SCIENCE CHINA
Mathematics

• ARTICLES •

January 2013 Vol. 56 No. 1: 177–200
doi: 10.1007/s11425-012-4547-z

On random coefficient INAR(1) processes

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Received April 28, 2012; accepted September 20, 2012

Abstract The random coefficient integer-valued autoregressive process was introduced by Zheng, Basawa, and Datta in 2007. In this paper we study the asymptotic behavior of this model (in particular, weak limits of extreme values and the growth rate of partial sums) in the case where the additive term in the underlying random linear recursion belongs to the domain of attraction of a stable law.

Keywords models for count data, thinning models, branching processes, random environment, limit theorems

MSC(2010) 60J80, 60K37, 60F05, 60J20

Citation: Roitershtein A, Zhong Z. On random coefficient INAR(1) processes. Sci China Math, 2013, 56: 177–200, doi: 10.1007/s11425-012-4547-z

1 Introduction

In this paper we consider a *first-order random coefficient integer-valued autoregressive* (abbreviated as RCINAR(1)) process that was introduced by Zheng et al. [53]. While [53] as well as a subsequent work have been focused mostly on direct statistical applications of the model, the primary goal of this paper is to contribute to the understanding of its probabilistic structure.

Let $\Phi := (\phi_n)_{n\in\mathbb{Z}}$ be an i.i.d. sequence of reals, each one taking values in the closed interval $[0,1]$. Further, let $\mathcal{Z} := (Z_n)_{n\in\mathbb{Z}}$ be a sequence of i.i.d. integer-valued non-negative random variables, independent of Φ. The pair $(\Phi,\mathcal{Z})$ is referred to in [53] as a sequence of random coefficients associated with the model.

Let $\mathbb{Z}_+$ denote the set of non-negative integers $\{n\in\mathbb{Z}: n\geqslant 0\}$. The RCINAR(1) process $\mathcal{X} := (X_n)_{n\in\mathbb{Z}_+}$ is then defined as follows. Let $B := (B_{n,k})_{n\in\mathbb{Z},k\in\mathbb{Z}}$ be a collection of Bernoulli random variables independent of $\mathcal{Z}$ and such that, given a realization of Φ, the variables $B_{n,k}$ are independent and

$$P_\Phi(B_{n,k}=1)=\phi_n \quad \text{and} \quad P_\Phi(B_{n,k}=0)=1-\phi_n, \quad \forall\, k\in\mathbb{N},$$

where P_Φ stands for the underlying probability measure conditional on Φ. Let $X_0=0$ and consider the following linear recursion:

$$X_n=\sum_{k=1}^{X_{n-1}} B_{n,k}+Z_n, \quad n\in\mathbb{N}, \tag{1.1}$$

where we make the usual convention that an empty sum is equal to zero. To emphasize the formal dependence on the initial condition, we will denote the underlying probability measure (i.e., the joint law of $\Phi,\mathcal{Z},B$, and $\mathcal{X}$) conditional on $\{X_0=0\}$ by P_0 and denote the corresponding expectation by E_0. For

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the most of the paper we will consider a natural initial assumption $X_0 = 0$ and hence consistently state our results for the measure P_0. We remark however that all our results (stated below in Section 2) are robust with respect to the initial condition X_0.

The RCINAR(1) process $\mathcal{X}$ defined by (1.1) is a generalization of the *integer-valued autoregressive of order one* (abbreviated as INAR(1)) model, in which the parameters ϕ_n are deterministic and identical for all $n \in \mathbb{Z}$. The model introduced in [53] has been further extended in [18, 46, 49–53]. We refer the reader to [25, 33, 35, 47] for a general review of integer-valued (data counting) time series models and their applications.

Formally, RCINAR(1) can be classified as a special kind of branching processes with immigration in the random environment Φ (cf. [27]). In particular, the process can be rigorously constructed on the state space of "genealogical trees" (see [22, Chapter VI]). The random variable X_n is then interpreted as the total number of individuals present at generation n. At the beginning of the n-th period of time, Z_n immigrants enter the system. Simultaneously and independently of it, each particle from the previous generation exits the system, producing in the next generation either one child (with probability ϕ_n) or none (with the complementary probability $1 - \phi_n$)[1)]. The branching processes interpretation is a useful point of view on RCINAR(1) which provides powerful tools for the asymptotic analysis of the model.

In this paper we focus on the case where production and immigration mechanisms are both defined by an i.i.d. environment and, furthermore, are independent of each other. More general type of branching process with immigration in random environment is considered, for instance, in [24, 27, 40]. Assuming suitable moment conditions and ergodic/mixing properties of the environment, a law of large numbers and a central limit theorem for such processes are obtained in [40]. It would be interesting to carry over to a more general setting the results of this paper which rest on the regular variation property of the coefficients when the moment conditions of [40] are not satisfied. It is plausible to assume and we leave this as a topic for future research that such an extension can be obtained by an adaptation of the techniques exploited in this paper for the case of Markovian coefficients with a possible correlation between production and immigration mechanisms. We remark that a bottleneck for such a generalization of our results appears to be a suitable extension to a more general setup of the identity (3.1) and Lemma 3.1 below.

Let $\mathcal{N}_+$ denote the set of non-negative integer-valued random variables in the underlying probability space. The first term on the right-hand side of (1.1) can be thought of as the result of applying to X_n a *binomial thinning operator* which is associated with ϕ_n. More precisely, using the following operator notation introduced by Steutel and van Harn [44]:

$$\phi_n \circ X := \sum_{n=1}^{X} B_{n,k}, \quad X \in \mathcal{N}_+,$$

(1.1) can be written as

$$X_n = \phi_n \circ X_{n-1} + Z_n, \quad n \in \mathbb{N}. \tag{1.2}$$

This form of the recursion indicates that an insight into the probabilistic structure of the RCINAR(1) process can be gained by comparing it to the classical AR(1) (*first-order autoregressive*) model for real-valued data. The latter is defined by means of i.i.d. pairs $(\phi_n, Z_n)_{n\in\mathbb{Z}}$ of real-valued random coefficients, through the following linear recursion:

$$Y_n = \phi_n Y_{n-1} + Z_n, \quad n \in \mathbb{N}. \tag{1.3}$$

In this paper we explore one of the aspects of the similarity between the RCINAR(1) and AR(1) processes. Namely, we show in Theorem 2.5 below that if Z_n are in the domain of attraction of a stable law so is the limiting distribution of X_n, and then consider some implications of this result for the asymptotic behavior

[1)] Alternatively, one can think that each particle either survives to the next generation (with probability ϕ_n) or dies out (with probability $1 - \phi_n$).

of the sequence X_n. A prototype of Theorem 2.5 for AR(1) processes has been obtained in [19, 21]. Our proof of Theorem 2.5 relies on an adaptation of the technique which has been developed in [19].

