, \tag{3.11}$$

provided that $\|\partial_x^{r-1}(\mathrm{e}^{\frac{1}{2}\beta x}\partial_x v)\|_{\omega^{(r+1,\beta)}}$ is finite.

Wang et al. [139] studied the interpolations $\widetilde{\mathcal{I}}_{Z,N,\alpha,\beta}v \in Q_{N,\alpha,\beta}(\mathbb{R}^+)$, defined by

$$\widetilde{\mathcal{I}}_{Z,N,\alpha,\beta}v(\xi^{(\alpha,\beta)}_{Z,N,j}) = v(\xi^{(\alpha,\beta)}_{Z,N,j}), \quad Z = G, R, \quad 0 \leqslant j \leqslant N,$$

where $z = G, R$ correspond to the generalized Laguerre-Gauss interpolation, and the generalized Laguerre-Radau interpolation, respectively.

If $\partial_x^k v \in L^2_{\omega^{(\alpha)}}(\mathbb{R}^+)$ and integers $1 \leqslant r \leqslant N+1$, then

$$\begin{aligned}\|\partial_x^k(\widetilde{\mathcal{I}}_{G,N,\alpha,\beta}v - v)\|_{\omega^{(\alpha)}} \leqslant{}& c(\beta N)^{\frac{2k+1-r}{2}}\Bigg((\beta^{-1} + (\beta N)^{\frac{-1-k}{2}})N^{-\frac{1}{2}}\sum_{\mu=0}^{k}\|\partial_x^\mu(\mathrm{e}^{\frac{1}{2}\beta x}v)\|_{\omega^{(\alpha+r-k,\beta)}}\\ &+\beta^{-1}\|\partial_x^r(\mathrm{e}^{\frac{1}{2}\beta x}v)\|_{\omega^{(\alpha+r-1,\beta)}} + (1+\beta^{-\frac{1}{2}})(\ln N)^{\frac{1}{2}}\|\partial_x^r(\mathrm{e}^{\frac{1}{2}\beta x}v)\|_{\omega^{(\alpha+r,\beta)}}\Bigg).\end{aligned} \tag{3.12}$$

If, in addition, $r > \alpha + 1$ or $|\alpha| < 1$, then we have also

$$\begin{aligned}\|\partial_x^k(\widetilde{\mathcal{I}}_{R,N,\alpha,\beta}v - v)\|_{\omega^{(\alpha)}} \leqslant{}& c(\beta N)^{\frac{2k+1-r}{2}}\Bigg((\beta^{-1} + (\beta N)^{\frac{-1-k}{2}})N^{-\frac{1}{2}}\sum_{\mu=0}^{k}\|\partial_x^\mu(\mathrm{e}^{\frac{1}{2}\beta x}v)\|_{\omega^{(\alpha+r-k,\beta)}}\\ &+\beta^{-1}\|\partial_x^r(\mathrm{e}^{\frac{1}{2}\beta x}v)\|_{\omega^{(\alpha+r-1,\beta)}} + (1+\beta^{-\frac{1}{2}})(\ln N)^{\frac{1}{2}}\|\partial_x^r(\mathrm{e}^{\frac{1}{2}\beta x}v)\|_{\omega^{(\alpha+r,\beta)}}\Bigg).\end{aligned} \tag{3.13}$$

The above two results are valid provided that all norms involved at the right-hand side of the above inequalities are finite. The special result with $\alpha = 0$ and $\beta = 1$ was due to Guo and Wang [57].

Shen [102] proposed the spectral method using the Laguerre functions, i.e., $\alpha = 0, \beta = 1$. Meanwhile, Guo and Ma [43], and Ma and Guo [93] used such basis functions for the related domain decomposition spectral methods. More precisely, they used the Legendre approximation on the subinterval near $x = 0$, where the solutions change rapidly, and adopted the Laguerre approximation on the remaining subinterval. If the solutions vary fast or have several peaks at the large points, then we may refine the numerical results by the multidomain Legendre approximation between the Laguerre interpolation nodes. This technique is very suitable for parallel computation, and recovers the structures of solutions between the interpolation nodes, see [57]. Azaiez et al. [1], and Zhuang et al. [160] considered the mixed Legendre-Laguerre pseudospectral and pseudospectral element methods for the Stokes equation in an infinite strip.

The generalized Laguerre functions are powerful tools for a number of important problems. For example, let x and $V(x,s)$ be the value of underlying security and the value of derivative security, respectively. We consider the Black-Scholes type equation

$$\begin{cases} \partial_s V(x,s) + xB\partial_x V(x,s) + \partial_x(x^2 A\partial_x V(x,s)) - GV(x,s) = F(x,s), & x \in \mathbb{R}^+, \quad 0 \leqslant s < T,\\ V(x,T) = V_0(x), & x \in \mathbb{R}^+, \end{cases} \tag{3.14}$$

where $0 < a_0 \leqslant A(x,s) \leqslant a_1$, and $|B(x,s)|$ is bounded as $x \to 0$. Clearly, the coefficients $xB(x,s)$ and $x^2A(x,s)$ degenerate as $x \to 0$. According to Fichera's theory, we could not impose any boundary condition at $x = 0$. The equation (3.14) includes many important models in financial mathematics, such as the Black-Scholes, Dothan, and Black-Derman-Toy models. In practical cases, the asymptotic behavior of solution of (3.14) depends on the final state $V_0(x)$. There are two types of the most interesting solutions. The first type of solutions is the call-option, growing to infinity as $x \to \infty$. The second type of solutions is the put-option, decaying to zero rapidly as $x \to \infty$. There exists an explicit relation between these two types of solutions. Guo and Zhang [81] proposed the spectral method for solving the put-options.

Zhang et al. [156] provided the mixed generalized Laguerre-spherical harmonic spectral method for the three-dimensional nonlinear Klein-Gordon equation. Zhang et al. [154] used the generalized Laguerre-Fourier spectral and pseudospectral methods for two-dimensional exterior problems. Wang et al. [140] developed the generalized Laguerre-spherical harmonic spectral method for three-dimensional exterior problems. Furthermore, Guo and Wang [65], Guo et al. [69], and Yan and Guo [148, 149] designed the new collocation methods for initial value problems of differential equations of first and second orders, based on the interpolation by using the Laguerre polynomials and the Laguerre functions.

3.4 Laguerre approximation with arbitrary parameter α

Guo et al. [53] developed the generalized Laguerre approximation with arbitrary real parameter α, and proposed the Laguerre quasi-orthogonal approximation (also see [76]). Everitt et al. [22] considered the Laguerre approximation with negative integer α, without the error estimate.

Denote by $[\alpha]$ the largest integer $\leqslant \alpha$. Let $\bar{l}_\alpha = [-\alpha]$ for $\alpha \leqslant -1$, and $\bar{l}_\alpha = 0$ for $\alpha > -1$. Meanwhile, $l_\alpha = l - [-\alpha]$ for $\alpha \leqslant -1$, and $l_\alpha = l$ for $\alpha > -1$. The new generalized Laguerre functions are defined by

$$\bar{\mathcal{L}}_l^{(\alpha,\beta)}(x) = \begin{cases} x^{-\alpha}\mathcal{L}_{l_\alpha}^{(-\alpha,\beta)}(x), & \alpha \leqslant -1, \quad l \geqslant \bar{l}_\alpha, \\ \mathcal{L}_l^{(\alpha,\beta)}(x), & \alpha > -1, \quad l \geqslant \bar{l}_\alpha, \end{cases}$$

which conform a complete $L^2_{\omega^{(\alpha,\beta)}}(\mathbb{R}^+)$-orthogonal system.

