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Fujita phenomena in nonlinear pseudo-parabolic system

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Abstract This paper deals with the Cauchy problem to the nonlinear pseudo-parabolic system $u_t - \Delta u - \alpha\Delta u_t = v^p$, $v_t - \Delta v - \alpha\Delta v_t = u^q$ with $p, q \geqslant 1$ and $pq > 1$, where the viscous terms of third order are included. We first find the critical Fujita exponent, and then determine the second critical exponent to characterize the critical space-decay rate of initial data in the co-existence region of global and non-global solutions. Moreover, time-decay profiles are obtained for the global solutions. It can be found that, different from those for the situations of general semilinear heat systems, we have to use distinctive techniques to treat the influence from the viscous terms of the highest order. To fix the non-global solutions, we exploit the test function method, instead of the general Kaplan method for heat systems. To obtain the global solutions, we apply the L^p-L^q technique to establish some uniform L^m time-decay estimates. In particular, under a suitable classification for the nonlinear parameters and the initial data, various L^m time-decay estimates in the procedure enable us to arrive at the time-decay profiles of solutions to the system. It is mentioned that the general scaling method for parabolic problems relies heavily on regularizing effect to establish the compactness of approximating solutions, which cannot be directly realized here due to absence of the smooth effect in the pseudo-parabolic system.

Keywords semilinear pseudo-parabolic system, critical Fujita exponent, second critical exponent, global profile

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1 Introduction

In this paper, we consider the Cauchy problem to coupled nonlinear pseudo-parabolic equations

$$\begin{cases} u_t - \Delta u - \alpha\Delta u_t = v^p, & (x,t) \in \mathbb{R}^N \times (0,T), \\ v_t - \Delta v - \alpha\Delta v_t = u^q, & (x,t) \in \mathbb{R}^N \times (0,T), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $p, q \geqslant 1$ with $pq > 1$, $\alpha > 0$, $u_0(x)$ and $v_0(x)$ are nonnegative, bounded and appropriately smooth. This kind of equations models a variety of important physical processes, for example, unidirectional propagation of non-linear dispersive long waves [1, 4], seepage of homogeneous fluids through a fissured rock [2] and discrepancy between the conductive and thermodynamic temperatures [6].

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It is well known that the heat equation

$$\begin{cases} u_t = \Delta u + u^p, & x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N \end{cases} \tag{1.2}$$

has no non-trivial global solutions whenever $1 < p \leqslant p_c = 1 + \frac{2}{N}$, whereas admits both global and non-global solutions if $p > p_c$, depending on the size of initial data [9, 30]. The critical exponents like such p_c have been established for various nonlinear PDEs. See the surveys [7, 17] and the recent papers [24, 25, 29, 32, 34] for example. In addition, the so-called second critical exponent was introduced for (1.2) to describe the critical space-decay rate of initial data in the co-existence parameter region of global and non-global solutions [16]. It was shown with $u_0(x) \sim |x|^{-a}$, $|x| \to \infty$ that in the region $p > p_c$, there exist global and non-global solutions to (1.2) for $a > a_0 = \frac{2}{p-1}$ and $0 < a < a_0$ respectively. As for the coupled heat system

$$u_t = \Delta u + v^p, \quad v_t = \Delta v + u^q, \tag{1.3}$$

Escobedo and Herrero [8] determined the critical Fujita curve as $(pq)_c = 1 + \frac{2}{N}\max\{p+1, q+1\}$, namely, every solution blows up in finite time if $1 < pq \leqslant (pq)_c$, and there exist both global and non-global solutions if $pq > (pq)_c$. With $u_0(x) \sim |x|^{-a}$, $v_0(x) \sim |x|^{-b}$, $|x| \to \infty$, Mochizuki [21] obtained in the coexistence region $pq > (pq)_c$ that there are global solutions to (1.3) if $a > a_0 = \frac{2(p+1)}{pq-1}$ and $b > b_0 = \frac{2(q+1)}{pq-1}$, while there are no global solutions if $0 < a < \frac{2(p+1)}{pq-1}$ or $0 < b < \frac{2(q+1)}{pq-1}$. The time-decay profiles for the global solutions were studied as well. For example, when $p > p_c$ the global solutions to (1.2) behave like the fundamental solution $G(x,t) = (4\pi t)^{-\frac{N}{2}} \mathrm{e}^{-\frac{|x|^2}{4t}}$ in the sense of $\lim_{t\to\infty} t^{\frac{N}{2}} \|u(\cdot,t) - BG(\cdot,t)\|_\infty = 0$ with $B = \lim_{t\to\infty} \|u(\cdot,t)\|_1$ [3], and $\lim_{t\to\infty} t^{\frac{a}{2}} \|u(\cdot,t) - AG(x,t) * (1+|x|)^{-a}\|_\infty = 0$ with $\lim_{|x|\to\infty} |x|^a u_0(x) = A > 0$ and $a \in (a_0, N)$ [23]. The time-decay profiles for the coupled heat system (1.3) were obtained also by Mochizuki [21].

The critical Fujita exponent to the pseudo-parabolic equation

$$u_t - \Delta u - \alpha \Delta u_t = u^p \tag{1.4}$$

was determined as $p_c = 1 + \frac{2}{N}$ in recent years, i.e., by Kaikina et al. [12] for $p > p_c$ and Cao et al. [5] for $p \leqslant p_c$. Currently, the second critical exponent for (1.4) was obtained as well [31]. In addition, Kaikina et al. [12] obtained that if $u_0 \in L^1_a(\mathbb{R}^N) = \{f(x) \mid \int_{\mathbb{R}^N} |x|^a |f(x)| dx < \infty\}$ and $0 < a < 1$, then

$$u(x,t) = BG(x,t) + o(t^{-\frac{N}{2}-\gamma}), \quad t \to \infty,$$

with $0 < \gamma < \min\{\frac{a}{2}, \frac{N}{2}(p-1) - 1\}$. The other studies for pseudo-parabolic equations can be found in, e.g., [10, 14, 15, 18, 19, 35]. Based on the above work, this paper investigates the asymptotic behavior of solutions to the pseudo-parabolic system (1.1), such as the critical Fujita exponent, the second critical exponent, as well as the time-decay profiles of solutions.

Denote by $\overline{F}_{\xi\to x}\phi$ the inverse Fourier transform of ϕ, and $\|u\|_m := \|u\|_{L^m(\mathbb{R}^N)}$.

We deal with mild solutions to (1.1) to treat global solutions. We call $(u,v) \in C([0,T]; C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))$ a mild solution to (1.1), if (u,v) satisfies

$$\begin{cases} u(x,t) = \mathcal{G}(t)u_0(x) + \displaystyle\int_0^t \mathcal{G}(t-s)\mathcal{B}v^p(x,s)ds, & x \in \mathbb{R}^N, \quad t > 0, \\ v(x,t) = \mathcal{G}(t)v_0(x) + \displaystyle\int_0^t \mathcal{G}(t-s)\mathcal{B}u^q(x,s)ds, & x \in \mathbb{R}^N, \quad t > 0, \end{cases} \tag{1.5}$$

with

$$\mathcal{G}(t) = \exp(-t(\alpha\Delta - I)^{-1}\Delta), \quad \mathcal{B} = -(\alpha\Delta - I)^{-1}.$$

We have

$$\mathcal{G}(t)\phi = G(x,t) * \phi(x) + \mathrm{e}^{-\frac{t}{\alpha}} \sum_{m=0}^{N} \frac{(\frac{t}{\alpha})^m}{m!} \mathcal{B}^m \phi + \mathcal{R}(t)\phi, \tag{1.6}$$

where

$$\mathcal{B}^m\phi = \int_{\mathbb{R}^N} B_m(x-y,t)\phi(y)dy, \quad \mathcal{R}(t)\phi = \int_{\mathbb{R}^N} R(x-y,t)\phi(y)dy,$$

and

$$B_m(x,t) = (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} \mathrm{e}^{\mathrm{i}\xi x}(1+|\xi|^2)^{-m}d\xi,$$

$$R(x,t) = \overline{F}_{\xi\to x}\left(\mathrm{e}^{-\frac{t\xi^2}{1+\alpha\xi^2}} - \mathrm{e}^{-t\xi^2} - \mathrm{e}^{-\frac{t}{\alpha}}\sum_{m=0}^{N}\frac{(\frac{t}{\alpha})^m}{m!}(1+\alpha\xi^2)^{-m}\right).$$

Similarly to [5], it is easy to know that there exist mild solutions to system (1.1), which are the classical solutions in fact if the initial data are appropriately smooth. In addition, the uniqueness of such solutions is valid with the comparison principle. It seems that the pseudo-parabolic equations like system (1.1) with viscous terms of highest order do not admit self-similar subsolutions. Moreover, the general energy blow-up method for scalar equations [5] is difficult to treat the coupled system (1.1). In this paper, we adopt the rescaled test function method (see, e.g., [20,33]) to fix the finite time blow-up of solutions. The viscous highest order terms contribute additional dispersal mechanism to the system. In order to treat the influence of these terms for non-global solutions, it is assumed that the initial data are compactly supported, or behave negative powers in space infinity. To deal with global solutions, we use the L^p-L^q technique to establish some uniform L^m time-decay estimates, with m being in a wider scope, and more precise even compared with those previously obtained for semilinear heat systems. Corresponding to a suitable classification to the nonlinear parameters and initial data, the related L^m estimates in the procedure enable us to arrive at the time-decay profiles of solutions to (1.1). It is mentioned that the general scaling method for parabolic problems [11, 13, 22] heavily relies on regularizing effect to establish the compactness of approximating solutions, which cannot be directly realized here due to absence of the smooth effect in the pseudo-parabolic system, for which the singular Dirac function presents in the fundamental function [14].