We conclude the introduction with the following remarks on the motivation for our study. Although it appears that most of our results (stated in Section 2 below) could be extended to a more general type of processes than is considered here, we prefer to focus on one important model. It is well known that certain *quenched* characteristics of branching processes in random environment satisfy the linear difference (1.3). In two different settings, both yielding stationary solutions to (1.3) with regularly varying tails, this observation has been used to obtain the asymptotic behavior of the extinction probabilities in a branching processes in random environment [19, 20] and the cumulative population for branching processes in random environment with immigration [26, 27]. These studies make it appealing to consider a model like (1.2) which evidently combines features of both branching processes in random environment (with immigration) and AR(1) time series.

In general, probabilistic analysis of the future behavior of average and extreme value characteristics of the underlying system might be handy for typical real-world applications of a counting data model. Our results thus constitute a natural complement to the statistical inference tools developed for the RCINAR(1) processes in [53]. For the sake of example, consider

1) maximal number of unemployed per month in an economy, according to the model discussed in [53, Section 1];

2) a variation of the model for city size distributions studied in [16, 17] where the underlying AR(1) equation is replaced by its suitable integer-valued analogue. More precisely, while it is argued in [16, 17] that the evolution of the normalized (to the total size of the population) size of a city Y_n obeys (1.3), we propose (1.2) as a possible alternative model for non-normalized size of the city population X_n, where ϕ_n is an average proportion of the population which will continue to live in the city in the observation epoch $n+1$ and Z_n is the factor accumulating both the natural population growth and migration;

3) total number of arrivals in the random coefficient variation of the queueing system proposed in [1, Subsection 3.2].

On the technical side, in contrary to [53], we do not restrict ourselves to a setup with $E[Z_0^2] < \infty$. This finite variance condition apparently does not pose a real limitation on the possibility of applications of RCINAR(1) to, say, the unemployment rate and the cities growth models mentioned above. In both the cases, it is reasonable to assume that the innovations Z_n are typically relatively small comparing to X_n and, furthermore, large fluctuations of their values are not very likely to occur. However, the situation seems to be quite different if one wishes to apply the theory of RCINAR(1) processes to the models of queueing theory (as it has been done in [1]) when the latter are assumed to operate under a heavy traffic regime. See, for instance, [3, 5, 10, 13, 37, 54] and [6, 38, 41] for queueing network models where it is assumed that the network input has sub-exponential or, more specifically, regularly varying distribution tails (typically resulting from the distribution of the length of ON/OFF periods). We remark that the extensive literature on queueing networks in a heavy traffic regime is partially motivated by the research on the Internet network activity where it has been shown that in many instances a web traffic is well-described by heavy-tailed random patterns; see, for instance, [11, 31, 39, 48].

2 Statement of results

This section contains the statement of our main results, and is structured as follows. We start with a formulation of our specific assumptions on the coefficients $(\Phi, \mathcal{Z})$ of the model (see Assumptions 2.1 and 2.2 below). Proposition 2.3 then ensures the existence of the limiting distribution of X_n and also states formally some related basic properties of this Markov chain. Theorem 2.5 is concerned with the asymptotic of the tail of the limiting distribution in the case where the additive coefficients Z_n belong to the domain of attraction of a stable law. The theorem shows that in this case, the tails of the limiting distribution inherit the structure of the tails of Z_0. This observation leads us to Theorem 2.6, which is an extreme value limit theorem for the sequence $(X_n)_{n\in\mathbb{Z}_+}$. Weak convergence of suitably normalized partial sums of X_n is the content of Theorems 2.11 and 2.12. The proof of these limit theorems exploit the branching process representation of a regenerative structure which is described by Proposition 2.7. Two curious implications of the existence of this regenerative structure are stated in Propositions 2.8 and 2.9. The proofs of main theorems stated below in this section are given in Section 3 while the proofs of two auxiliary propositions are deferred to the Appendix.

Specific assumptions on the random coefficients. Recall that a function $f:\mathbb{R}\to\mathbb{R}$ is called regularly varying if $f(t)=t^{\alpha}L(t)$ for some $\alpha\in\mathbb{R}$ and a function L such that $\lim_{t\to\infty}L(\lambda t)/L(t)=1$ for all $\lambda>0$. The parameter α is called the index of the regular variation. If $\alpha=0$, then f is said to be slowly varying. We will denote by $\mathcal{R}_\alpha$ the class of all regularly varying real-valued functions with index α. We will impose the following assumption on the coefficients of the model defined by (1.1).

Assumption 2.1. (A1) $P(\phi_0=1)<1$.

(A2) *For some* $\alpha>0$, *there exists* $h\in\mathcal{R}_\alpha$ *such that* $\lim_{t\to\infty}h(t)\cdot P(Z_n>t)=1$.

Throughout the paper we will assume (actually, without loss of generality in view of (A2) and [4, Theorem 1.5.4] which ensures the existence of a non-decreasing equivalent for h) that the following condition is included in Assumption 2.1:

(A3) *Let* $h:(0,\infty)\to\mathbb{R}$ *be as in* (A2). *Then* $\sup_{t>0}h(t)\cdot P(Z_n>t)<\infty$.

The assumption of heavy-tailed innovations (noise terms) in autoregressive models is quite common in the applied probability literature. It is a well-known paradigm that such an assumption yields a rich probabilistic structure of the stationary solution and allows for a great flexibility in the modeling of its asymptotic behavior. See for instance [19, 21], more recent articles [8, 9, 23, 29, 36, 42, 43], and references therein.

In a few occasions (including a central limit theorem stated below in Theorem 2.14) we will use the following weaker version of Assumption 2.1:

Assumption 2.2. *Condition* (A1) *of Assumption* 2.1 *is satisfied and, in addition, the following holds*:

(A4) $E[Z_0^\beta]<\infty$ *for some* $\beta>0$.

Assumption 2.2 is stronger than the usual $E(\log^+|Z_0|)<+\infty$, where $x^+:=\max\{x,0\}$ for $x\in\mathbb{R}$, which is essentially required for the existence and uniqueness of the stationary solution to (1.2). It can be seen through the formula $E[Z_0^\beta]=\int_0^\infty\beta x^{\beta-1}P(Z_0>x)dx$ (recall that $Z_0\geqslant 0$) that (A4) is basically equivalent to the assumption that the distribution tails of Z_0 are "not too thick".