Let

$$\bar{Q}_{N,\alpha,\beta}(\mathbb{R}^+) = \mathrm{span}\{\bar{\mathcal{L}}_l^{(\alpha,\beta)}(x),\ \bar{l}_\alpha \leqslant l \leqslant N\}.$$

The orthogonal projection $\bar{P}_{N,\alpha,\beta} : L^2_{\omega^{(\alpha,\beta)}}(\mathbb{R}^+) \to \bar{Q}_{N,\alpha,\beta}(\mathbb{R}^+)$ is defined by

$$(\bar{P}_{N,\alpha,\beta}v - v, \phi)_{\omega^{(\alpha,\beta)}} = 0, \quad \forall\, \phi \in \bar{Q}_{N,\alpha,\beta}(\mathbb{R}^+).$$

For estimating the approximation error, we introduce the Sturm-Liouville operator

$$\mathcal{A}_{\alpha,\beta}v(x) = -x^{-\alpha}\mathrm{e}^{\beta x}\partial_x(x^{\alpha+1}\mathrm{e}^{-\beta x}\partial_x v(x)).$$

Furthermore, we define the following Sobolev-type spaces with integer $r \geqslant 0$,

$$D(\mathcal{A}^r_{\alpha,\beta}) = \{v \mid \mathcal{A}^k_{\alpha,\beta}v \in L^2_{\omega^{(\alpha,\beta)}}(\mathbb{R}^+) \text{ for } 0 \leqslant k \leqslant r\},$$
$$D(\mathcal{A}^{r+\frac{1}{2}}_{\alpha,\beta}) = \{v \mid v \in D(\mathcal{A}^r_{\alpha,\beta}) \text{ and } \partial_x\mathcal{A}^r_{\alpha,\beta}v \in L^2_{\omega^{(\alpha,\beta)}}(\mathbb{R}^+)\},$$

equipped with the following semi-norms and norms,

$$|v|_{D(\mathcal{A}^r_{\alpha,\beta})} = \|\mathcal{A}^r_{\alpha,\beta}v\|_{\omega^{(\alpha,\beta)}}, \quad |v|_{D(\mathcal{A}^{r+\frac{1}{2}}_{\alpha,\beta})} = \|\partial_x\mathcal{A}^r_{\alpha,\beta}v\|_{\omega^{(\alpha+1,\beta)}},$$

$$\|v\|_{D(\mathcal{A}^r_{\alpha,\beta})} = \left(\sum_{k=0}^{r}|v|^2_{D(\mathcal{A}^k_{\alpha,\beta})}\right)^{\frac{1}{2}}, \quad \|v\|_{D(\mathcal{A}^{r+1/2}_{\alpha,\beta})} = (\|v\|^2_{D(\mathcal{A}^r_{\alpha,\beta})} + |v|^2_{D(\mathcal{A}^{r+1/2}_{\alpha,\beta})})^{\frac{1}{2}}.$$

If $v \in D(\mathcal{A}^{\frac{r}{2}}_{\alpha,\beta})$ and integers $r \geqslant 0$, $0 \leqslant k \leqslant r \leqslant N+1$, then

$$|\bar{P}_{N,\alpha,\beta}v - v|_{D(\mathcal{A}^{\frac{k}{2}}_{\alpha,\beta})} \leqslant c(\beta N)^{\frac{k-r}{2}}|v|_{D(\mathcal{A}^{\frac{r}{2}}_{\alpha,\beta})}. \tag{3.15}$$

In practice, the specific projection $\bar{P}_{N,-m,\beta}v$ is the most useful, where m is any positive integer. If $\partial_x^k v \in L^2_{\omega^{(-m+k,\beta)}}(\mathbb{R}^+)$, $\partial_x^r v \in L^2_{\omega^{(-m+r,\beta)}}(\mathbb{R}^+)$, and integers $1 \leqslant m \leqslant \min(r,N)$, $0 \leqslant k \leqslant r \leqslant N+1$, $k \leqslant m$, then

$$\|\partial_x^k(\bar{P}_{N,-m,\beta}v - v)\|_{\omega^{(-m+k,\beta)}} \leqslant c(\beta N)^{\frac{k-r}{2}}\|\partial_x^r v\|_{\omega^{(-m+r,\beta)}}. \tag{3.16}$$

For numerical solutions of high order differential equations, we need another orthogonal projection. For this purpose, we introduce the space $\bar{H}^m_{\beta,A}(\mathbb{R}^+)$, equipped with norm $\|v\|_{\bar{H}^m_{\beta,A}} = (\sum_{k=0}^m \|\partial_x^k v\|^2_{\omega^{(-m+k,\beta)}})^{\frac{1}{2}}$. Moreover,

$$ {}_0\bar{H}^m_{\beta,A}(\mathbb{R}^+) = \{v \mid v \in \bar{H}^m_{\beta,A}(\mathbb{R}^+),\ \text{and } \partial_x^k v(0) = 0 \text{ for } 0 \leqslant k \leqslant m-1\}.$$

The projection ${}_0\bar{P}^m_{N,-m,\beta} : {}_0H^m_{\beta,A}(\mathbb{R}^+) \to \bar{Q}_{N,-m,\beta}(\mathbb{R}^+)$ is defined by

$$(\partial_x^m(v - {}_0\bar{P}^m_{N,-m,\beta}v), \partial_x^m\phi)_{\omega^{(0,\beta)}} = 0, \quad \forall\, \phi \in \bar{Q}_{N,-m,\beta}(\mathbb{R}^+).$$

In fact, ${}_0\bar{P}^m_{N,-m,\beta}v(x) = \bar{P}_{N,-m,\beta}v(x)$ for any $v \in {}_0\bar{H}^m_{\beta,A}(\mathbb{R}^+)$. Accordingly, if $\partial_x^k v \in L^2_{\omega^{(-m+k,\beta)}}(\mathbb{R}^+)$, $\partial_x^r v \in L^2_{\omega^{(-m+r,\beta)}}(\mathbb{R}^+)$, and integers $1 \leqslant m \leqslant \min(r,N)$, $0 \leqslant k \leqslant r \leqslant N+1$, $k \leqslant m$, then

$$\|\partial_x^k({}_0\bar{P}^m_{N,-m,\beta}v - v)\|_{\omega^{(-m+k,\beta)}} \leqslant c(\beta N)^{\frac{k-r}{2}}\|\partial_x^r v\|_{\omega^{(-m+r,\beta)}}. \tag{3.17}$$

We now turn to the Laguerre quasi-orthogonal approximation. Let

$$\bar{v}_{b,m}(x) = \sum_{j=0}^{m-1} \partial_x^j v(0)\frac{x^j}{j!}.$$

For any $v \in \bar{H}^m_{\beta,A}(\mathbb{R}^+)$, we set $\bar{v}(x) = v(x) - \bar{v}_{b,m}(x)$. Evidently, $\bar{v}(x) \in {}_0\bar{H}^m_{\beta,A}(\mathbb{R}^+)$. Thus, we could define the generalized Laguerre quasi-orthogonal projection by

$$\bar{P}^m_{*,N,-m,\beta}v(x) = {}_0\bar{P}^m_{N,-m,\beta}\bar{v}(x) + \bar{v}_{b,m}(x).$$

Obviously, $\bar{P}^m_{*,N,-m,\beta}v \in \mathcal{P}_N(\mathbb{R}^+)$. Moreover,

$$\partial_x^k \bar{P}^m_{*,N,-m,\beta}v(0) = \partial_x^k v(0), \quad 0 \leqslant k \leqslant m-1.$$