2 Critical Fujita exponent

We deal with the critical Fuijita curve for the system (1.1) in this section.

Theorem 2.1. *The critical Fujita curve to the system* (1.1) *is*

$$(pq)_c = 1 + \frac{2}{N}\max\{p+1, q+1\}, \tag{2.1}$$

namely, the system (1.1) *has no nontrivial global solution if* $1 < pq \leqslant (pq)_c$*, but admits both global and non-global solutions if* $pq > (pq)_c$*, depending on the size of the initial data.*

Proof. Let $1 < pq \leqslant (pq)_c$, and assume $p \geqslant q$ without loss of generality. To prove the finite time blow-up of solutions, it suffices to treat the initial data with compact support by the comparison principle.

We first deal with the case $p \geqslant q > 1$. Choose sufficiently smooth and nonincreasing functions $0 \leqslant \xi(r), \eta(r) \leqslant 1$ satisfying

$$\xi(r) = \eta(r) \equiv 1, \quad 0 \leqslant r \leqslant 1/2; \quad \xi(r) = \eta(r) \equiv 0, \quad r \geqslant 1.$$

Assume for contradiction that (u, v) is a nonnegative nontrivial global solution to (1.1), and denote

$$I_p = \int_0^T\int_{B_R} v^p\xi^l\left(\frac{|x|}{R}\right)\eta^l\left(\frac{t}{R^2}\right)dxdt, \quad J_q = \int_0^T\int_{B_R} u^q\xi^l\left(\frac{|x|}{R}\right)\eta^l\left(\frac{t}{R^2}\right)dxdt,$$

where $R > 1$, $T > R^2$, $B_R = \{x \in \mathbb{R}^N \mid |x| \leqslant R\}$ and $l \geqslant \max\{\frac{p}{p-1}, \frac{q}{q-1}\}$. Then,

$$I_p = \int_0^T\int_{B_R}(u_t - \Delta u - \alpha\Delta u_t)\xi^l\left(\frac{|x|}{R}\right)\eta^l\left(\frac{t}{R^2}\right)dxdt$$

$$
\begin{aligned}
&= -\int_{B_R} u_0\xi^l\left(\frac{|x|}{R}\right)dx - lR^{-2}\int_{\frac{R^2}{2}}^{R^2}\int_{B_R} u\xi^l\left(\frac{|x|}{R}\right)\eta^{l-1}\left(\frac{t}{R^2}\right)\eta'\left(\frac{t}{R^2}\right)dxdt \\
&\quad - \int_0^{R^2}\int_{B_R\setminus B_{\frac{R}{2}}} u\Delta\xi^l\left(\frac{|x|}{R}\right)\eta^l\left(\frac{t}{R^2}\right)dxdt + \alpha\int_{B_R\setminus B_{\frac{R}{2}}} u_0\Delta\xi^l\left(\frac{|x|}{R}\right)dx \\
&\quad + \alpha lR^{-2}\int_{\frac{R^2}{2}}^{R^2}\int_{B_R\setminus B_{\frac{R}{2}}} u\Delta\xi^l\left(\frac{|x|}{R}\right)\eta^{l-1}\left(\frac{t}{R^2}\right)\eta'\left(\frac{t}{R^2}\right)dxdt. \qquad (2.2)
\end{aligned}
$$

When u_0 is compactly supported, let R be large enough such that

$$
\int_{B_R\setminus B_{\frac{R}{2}}} u_0\Delta\xi^l\left(\frac{|x|}{R}\right)dx = 0, \qquad (2.3)
$$

and hence,

$$
\begin{aligned}
I_p &\leqslant -lR^{-2}\int_{\frac{R^2}{2}}^{R^2}\int_{B_R} u\xi^l\left(\frac{|x|}{R}\right)\eta^{l-1}\left(\frac{t}{R^2}\right)\eta'\left(\frac{t}{R^2}\right)dxdt \\
&\quad - \int_0^{R^2}\int_{B_R\setminus B_{\frac{R}{2}}} u\Delta\xi^l\left(\frac{|x|}{R}\right)\eta^l\left(\frac{t}{R^2}\right)dxdt \\
&\quad + \alpha lR^{-2}\int_{\frac{R^2}{2}}^{R^2}\int_{B_R\setminus B_{\frac{R}{2}}} u\Delta\xi^l\left(\frac{|x|}{R}\right)\eta^{l-1}\left(\frac{t}{R^2}\right)\eta'\left(\frac{t}{R^2}\right)dxdt \\
&=: K_1 + K_2 + K_3. \qquad (2.4)
\end{aligned}
$$

By Hölder's inequality with $l \geqslant \frac{2q}{q-1}$, we have

$$
\begin{aligned}
K_1 &\leqslant CR^{-2}\int_{\frac{R^2}{2}}^{R^2}\int_{B_R} u\xi^{\frac{l}{q}}\left(\frac{|x|}{R}\right)\eta^{\frac{l}{q}}\left(\frac{t}{R^2}\right)dxdt \\
&\leqslant CR^{-2}\left(\int_{\frac{R^2}{2}}^{R^2}\int_{B_R} u^q\xi^l\left(\frac{|x|}{R}\right)\eta^l\left(\frac{t}{R^2}\right)dxdt\right)^{\frac{1}{q}}\left(\int_{\frac{R^2}{2}}^{R^2}\int_{B_R} 1dxdt\right)^{\frac{q-1}{q}} \\
&\leqslant CR^{N-\frac{N+2}{q}}\left(\int_{\frac{R^2}{2}}^{R^2}\int_{B_R} u^q\xi^l\left(\frac{|x|}{R}\right)\eta^l\left(\frac{t}{R^2}\right)dxdt\right)^{\frac{1}{q}}, \qquad (2.5)
\end{aligned}
$$

where and throughout the paper, C denotes positive constants independent of R, u and v, and may be different from line to line. Since $\Delta\xi^l(|x|) = l\xi^{l-1}(\xi_{rr} + \frac{N-1}{r}\xi_r) + l(l-1)\xi^{l-2}|\xi_r|^2$, by Hölder's inequality again,

$$
\begin{aligned}
K_2 &\leqslant CR^{-2}\int_0^{R^2}\int_{B_R\setminus B_{\frac{R}{2}}} u\xi^{l-1}\left(\frac{|x|}{R}\right)\eta^l\left(\frac{t}{R^2}\right)dxdt \\
&\leqslant CR^{N-\frac{N+2}{q}}\left(\int_0^{R^2}\int_{B_R\setminus B_{\frac{R}{2}}} u^q\xi^l\left(\frac{|x|}{R}\right)\eta^l\left(\frac{t}{R^2}\right)dxdt\right)^{\frac{1}{q}}. \qquad (2.6)
\end{aligned}
$$

Similarly,

$$
K_3 \leqslant \alpha CR^{N-2-\frac{N+2}{q}}\left(\int_{\frac{R^2}{2}}^{R^2}\int_{B_R\setminus B_{\frac{R}{2}}} u^q\xi^l\left(\frac{|x|}{R}\right)\eta^l\left(\frac{t}{R^2}\right)dxdt\right)^{\frac{1}{q}}. \qquad (2.7)
$$