Limiting distribution of X_n. Let $Y_n\Rightarrow Y_\infty$ stand for the convergence in distribution of a sequence of random variables $(Y_n)_{n\in\mathbb{N}}$ to a random variable Y_∞ (we will usually omit the indication "as $n\to\infty$"). We will use the notation $X=_D Y$ to indicate that the distributions of random variables X and Y coincide under the law P_0. For $X\in\mathcal{N}_+$ define $\Pi_0\circ X:=X$ and, recursively, $\Pi_{k+1}\circ X:=\phi_{k+1}\circ(\Pi_k\circ X)$. This defines a sequence of random operators acting in $\mathcal{N}_+$ as follows:

$$\Pi_k\circ X=\phi_k\circ\phi_{k-1}\circ\cdots\circ\phi_1\circ X,\quad X\in\mathcal{N}_+. \tag{2.1}$$

The existence of the stationary distribution for the sequence $\mathcal{X}=(X_n)_{n\geqslant 0}$ introduced in (1.1) is the content of the following proposition.

Proposition 2.3. *Let Assumption* 2.2 *hold. Then,*

(a) *The following series converges to a finite limit with probability one*:

$$X_\infty:=\sum_{k=0}^{\infty}X_{0,k}, \tag{2.2}$$

where the random variables $(X_{0,k})_{k\in\mathbb{Z}_+}$ *are independent, and* $X_{0,k}=_D\Pi_k\circ Z_0$ *for any* $k\in\mathbb{N}$.

(b) $X_n\Rightarrow X_\infty$ *for any* $X_0\in\mathcal{N}_+$. *Here* $(X_n)_{n\in\mathbb{Z}_+}$ *is understood as the sequence produced by the recursion rule* (1.1) *with an arbitrary initial value* X_0.

(c) *The distribution of* X_∞ *is the unique distribution of* X_0 *which makes* $(X_n)_{n\in\mathbb{Z}_+}$ *into a stationary sequence.*

The proof of the proposition is deferred to the appendix. We remark that if $E[Z_0^2] < \infty$ is assumed, the above statement is essentially [53, Proposition 2.2]. For a counterpart of this result for AR(1) processes see, for instance, [7, Theorem 1]. It is not hard to deduce from the above proposition the following corollary, whose proof is omitted:

Corollary 2.4. *Suppose that Assumption* 2.2 *holds, and let* $\mathcal{X} = (X_n)_{n\in\mathbb{Z}_+}$ *be a random sequence defined by* (1.1). *Then* $\mathcal{X}$ *is an irreducible, aperiodic, and positive-recurrent Markov chain whose stationary measure is supported on a set of integers* $\{k \in \mathbb{Z}_+ : k \geqslant k_{\min}\}$, *where* $k_{\min} := \min\{k \in \mathbb{Z}_+ : P(Z_0 = k) > 0\}$. *In particular,* $\mathcal{X}$ *is an ergodic sequence.*

It follows from the above proposition that X_∞ is the unique solution to the distributional fixed point equation $X =_D \phi_0 \circ X + Z_0$ which is independent of (ϕ_0, B_0, Z_0), where B_0 denotes the sequence $(B_{0,k})_{k\in\mathbb{N}}$. In fact, the explicit form (2.2) of the stationary distribution along with the identity $(\phi_n, Z_n)_{n\in\mathbb{Z}} =_D (\phi_{-n}, Z_{-n})_{n\in\mathbb{Z}}$, implies that the unique stationary solution to (1.1) is given by the following infinite series:

$$X_n = \sum_{k=-\infty}^{n} X_{k,n}, \tag{2.3}$$

where the random variables $(X_{k,n})_{k\in\mathbb{Z}}$ are independent, and

$$X_{k,n} =_P \phi_{n-1} \circ \phi_{n-2} \circ \cdots \circ \phi_{k+1} \circ Z_k, \quad k \leqslant n.$$

By means of the branching process interpretation,

$$X_{k,n} = \#\{\text{progeny alive at time } n \text{ of all the immigrants who arrived at time } k\}, \tag{2.4}$$

with the convention that $X_{n,n} = Z_n$ and $X_{k,n} = 0$ for $k > n$. Thus (2.3) states that the stationary solution to (1.1) is formally obtained by letting the zero generation to be formed as a union of the following two groups of individuals:

1. Z_0 immigrants arriving at time zero, and
2. descendants, present in the population at time zero, of all "demo-immigrants" who has entered the system at the negative times $k = -1, -2, \ldots$

The random variables $X_{k,n}$ can be defined rigorously on the natural state space of the branching process, which is a space of family trees describing the "genealogy" of the individuals (see [22, Chapter VI]). To distinguish between the branching process starting at time zero with $X_0 = 0$ and its stationary version "starting at time $-\infty$", we will denote by P the distribution of the latter, while continuing to use P_0 for the probability law of the former. We will denote by E the expectation operator associated with the probability measure P. We will use the notation $X =_P Y$ to indicate that the distributions of random variables X and Y coincide under the stationary law P. As it has been mentioned earlier, we will consistently state our results for the underlying process under the law P_0 and thus will consider measure P as an auxiliary tool rather than a primary object of interest.

In the case when the additive term in the underlying random linear recursion belongs to the domain of attraction of a stable law we have the following theorem:

Theorem 2.5. *Let Assumption* 2.1 *hold. Then,*

$$\lim_{t\to\infty} h(t) \cdot P(X_\infty > t) = (1 - E[\phi_0^\alpha])^{-1} \in (0, \infty).$$

A prototype of this result for AR(1) processes has been obtained in [19, 21]. The proof of Theorem 2.5 given in Subsection 3.1 relies on an adaptation to our setup of a technique which has been developed in [19].

Extreme values of $\mathcal{X}$. We next show that the running maximum of the sequence $\mathcal{X}$ exhibits the same asymptotic behavior as that of $\mathcal{Z} = (Z_n)_{n\in\mathbb{Z}_+}$. Let

$$M_n = \max\{X_1, \ldots, X_n\}, \quad n \in \mathbb{N}, \tag{2.5}$$

and

$$b_n = \inf\{t > 0 : h(t) \geqslant n\}, \tag{2.6}$$

where $h(t)$ is the function introduced in Assumption 2.1.

The proof of the following theorem is given in Subsection 3.2 below.

Theorem 2.6. *Let Assumption* 2.1 *hold. Then, under the law* P_0, $M_n/b_n \Rightarrow M_\infty$, *where* M_∞ *is a proper random variable with the following distribution function*:

$$P_0(M_\infty > x) = \mathrm{e}^{-x^{-1/\alpha}}, \quad x > 0,$$

where $\alpha > 0$ *is the constant introduced in Assumption* 2.1.

The distribution of M_∞ belongs to the class of the so-called Fréchet extreme value distributions and in fact (see, for instance, [14, Subsection 3.3]),

$$P_0(M_\infty > x) = \lim_{n\to\infty} P\Big(\max_{1\leqslant k\leqslant n} Z_k > xb_n\Big), \quad x > 0.$$

It is quite remarkable that the distribution of ϕ_0 does not play any role in the result of Theorem 2.6. An intuitive explanation for this phenomenon, which can be derived from the proof, is as follows. Due to the basic property of regular variation, two independent terms $\phi_n \circ X_{n-1}$ and Z_n are unlikely to "help" each other in creating a large value of the sum $X_{n+1} = \phi_n \circ X_{n-1} + Z_n$. Moreover, the law of large numbers ensures that the ratio $\phi_n \circ X_{n-1}/X_{n-1}$ is bounded away from one with an overwhelming probability whenever $\phi_n \circ X_{n-1}$ is large. Therefore, the asymptotic of the extreme value of the sequence X_n follows that of Z_n.