If $\partial_x^k v \in L^2_{\omega^{(-m+k,\beta)}}(\mathbb{R}^+), \partial_x^r v \in L^2_{\omega^{(-m+r,\beta)}}(\mathbb{R}^+)$, and integers $1 \leqslant m \leqslant \min(r,N)$, $0 \leqslant k \leqslant r \leqslant N+1$, $k \leqslant m$, then

$$\|\partial_x^k(\bar{P}^m_{*,N,-m,\beta}v - v)\|_{\omega^{(-m+k,\beta)}} \leqslant c(\beta N)^{\frac{k-r}{2}}\|\partial_x^r v\|_{\omega^{(-m+r,\beta)}}. \tag{3.18}$$

Guo et al. [53] proposed the new generalized Laguerre-Gauss-Radau interpolation. For any integer $r \geqslant m-1$, we set

$${}_0C^r_m(\bar{\mathbb{R}}^+) = \{v \in C^r(\bar{\mathbb{R}}^+) \mid \partial_x^k u(0) = 0 \text{ for } 0 \leqslant k \leqslant m-1\}.$$

Let $m \geqslant 0$, and

$$\xi^{(m,\beta)}_{N,j} = \xi^{(m,\beta)}_{G,N-m,j}, \quad 0 \leqslant j \leqslant N-m.$$

For any $v \in {}_0C^{m-1}_m(\bar{\mathbb{R}}^+)$ and $m \geqslant 1$, we introduce the auxiliary interpolation by

$$\bar{\mathcal{I}}_{N,-m,\beta}v(\xi^{(m,\beta)}_{N,j}) = v(\xi^{(m,\beta)}_{N,j}), \quad 0 \leqslant j \leqslant N-m.$$

For any $v \in C^{m-1}_m(\bar{\mathbb{R}}^+)$, we define the new interpolation as

$$\bar{\mathcal{I}}_{R,N,-m,\beta}v(x) = \bar{\mathcal{I}}_{N,-m,\beta}\bar{v}(x) + \bar{v}_{b,m}(x).$$

It can be checked that

$$\bar{\mathcal{I}}_{R,N,-m,\beta}v(\xi^{(m,\beta)}_{N,j}) = v(\xi^{(m,\beta)}_{N,j}), \quad 0 \leqslant j \leqslant N-m,$$
$$\bar{\mathcal{I}}_{R,N,-m,\beta}v(0) = \partial_x^k v(0), \quad 0 \leqslant k \leqslant m-1.$$

This interpolation is the same as the generalized Laguerre-Gauss-Radau interpolation, since both of them are polynomials of degree N, satisfying the same condition at the same interpolation nodes.

If $\partial_x^k v \in L^2_{\omega^{(-m+k,\beta)}}(\mathbb{R}^+), \partial_x^r v \in L^2_{\omega^{(-m+r,\beta)}}(\mathbb{R}^+)$, and integers $1 \leqslant m \leqslant \min(r, N)$, $0 \leqslant k \leqslant r \leqslant N+1$, $k \leqslant m$, then

$$\|\partial_x^k(\bar{\mathcal{I}}_{R,N,-m,\beta}v - v)\|_{\omega^{(-m+k,\beta)}} \leqslant c(\beta^{-\frac{1}{2}}+1)(\ln N)^{\frac{1}{2}}(\beta N)^{\frac{k+1-r}{2}}\|\partial_x^r v\|_{\omega^{(-m+r,\beta)}}. \tag{3.19}$$

The above result improves and generalizes the existing results essentially. For example, the result (3.19) with $m = 1$ implies $\|\partial_x^k(\bar{\mathcal{I}}_{R,N,-1,\beta}v - v)\|_{\omega^{(-1+k,\beta)}} \leqslant c(\beta^{-\frac{1}{2}}+1)(\ln N)^{\frac{1}{2}}(\beta N)^{\frac{k+1-r}{2}}\|\partial_x^r v\|_{\omega^{(-1+r,\beta)}}$. This estimate is better than (3.8).

The Laguerre quasi-orthogonal approximation and the generalized Laguerre-Gauss-Radau interpolation play important roles in the spectral and collection methods for high order problems with mixed inhomogeneous boundary conditions, such as some problems similar to the steady beam equation and extended Fisher-Kolmogorov equation.

3.5 Approximation using Laguerre functions with arbitrary parameter α

Zhang and Guo [152] proposed the new orthogonal approximation using the Laguerre functions with arbitrary real parameter α, and the corresponding quasi-orthogonal approximation.

Let $[\alpha], \bar{l}_\alpha, \omega^{(\alpha)}$ and $\bar{\mathcal{L}}_l^{(\alpha,\beta)}(x)$ be the same as before. The new generalized Laguerre functions are given by

$$\hat{\mathcal{L}}_l^{(\alpha,\beta)}(x) = \mathrm{e}^{-\frac{\beta}{2}x}\bar{\mathcal{L}}_l^{(\alpha,\beta)}(x), \qquad l \geqslant \bar{l}_\alpha,$$

which form a complete $L^2_{\omega^{(\alpha)}}(\mathbb{R}^+)$-orthogonal system.

Let

$$\hat{Q}_{N,\alpha,\beta}(\mathbb{R}^+) = \{\mathrm{e}^{-\frac{\beta}{2}x}\phi \mid \phi \in \bar{Q}_{N,\alpha,\beta}(\mathbb{R}^+)\}.$$

The $L^2_{\omega^{(\alpha)}}(\mathbb{R}^+)$-orthogonal projection $\hat{P}_{N,\alpha,\beta,\Lambda} : L^2_{\omega^{(\alpha)}}(\mathbb{R}^+) \to \hat{Q}_{N,\alpha,\beta}(\mathbb{R}^+)$ is defined by

$$(\hat{P}_{N,\alpha,\beta,\Lambda}v - v, \phi)_{\omega^{(\alpha)}} = 0, \quad \forall\, \phi \in \hat{Q}_{N,\alpha,\beta}(\mathbb{R}^+).$$

For description of the approximation error, we introduce the Sturm-Liouville operator

$$\hat{\mathcal{A}}_{\alpha,\beta}v(x) = -x^{-\alpha}\mathrm{e}^{\frac{\beta}{2}x}\partial_x(x^{\alpha+1}\mathrm{e}^{-\beta x}\partial_x(\mathrm{e}^{\frac{\beta}{2}x}v(x))).$$

Accordingly, we define the following Sobolev-type space with integer $r \geqslant 0$,

$$D(\hat{\mathcal{A}}^r_{\alpha,\beta}) = \{v \mid \hat{\mathcal{A}}^k_{\alpha,\beta}v \in L^2_{\omega^{(\alpha)}}(\mathbb{R}^+) \text{ for } 0 \leqslant k \leqslant r\},$$
$$D(\hat{\mathcal{A}}^{r+\frac{1}{2}}_{\alpha,\beta}) = \{v \mid v \in D(\hat{\mathcal{A}}^r_{\alpha,\beta}) \text{ and } \partial_x\hat{\mathcal{A}}^r_{\alpha,\beta}v \in L^2_{\omega^{(\alpha)}}(\mathbb{R}^+)\},$$

equipped with the following semi-norms and norms,

$$|v|_{D(\hat{\mathcal{A}}^r_{\alpha,\beta})} = \|\hat{\mathcal{A}}^r_{\alpha,\beta}v\|_{\omega^{(\alpha)}}, \quad |v|_{D(\hat{\mathcal{A}}^{r+\frac{1}{2}}_{\alpha,\beta})} = \|\partial_x\hat{\mathcal{A}}^r_{\alpha,\beta}v\|_{\omega^{(\alpha+1)}},$$
$$\|v\|_{D(\hat{\mathcal{A}}^r_{\alpha,\beta})} = \left(\sum_{k=0}^{r}|v|^2_{D(\hat{\mathcal{A}}^k_{\alpha,\beta})}\right)^{\frac{1}{2}}, \quad \|v\|_{D(\hat{\mathcal{A}}^{r+1/2}_{\alpha,\beta})} = (\|v\|^2_{D(\hat{\mathcal{A}}^r_{\alpha,\beta})} + |v|^2_{D(\hat{\mathcal{A}}^{r+1/2}_{\alpha,\beta})})^{\frac{1}{2}}.$$