It follows from (2.4)–(2.7) that

$$
I_p \leqslant CR^{N-\frac{N+2}{q}}\left(\int_{\frac{R^2}{2}}^{R^2}\int_{B_R} u^q\xi^l\left(\frac{|x|}{R}\right)\eta^l\left(\frac{t}{R^2}\right)dxdt\right)^{\frac{1}{q}}
$$

$$
\begin{aligned}
&+ CR^{N-\frac{N+2}{q}}\left(\int_0^{R^2}\int_{B_R\setminus B_{\frac{R}{2}}} u^q\xi^l\left(\frac{|x|}{R}\right)\eta^l\left(\frac{t}{R^2}\right)dxdt\right)^{\frac{1}{q}}\\
&+ \alpha CR^{N-2-\frac{N+2}{q}}\left(\int_{\frac{R^2}{2}}^{R^2}\int_{B_R\setminus B_{\frac{R}{2}}} u^q\xi^l\left(\frac{|x|}{R}\right)\eta^l\left(\frac{t}{R^2}\right)dxdt\right)^{\frac{1}{q}}\\
\leqslant{}& (2+\alpha)CR^{N-\frac{N+2}{q}}J_q^{\frac{1}{q}}, \qquad (2.8)
\end{aligned}
$$

and similarly,

$$
\begin{aligned}
J_q \leqslant{}& CR^{N-\frac{N+2}{p}}\left(\int_{\frac{R^2}{2}}^{R^2}\int_{B_R} v^p\xi^l\left(\frac{|x|}{R}\right)\eta^l\left(\frac{t}{R^2}\right)dxdt\right)^{\frac{1}{p}}\\
&+ CR^{N-\frac{N+2}{p}}\left(\int_0^{R^2}\int_{B_R\setminus B_{\frac{R}{2}}} v^p\xi^l\left(\frac{|x|}{R}\right)\eta^l\left(\frac{t}{R^2}\right)dxdt\right)^{\frac{1}{p}}\\
&+ \alpha CR^{N-2-\frac{N+2}{p}}\left(\int_{\frac{R^2}{2}}^{R^2}\int_{B_R\setminus B_{\frac{R}{2}}} v^p\xi^l\left(\frac{|x|}{R}\right)\eta^l\left(\frac{t}{R^2}\right)dxdt\right)^{\frac{1}{p}}\\
\leqslant{}& (2+\alpha)CR^{N-\frac{N+2}{p}}I_p^{\frac{1}{p}}. \qquad (2.9)
\end{aligned}
$$

This yields

$$
I_p \leqslant (2+\alpha)^{1+\frac{1}{q}}CR^{N-\frac{N+2}{q}}(R^{N-\frac{N+2}{p}}I_p^{\frac{1}{p}})^{\frac{1}{q}} = (2+\alpha)^{1+\frac{1}{q}}CR^{N-\frac{2}{q}-\frac{N+2}{pq}}I_p^{\frac{1}{pq}}. \qquad (2.10)
$$

By Young's inequality,

$$
I_p \leqslant \frac{1}{2}I_p + (2+\alpha)^{\frac{pq+p}{pq-1}}CR^{N-\frac{2(p+1)}{pq-1}}. \qquad (2.11)
$$

Noticing $pq < (pq)_c = 1 + \frac{2(p+1)}{N}$ implies $N - \frac{2(p+1)}{pq-1} < 0$, let $R\to\infty$ in (2.11) to lead a contradiction. If $pq = (pq)_c = 1 + \frac{2(p+1)}{N}$, i.e., $N - \frac{2(p+1)}{pq-1} = 0$, we have $\lim_{R\to\infty} I_p = \int_0^\infty\int_{\mathbb{R}^N} v^p(x,t)dxdt \leqslant (2+\alpha)^{\frac{pq+p}{pq-1}}C$. It follows from (2.9) that for any $\varepsilon > 0$, there exists $R_1 > 0$ such that

$$
J_q \leqslant (2+\alpha)C_1\varepsilon^{\frac{1}{p}}R^{N-\frac{N+2}{p}} \quad \text{for } R > R_1, \qquad (2.12)
$$

with $C_1 > 0$ independent of R and ε. Combining (2.8) and (2.12), we get with $N - \frac{2(p+1)}{pq-1} = 0$ that $\lim_{R\to\infty} I_p \leqslant (2+\alpha)^{1+\frac{1}{q}}\widetilde{C}_1\varepsilon^{\frac{1}{pq}}$, where the constant $\widetilde{C}_1$ is also independent of ε. The arbitrariness of ε yields a contradiction.

If $p > q = 1$, set

$$
I_p = \int_0^T\int_{B_R} v^p\xi^{l_1}\left(\frac{|x|}{R}\right)\eta^{l_1}\left(\frac{t}{R^2}\right)dxdt, \quad J_1 = \int_0^T\int_{B_R} u\xi^{l_2}\left(\frac{|x|}{R}v\right)\eta^{l_2}\left(\frac{t}{R^2}\right)dxdt
$$

with $l_1 - 1 > l_2 > \frac{l_1}{p} + 1$. It is easy to know

$$
\int_0^{R^2}\int_{B_R} u\Delta\xi^{l_1-1}\left(\frac{|x|}{R}\right)\eta^{l_1-1}\left(\frac{t}{R^2}\right)\eta'\left(\frac{t}{R^2}\right)dxdt \leqslant CJ_1.
$$

Together with (2.2) and (2.3), we conclude (2.8). Then, we can follow the same argument for the case of $p \geqslant q > 1$ to get the finite time blow-up.

The coexistence of global and non-global solutions under $pq > (pq)_c$ will be quantitatively treated in the next section for the second critical exponent. □

3 Second critical exponent

In this section, we will determine the second critical exponent for (1.1), i.e., the critical space-decay rate of the initial data in the co-existence region $pq > (pq)_c$ for global and non-global solutions. We need the following notation:

$$\mathbb{I}_a = \Big\{\phi(x) \in C_b(\mathbb{R}^N) \;\Big|\; \phi(x) \geqslant 0, \liminf_{|x|\to\infty} |x|^a \phi(x) > C_l\Big\},$$
$$\mathbb{I}^a = \Big\{\phi(x) \in C_b(\mathbb{R}^N) \;\Big|\; \phi(x) \geqslant 0, \limsup_{|x|\to\infty} |x|^a \phi(x) < C_s\Big\},$$

where $C_b(\mathbb{R}^N)$ denotes bounded continuous functions in $\mathbb{R}^N$, $C_l, C_s > 0$. Let

$$a_0 = \frac{2(p+1)}{pq-1}, \quad b_0 = \frac{2(q+1)}{pq-1}. \tag{3.1}$$

The second critical exponent of (1.1) can be stated in the following theorem:

Theorem 3.1. *Let $pq > (pq)_c$, with nontrivial initial data $u_0(x) = \lambda\psi(x)$, $v_0(x) = \mu\varphi(x)$, $x \in \mathbb{R}^N$.*

(i) *If $\psi(x) \in \mathbb{I}^a$ and $\varphi(x) \in \mathbb{I}^b$ for some $a > a_0$ and $b > b_0$, then there exist $\lambda_1 \geqslant \lambda_0 > 0$ and $\mu_1 \geqslant \mu_0 > 0$ such that the solutions to* (1.1) *are global if* $\max\{\lambda - \lambda_0, \mu - \mu_0\} < 0$, *and non-global if* $\max\{\lambda - \lambda_1, \mu - \mu_1\} \geqslant 0$.

(ii) *If $\psi(x) \in \mathbb{I}_a$ for some $a \in (0, a_0)$ or $\varphi(x) \in \mathbb{I}_b$ for some $b \in (0, b_0)$, then the solutions to* (1.1) *blow up in finite time.*

We first give a preliminary proposition:

Proposition 3.2. *For $pq > (pq)_c$, there is $\eta > 0$ small such that if*

$$\|u_0\|_\infty + \|v_0\|_\infty + \|u_0\|_{\frac{N}{a_0}} + \|v_0\|_{\frac{N}{b_0}} < \eta, \tag{3.2}$$

then the solution to (1.1) *is global, satisfying*

$$\|u\|_m \leqslant C(1+t)^{-\frac{a_0}{2}(1-\frac{N}{ma_0})}, \quad \|v\|_n \leqslant C(1+t)^{-\frac{b_0}{2}(1-\frac{N}{nb_0})}, \quad t > 0, \tag{3.3}$$

with $\frac{N}{a_0} \leqslant m \leqslant \infty$ and $\frac{N}{b_0} \leqslant n \leqslant \infty$.