Regenerative structure of $\boldsymbol{\mathcal{X}}$. Let

$$\nu_0 = 1 \quad \text{and} \quad \nu_n = \inf\{i > \nu_{n-1} : \phi_i \circ X_{i-1} = 0\}, \tag{2.7}$$

with the usual convention that the infimum over an empty set is ∞. We will refer to ν_n as a *regeneration time* and to the time elapsing from ν_{n-1} until $\nu_n - 1$ as *the n-th renewal epoch*. In the language of branching processes, at the regeneration times the extinction occurs and and the process starts again with the next wave of the immigration. For $n \in \mathbb{N}$, let

$$\sigma_n = \nu_n - \nu_{n-1} \quad \text{and} \quad R_n = (X_i : \nu_{n-1} \leqslant i < \nu_n)$$

be, respectively, the length of the n-th renewal epoch and the list of the values of X_i recorded during the n-th renewal epoch.

The proof of the following proposition is given in the Appendix.

Proposition 2.7. *Let Assumption* 2.2 *hold. Then,*

(a) $P_0(\nu_n < \infty) = 1$ *for all* $n \in \mathbb{N}$. *Moreover, the pairs* $(\sigma_n, R_n)_{n\in\mathbb{N}}$ *form an i.i.d. sequence.*

(b) *There exist positive constants* $K_1 > 0$ *and* $K_2 > 0$ *such that*

$$P_0(\sigma_1 > t) \leqslant K_1 \mathrm{e}^{-K_2 t}, \quad \forall\, t > 0. \tag{2.8}$$

While the first part of the proposition is a standard Markov chain exercise, the exponential bound in (2.8) is a delicate result. A similar bound has been proved for a general type of branching processes with immigration in [27]. An argument which is due to Kozlov and which has been adapted for the proof of [27, Theorem 4.2] goes through almost verbatim for our setting. We provide a suitable variation of this argument in the appendix.

The existence of the "life-cycles" (i.e., renewal epochs) for the branching process implies, for instance, the following. Recall $X_{k,n}$ from (2.4). Let

$$\lambda_n = n - \max\{k < n : X_{k,n} > 0\} \quad \text{and} \quad \eta_n = \frac{\sum_{k=1}^n X_{k,n} \cdot (n-k)}{\sum_{k=1}^n X_{k,n}}$$

be, respectively, the maximal and the average age of the individuals present at generation n (see the above footnote remark on page 178).

Proposition 2.8. *Let Assumption* 2.2 *hold. Then both* λ_n *and* η_n *converge weakly, as* $n \to \infty$, *to proper distributions. More precisely, under the law* P_0,

$$\lambda_n \Rightarrow \sigma_1 \cdot U \quad \text{and} \quad \eta_n \Rightarrow -\frac{\sum_{k=-\infty}^{0} X_{k,0} \cdot k}{\sum_{k=-\infty}^{0} X_{k,0}},$$

where U *is a random variable which is independent of* σ_1 *and is distributed uniformly over the interval* $[0,1]$.

The first result in the above proposition is a direct implication of the renewal theorem whereas the second one is a consequence of the explicit formula for η_n given above and the fact that $X_{k,0} = 0$ for $k < \nu_{-1}$ and $P(\nu_{-1} < \infty) = 1$. Here ν_{-1} is time of the last renewal up to time zero for the process starting at $-\infty$. We leave details to the reader.

Another interesting implication of the existence of the regenerative structure is the convergence in distribution of the coalescence time at generation n. Suppose that $X_n > 2$ and sample at random two individuals present at generation n. Then the coalescence time T_n is defined as $n - k$ if the immigrant ancestors of both individuals have entered the system at the same time $k \in \mathbb{Z}_+$, and is set to be infinity otherwise (cf., for instance, [30]). Since the probability of sampling of both individuals among the descendants of the immigration wave k is $\frac{X_{k,n}(X_{k,n}-1)}{X_n(X_n-1)}$,

$$P_0(T_n \leqslant t) = E\left[\frac{\sum_{k=n-t}^{n} X_{k,n}(X_{k,n}-1)}{\sum_{k=1}^{n} X_{k,n}(\sum_{k=1}^{n} X_{k,n}-1)}\right].$$

We have thus obtained the following proposition:

Proposition 2.9. *Let Assumption* 2.2 *hold. Then* T_n *converges weakly under* P_0, *as* $n \to \infty$, *to a proper random variable with the following distribution function on* $\mathbb{N} \cup \{0, +\infty\}$:

$$F(t) = E\left[\frac{\sum_{k=-t}^{0} X_{k,0}(X_{k,0}-1)}{\sum_{k=-\infty}^{0} X_{k,0}(\sum_{k=-\infty}^{0} X_{k,0}-1)}\right], \quad t < \infty,$$

where $\frac{0}{0}$ *inside the expectation is interpreted as* 0.

Growth rate and fluctuations of the partial sums of $\boldsymbol{\mathcal{X}}$**.** Let $S_n = \sum_{k=1}^{n} X_k$. The following law of large numbers is a direct consequence of Corollary 2.4.

Proposition 2.10. *Let Assumption* 2.2 *hold with* $\beta = 1$. *Then*

$$\lim_{n\to\infty} \frac{S_n}{n} = E[X_0] = \frac{E[Z_0]}{1 - E[\phi_0]}, \quad P_0\text{-}a.s.$$

The next theorem is concerned with the rate of the growth of the partial sums when Z_0 has infinite mean. For $\alpha \in (0,2]$ and $b > 0$ denote by $\mathcal{L}_{\alpha,b}$ the strictly asymmetric stable law of index α with the characteristic function

$$\log \widehat{\mathcal{L}}_{\alpha,b}(t) = -b|t|^{\alpha}\left(1 + \mathrm{i}\frac{t}{|t|} f_\alpha(t)\right), \tag{2.9}$$

where $f_\alpha(t) = -\tan \frac{\pi}{2}\alpha$ if $\alpha \neq 1$, $f_1(t) = 2/\pi \log t$. With a slight abuse of notation we use the same symbol for the distribution function of this law. If $\alpha < 1$, $\mathcal{L}_{\alpha,b}$ is supported on the positive reals, and if $\alpha \in (1,2]$, it has zero mean [14, Section 2.2].