If $v \in D(\hat{\mathcal{A}}^{\frac{r}{2}}_{\alpha,\beta})$ and integers $r \geqslant 0$, $0 \leqslant k \leqslant r \leqslant N+1$, then

$$|\hat{P}_{N,\alpha,\beta}v - v|_{D(\mathcal{A}^{\frac{k}{2}}_{\alpha,\beta})} \leqslant c(\beta N)^{\frac{k-r}{2}}|v|_{D(\hat{\mathcal{A}}^{\frac{r}{2}}_{\alpha,\beta})}. \tag{3.20}$$

In practice, the specific projection $\hat{P}_{N,-m,\beta}v$ with positive m is the most useful. If $\partial_x^k v \in L^2_{\omega^{(-m+k)}}(\mathbb{R}^+)$, $\partial_x^r(\mathrm{e}^{\frac{\beta}{2}x}v) \in L^2_{\omega^{(-m+r,\beta)}}(\mathbb{R}^+)$, and integers $1 \leqslant m \leqslant \min(r, N)$, $0 \leqslant k \leqslant r \leqslant N+1$, $k \leqslant m$, then

$$\|\partial_x^k(\hat{P}_{N,-m,\beta}v - v)\|_{\omega^{(-m+k)}} \leqslant c(\beta N)^{\frac{k-r}{2}}\|\partial_x^r(\mathrm{e}^{\frac{\beta}{2}x}v)\|_{\omega^{(-m+r,\beta)}}. \tag{3.21}$$

We now introduce the space $\hat{H}^m_A(\mathbb{R}^+)$, equipped with the norm $\|v\|_{\hat{H}^m_A} = (\sum_{k=0}^m \|\partial_x^k v\|^2_{\omega^{(-m+k)}})^{\frac{1}{2}}$. Moreover, for $1 \leqslant m \leqslant r$,

$$_0\hat{H}^m_A(\mathbb{R}^+) = \{v \mid v \in \hat{H}^m_A(\mathbb{R}^+), \text{ and } \partial_x^k v(0) = 0 \text{ for } 0 \leqslant k \leqslant m-1\}.$$

Let $_0\bar{P}^m_{N,-m,\beta}v$ be the same as in the last subsection. If $v \in {}_0\hat{H}^m_A(\mathbb{R}^+)$, then $_0\bar{P}^m_{N,-m,\beta}(\mathrm{e}^{\frac{\beta}{2}x}v)$ is meaningful. Furthermore, let

$$\hat{v}_{b,m}(x) = \mathrm{e}^{-\frac{\beta}{2}x}\sum_{j=0}^{m-1}\left(\frac{1}{j!}\sum_{i=0}^{j} C_j^i\left(\frac{\beta}{2}\right)^{j-i}\partial_x^i v(0)\right)x^j.$$

For any $v \in \hat{H}^m_A(\mathbb{R}^+)$, we set $\hat{v}(x) = v(x) - \hat{v}_{b,m}(x)$, and define the generalized Laguerre quasi-orthogonal projection as

$$\hat{P}^m_{*,N,-m,\beta}v(x) = \mathrm{e}^{-\frac{\beta}{2}x}{}_0\bar{P}^m_{N,-m,\beta}(\mathrm{e}^{\frac{\beta}{2}x}\hat{v}(x)) + \hat{v}_{b,m}(x).$$

Clearly, $\hat{P}^m_{*,N,-m,\beta}v \in \hat{Q}_{N,-m,\beta}(\mathbb{R}^+)$. Moreover, a direct calculation shows

$$\partial_x^k \hat{P}^m_{*,N,-m,\beta,\Lambda}v(0) = \partial_x^k v(0), \quad 0 \leqslant k \leqslant m-1.$$

If $\partial_x^k v \in L^2_{\omega^{(-m+k)}}(\mathbb{R}^+), \partial_x^r(\mathrm{e}^{\frac{\beta}{2}x}v) \in L^2_{\omega^{(-m+r,\beta)}}(\mathbb{R}^+)$, and integers $1 \leqslant m \leqslant \min(r,N)$, $0 \leqslant k \leqslant r \leqslant N+1$, $k \leqslant m$, then

$$\|\partial_x^k(\hat{P}^m_{*,N,-m,\beta}v - v)\|_{\omega^{(-m+k)}} \leqslant c(\beta N)^{\frac{k-r}{2}}\|\partial_x^r(\mathrm{e}^{\frac{\beta}{2}x}v)\|_{\omega^{(-m+r,\beta)}}. \tag{3.22}$$

We now turn to the interpolation. Let $\xi^{(m,\beta)}_{N,j}$ be the same as in the last subsection. For any $v \in {}_0C^{m-1}_m(\mathbb{R}^+)$ and $m \geqslant 1$, the auxiliary interpolation $\hat{\mathcal{I}}_{N,-m,\beta}v \in \hat{Q}^{(m,\beta)}_N(\bar{\mathbb{R}}^+)$ is determined uniquely by

$$\hat{\mathcal{I}}_{N,-m,\beta}v(\xi^{(m,\beta)}_{N,j}) = v(\xi^{(m,\beta)}_{N,j}), \quad 0 \leqslant j \leqslant N-m.$$

For any $v \in C^{m-1}_m(\bar{\mathbb{R}}^+)$, we put $\hat{v}(x) = v(x) - \hat{v}_{b,m}(x)$, and define the new Laruerre-Gauss-Radau interpolation by

$$\hat{\mathcal{I}}_{R,N,-m,\beta}v(x) = \hat{\mathcal{I}}_{N,m,\beta}\hat{v}(x) + \hat{v}_{b,m}(x).$$

It can be checked that

$$\hat{\mathcal{I}}_{R,N,-m,\beta}v(\xi^{(m,\beta)}_{N,j}) = v(\xi^{(m,\beta)}_{N,j}), \quad 0 \leqslant j \leqslant N-m,$$
$$\partial_x^k\hat{\mathcal{I}}_{R,N,-m,\beta,\Lambda}v(0) = \partial_x^k v(0), \quad 0 \leqslant k \leqslant m-1.$$

If $\partial_x^k v \in L^2_{\omega^{(-m+k,\beta)}}(\mathbb{R}^+), \partial_x^r(\mathrm{e}^{\frac{\beta}{2}x}v) \in L^2_{\omega^{(-m+r,\beta)}}(\mathbb{R}^+)$, and integers $1 \leqslant m \leqslant \min(r,N)$, $0 \leqslant k \leqslant r \leqslant N+1$, $k \leqslant m$, then,

$$\|\partial_x^k(\hat{\mathcal{I}}_{R,N,-m,\beta}v - v)\|_{\tilde{\omega}^{(-m+k)}} \leqslant c(\beta^{-\frac{1}{2}}+1)(\ln N)^{\frac{1}{2}}(\beta N)^{\frac{k+1-r}{2}}\|\partial_x^r(\mathrm{e}^{\frac{\beta}{2}x}v)\|_{\omega^{(-m+r,\beta)}}. \tag{3.23}$$

Recently, Guo and Zhang [77] introduced the more general Lagurre functions

$$\hat{\mathcal{L}}^{(\alpha,\beta,\gamma,\delta)}_l(x) = (\delta+x)^{\frac{\gamma}{2}}\mathrm{e}^{-\frac{\beta}{2}x}\mathcal{L}^{(\alpha,\beta)}_l(x), \quad l \geqslant \bar{l}_\alpha,$$

where $\delta > 0$, α and γ are any real numbers. They are mutually orthogonal with the weight function $x^\alpha(\delta+x)^{-\gamma}$. The basic results on the corresponding quasi-orthogonal approximation and interpolation were established. By adjusting the parameters α and γ suitably, these approaches not only fit the boundary conditions of considered functions at the fixed boundary exactly, but also simulate their asymptotic behaviors at infinity reasonably.