Proof. It is known that the following L^p-L^q estimates hold for $\phi \in L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ with $1 \leqslant q \leqslant p \leqslant \infty$ [12, 14]:

$$\|\mathcal{B}\phi\|_p \leqslant C\|\phi\|_p, \tag{3.4}$$
$$\|\mathcal{R}(t)\phi\|_p \leqslant C(1+t)^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})-1}\|\phi\|_q, \tag{3.5}$$
$$\|\mathcal{G}(t)\phi\|_p \leqslant Ce^{-\frac{\kappa t}{\alpha}}\|\phi\|_p + C(1+t)^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})}\|\phi\|_q, \tag{3.6}$$

where $0 < \kappa < 1$. Similar to [28], denote

$$X = C([0,\infty); C(\mathbb{R}^N) \cap L^{\frac{N}{a_0}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)) \times C([0,\infty); C(\mathbb{R}^N) \cap L^{\frac{N}{b_0}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)),$$

with

$$\|(u,v)\|_X = \sup_{t\in[0,\infty)} \{\|u(t)\|_{\frac{N}{a_0}} + \|v(t)\|_{\frac{N}{b_0}} + (1+t)^{\frac{a_0}{2}}\|u(t)\|_\infty + (1+t)^{\frac{b_0}{2}}\|v(t)\|_\infty\}. \tag{3.7}$$

Set $(u,v) \in X \cap B_\varepsilon$ with $B_\varepsilon = \{(u,v) \mid \|(u,v)\|_X \leqslant \varepsilon\}$, and $\varepsilon > 0$ to be determined. Let

$$\mathcal{M}_1[u](t) = \mathcal{G}(t)u_0 + \int_0^t \mathcal{G}(t-s)\mathcal{B}v^p(x,s)ds, \quad \mathcal{M}_2[v](t) = \mathcal{G}(t)v_0 + \int_0^t \mathcal{G}(t-s)\mathcal{B}u^q(x,s)ds,$$

and denote $\mathcal{M}[(u,v)] = (\mathcal{M}_1[u], \mathcal{M}_2[v])$.

Consider the case with $p,q>1$ only. If $\min\{p,q\}=1$, for example $p=1$, we can substitute the representation of v in (1.5) into the equation for u there to treat. Choosing $p=\infty, q=\frac{N}{a_0}$; $p=\infty, q=\frac{N}{\delta a_0}$ with $1<\delta<\min\{\frac{N}{a_0},\frac{a_0+2}{a_0},\frac{N}{b_0},\frac{b_0+2}{b_0}\}$, and $p=q=\infty$ respectively in (3.6), and then combining with (3.4), we have

$$\begin{aligned}\|\mathcal{M}_1[u]\|_\infty &\leqslant \|\mathcal{G}(t)u_0\|_\infty+\int_0^t\|\mathcal{G}(t-s)\mathcal{B}v^p(s)\|_\infty ds\\ &\leqslant C\mathrm{e}^{-\frac{\kappa t}{\alpha}}\|u_0\|_\infty+C(1+t)^{-\frac{a_0}{2}}\|u_0\|_{\frac{N}{a_0}}+C\int_0^{\frac{t}{2}}\mathrm{e}^{-\frac{\kappa(t-s)}{\alpha}}\|\mathcal{B}v^p(s)\|_\infty ds\\ &\quad+C\int_0^{\frac{t}{2}}(1+t-s)^{\frac{-\delta a_0}{2}}\|\mathcal{B}v^p(s)\|_{\frac{N}{\delta a_0}}ds+C\int_{\frac{t}{2}}^t\|\mathcal{B}v^p(s)\|_\infty ds\\ &\leqslant C(1+t)^{-\frac{a_0}{2}}(\|u_0\|_\infty+\|u_0\|_{\frac{N}{a_0}})+C\int_0^{\frac{t}{2}}\mathrm{e}^{-\frac{\kappa(t-s)}{\alpha}}\|v^p(s)\|_\infty ds\\ &\quad+C\int_0^{\frac{t}{2}}(1+t-s)^{-\frac{\delta a_0}{2}}\|v^p(s)\|_{\frac{N}{\delta a_0}}ds+C\int_{\frac{t}{2}}^t\|v^p(s)\|_\infty ds\\ &=: C(1+t)^{-\frac{a_0}{2}}(\|u_0\|_\infty+\|u_0\|_{\frac{N}{a_0}})+L_1+L_2+L_3. \end{aligned}\tag{3.8}$$

Noticing $pb_0=a_0+2$, we obtain with (3.7) that

$$\begin{aligned}L_1&\leqslant\int_0^{\frac{t}{2}}\mathrm{e}^{-\frac{\kappa(t-s)}{\alpha}}(1+s)^{-\frac{pb_0}{2}}ds\|(u,v)\|_X^p\leqslant C\mathrm{e}^{-\frac{\kappa t}{2\alpha}}\int_0^{\frac{t}{2}}(1+s)^{-\frac{a_0}{2}-1}ds\|(u,v)\|_X^p\\ &\leqslant C(1+t)^{-\frac{a_0}{2}}\|(u,v)\|_X^p,\end{aligned}\tag{3.9}$$

$$L_3\leqslant C\int_{\frac{t}{2}}^t(1+s)^{-\frac{a_0}{2}-1}ds\|(u,v)\|_X^p\leqslant C(1+t)^{-\frac{a_0}{2}}\|(u,v)\|_X^p.\tag{3.10}$$

Moreover, we have with $\frac{N}{\delta a_0}>1$, $p-\frac{\delta a_0}{b_0}>0$ and $\frac{pb_0-\delta a_0}{2}<1$ that

$$\begin{aligned}L_2&\leqslant C(1+t)^{-\frac{\delta a_0}{2}}\int_0^{\frac{t}{2}}\|v(s)\|_\infty^{p-\frac{\delta a_0}{b_0}}\|v(s)\|_{\frac{N}{b_0}}^{\frac{\delta a_0}{b_0}}ds\\ &\leqslant C(1+t)^{-\frac{\delta a_0}{2}}\int_0^{\frac{t}{2}}(1+s)^{-\frac{pb_0-\delta a_0}{2}}ds\|(u,v)\|_X^p\\ &\leqslant C(1+t)^{-\frac{\delta a_0}{2}}(1+t)^{\frac{\delta a_0-pb_0}{2}+1}\|(u,v)\|_X^p\\ &\leqslant C(1+t)^{-\frac{a_0}{2}}\|(u,v)\|_X^p.\end{aligned}\tag{3.11}$$

Combine (3.9)–(3.11) to get

$$\|\mathcal{M}_1[u]\|_\infty\leqslant C(1+t)^{-\frac{a_0}{2}}(\|u_0\|_\infty+\|u_0\|_{\frac{N}{a_0}})+C(1+t)^{-\frac{a_0}{2}}\|(u,v)\|_X^p,$$

and similarly, $\|\mathcal{M}_2[v]\|_\infty\leqslant C(1+t)^{-\frac{b_0}{2}}(\|v_0\|_\infty+\|v_0\|_{\frac{N}{b_0}})+C(1+t)^{-\frac{b_0}{2}}\|(u,v)\|_X^q$. By the same procedure, we have

$$\|\mathcal{M}_1[u]\|_{\frac{N}{a_0}}\leqslant C\|u_0\|_{\frac{N}{a_0}}+C\|(u,v)\|_X^p,\quad \|\mathcal{M}_2[v]\|_{\frac{N}{b_0}}\leqslant C\|v_0\|_{\frac{N}{b_0}}+C\|(u,v)\|_X^q.$$

Consequently,

$$\|\mathcal{M}[(u,v)]\|_X\leqslant C(\|u_0\|_\infty+\|v_0\|_\infty+\|u_0\|_{\frac{N}{a_0}}+\|v_0\|_{\frac{N}{b_0}})+C(\|(u,v)\|_X^p+\|(u,v)\|_X^q).$$

Since $p,q>1$, we have $C(\|(u,v)\|_X^p+\|(u,v)\|_X^q)\leqslant\frac{\varepsilon}{2}$ provided $\varepsilon>0$ small enough. Take $\eta=\eta(\varepsilon)>0$ in (3.2) small such that $C(\|u_0\|_\infty+\|v_0\|_\infty+\|u_0\|_{\frac{N}{a_0}}+\|v_0\|_{\frac{N}{b_0}})\leqslant\frac{\varepsilon}{2}$. Thus, $\mathcal{M}$ maps $X\cap B_\varepsilon$ into itself.