Recall b_n from (2.6). The following result is proved in Subsection 3.3 below by using an approximation of the partial sums of the process by those of a stationary strongly mixing sequence for which we are able to verify the conditions of a general stable limit theorem.

Theorem 2.11. *Let Assumption* 2.1 *hold with* $\alpha \in (0,1)$. *Then* $b_n^{-1} S_n \Rightarrow \mathcal{L}_{\alpha,b}$.

We next study the fluctuations of the partial sums in the case where nontrivial centering of X_n is required to obtain a proper weak limit for the partial sums.

Theorem 2.12. *Let Assumption* 2.1 *hold with* $\alpha \in [1,2]$. *For* $n \in \mathbb{N}$, *define*

$$a_n = \begin{cases} b_n, \ \text{where } b_n \text{ is defined in } (2.6), & \text{if } \alpha < 2, \\ \inf\{t > 0 : nt^{-2} \cdot E[X_0^2;\, X_0 \leqslant t] \leqslant 1\}, & \text{if } \alpha = 2. \end{cases}$$

Denote $\mu := E[X_0]$. *Then the following holds for some* $b > 0$:

(i) *If* $\alpha = 1$, *then* $a_n^{-1}(S_n - c_n) \Rightarrow \mathcal{L}_{1,b}$ *with* $c_n = nE[X_0; X_0 \leqslant a_n]$.

(ii) *If* $\alpha \in (1,2)$, *then* $a_n^{-1}(S_n - n\mu) \Rightarrow \mathcal{L}_{\alpha,b}$.

(iii) *If* $\alpha = 2$ *and* $E[Z_0^2] = \infty$, *then* $a_n^{-1}(S_n - n\mu) \Rightarrow \mathcal{L}_{2,b}$.

Recall ν_n from (2.7) and define

$$W_n = \sum_{i=\nu_{n-1}}^{\nu_n - 1} X_i, \quad n \in \mathbb{N}.$$

Theorem 2.12 can be derived from stable limit theorems for partial sums of i.i.d. variables, using the regenerative structure and the following lemma.

Lemma 2.13. *Let Assumption* 2.1 *hold. Then the following limit exists*:

$$\lim_{t\to\infty} h(t) \cdot P_0(W_1 > t).$$

Moreover, the limit is finite and strictly positive.

The proof of the lemma given below in Subsection 3.4 is (although technical details are quite different) along the line of the proof of a similar result given for a different branching process in [26]. We remark that though a similar technique can be used to obtain Theorem 2.11, we prefer to employe a more direct approach in the case $\alpha \in (0,1)$. Theorem 2.12 follows from the above lemma by using a standard argument, which is outlined in [26] for the case $h(x) = x^{-1}$. Since only an obvious minor modification is required to extend the argument to a general h (see, for instance, [32] for $h(x) = x^{-2}$), we omit details of this argument here.

If an appropriate second moment condition is assumed, one can establish the following functional limit theorem for normalized partial sums of $\mathcal{X}$. Let $D(\mathbb{R}_+, \mathbb{R})$ denote the set of real-valued càdlàg functions on $\mathbb{R}_+ := [0, \infty)$, endowed with the Skorokhod J_1-topology. Let $\lfloor x \rfloor$ denote the integer part of $x \in \mathbb{R}$. We have the following theorem:

Theorem 2.14. *Let Assumption* 2.2 *hold with a constant* $\beta > 2$. *Then, as* $n \to \infty$, *the sequence of processes*

$$S_t^{(n)} = n^{-1/2}(S_{\lfloor nt \rfloor} - nt\mu), \quad t \in [0,1],$$

in $D(\mathbb{R}_+, \mathbb{R})$ *converges weakly to a non-degenerate Brownian motion* W_t, $t \in [0,1]$.

Theorem 2.14 is a particular case of [40, Theorem 1.5], and therefore its proof is omitted. Notice that the conditions of the theorem are satisfied if Assumption 2.1 holds with $\alpha > 2$.

3 Proof of the main results

This section is devoted to the proof of the theorems stated in Section 2 (namely, Theorems 2.5, 2.6, 2.11 and 2.12), and is divided into four subsections correspondingly.

3.1 Proof of Theorem 2.5

First, we observe the following lemma:

Lemma 3.1. *Let $X \in \mathcal{N}_+$ be a random variable in the underlying probability space such that*

(i) *X is independent of $(\phi_n, Z_n, B_n)_{n\in\mathbb{Z}_+}$, where $B_n := (B_{n,k})_{k\in\mathbb{N}}$.*

(ii) *$\lim_{t\to\infty} h(t) \cdot P_\Phi(X > t) = 1$ for some $h \in \mathcal{R}_\alpha$, $\alpha > 0$.*

Then $\lim_{t\to\infty} h(t) \cdot P_\Phi(\phi_0 \circ X > t) = \phi_0^\alpha$.

Proof. Fix a constant $\varepsilon \in (0, 1)$. For $t > 0$ define the following three events:

$$A_{t,\varepsilon} = \{X > t \cdot (\phi_0^{-1} + \varepsilon)\},$$
$$B_{t,\varepsilon} = \{t \cdot (\phi_0^{-1} - \varepsilon) < X \leqslant t \cdot (\phi_0^{-1} + \varepsilon)\},$$
$$C_{t,\varepsilon} = \{X \leqslant t \cdot (\phi_0^{-1} - \varepsilon)\}.$$

We will use the following splitting formula:

$$P_\Phi(\phi_0 \circ X > t) = P_\Phi(\phi_0 \circ X > t;\, A_{t,\varepsilon}) + P_\Phi(\phi_0 \circ X > t;\, B_{t,\varepsilon}) + P_\Phi(\phi_0 \circ X > t;\, C_{t,\varepsilon}).$$

By the law of large numbers,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} B_{1,k} = \phi_0, \quad \text{P-a.s.}$$

Since $h(t)$ is regularly varying, Chernoff's bound (Cramér's large deviation theorem for coin flipping, see [12]) applied to the partial sums $\sum_{k=1}^n B_k$ implies that

$$0 \leqslant \limsup_{t\to\infty} h(t) \cdot P_\Phi(\phi_0 \circ X > t;\, C_{t,\varepsilon}) \leqslant \limsup_{t\to\infty} h(t) \cdot P_\Phi\Big(\sum\nolimits_{k=1}^{\lfloor t(\phi_0^{-1}-\varepsilon)\rfloor} B_k > t\Big) = 0.$$

Next, by the conditions of the lemma,

$$\begin{aligned}\lim_{t\to\infty} h(t) \cdot P_\Phi(\phi_0 \circ X > t;\, B_{t,\varepsilon}) &\leqslant \lim_{t\to\infty} h(t) \cdot P_\Phi(B_{t,\varepsilon}) \\ &= [(\phi_0^{-1} - \varepsilon)^{-\alpha} - (\phi_0^{-1} + \varepsilon)^{-\alpha}] \to_{\varepsilon\to 0} 0.\end{aligned}$$