The previous quasi-orthogonal approximations and the corresponding interpolations play the essential roles in the multidomain spectral and pseudospectral methods of high order differential equations defined on unbounded domains, with mixed inhomogeneous boundary conditions, see [77, 152].

The above approaches are also very helpful for solving problems of non-standard type. For example, let x be the velocity of particles. Denote by $U(x, y, t)$ the probability density. The positive constants

k, T and m stand for the Boltzmann's constant, the absolute temperature, and the mass of particles, respectively. Let $\mu = \frac{kT}{m}$ and $\beta^{-1} > 0$ be the particle relaxation time. We consider the Fokker-Planck equation describing the Brownian motion of particles in an infinite channel,

$$\begin{cases} \partial_t U(x,y,t) + x\partial_y U(x,y,t) - \beta\partial_x(xU(x,y,t)) + y\partial_x U(x,y,t) \\ \quad -\beta\mu\partial_x^2 U(x,y,t) = 0, \quad x \in \mathbb{R}, \quad |y| < 1, \quad 0 < t \leqslant T, \\ U(x,y,t) = 0, \quad x \geqslant 0, \quad y = -1 \text{ or } x \leqslant 0, \quad y = 1, \quad 0 < t \leqslant T, \\ U(x,y,t) \to 0, \quad |x| \to \infty, \quad |y| < 1, \quad 0 < t \leqslant T, \\ U(x,y,0) = U_0(x,y), \quad x \in \mathbb{R}, \quad |y| \leqslant 1. \end{cases} \tag{3.24}$$

There exist several difficulties for solving (3.24) numerically. Indeed, the above equation behaves like parabolic equation in the x-direction, and like hyperbolic equation in the y-direction. Next, the coefficient of the convective term $\partial_y U$ changes the sign at $x = 0$, and so different kinds of boundary conditions are imposed on different subdomains with $x < 0$ or $x > 0$, respectively. Furthermore, the terms $\partial_x U(x,y,t)$ and $\partial_y U(x,y,t)$ possess the coefficients x, which varies from $-\infty$ to ∞. Consequently, we could not use the usual spectral method for solving (3.24). Guo and Wang [61] introduced a proper composite approximation, which is a set of two mixed generalized Laguerre-Legendre quasi-orthogonal approximations on the subdomains. The numerical solution of the corresponding spectral method keeps the continuity at $x = 0$, and the global spectral accuracy. Wang and Guo [132] also provided the pseudospectral method for (3.24).

Another challenging problem is how to design reasonable spectral and pseudospectral methods for exterior problems with polygon obstacles. Guo and Wang [61, 62], Guo and Yu [75], and Wang and Guo [133] divided the unbounded exterior domain into several subdomains, and constructed the different Jacobi, Laguerre and mixed Laguerre-Legendre quasi-orthogonal approximations on different subdomains. They form the composite approximations on the whole exterior domain, keeping the continuity and possessing the global spectral accuracy. The corresponding spectral and pseudospectral methods provided accurate numerical results of second and fourth order problems.

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant No. 11171227), Fund for Doctoral Authority of China (Grant No. 20123127110001), Fund for E-institute of Shanghai Universities (Grant No. E03004), and Leading Academic Discipline Project of Shanghai Municipal Education Commission (Grant No. J50101).

References

1 Azaiez M, Shen J, Xu C J, et al. A Laguerre-Legendre spectral method for the Stokes problem in a semi-infinite channel. SIAM J Numer Anal, 2008, 47: 271–292

2 Babuška I, Guo B Q. Optimal estimates for lower and upper bounds of approximation errors in the p-version of finite element method in two dimensions. Numer Math, 2000, 85: 219–255

3 Babuška I, Guo B Q. Direct and inverse approximation theorems for the p-version of the finite element method in the framework of weighted Besov spaces, Part I: Approximability of functions in the weighted Besov spaces. SIAM J Numer Anal, 2001, 39: 1512–1538

4 Babuška I, Janik T. The h-p version of the finite element method for parabolic equations, Part 1: The p-version in time. Numer Methods Partial Differential Equations, 1989, 5: 363–399

5 Bernardi C, Coppoletta G, Maday Y. Some spectral approximations of multi-dimensional fourth-order problems. Internal Report 90021. Paris: Laboratoire d'Analyse Numérique, Université Pierre et Marie Curie, 1990

6 Bernardi C, Dauge M, Maday Y. Sectral Methods for Axisymmetric Domains. In: Ciarlet P G, Lions P L, eds. Series in Appl Math. Paris: Gauhtier-Villars/North-Holland, 1999

7 Bernardi C, Maday Y. Basic Results on Spectral Methods, R94022. Paris: Université Pierre et Marie Curie, 1994

8 Bernardi C, Maday Y. Approximations Spectrales de Problèmes aux Limites Elliptiques. Berlin: Springer-Verlag, 1992

9 Bernardi C, Maday Y. Spectral methods. In: Ciarlet P G, Lions J L, eds. Handbook of Numerical Analysis, vol. 5. Techniques of Scientific Computing. Amsterdam: Elsevier, 1997, 209–486

10 Bernardi C, Maday Y, Rapetti F. Discretisations Variationnelles de Problemes aux Limites Elliptique. Collection: Mathematique et Applications, vol. 45. Berlin: Springer-Verlag, 2004

11 Boyd J P. Orthogonal rational functions on a semi-infinite interval. J Comp Phys, 1987, 70: 63–88

12 Boyd J P. Spectral method using rational basis functions on infinite intervals. J Comp Phys, 1987, 69: 112–142

13 Boyd J P. Chebyshev and Fourier Spectral Methods, 2nd edition. New York: Dover Publication Inc, 2001

14 Canuto C, Hussaini M Y, Quarteroni A, et al. Spectral Methods in Fluid Dynamics. Berlin: Springer-Verlag, 1988

15 Canuto C, Hussaini M Y, Quarteroni A, et al. Spectral Methods: Fundamentals in Single Domains. Berlin: Springer-Verlag, 2006

16 Canuto C, Hussaini M Y, Quarteroni A, et al. Spectral Methods: Evolution to Complex Geometries and Applications to Fluid Dynamics. Berlin: Springer-Verlag, 2007

17 Chen Y P, Tang T. Spectral methods for weakly singular Volterra integral equations with smooth solutions. J Comp Appl Math, 2009, 233: 938–950

18 Chen Y P, Tang T. Convergence analysis of the Jacobi spectral-collocation methods for Volterra integral equations with a weakly singular kernel. Math Comp, 2010, 79: 147–167

19 Christov C I. Complete orthogonal system of functions in $L^2(-\infty,\infty)$ space. SIAM J Appl Math, 1982, 42: 1337–1344

20 Coulaud O, Funaro D, Kavian O. Laguerre spectral approximation of elliptic problems in exterior domains. Comp Mech Appl Mech Eng, 1990, 80: 451–458

21 Dubiner M. Spectral methods on triangles and other domains. J Sci Comput, 1991, 6: 345–390

22 Everitt W N, Littlejohn L L, Wellman R. The Sobolev orthogonality and spectral analysis of the Laguerre polynomials $\{L_n^{-k}\}$ for positive integers k. J Comp Appl Math, 2004, 171: 199–234