Furthermore, for $(u_1,v_1),(u_2,v_2)\in X$, we know by (3.4) and (3.6) that

$$\begin{aligned}\|\mathcal{M}_1[u_1]-\mathcal{M}_1[u_2]\|_{\frac{N}{a_0}}&\leqslant\int_0^t\|\mathcal{G}(t-s)\mathcal{B}(v_1^p(s)-v_2^p(s))\|_{\frac{N}{a_0}}ds\\&\leqslant\int_0^{\frac{t}{2}}\mathrm{e}^{-\frac{\kappa(t-s)}{\alpha}}\|v_1^p(s)-v_2^p(s)\|_{\frac{N}{a_0}}ds\\&\quad+\int_0^{\frac{t}{2}}(1+t-s)^{-\frac{(\delta-1)a_0}{2}}\|v_1^p(s)-v_2^p(s)\|_{\frac{N}{\delta a_0}}ds\\&\quad+\int_{\frac{t}{2}}^t\|v_1^p(s)-v_2^p(s)\|_{\frac{N}{a_0}}ds.\end{aligned}$$

By Hölder's inequality,

$$\begin{aligned}&\int_0^{\frac{t}{2}}\mathrm{e}^{-\frac{\kappa(t-s)}{\alpha}}\|v_1^p(s)-v_2^p(s)\|_{\frac{N}{a_0}}ds\\&\quad\leqslant\int_0^{\frac{t}{2}}\mathrm{e}^{-\frac{\kappa(t-s)}{\alpha}}\|v_1(s)-v_2(s)\|_{\frac{pN}{a_0}}\|v_1^{p-1}(s)+v_2^{p-1}(s)\|_{\frac{pN}{(p-1)a_0}}ds\\&\quad\leqslant C\int_0^{\frac{t}{2}}\mathrm{e}^{-\frac{\kappa(t-s)}{\alpha}}\|v_1(s)-v_2(s)\|_{\frac{pN}{a_0}}(\|v_1(s)\|_{\frac{pN}{a_0}}^{p-1}+\|v_2(s)\|_{\frac{pN}{a_0}}^{p-1})ds\\&\quad\leqslant\int_0^{\frac{t}{2}}\mathrm{e}^{-\frac{\kappa(t-s)}{\alpha}}\|v_1(s)-v_2(s)\|_{\frac{N}{b_0}}^{\frac{a_0}{pb_0}}\|v_1(s)-v_2(s)\|_\infty^{1-\frac{a_0}{pb_0}}\\&\qquad\times(\|v_1\|_{\frac{N}{b_0}}^{\frac{(p-1)a_0}{pb_0}}\|v_1(s)\|_\infty^{p-1-\frac{(p-1)a_0}{pb_0}}+\|v_2(s)\|_{\frac{N}{b_0}}^{\frac{(p-1)a_0}{pb_0}}\|v_2(s)\|_\infty^{p-1-\frac{(p-1)a_0}{pb_0}})ds\\&\quad\leqslant\int_0^{\frac{t}{2}}\mathrm{e}^{-\frac{\kappa(t-s)}{\alpha}}(1+s)^{-1}ds(\|(u_1,v_1)\|_X^{p-1}+\|(u_2,v_2)\|_X^{p-1})\|(u_1-u_2,v_1-v_2)\|_X\\&\quad\leqslant C(\|(u_1,v_1)\|_X^{p-1}+\|(u_2,v_2)\|_X^{p-1})\|(u_1-u_2,v_1-v_2)\|_X,\end{aligned}\tag{3.12}$$

and similarly,

$$\int_{\frac{t}{2}}^t\|v_1^p(s)-v_2^p(s)\|_{\frac{N}{a_0}}ds\leqslant C(\|(u_1,v_1)\|_X^{p-1}+\|(u_2,v_2)\|_X^{p-1})\|(u_1-u_2,v_1-v_2)\|_X.\tag{3.13}$$

In the same way, we also have

$$\begin{aligned}&\int_0^{\frac{t}{2}}(1+t-s)^{-\frac{(\delta-1)a_0}{2}}\|v_1^p(s)-v_2^p(s)\|_{\frac{N}{\delta a_0}}ds\\&\quad\leqslant C(1+t)^{-\frac{(\delta-1)a_0}{2}}\int_0^{\frac{t}{2}}(1+s)^{-\frac{b_0p-\delta a_0}{2}}ds(\|(u_1,v_1)\|_X^{p-1}+\|(u_2,v_2)\|_X^{p-1})\|(u_1-u_2,v_1-v_2)\|_X\\&\quad\leqslant C(\|(u_1,v_1)\|_X^{p-1}+\|(u_2,v_2)\|_X^{p-1})\|(u_1-u_2,v_1-v_2)\|_X.\end{aligned}\tag{3.14}$$

Combine (3.12)–(3.14) to conclude

$$\|\mathcal{M}_1[u_1]-\mathcal{M}_1[u_2]\|_{\frac{N}{a_0}}\leqslant C(\|(u_1,v_1)\|_X^{p-1}+\|(u_2,v_2)\|_X^{p-1})\|(u_1-u_2,v_1-v_2)\|_X,\tag{3.15}$$

and similarly,

$$\|\mathcal{M}_2[v_1]-\mathcal{M}_2[v_2]\|_{\frac{N}{b_0}}\leqslant C(\|(u_1,v_1)\|_X^{q-1}+\|(u_2,v_2)\|_X^{q-1})\|(u_1-u_2,v_1-v_2)\|_X.\tag{3.16}$$

Moreover, it can be obtained that

$$\|\mathcal{M}_1[u_1]-\mathcal{M}_1[u_2]\|_\infty\leqslant C(1+t)^{-\frac{a_0}{2}}(\|(u_1,v_1)\|_X^{p-1}+\|(u_2,v_2)\|_X^{p-1})\|(u_1-u_2,v_1-v_2)\|_X,\tag{3.17}$$

$$\|\mathcal{M}_2[v_1]-\mathcal{M}_2[v_2]\|_\infty\leqslant C(1+t)^{-\frac{b_0}{2}}(\|(u_1,v_1)\|_X^{q-1}+\|(u_2,v_2)\|_X^{q-1})\|(u_1-u_2,v_1-v_2)\|_X.\tag{3.18}$$

We conclude from (3.15)–(3.18) that $\mathcal{M}$ is a strict contraction in $X \cap B_\varepsilon$ provided ε is small enough, and so, there is a unique global solution to (1.1).

The estimate (3.3) follows by interpolation. □

Proof of Theorem 3.1. First, prove the global existence in the part (i). Since $a > a_0$, $b > b_0$, $u_0(x) = \lambda\psi(x)$, $v_0(x) = \mu\varphi(x)$ with $\psi(x) \in \mathbb{I}^a$ and $\varphi(x) \in \mathbb{I}^b$, there exist $\lambda_0 > 0$ and $\mu_0 > 0$ such that (3.2) holds provided $\max\{\lambda - \lambda_0, \mu - \mu_0\} \leqslant 0$. The global existence of solutions to (1.1) comes from Proposition 3.2.

Next, we deal with the blow-up of solutions under the mentioned large initial data. Suppose for contradiction that there is a nontrivial global solution (u, v) to (1.1), and treat the large u_0 only. It follows from (2.2) and (2.8) that

$$I_p + \int_{B_{\frac{R}{2}}} u_0 dx \leqslant C_2(2+\alpha) R^{N-\frac{N+2}{q}} J_q^{\frac{1}{q}} + C_2 R^{-2} \int_{B_R \setminus B_{\frac{R}{2}}} u_0 dx,$$

provided R large enough, where and throughout this section $C_2 > 0$ denotes constants independent of α, R and λ. For simplicity, assume $u_0 \sim \lambda |x|^{-a}$ with $\lambda > 1$, $|x| \to \infty$. Thus, for sufficiently large R,

$$\begin{aligned} I_p + \frac{1}{2}\int_{B_{\frac{R}{2}}} u_0 dx &\leqslant C_2(2+\alpha) R^{N-\frac{N+2}{q}} J_q^{\frac{1}{q}} + C_2 R^{-2} \int_{B_R \setminus B_{\frac{R}{2}}} u_0 dx - \frac{1}{2}\int_{B_{\frac{R}{2}} \setminus B_{\frac{R}{4}}} u_0 dx \\ &\leqslant C_2(2+\alpha) R^{N-\frac{N+2}{q}} J_q^{\frac{1}{q}} + C_2 R^{N-2-a} - C_2 R^{N-a} \\ &\leqslant C_2(2+\alpha) R^{N-\frac{N+2}{q}} J_q^{\frac{1}{q}}, \end{aligned}$$

and hence, $I_p + \frac{1}{2}\int_{B_{\frac{R}{2}}} u_0 dx \leqslant C_2(2+\alpha) R^{N-\frac{N+2}{q}} J_q^{\frac{1}{q}}$. Together with (2.9), we get

$$I_p + \frac{1}{2}\int_{B_{\frac{R}{2}}} u_0 dx \leqslant C_2(2+\alpha)^{1+\frac{1}{q}} R^{N-\frac{2}{q}-\frac{N+2}{pq}} I_p^{\frac{1}{pq}}. \tag{3.19}$$