Finally, using again the large deviation principle for $\sum_{k=1}^n B_k$,

$$\begin{aligned}\liminf_{t\to\infty} h(t) \cdot P_\Phi(\phi_0 \circ X > t;\, A_{t,\varepsilon}) &= \liminf_{t\to\infty} h(t) \cdot [P_\Phi(A_{t,\varepsilon}) - P_\Phi(\phi_0 \circ X \leqslant t;\, A_{t,\varepsilon})] \\ &\geqslant \liminf_{t\to\infty} h(t) \cdot P_\Phi(A_{t,\varepsilon}) = (\phi_0^{-1} + \varepsilon)^{-\alpha}.\end{aligned}$$

On the other hand, clearly,

$$\liminf_{t\to\infty} h(t) \cdot P_\Phi(\phi_0 \circ X > t;\, A_{t,\varepsilon}) \leqslant \liminf_{t\to\infty} h(t) \cdot P_\Phi(A_{t,\varepsilon}) = (\phi_0^{-1} + \varepsilon)^{-\alpha}.$$

Since $\varepsilon > 0$ is arbitrary and $(\phi_0^{-1}+\varepsilon)^{-\alpha} \to \phi_0^\alpha$ as ε goes to zero, this completes the proof of the lemma. □

Remark 3.2. The above proof of Lemma 3.1 can be adopted without modification for a more general type of sums $\sum_{k=1}^X B_k$, where $X \in \mathcal{N}_+$ has regularly varying distribution tails and $(B_k)_{k\in\mathbb{N}}$ are independent of X. In fact, the only property of the sequence B_k required by the proof is the availability of a non-trivial large deviations upper bound for its partial sums. Note that if $f(\lambda) := E_\Phi[\mathrm{e}^{\lambda B_1}]$ is finite in a neighborhood of zero, such a bound in the form $P_\Phi(|\frac{1}{n}\sum_{k=1}^n B_k - E_\Phi[B_1]| > x) \leqslant c(x)\mathrm{e}^{-nI(x)}$ with suitable constants $c(x), I(x) > 0$ holds for any $x > 0$ (see, for instance, the first inequality in the proof of [12, Lemma 2.2.20]).

Recall (see, for instance, [14, Lemma 1.3.1]) that if X and Y are two independent random variables such that $\lim_{x\to\infty} h(x) \cdot P(X > x) = c_1 > 0$ and $\lim_{x\to\infty} h(x) \cdot P(Y > x) = c_2 > 0$ for some $h \in \mathcal{R}_\alpha$, $\alpha > 0$, then

$$\lim_{x\to\infty} h(x) \cdot P(X + Y > x) = c_1 + c_2. \tag{3.1}$$

Using this property and iterating (1.1), one can deduce from Lemma 3.1 the following corollary. Consider (in an enlarged probability space, if needed) a sequence $\widetilde{\mathcal{X}} = (\widetilde{X}_n)_{n\in\mathbb{Z}_+}$ which solves (1.1), that is a sequence such that

$$\widetilde{X}_n = \sum_{k=1}^{\widetilde{X}_{n-1}} B_{n,k} + Z_n, \quad n \in \mathbb{N}, \tag{3.2}$$

for some initial (not necessarily equal to zero) random value $\widetilde{X}_0$.

Corollary 3.3. *Let Assumption* 2.1 *hold and suppose in addition that the following two conditions are satisfied*:

(i) $\widetilde{X}_0$ *is independent of* $(\phi_k, B_k, Z_k)_{k>0}$, *where* $B_k = (B_{k,j})_{j\in\mathbb{N}}$.

(ii) $\lim_{t\to\infty} h(t)\cdot P_\Phi(\widetilde{X}_0 > t) = c_0$ *for some random variable* $c_0 = c_0(\Phi)$.

Then $\lim_{t\to\infty} h(t)\cdot P_\Phi(\widetilde{X}_n > t) = c_n$ *for any* $n\in\mathbb{N}$, *where the random variables* $c_n = c_n(\Phi)$ *are defined recursively by*

$$c_{n+1} = c_n\phi_{n+1}^{\alpha} + 1, \quad n\in\mathbb{Z}_+. \tag{3.3}$$

The recursive relation (3.3) implies that

$$c_n = \bar{c}_n + c_0\prod_{j=1}^{n}\phi_j^{\alpha}, \quad \text{where} \quad \bar{c}_n = 1 + \sum_{k=2}^{n}\prod_{j=k}^{n}\phi_j^{\alpha}, \tag{3.4}$$

and hence (see, for instance, [7, Theorem 1]) the random variables c_n converge in distribution, as $n\to\infty$, to $c_\infty := 1 + \sum_{k=0}^{\infty}\prod_{i=0}^{k}\phi_{-i}^{\alpha}$. Furthermore, we have the following:

Corollary 3.4. *Suppose that the conditions of Corollary* 3.3 *are satisfied and, in addition, there exists a positive constant* $C>0$ *such that the following holds*:

$$P\Big(\sup_{t>0}\{h(t)\cdot P_\Phi(\widetilde{X}_0 > t)\} < C\Big) = 1.$$

Then the following limit exists and the identity holds:

$$\lim_{n\to\infty} h(t)\cdot P(\widetilde{X}_n > t) = E[c_n], \quad n\in\mathbb{N},$$

where c_n *are random variables defined in* (3.4).

Proof. Corollary 3.3 and the bounded convergence theorem imply that

$$\begin{aligned}\lim_{t\to\infty} h(t)\cdot P(\widetilde{X}_n > t) &= \lim_{t\to\infty} h(t)\cdot E[P_\Phi(\widetilde{X}_n > t)]\\ &= E\Big[\lim_{t\to\infty} h(t)\cdot P_\Phi(\widetilde{X}_n > t)\Big] = E[c_n].\end{aligned} \tag{3.5}$$

To justify interchanging of the limit with the expectation, observe that $\widetilde{X}_n \leqslant \widetilde{X}_0 + \sum_{k=1}^{n} Z_k$ and hence, by virtue of (A3) in Assumption 2.1, the following inequalities hold with probability one for some positive constant $C_1>0$:

$$\begin{aligned}h(t)\cdot P_\Phi(\widetilde{X}_n > t) &\leqslant h(t)\cdot P_\Phi(\widetilde{X}_0 > t/2) + h(t)\cdot P_\Phi\Big(\sum_{k=1}^{n} Z_k > t/2\Big)\\ &\leqslant h(t)\cdot P_\Phi(\widetilde{X}_0 > t/2) + nh(t)\cdot P(Z_0 > t/(2n))\\ &\leqslant C\frac{h(t)}{h(t/2)} + C_1 n\frac{h(t)}{h(t/(2n))}.\end{aligned}$$