23 Johnson Fox C M, Guo B Y, Tang T. Combined Hermite spectral-finite difference method for the Fokker-Planck equations. Math Comp, 2001, 71: 1497–1528

24 Funaro D. Estimates of Laguerre spectral projectors in Sobolev spaces. In: Brezinski C, Gori L, Ronveaux A, eds. Orthogonal Polynomials and Their Applications. New Brunswick: IMACS, 1991

25 Funaro D. Polynomial Approxiamtions of Differential Equations. Berlin: Springer-Verlag, 1992

26 Funaro D, Kavian O. Approximation of some diffusion evolution equations in unbounded domains by Hermite functions. Math Comp, 1990, 57: 597–619

27 Gottlieb D, Orszag S A. Numerical Analysis of Spectral Methods: Theory and Applications. Philadelphia: SIAM-CBMS, 1977

28 Gottlieb D, Shu C W. On the Gibbs phenomenon IV: Resolution exponential accuracy in a subinterval from a Gegenbauer partial sum of a piecewise analytic functions. Math Comp, 1995, 64: 1081–1095

29 Gottlieb D, Shu C W. Resolution exponential accuracy from collocation point values of piecewise analytic function. Numer Math, 1995, 71: 511–526

30 Guo B Y. Spectral Methods and Their Applictions. Singapore: World Scientific, 1998

31 Guo B Y. Gegenbauer approximation and its applications to differential equations on the whole line. J Math Anal Appl, 1998, 22: 180–206

32 Guo B Y. Error estimation of Hermite spectral method for nonlinear partial differential equations. Math Comp, 1999, 68: 1067–1078

33 Guo B Y. Jacobi approximation and its applications to differential equations on the half line. J Comp Math, 2000, 18: 95–112

34 Guo B Y. Gegenbauer approximation in certain Hilbert spaces and its applications to singular differential equations. SIAM J Numer Anal, 2000, 37: 621–645

35 Guo B Y. Jacobi approximations in certain Hilbert spaces and their applications to singular differential equations. J Math Anal Appl, 2000, 243: 373–406

36 Guo B Y. Gegenbauer approximation and its applications to differential equations with rough asymptotic behaviors at infinity. Appl Numer Math, 2001, 38: 403–425

37 Guo B Y. Jacobi spectral method for differential equations with rough asymptotic behaviors at infinity. J Comp Math Appl, 2003, 46: 95–104

38 Guo B Y. Some developments in spectral methods for nonlinear partial differential equations in unbounded domains. In: Gu C H, Hu H S, Li T T, eds. Differential Geometry and Related Topics. Singapore: World Scientific, 2002, 68–90

39 Guo B Y, Huang W. Mixed Jacobi-spherical harmonic spectral method for Navier-Stokes equations. Appl Numer Math, 2007, 57: 939–961

40 Guo B Y, Jia H L. Spectral method on quadrilaterals. Math Comp, 2010, 79: 2237–2264

41 Guo B Y, Jiao Y J. Mixed generalized Laguerre-Fourier spectral method for exterior problem of Navier-Stokes equations. Discrete Contin Dyn Syst Ser B, 2009, 11: 315–345

42 Guo B Y, Jiao Y J. Spectral method for Navier-Stokes equations with slip boundary conditions. J Sci Comput, in press, doi: 10.1007/s10915-013-9729-5

43 Guo B Y, Ma H P. Composite Legendre-Laguerre approximation in unbounded domains. J Comp Math, 2001, 19: 101–112

44 Guo B Y, Shen J. Laguerre-Galerkin method for nonlinear partial differential equations on a semi-infinite interval. Numer Math, 2000, 86: 635–654

45 Guo B Y, Shen J. On spectral approximations using modified Legendre rational functions: application to the Korteweg de Vries equation on the half line. Indiana Univ Math J, 2001, 50: 181–204

46 Guo B Y, Shen J. Irrational approximations and their applications to partial differential equations in exterior domains. Adv Comp Math, 2008, 28: 237–267

47 Guo B Y, Shen J, Wang L L. Optimal spectral-Galerkin methods using generalized Jacobi polynomials. J Sci Comput, 2006, 27: 305–322

48 Guo B Y, Shen J, Wang L L. Generalized Jacobi polynomials/functions and their Applications. Appl Numer Math, 2009, 59: 1011–1028

49 Guo B Y, Shen J, Wang Z Q. A rational approximation and its applications to differential equations on the half line. J Sci Comput, 2000, 15: 117–148

50 Guo B Y, Shen J, Wang Z Q. Chebyshev rational spectral and pseudospectral methods on a semi-infinite interval. Int J Numer Meth Eng, 2002, 53: 65–84

51 Guo B Y, Shen J, Xu C L. Spectral and pseudospectral approximations using Hermite functions: Application to the Dirac equation. Adv Comp Math, 2003, 19: 35–55

52 Guo B Y, Shen J, Xu C L. Generalized Laguerre approximation and its applications to exterior problems. J Comp Math, 2005, 23: 113–130

53 Guo B Y, Sun T, Zhang C. Jacobi and Laguerre quasi-orthogonal approximations and related interpolations. Math Comp, 2013, 82: 413–441

54 Guo B Y, Wang L L. Jacobi interpolation approximations and their applications to singular differential equations. Adv Comp Math, 2001, 14: 227–276

55 Guo B Y, Wang L L. Non-isotropic Jacobi spectral method. Contemp Math, 2003 329: 157–164

56 Guo B Y,Wang L L. Jacobi approximation in non-uniformly Jacobi-weighted Sobolev spaces. J Appro Theor, 2004, 128: 1–41

57 Guo B Y, Wang L L. Modified Laguerre pseudospectral method refined by multidomain Legendre approximation for differential equations on the half line. J Comp Appl Math, 2006, 190: 304–324

58 Guo B Y, Wang L L. Error analysis of spectral method on a triangle. Adv Comp Math, 2007, 26: 473–496

59 Guo B Y, Wang L L, Wang Z Q. Generalized Laguerre interpolation and pseudospectral method for unbounded domains. SIAM J Numer Anal, 2006, 43: 2567–2589

60 Guo B Y, Wang T J. Composite generalized Laguerre-Legendre spectral method with domain decomposition and its application to Fokker-Planck equation in an infinite channel. Math Comp, 2009, 78: 129–151

61 Guo B Y, Wang T J. Composite Laguerre-Legendre spectral method for exterior problems. Adv Comp Math, 2010, 32: 393–429

62 Guo B Y, Wang T J. Composite Laguerre-Legendre spectral method for fourth-order exterior problems. J Sci Comput, 2010, 44: 255–285

63 Guo B Y, Wang Z Q. Modified Chebyshev rational spectral method for the whole line. Discrete Contin Dyn Syst, 2003, 9: 365–374

64 Guo B Y, Wang Z Q. Legendre rational approximation on the whole line. Sci China Ser A, 2004, 47: 155–164

65 Guo B Y, Wang Z Q. Numerical integration based on Laguerre-Gauss interpolation. Comp Meth Appl Mech Eng, 2007, 196: 3726–3741

66 Guo B Y, Wang Z Q. Legendre-Gauss collocation methods for ordinary differential equations. Adv Comp Math, 2009, 30: 249–280

67 Guo B Y,Wang Z Q. A spectral collocation method for solving initial value problems of first order ordinary differential equations. Discrete Contin Dyn Syst Ser B, 2010, 14: 1029–1054

68 Guo B Y, Wang Z Q. A collocation method for generalized nonlinear Klein-Gordon equation. Adv Comp Math, submitted

69 Guo B Y, Wang Z Q, Tian H J, et al. Integration processes of ordinary differential equations based on Laguerre-Gauss interpolations. Math Comp, 2008, 77: 181–199