By Young's inequality,

$$I_p + \frac{1}{2}\int_{B_{\frac{R}{2}}} u_0 dx \leqslant \frac{1}{2} I_p + C_2(2+\alpha)^{\frac{p(q+1)}{pq-1}} R^{N-\frac{2(p+1)}{pq-1}}. \tag{3.20}$$

Due to $\int_{B_{\frac{R}{2}}} u_0 dx \geqslant \int_{B_{\frac{R}{2}} \setminus B_{\frac{R}{4}}} u_0 dx \geqslant C_2 \lambda R^{N-a}$, we conclude from (3.20) that

$$I_p + C_2 \lambda R^{N-a} \leqslant C_2(2+\alpha)^{\frac{p(q+1)}{pq-1}} R^{N-\frac{2(p+1)}{pq-1}}. \tag{3.21}$$

If $0 < a < a_0 = \frac{2(p+1)}{pq-1}$, (3.21) with R large enough leads to a contradiction. If $a > a_0 = \frac{2(p+1)}{pq-1}$, another contradiction comes from (3.21) provided $\lambda > \lambda_1(\alpha) =: 1 + C_2(2+\alpha)^{\frac{p(q+1)}{pq-1}} R^{a_0 - a}$. □

Remark 3.3. Theorem 3.1 means that in the co-existence region of global and non-global solutions, the space-decays of the initial data in the item (ii) are slow enough to yield a finite time blow-up of solutions. Differently, for realizing the blow-up of solutions in the case (i), the coefficients associated with the initial data should be large with e.g. $\lambda > \lambda_1(\alpha)$ to overcome the additional dispersal mechanism due to the third order term, since $\lambda_1(\alpha)$ is obviously increasing with α.

4 Global profile of solutions

In this section, we further investigate time-decay profiles for the global solutions to (1.1). This is stated in the following theorem:

Theorem 4.1. *Assume* $pq > (pq)_c$. *Let* (u, v) *be a solution to* (1.1) *with* $u_0(x) = \lambda\psi(x)$, $v_0(x) = \mu\varphi(x)$, $\psi(x) \in \mathbb{I}^a$, *and* $\varphi(x) \in \mathbb{I}^b$.

(i) *If* $a_0 < a < \min\{N, bp-2\}$ *and* $b_0 < b < \min\{N, aq-2\}$, *with* a_0 *and* b_0 *defined in* (3.1), λ *and* μ *small enough,* $\lim_{|x|\to\infty}(1+|x|)^a u_0(x) = A > 0$, $\lim_{|x|\to\infty}(1+|x|)^b v_0(x) = B > 0$, *then*

$$\lim_{t\to\infty} t^{\frac{a}{2}}\|u(\cdot,t) - AG(x,t) * (1+|x|)^{-a}\|_\infty = 0, \tag{4.1}$$

$$\lim_{t\to\infty} t^{\frac{b}{2}}\|v(\cdot,t) - BG(x,t) * (1+|x|)^{-b}\|_\infty = 0. \tag{4.2}$$

(ii) *If* $a_0 < a < \min\{N, Np-2\}$, $N < b < aq-2$, *with* λ *and* μ *small enough,* $\lim_{|x|\to\infty}(1+|x|)^a u_0(x) = A > 0$, *then* (u,v) *satisfies* (4.1) *and*

$$\lim_{t\to\infty} t^{\frac{N}{2}}\|v(\cdot,t) - M_0 G(\cdot,t)\|_\infty = 0, \tag{4.3}$$

with

$$M_0 = \lim_{t\to\infty}\|v(\cdot,t)\|_1 = \|v_0\|_1 + \int_0^\infty \|u^q(s)\|_1 ds.$$

(iii) *If* $a, b > N$, $p, q > 1 + \frac{2}{N}$, *with* λ *and* μ *small enough, then* (u,v) *satisfies* (4.3) *and*

$$\lim_{t\to\infty} t^{\frac{N}{2}}\|u(\cdot,t) - N_0 G(\cdot,t)\|_\infty = 0, \tag{4.4}$$

with

$$N_0 = \lim_{t\to\infty}\|u(\cdot,t)\|_1 = \|u_0\|_1 + \int_0^\infty \|v^p(s)\|_1 ds.$$

To prove the theorem we need more precise time-decay rates for the global solutions, namely, the following lemma.

Lemma 4.2. *Assume the conditions of Theorem* 4.1 *for the items* (i)–(iii) *are satisfied respectively.*

(i) *It is true with* $\gamma \in (\max\{\frac{2+a}{pb}, \frac{2+b}{qa}\}, 1)$ *that*

$$\|u\|_{\frac{N}{\gamma a}}, \|v\|_{\frac{N}{\gamma b}} \leqslant C, \quad \|u\|_\infty \leqslant C(1+t)^{-\frac{\gamma a}{2}}, \quad \|v\|_\infty \leqslant C(1+t)^{-\frac{\gamma b}{2}}. \tag{4.5}$$

(ii) *It is true with* $\gamma_1 \in (\max\{\frac{2+a}{pN}, \frac{2+N}{qa}\}, 1)$ *that*

$$\|u\|_{\frac{N}{\gamma_1 a}}, \|v\|_1 \leqslant C, \quad \|u\|_\infty \leqslant C(1+t)^{-\frac{\gamma_1 a}{2}}, \quad \|v\|_\infty \leqslant C(1+t)^{-\frac{N}{2}}. \tag{4.6}$$

(iii) *It holds that*

$$\|u\|_1, \|v\|_1 \leqslant C, \quad \|u\|_\infty \leqslant C(1+t)^{-\frac{N}{2}}, \quad \|v\|_\infty \leqslant C(1+t)^{-\frac{N}{2}}. \tag{4.7}$$

Proof. Similarly to the proof of Proposition 3.2, assume $p, q > 1$. Under the conditions of the case (i), set

$$Y = C([0,\infty); C(\mathbb{R}^N) \cap L^{\frac{N}{\gamma a}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)) \times C([0,\infty); C(\mathbb{R}^N) \cap L^{\frac{N}{\gamma b}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)),$$

with

$$\|(u,v)\|_Y = \sup_{t\in[0,\infty)}\{\|u(t)\|_{\frac{N}{\gamma a}} + \|v(t)\|_{\frac{N}{\gamma b}} + (1+t)^{\frac{\gamma a}{2}}\|u(t)\|_\infty + (1+t)^{\frac{\gamma b}{2}}\|v(t)\|_\infty\}. \tag{4.8}$$

Set $(u,v) \in Y \cap B_{\varepsilon_1}$ with $B_{\varepsilon_1} = \{(u,v) \mid \|(u,v)\|_Y \leqslant \varepsilon_1\}$, and $\varepsilon_1 > 0$ to be determined. By (3.4), (3.6), and (4.8), with $pb > a+2$ and $\max\{\frac{2+a}{pb}, \frac{2+b}{qa}\} < \gamma < 1$, we have

$$\begin{aligned}\|\mathcal{M}_1[u]\|_\infty &\leqslant \|\mathcal{G}(t)u_0\|_\infty + \int_0^t \|\mathcal{G}(t-s)\mathcal{B}v^p(s)\|_\infty ds\\ &\leqslant C(1+t)^{-\frac{\gamma a}{2}}(\|u_0\|_\infty + \|u_0\|_{\frac{N}{\gamma a}}) + C\int_0^{\frac{t}{2}} \mathrm{e}^{-\frac{\kappa(t-s)}{\alpha}}\|v(s)\|_\infty^p ds\\ &\quad + C\int_0^{\frac{t}{2}}(1+t-s)^{-\frac{\gamma a}{2}}\|v^p(s)\|_{\frac{N}{\gamma a}} ds + C\int_{\frac{t}{2}}^t \|v(s)\|_\infty^p ds\end{aligned}$$