It follows (see, for instance, [19, Lemma 1]) that there exists a constant $C_2>0$ such that

$$P\Big(\sup_{t>t_0}\{h(t)\cdot P_\Phi(\widetilde{X}_n > t)\} < C_2\Big) = 1.$$

This enables one to apply the bounded convergence theorem in (3.5) and thus completes the proof of the corollary. □

In what follows notations $X \leqslant_D Y$ and $X \geqslant_D Y$ for random variables X and Y are used to indicate that $P(X > t) \leqslant P(Y > t)$ or, respectively, $P(X > t) \geqslant P(Y > t)$ holds for all $t \in \mathbb{R}$. In order to exploit Corollary 3.4 in the proof of Theorem 2.5, we need the following lemma:

Lemma 3.5. *Suppose that the conditions of Corollary* 3.4 *are satisfied. Then*
(a) *If* $\widetilde{X}_0 \leqslant_D \widetilde{X}_1$*, then* $\widetilde{X}_n \leqslant_D \widetilde{X}_{n+1}$ *for all* $n \in \mathbb{N}$.
(b) *If* $\widetilde{X}_0 \geqslant_D \widetilde{X}_1$*, then* $\widetilde{X}_n \geqslant_D \widetilde{X}_{n+1}$ *for all* $n \in \mathbb{N}$.

Proof. The proof is by induction. Suppose first that $\widetilde{X}_{n-1} \leqslant_D \widetilde{X}_n$ for some $n \in \mathbb{N}$, $\widetilde{X}_{n-1}$ is independent of $(\phi_k, B_k, Z_k)_{k>n-1}$, and $\widetilde{X}_n$ is independent of $(\phi_k, B_k, Z_k)_{k>n}$. We will now use the following standard trick to construct an auxiliary random pair (V_{n-1}, V_n) such that

$$P(V_{n-1} \leqslant V_n = 1), \quad V_{n-1} =_P \widetilde{X}_{n-1}, \quad \text{and} \quad V_n =_P \widetilde{X}_n. \tag{3.6}$$

Let U be a uniform random variable on $[0,1]$, independent of the random coefficients sequence $(\Phi, \mathcal{Z})$. Denote by F_n and F_{n-1}, respectively, the distribution functions of X_n and X_{n-1}. Set $V_n = F_n^{-1}(U)$ and $V_{n-1} = F_{n-1}^{-1}(U)$, where $F^{-1}(y) := \inf\{x \in \mathbb{R} : F(x) \geqslant y\}$, $y \in [0,1]$, with the convention that $\inf \emptyset = \infty$.

Let $\widetilde{X}_{n+1} = \phi_{n+1} \circ \widetilde{X}_n + Z_{n+1}$. Then $\widetilde{X}_{n+1}$ is independent of $(\phi_k, B_k, Z_k)_{k>n+1}$. Furthermore, since (V_{n-1}, V_n) is independent of $(\Phi, \mathcal{Z})$, we obtain for any $t > 0$,

$$\begin{aligned} P(\widetilde{X}_{n+1} > t) &= P(\phi_{n+1} \circ \widetilde{X}_n + Z_{n+1} > t) = P(\phi_{n+1} \circ V_n + Z_{n+1} > t) \\ &\geqslant P(\phi_{n+1} \circ V_{n-1} + Z_{n+1} > t) = P(\phi_n \circ V_{n-1} + Z_n > t) \\ &= P(\phi_n \circ \widetilde{X}_{n-1} + Z_n > t) = P(\widetilde{X}_n > t). \end{aligned} \tag{3.7}$$

This shows that part (a) of the lemma holds true. The same argument, but with $\leqslant$ replaced by $\geqslant$ and vice versa in the base of induction, (3.6) and (3.7), yields part (b). □

We are now in a position to complete the proof of Theorem 2.5. First, we have the following lemma:

Lemma 3.6. *There exists a random variable* $\widetilde{X}_0 \geqslant 0$ *satisfying the conditions of Corollary* 3.4*, such that* $\widetilde{X}_1 \geqslant_D \widetilde{X}_0$.

Proof. Set $\widetilde{X}_0 = Z_{-1}$. □

In view of Lemma 3.5, this implies that we can find a sequence $\widetilde{X}_n$ that solves (1.1) and such that $\widetilde{X}_n \leqslant_D \widetilde{X}_\infty$, while $\widetilde{X}_0$ satisfies the conditions of Corollary 3.4. Combining this result with the conclusion of the corollary yields

$$\liminf_{t\to\infty} h(t) \cdot P(X_\infty > t) \geqslant \lim_{t\to\infty} h(t) \cdot P(\widetilde{X}_n > t) = E[c_n], \quad n \in \mathbb{N}.$$

Hence

$$\liminf_{t\to\infty} h(t) \cdot P(X_\infty > t) \geqslant \lim_{n\to\infty} E[c_n] = \frac{1}{1 - E[\phi_0^\alpha]}. \tag{3.8}$$

On the other hand, we have the following lemma:

Lemma 3.7. *Let Assumption* 2.1 *hold. There exists a random variable* $\widetilde{X}_0 \geqslant 0$ *satisfying the conditions of Lemma* 3.1 *such that* $\widetilde{X}_1 \leqslant_D \widetilde{X}_0$.

Proof. Given a realization of the sequence Φ, choose a constant c_0 in such a way that

$$c_0 > \frac{1}{1 - E[\phi_0^\alpha]}.$$

Let $Y_0 = c_0^{1/\alpha} Z_{-1}$. Then $\lim_{t\to\infty} h(t) \cdot P(Y_0 > t) = c_0$. If we would choose $\widetilde{X}_0 = Y_0$, we would have $c_1 := \lim_{t\to\infty} h(t) \cdot P(\widetilde{X}_1 > t) < c_0$ by virtue of (3.3) and Corollary 3.4. This would imply that $P(\widetilde{X}_1 > t) < P(\widetilde{X}_0 > t)$ for $t > t_0$, where $t_0 > 0$ is a positive constant which depends on c_0. Consider now (in an enlarged probability space, if needed) a random variable $\widetilde{X}_0$ such that $\widetilde{X}_0$ is independent of $(\phi_k, B_k, Z_k)_{k\in\mathbb{Z}}$ and

$$P(\widetilde{X}_0 > t) = P(Y_0 > t \mid Y_0 > t_0).$$

Note that such $\widetilde{X}_0$ satisfies the conditions of Corollary 3.4 because $P_\Phi(\widetilde{X}_0 > t) = P(\widetilde{X}_0 > t)$ with probability one, and for $t > t_0$,

$$h(t) \cdot P(\widetilde{X}_0 > t) \leqslant \frac{1}{P(c_0^\alpha Z_0 > t_0)} \cdot \frac{h(t)}{h(tc_0^{-\alpha})}(h(tc_0^{-\alpha}) \cdot P(Z_0 > tc_0^{-\alpha})),$$

and $\sup_{t>0} h(t)/h(tc_0^{-\alpha}) < \infty$ (see, for instance, [19, Lemma 1]). Then, for $t > t_0$,

$$\begin{aligned} P(\phi_1 \circ \widetilde{X}_0 + Z_1 > t) &= P(\phi_1 \circ Y_0 + Z_1 > t \mid Y_0 > t_0) \\ &= \frac{P(\phi_1 \circ Y_0 + Z_1 > t; Y_0 > t_0)}{P(Y_0 > t_0)} \leqslant \frac{P(\phi_1 \circ Y_0 + Z_1 > t)}{P(Y_0 > t_0)} \\ &\leqslant \frac{P(Y_0 > t)}{P(Y_0 > t_0)} = P(Y_0 > t \mid Y_0 > t_0) = P(\widetilde{X}_0 > t). \end{aligned}$$