70 Guo B Y, Wang Z Q, Wan Z S, et al. Second order Jacobi approximation with applications to fourth order differential equations. Appl Numer Math, 2005, 55: 480–502

71 Guo B Y, Yan J P. Legendre-Gauss collocation methods for initial value problems of second order ordinary differential equations. Appl Numer Math, 2009, 59: 1386–1408

72 Guo B Y, Yi Y G. Generalized Jacobi rational spectral method and its applications. J Sci Comput, 2010, 43: 201–238

73 Guo B Y, Xu C L. Hermite pseudospectral method for nonlinear partial differential equations. RAIRO Math Model Numer Anal, 2000, 34: 859–872

74 Guo B Y, Xu C L. Mixed Laguerre-Lagendre pseudospectral method for incompressible fluid flow in an infinite strip. Math Comp, 2003, 73: 95–125

75 Guo B Y, Yu X H. Composite spectral method for exterior problems with polygonal obstacles. Submitted

76 Guo B Y, Zhang C, Sun T. Some developments in spectral methods. Stud Adv Math, 2012, 51: 561–574

77 Guo B Y, Zhang C. Spectral method for high order problems with proper simulations of asymptotic behaviors at infinity. J Comp Appl Math, 2013, 237: 269–294

78 Guo B Y, Zhang C. Generalized Hermite spectral method matching different asymptotic behaviors at different end-points. Unpublished

79 Guo B Y, Zhang K J. On non-isotropic Jacobi pseudospectral method. J Comp Math, 2008, 26: 511–535

80 Guo B Y, Zhang X Y. A new generalized Laguerre spectral approximation and its applications. J Comp Appl Math, 2005, 181: 342–363

81 Guo B Y, Zhang X Y. Spectral method for differential equations of degenerate type on unbounded domains by using generalized Laguerre functions. Appl Numer Math, 2007, 57: 455–471

82 Hesthaven J S, Gottlieb S, Gottlieb D. Spectral Methods for Time-Dependent Problems. Cambridge Monographs on Applied and Computational Mathematics, vol. 21. Cambridge: Cambridge University Press, 2007

83 Hesthaven J S, Warburton T. Nodal Discontinuous Galerkin Methods: Algorithms, Analysis and Applications. Springer Texts in Applied Mathematics, vol. 54. Berlin: Springer Verlag, 2008

84 Jia H L, Guo B Y. Petrov-Galerkin spectral element method for mixed inhomogeneous boundary value problems on polygons. Chinese Ann Math Ser B, 2010, 31: 855–878

85 Junghanns V P. Uniform convergence of approximate methods for Cauchy type singular equation over $(-1, 1)$. Wiss Z Tech Hocsch Karl-Mars Stadt, 1984, 26: 250–256

86 Kanyamee N, Zhang Z M. Comparison of a spectral collocation method and symplectic methods for Hamiltonian systems. Int J Numer Anal Model, 2011, 8: 86–104

87 Karniadakis G E, Sherwin S T. Spectral/hp Element Methods for CFD. Oxford: Oxford University Press, 1999

88 Li H Y, Shen J. Optimal error estimates in Jacobi-weighted Sobolev spaces for polynomial approximations on the triangle. Math Comp, 2010, 79: 1621–1646

89 Li H Y, Wang L L. A spectral method on tetrahedra using rational basis functions. Inter J Numer Anal Model, 2010, 7: 330–355

90 Li X J, Tang T. Convergence analysis of the Jacobi spectral-collocation methods for Abel-Volterra integral equations of the second kind. Front Math China, 2012, 7: 69–84

91 Li Y Y, Wang L L, Li H Y, et al. A new spectral method on triangles. In: Lecture Notes in Computational Sciences and Engineering. Proceeding of International Conference on Spectral and High-Order Methods (ICOSAHOM09). New York: Springer-Verlag, 2010, 237–246

92 Lin Y M, Li X J, Xu C J. Finite difference/spectral approximations for the fractional cable equation. Math Comp, 2011, 80: 1369–1396

93 Ma H P, Guo B Y. Composite Legendre-Laguerre pseudospectral approximation in unbounded domains. IMA J Numer Anal, 2001, 21: 587–602

94 Ma H P, Sun W W. A Legendre-Petrov-Galerkin and Chebyshev collocation method for third-order differential equations. SIAM J Numer Anal, 2000, 38: 1425–1438

95 Ma H P, Sun W W. Optimal error estimates of the Legendre-Petrov-Galerkin method for the Korteweg-de Vries equation. SIAM J Numer Anal, 2001, 39: 1380–1394

96 Ma H P, Sun W W, Tang T. Hermite spectral methods with a time-dependent scaling for parabolic equations in unbounded domains. SIAM J Numer Anal, 2006, 43: 58–75

97 Ma H P, Zhao T G. A stabilized Hermite spectral method for second-order differential equations in unbounded domain. Numer Methods Partial Differential Equations, 2007, 23: 968–983

98 Maday Y, Pernaud-Thomas B, Vandeven H. One réhabilitation des méthods spèctrales de type Laguerre. Rech Aérospat, 1985, 6: 353–379

99 Mastroianni G, Monegato G. Nyström interpolants based on zeros of Laguerre polynomials for some Weiner-Hopf equations. IMA J Numer Anal, 1997, 17: 621–642

100 Owens R G. Spectral Approximation on the triangle. Proc R Soc Lond Ser A, 1998, 454: 857–872

101 Samson M D, Li H Y, Wang L L. A new triangular spectral element method I: implementation and analysis on a triangle. Numer Algor, in press, doi: 10.1007/s11075-012-9677-4

102 Shen J. Stable and efficient spectral methods in unbounded domains using Laguerre functions. SIAM J Numer Anal, 2000, 38: 1113–1133

103 Shen J. A new dual-Petrov-Galerkin method for third and higher odd-order differential equations: Application to the KDV equation. SIAM J Numer Anal, 2003, 41: 1595–1619

104 Shen J, Tang T. Spectral and High-Order Methods with Applications. Beijing: Science Press, 2006

105 Shen J, Tang T, Wang L L. Spectral Methods: Algorithms, Analysis and Applications. Berlin: Springer Verlag, 2011

106 Shen J, Wang L L. Error analysis for mapped Legendre spectral and pseudospectral methods. SIAM J Numer Anal, 2004, 42: 326–349

107 Shen J, Wang L L. Laguerre and composite Legendre-Laguerre dual-Petrov-Galerkin methods for third-order equations. Disc Contin Dyn Syst Ser B, 2006, 6: 1381–1402

108 Shen J, Wang L L. Legendre and Chebyshev dual-Petrov-Galerkin methods for hyperbolic equations. Comp Meth Appl Mech Eng, 2007, 196: 3785–3797

109 Shen J, Wang L L. Fourierization of the Legendre-Galerkin method and a new space-time spectral method. Appl Numer Math, 2007, 57: 710–720

110 Shen J, Wang L L. On spectral approximations in elliptic geometries using Mathieu functions. Math Comp, 2009, 78: 815–844.