$$
\begin{aligned}
&\leqslant C(1+t)^{-\frac{\gamma a}{2}}(\|u_0\|_\infty+\|u_0\|_{\frac{N}{\gamma a}})+C\int_0^{\frac{t}{2}}\mathrm{e}^{-\frac{\kappa(t-s)}{\alpha}}(1+s)^{-\frac{\gamma bp}{2}}ds\|(u,v)\|_Y^p\\
&\quad+C(1+t)^{-\frac{\gamma a}{2}}\int_0^{\frac{t}{2}}\|v(s)\|_\infty^{p-\frac{a}{b}}\|v(s)\|_{\frac{N}{\gamma b}}^{\frac{a}{b}}ds+C\int_{\frac{t}{2}}^t(1+s)^{-\frac{\gamma bp}{2}}ds\|(u,v)\|_Y^p\\
&\leqslant C(1+t)^{-\frac{\gamma a}{2}}(\|u_0\|_\infty+\|u_0\|_{\frac{N}{\gamma a}})+C\mathrm{e}^{-\frac{\kappa t}{2\alpha}}\int_0^{\frac{t}{2}}(1+s)^{-\frac{\gamma bp}{2}}ds\|(u,v)\|_Y^p\\
&\quad+C(1+t)^{-\frac{\gamma a}{2}}\int_0^{\frac{t}{2}}(1+s)^{-\frac{\gamma(pb-a)}{2}}ds\|(u,v)\|_Y^p+C(1+t)^{-\frac{\gamma bp}{2}+1}\|(u,v)\|_Y^p,
\end{aligned}
$$

which indicates that

$$
\|\mathcal{M}_1[u]\|_\infty\leqslant C(1+t)^{-\frac{\gamma a}{2}}(\|u_0\|_\infty+\|u_0\|_{\frac{N}{\gamma a}})+C(1+t)^{-\frac{\gamma a}{2}}\|(u,v)\|_Y^p,
$$

and similarly,

$$
\begin{aligned}
&\|\mathcal{M}_2[v]\|_\infty\leqslant C(1+t)^{-\frac{\gamma b}{2}}(\|v_0\|_\infty+\|v_0\|_{\frac{N}{\gamma b}})+C(1+t)^{-\frac{\gamma b}{2}}\|(u,v)\|_Y^q,\\
&\|\mathcal{M}_1[u]\|_{\frac{N}{\gamma a}}\leqslant C\|u_0\|_{\frac{N}{\gamma a}}+C\|(u,v)\|_Y^p,\quad \|\mathcal{M}_2[v]\|_{\frac{N}{\gamma b}}\leqslant C\|v_0\|_{\frac{N}{\gamma b}}+C\|(u,v)\|_Y^q.
\end{aligned}
$$

In addition, we can also have the estimates like (3.15)–(3.18). By similar arguments as in the proof of Proposition 3.2, we can prove that for ε_1 small enough and $\|u_0\|_\infty+\|v_0\|_\infty+\|u_0\|_{\frac{N}{\gamma a}}+\|v_0\|_{\frac{N}{\gamma b}}$ small enough, $\mathcal{M}:Y\cap B_{\varepsilon_1}\to Y\cap B_{\varepsilon_1}$ is a contraction, and the global existence of solutions with the estimates in (4.5) follows immediately.

The case (ii) can be treated in the same way. The proof for (iii) is similar to [5, Theorem 3.4]. We omit the details. □

Lemma 4.2 enables us to fix the time-decay profiles for the global solutions.

Proof of Theorem 4.1. First treat the case (i) with $a,b<N$. By the representation (1.5),

$$
\begin{aligned}
\|u-AG(x,t)*(1+|x|)^{-a}\|_\infty&\leqslant\|\mathcal{G}(t)u_0-AG(x,t)*(1+|x|)^{-a}\|_\infty+\int_0^t\|\mathcal{G}(t-s)\mathcal{B}v^p(s)\|_\infty ds\\
&=:\widetilde{K}_1+\widetilde{K}_2.
\end{aligned}
$$

It follows from (1.6) with (3.4)–(3.6) that

$$
\begin{aligned}
\widetilde{K}_1&\leqslant\|G(x,t)*u_0-AG(x,t)*(1+|x|)^{-a}\|_\infty+\mathrm{e}^{-\frac{t}{\alpha}}\sum_{i=0}^N\frac{t^i}{i!\alpha^i}\|\mathcal{B}^iu_0\|_\infty+\|\mathcal{R}(t)u_0\|_\infty\\
&\leqslant\|G(x,t)*u_0-AG(x,t)*(1+|x|)^{-a}\|_\infty+C\mathrm{e}^{-\frac{t}{\alpha}}(1+t)^N\|u_0\|_\infty+C(1+t)^{-\frac{(\gamma a+2)}{2}}\|u_0\|_{\frac{N}{\gamma a}},
\end{aligned}
$$

with $\gamma\in(\max\{\frac{2+a}{pb},\frac{2+b}{qa},\frac{a-2}{a}\},1)$. In addition, we know by [23] that $\lim_{t\to\infty}t^{\frac{a}{2}}\|G(x,t)*u_0-AG(x,t)*(1+|x|)^{-a}\|_\infty=0$. Consequently, $\lim_{t\to\infty}t^{\frac{a}{2}}\widetilde{K}_1=0$. On the other hand, choosing $\delta_1>1$ with $\gamma pb-\delta_1a>2$ and $\delta_1a<N$, we have by (3.4), (3.6) and (4.5) that

$$
\begin{aligned}
\widetilde{K}_2&\leqslant C\int_0^{\frac{t}{2}}\mathrm{e}^{-\frac{\kappa(t-s)}{\alpha}}\|v(s)\|_\infty^pds+C\int_0^{\frac{t}{2}}(1+t-s)^{-\frac{\delta_1a}{2}}\|v^p(s)\|_{\frac{N}{\delta_1a}}ds+\int_{\frac{t}{2}}^t\|v(s)\|_\infty^pds\\
&\leqslant C\int_0^{\frac{t}{2}}\mathrm{e}^{-\frac{\kappa(t-s)}{\alpha}}(1+s)^{-\frac{\gamma bp}{2}}ds\|(u,v)\|_Y^p+C\int_0^{\frac{t}{2}}(1+t-s)^{-\frac{\delta_1a}{2}}\|v(s)\|_\infty^{p-\frac{\delta_1a}{\gamma b}}\|v(s)\|_{\frac{N}{\gamma b}}^{\frac{\delta_1a}{\gamma b}}ds\\
&\quad+\int_{\frac{t}{2}}^t(1+s)^{-\frac{\gamma bp}{2}}ds\|(u,v)\|_Y^p\\
&\leqslant C(1+t)^{-\frac{\delta_1a}{2}}\|(u,v)\|_Y^p+C\int_0^{\frac{t}{2}}(1+t-s)^{-\frac{\delta_1a}{2}}(1+s)^{-\frac{\gamma pb-\delta_1a}{2}}ds\|(u,v)\|_Y^p\\
&\quad+C(1+t)^{-\frac{\delta_1a}{2}}\|(u,v)\|_Y^p
\end{aligned}
$$

$$\leqslant C(1+t)^{-\frac{\delta_1 a}{2}}\|(u,v)\|_Y^p + C(1+t)^{-\frac{\delta_1 a}{2}} ds\|(u,v)\|_Y^p + C(1+t)^{-\frac{\delta_1 a}{2}}\|(u,v)\|_Y^p,$$

which indicates $\lim_{t\to\infty} t^{\frac{a}{2}}\widetilde{K}_2 = 0$. This proves the part (i).

In what follows, we prove the part (ii) with $a < N < b$ by using the idea in [26]. We first assert that $\lim_{t\to\infty}\|v(x,t)\|_1$ exists. By (1.5),

$$\begin{aligned}
&\left|\|v(x,t)\|_1 - \|v_0\|_1 - \int_0^\infty \|\mathcal{B}u^q(s)\|_1 ds\right| \\
&\quad\leqslant |\|\mathcal{G}(t)v_0\|_1 - \|v_0\|_1| + \int_0^t |\|\mathcal{G}(t-s)\mathcal{B}u^q(s)\|_1 - \|\mathcal{B}u^q(s)\|_1| ds + \int_t^\infty \|\mathcal{B}u^q(s)\|_1 ds \\
&\quad=: \widetilde{L}_1 + \widetilde{L}_2 + \widetilde{L}_3.
\end{aligned} \tag{4.9}$$

Due to (1.6) and $\|G(x,t)*v_0\|_1 = \|v_0\|_1$, we obtain

$$\widetilde{L}_1 \leqslant |\|G(x,t)*v_0\|_1 - \|v_0\|_1| + C\mathrm{e}^{-\frac{t}{\alpha}}(1+t)^N\|v_0\|_1 + C(1+t)^{-1}\|v_0\|_1 \leqslant C(1+t)^{-1}\|v_0\|_1. \tag{4.10}$$

Similarly,

$$\|\mathcal{G}(t-s)\mathcal{B}u^q(s) - \mathcal{B}u^q(s)\|_1 \leqslant C(1+t-s)^{-1}\|\mathcal{B}u^q(s)\|_1.$$

Together with (3.4), (4.6) and $\gamma_1 aq > N+2$, we have

$$\begin{aligned}
\widetilde{L}_2 &\leqslant \int_0^{\frac{t}{2}} (1+t-s)^{-1}\|\mathcal{B}u^q(s)\|_1 ds + \int_{\frac{t}{2}}^t \|\mathcal{B}u^q(s)\|_1 ds \\
&\leqslant C(1+t)^{-1}\int_0^{\frac{t}{2}} \|u(s)\|_\infty^{q-\frac{N}{\gamma_1 a}}\|u(s)\|_{\frac{N}{\gamma_1 a}}^{\frac{N}{\gamma_1 a}} ds + \int_{\frac{t}{2}}^t \|u(s)\|_\infty^{q-\frac{N}{\gamma_1 a}}\|u(s)\|_{\frac{N}{\gamma_1 a}}^{\frac{N}{\gamma_1 a}} ds \\
&\leqslant C(1+t)^{-1}\int_0^{\frac{t}{2}} (1+s)^{-\frac{\gamma_1 aq-N}{2}} ds + \int_{\frac{t}{2}}^t (1+s)^{-\frac{\gamma_1 aq-N}{2}} ds \\
&\leqslant C(1+t)^{-1} + C(1+t)^{1-\frac{\gamma_1 aq-N}{2}},
\end{aligned} \tag{4.11}$$

$$\widetilde{L}_3 \leqslant \int_t^\infty (1+s)^{-\frac{\gamma_1 aq-N}{2}} ds \leqslant C(1+t)^{-\frac{\gamma_1 aq-N}{2}+1}. \tag{4.12}$$

It follows from (4.9)–(4.12) that

$$\lim_{t\to\infty}\left|\|v(t)\|_1 - \|v_0\|_1 - \int_0^\infty \|\mathcal{B}u^q(s)\|_1 ds\right| = 0,$$

which together with $\|\mathcal{B}\phi\|_1 = \|\phi\|_1$ for arbitrary $\phi \in L^1(\mathbb{R}^N)$ leads to

$$\lim_{t\to\infty}\|v(t)\|_1 = \|v_0\|_1 + \int_0^\infty \|u^q(s)\|_1 ds := M_0.$$

Write the second equation of (1.5) as $v(x,t) = \mathcal{G}(t-\tau)v(\tau) + \int_\tau^t \mathcal{G}(t-s)\mathcal{B}u^q(s)ds$. Then

$$\begin{aligned}
\|v(\cdot,t) - M_0 G(\cdot,t)\|_\infty &\leqslant \|v(\cdot,t) - \mathcal{G}(t-\tau)v(\tau)\|_\infty + \|\mathcal{G}(t-\tau)v(\tau) - \|v(\tau)\|_1 G(\cdot,t-\tau)\|_\infty \\
&\quad + \|\|v(\tau)\|_1 G(\cdot,t-\tau) - \|v(\tau)\|_1 G(\cdot,t)\|_\infty + \|(\|v(\tau)\|_1 - M_0)G(\cdot,t)\|_\infty \\
&\leqslant \int_\tau^t \|\mathcal{G}(t-s)\mathcal{B}u^q(s)\|_\infty ds + \|\mathcal{G}(t-\tau)v(\tau) - \|v(\tau)\|_1 G(\cdot,t-\tau)\|_\infty \\
&\quad + C(1+t)^{-\frac{N}{2}-1}\|v(\tau)\|_1 + C(1+t)^{-\frac{N}{2}}|\|v(\tau)\|_1 - M_0|.
\end{aligned}$$

By using (3.4) and (3.6), we have

$$\int_\tau^t \|\mathcal{G}(t-s)\mathcal{B}u^q(s)\|_\infty ds$$

$$\leqslant C\int_{\tau}^{\frac{t}{2}} \mathrm{e}^{-\frac{\kappa(t-s)}{\alpha}}\|\mathcal{B}u^q(s)\|_\infty ds + C\int_{\tau}^{\frac{t}{2}}(1+t-s)^{-\frac{N}{2}}\|u^q(s)\|_1 ds + C\int_{\frac{t}{2}}^{t}\|u^q(s)\|_\infty ds$$
$$\leqslant C(1+t)^{-\frac{\gamma_1 aq}{2}+1} + C(1+t)^{-\frac{N}{2}}\int_{\tau}^{t}(1+s)^{-\frac{\gamma_1 aq-N}{2}}ds. \tag{4.13}$$

In addition, by (1.6),

$$\|\mathcal{G}(t-\tau)v(\tau) - \|v(\tau)\|_1 G(\cdot,t-\tau)\|_\infty \leqslant \|G(\cdot,t-\tau)*v(\tau) - \|v(\tau)\|_1 G(\cdot,t-\tau)\|_\infty + \mathrm{e}^{-\frac{\kappa t}{\alpha}}(1+t)^N\|v(\tau)\|_\infty + (1+t)^{-1-\frac{N}{2}}\|v(\tau)\|_1.$$

It is well known that $\lim_{t\to\infty} t^{\frac{N}{2}}\|G(x,t-\tau)*v(\tau) - \|v(\tau)\|_1 G(\cdot,t-\tau)\|_\infty = 0$, and thus

$$\lim_{t\to\infty} t^{\frac{N}{2}}\|\mathcal{G}(t-\tau)v(\tau) - \|v(\tau)\|_1 G(x,t-\tau)\|_\infty = 0. \tag{4.14}$$

By (4.13) and (4.14) with $\gamma_1 aq > N+2$, we have

$$\lim_{t\to\infty} t^{\frac{N}{2}}\|v(\cdot,t) - M_0 G(\cdot,t)\|_\infty \leqslant C\int_{\tau}^{\infty}(1+s)^{-\frac{\gamma_1 aq-N}{2}}ds + C|\|v(\tau)\|_1 - M_0|.$$

Letting $\tau\to\infty$, we prove (4.3). The proof for (4.1) is similar to that in (i).

The case (iii) for $a,b>N$ can be proved via the arguments for (4.3) in (ii). □

Remark 4.3. It is pointed out that the assumptions in Theorem 4.1 are optimal. Consider (iii) with L^1 initial data as an example with $a,b>N$. If $p,q>1+\frac{2}{N}$ is not satisfied, e.g., $1<q<1+\frac{2}{N}$ with $pq>(pq)_c$, then for $u_0=v_0=\varepsilon(1+|x|)^{-\sigma}$ with $\sigma\in(N,\frac{N+2}{q})$ and ε small enough, it is known by constructing a subsolution (similar to [27]) that the unique global solution to (1.1) ensured by Theorem 3.1 satisfies $u\geqslant C(1+t+|x|^2)^{-\frac{\sigma}{2}}$. Hence, by (1.6) and (3.5), it holds for t large enough that

$$\begin{aligned}\|v(\cdot,t)\|_\infty &\geqslant \int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^N} G(y,t-s)u^q(y,s)dyds - \int_{\frac{t}{2}}^{t}(1+t)^{-\frac{N}{2}-1}\|u^q(s)\|_1 ds\\ &\geqslant C\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^N}\mathrm{e}^{-\frac{|y|^2}{4}}(1+s+(t-s)|y|^2)^{-\frac{\sigma q}{2}}dyds - \int_{\frac{t}{2}}^{t}(1+s)^{-\frac{N}{2}-1}\|u(s)\|_1 ds\\ &\geqslant C\int_{0}^{t}\int_{\mathbb{R}^N}\mathrm{e}^{-\frac{|y|^2}{4}}(1+s)^{-\frac{\sigma q}{2}}(1+|y|^2)^{-\frac{\sigma q}{2}}dyds - C(1+t)^{-\frac{N}{2}}\\ &\geqslant C(1+t)^{-\frac{\sigma q}{2}+1} - C(1+t)^{-\frac{N}{2}}\\ &\gg C(1+t)^{-\frac{N}{2}},\end{aligned}$$

which destroys (4.3), since $\|G(\cdot,t)\|_\infty = O(t^{-N/2})$ there.

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