111 Shen J, Wang L L. Some recent advances in spectral methods for unbounded domains. Comm Comp Phys, 2009, 5: 195–241

112 Shen J, Wang L L. Sparse spectral approximations of the high-dimensional problems based on hyperbolic cross. SIAM J Numer Anal, 2010, 48: 1087–1109

113 Shen J, Wang L L, Li H Y. A triangular spectral element method using fully tensorial rational basis functions. SIAM J Numer Anal, 2009, 47: 1619–1650

114 Sherwin S J, Karniadakis G E. A new triangular and tetrahedral basis for high-order finite element methods. Int J Numer Method Eng, 1995, 38: 3775–3802

115 Stephan E P, Suri M. On the convergence of the p-version of the boundary element Galerkin method. Math Comp, 1989, 52: 31–48

116 Sun T, Guo B Y. Generalized Jacobi approximation in multiple dimensions and its applications. J Sci Comput, 2013, 55: 327–350

117 Tang J G, Ma H P. Single and multi-interval Legendre τ-methods in time for parabolic equations. Adv Comput Math, 2002, 17: 349–367

118 Tang J G, Ma H P. Single and multi-interval Legendre spectral methods in time for parabolic equations. Numer Methods Partial Differential Equations, 2006, 22: 1007–1034

119 Tang J G, Ma H P. A Legendre spectral method in time for first-order hyperbolic equations. Appl Numer Math, 2007, 57: 1–11

120 Tang T. The Hermite spectral method for Gaussian type functions. SIAM J Sci Comput, 1993, 14: 594–606

121 Tang T, Mckee S, Reeks M W. A spectral method for a kinetic equation describing the dispersion of small particles in a turbulent flow. J Comp Phys, 1991, 103: 222–230

122 Tang T, Xu X, Cheng J. On spectral methods for Volterra type integral equations and the convergence analysis. J Comp Math. 2008, 26: 825–837

123 Wan Z S, Guo B Y, Wang Z Q. Jacobi pseudospectral method for fourth order problems. J Comp Math, 2006, 24: 481–500

124 Wang L L, Guo B Y. Jacobi spectral methods for multiple-dimensional singular differential equations. J Comp Math, 2003, 21: 325–338

125 Wang L L, Guo B Y. Non-isotropic Jacobi spectral methods for unbounded domains. Numer Math, 2004, 13: 204–224

126 Wang L L, Guo B Y. Stair Laguerre pseudospectral method for differential equations on the half line. Adv Comp Math, 2006, 25: 305–322

127 Wang L L, Guo B Y. Mixed Fourier-Jacobi spectral method. J Math Anal Appl, 2006, 315: 8–28

128 Wang L L, Guo B Y. Interpolation approximations based on Gauss-Lobatto-Legendre-Birkhoff quadrature. J Appro Theor, 2009, 161: 142–173

129 Wang L L, Shen J. Error analysis for mapped Jacobi spectral methods. J Sci Comput, 2005, 24: 183–218

130 Wang L L, Xie Z Q, Zhao X D. On exponential convergence of Gegenbauer interpolation and spectral differentiation. Math Comp, 2013, 82: 1017–1036

131 Wang T J, Guo B Y. Mixed Legendre-Hermite pseudospectral method for heat transfer in an infinite plate. J Comp Math, 2005, 23: 587–602

132 Wang T J, Guo B Y. Composite generalized Laguerre-Legendre pseudospectral method for Fokker-Planck equation in an infinite channel. Appl Numer Math, 2008, 58: 1448–1466

133 Wang T J, Guo B Y. Composite Laguerre-Legendre pseudospectral method for exterior problems. Comm Comp Phys, 2009, 5: 350–375

134 Wang T J, Wang Z Q. Error analysis of Legendre spectral method with essential imposition of Neumann boundary condition. Appl Numer Math, 2009, 59: 2444–2451

135 Wang Z Q, Guo B Y. A rational approximation and its applications to nonlinear differential equations on the whole line. J Math Anal Appl, 2002, 274: 374–403

136 Wang Z Q, Guo B Y. Modified Legendre rational spectral method for the whole line. J Comp Math, 2004, 22: 457–474

137 Wang Z Q, Guo B Y. Jacobi rational approximation and spectral method for differential equations of degenerate type. Math Comp, 2008, 77: 883–907

138 Wang Z Q, Guo B Y. Legendre-Gauss-Radau collocation method for solving initial value problems of first order ordinary differential equations. J Sci Comput, 2012, 52: 226–255

139 Wang Z Q, Guo B Y, Wu Y N. Pseudospectral method using generalized Laguerre functions for singular problems on unbounded domains. Discrete Contin Dyn Syst Ser B, 2009, 11: 1019–1038

140 Wang Z Q, Guo B Y, Zhang W. Mixed spectral method for three-dimensional exterior problems using spherical harmonic and generalized Laguerre functions. J Comp Appl Math, 2008, 217: 277–298

141 Wang Z Q, Wang L L. A Legendre-Gauss collocation method for nonlinear delay differential equations. Discrete Contin Dyn Syst Ser B, 2010, 13: 685–708

142 Wang Z Q, Wang L L. A collocation method with exact imposition of mixed boundary conditions. J Sci Comput, 2010, 42: 291–317

143 Xu C L, Guo B Y. Laguerre pseudospectral method for nonlinear partial differential equations. J Comp Math, 2002, 20: 413–428

144 Xu C L, Guo B Y. Mixed Laguerre-Legendre spectral method for incompressible flow in an infinite strip. Adv Comp Math, 2002, 16: 77–96

145 Xu C L, Guo B Y. Hermite spectral and pseudospectral methods for nonlinear partial differential equations in multiple dimensions. Comp Appl Math, 2003, 22: 167–193

146 Xu C L, Guo B Y. Modified Laguerre spectral and pseudospectral methods for nonlinear partial differential equations in multiple dimensions. Appl Math Mech, 2008, 29: 311–331

147 Xiang X M, Wang Z Q. Generalized Hermite spectral method and its applications to problems in unbounded domains. SIAM J Numer Anal, 2010, 48: 1231–1253

148 Yan J P, Guo B Y. Laguerre-Gauss collocation method for initial value problems of second-order ODEs. Appl Math Mech, 2011, 32: 1541–1564

149 Yan J P, Guo B Y. A collocation methods for initial value problems of second order ODEs by using Legendre functions. Numer Math Theo Meth Appl, 2011, 4: 282–294

150 Yi Y G, Guo B Y. Generalized Jacobi rational spectral method on the half line. Adv Comp Math, 2012, 37: 1–37

151 Yu X H, Guo B Y. Spectral element method for mixed inhomogeneous boundary value problems of fourth-order. Unpublished

152 Zhang C, Guo B Y. Domain decomposition spectral method for mixed inhomogeneous boundary value problems of high order differential equations on unbounded domains. J Sci Comput, 2012, 53: 451–480

153 Zhang C, Guo B Y. Generalized Hermite spectral method matching asymptotic behaviors. Submitted

154 Zhang R, Wang Z Q, Guo B Y. Mixed Fourier-Laguerre spectral and pseudospectral methods for exterior problems using generalized Laguerre functions. J Sci Comput, 2008, 36: 263–283.

155 Zhang X Y, Guo B Y. Spherical harmonic-generalized Laguerre spectral method for exterior problems. J Sci Comput, 2006, 27: 523–537

156 Zhang X Y, Guo B Y, Jiao Y J. Spectral method for three-dimensional nonlinear Klein-Gordon equation by generalized Laguerre and spherical harmonic functions. Numer Math Theo Meth Appl, 2009, 2: 43–64

157 Zhang Z M. Superconvergence points of polynomial spectral interpolation. SIAM J Numer Anal, 2012, 50: 2966–2985

158 Zhang Z M. Superconvergence of spectral collocation and p-version methods in one dimensional problems. Math Comp, 2005, 74: 1621–1636

159 Zhang Z M. Superconvergence of a Chebyshev spectral collocation method. J Sci Comput, 2008, 34: 237–246

160 Zhuang Q Q, Shen J, Xu C J. A coupled Legendre-Laguerre spectral-element method for the Navier-Stokes equations in unbounded domains. J Sci Comput, 2010, 42: 1–22