On the other hand, if $t \leqslant t_0$ then

$$P(\widetilde{X}_0 > t) = P(\widetilde{X}_0 > t \mid \widetilde{X}_0 > t_0) = 1.$$

Thus

$$P(\phi_1 \circ \widetilde{X}_0 + Z_1 > t) \leqslant P(\widetilde{X}_0 > t)$$

for all $t > t_0$, and we can set $\widetilde{X}_0$ as the initial value for the recursion. □

Combining this result with Corollary 3.4 yields

$$\limsup_{t\to\infty} h(t) \cdot P(X_\infty > t) \leqslant \lim_{t\to\infty} h(t) \cdot P_0(X_n > t) = E[c_n], \quad n \in \mathbb{N}.$$

Hence,

$$\limsup_{t\to\infty} h(t) \cdot P(X_\infty > t) \leqslant \lim_{n\to\infty} E[c_n] = \frac{1}{1 - E[\phi_0^\alpha]}.$$

The proof of Theorem 2.5 is completed in view of (3.8).

3.2 Proof of Theorem 2.6

For $n \in \mathbb{N}$, denote $K_n = \max_{1\leqslant k\leqslant n} Z_k$. It follows from (1.1) that $M_n \geqslant_D K_n$. To conclude the proof of the theorem, it thus suffices to show that

$$\limsup_{n\to\infty} P_0(M_n > xb_n) \leqslant \lim_{n\to\infty} P_0(K_n > xb_n) = \mathrm{e}^{-x^{-1/\alpha}}, \quad x > 0.$$

Observe that, under the stationary law P, the branching process (without immigration) originated by the initial X_0 individuals will eventually die out. Therefore, the total number of progeny of the individuals in the zero generation is P-a.s. finite. Furthermore, the branching process $X_n - \sum_{k=-\infty}^0 X_{k,n}$, $n \in \mathbb{N}$, obtained by excluding the contribution of these individuals from the original one, is distributed under P as X_n, $n \in \mathbb{N}$, under P_0. It thus suffices to show that

$$\limsup_{n\to\infty} P(M_n > xb_n) \leqslant \lim_{n\to\infty} P(K_n > xb_n) = \mathrm{e}^{-x^{-1/\alpha}}, \quad x > 0.$$

Toward this end, define the following events. For $x > 0$, $\delta > 0$, and $\varepsilon \in (0, 1/2)$, let

$$A_{x,\delta}^{(n)} = \{xb_n < M_n \leqslant x(1+\delta)b_n\}, \qquad n \in \mathbb{N},$$

$$
\begin{aligned}
B^{(n)}_{x,\delta,\varepsilon} &= A^{(n)}_{x,\delta} \cap \{x(1-\varepsilon)b_n < K_n \leqslant x(1+\delta)b_n\}, && n \in \mathbb{N}, \\
C^{(n,k)}_{x,\delta,\varepsilon} &= A^{(n)}_{x,\delta} \cap \{X_k > xb_n,\ \varepsilon x b_n < Z_k \leqslant x(1-\varepsilon)b_n\}, && n \in \mathbb{N}, \quad k = 1,2,\dots,n, \\
D^{(n,k)}_{x,\delta,\varepsilon} &= A^{(n)}_{x,\delta} \cap \{X_k > xb_n,\ Z_k \leqslant x\varepsilon b_n\}, && n \in \mathbb{N}, \quad k = 1,2,\dots,n.
\end{aligned}
$$

Then

$$
\begin{aligned}
P(A^{(n)}_{x,\delta}) &\leqslant P(B^{(n)}_{x,\delta,\varepsilon}) + P\bigg(\bigcup_{k=1}^n C^{(n,k)}_{x,\delta,\varepsilon}\bigg) + P\bigg(\bigcup_{k=1}^n D^{(n,k)}_{x,\delta,\varepsilon}\bigg) \\
&\leqslant P(x(1-\varepsilon)b_n < K_n \leqslant x(1+\delta)b_n) + nP(C^{(n,1)}_{x,\delta,\varepsilon}) + nP(D^{(n,1)}_{x,\delta,\varepsilon}).
\end{aligned} \tag{3.9}
$$

Taking into account the independence of the pair (ϕ_k, X_{k-1}) of Z_k, it follows from (1.2), Assumption 2.1, and Lemma 3.1 that for any positive constants $\delta, x, \varepsilon > 0$

$$
\limsup_{n\to\infty} n\,P(C^{(n,1)}_{x,\delta,\varepsilon}) \leqslant \lim_{n\to\infty} n\,P(\phi_1 \circ X_0 > \varepsilon x b_n, Z_1 > \varepsilon x b_n) = 0. \tag{3.10}
$$

Furthermore,

$$
\begin{aligned}
P(D^{(n,1)}_{x,\delta,\varepsilon}) &\leqslant P(\phi_1 \circ X_0 > (1-\varepsilon)xb_n, X_0 \leqslant x(1+\delta)b_n) \\
&\leqslant P(\phi_1 \circ X_0 > (1-\varepsilon)xb_n | X_0 \leqslant x(1+\delta)b_n) \\
&\leqslant P\bigg(\sum_{i=1}^{\lfloor x(1+\delta)b_n \rfloor} B_{0,i} > (1-\varepsilon)xb_n\bigg) \\
&= E\bigg[P_\Phi\bigg(\frac{1}{x(1+\delta)b_n}\sum_{i=1}^{\lfloor x(1+\delta)b_n \rfloor} B_{0,i} > \frac{1-\varepsilon}{1+\delta}\bigg)\bigg].
\end{aligned} \tag{3.11}
$$

Assume now that the constants $\delta > 0$ and $\varepsilon > 0$ are chosen so small that $\frac{1-\varepsilon}{1+\delta} > E[\phi_0]$, and hence

$$
\frac{1-\varepsilon}{1+\delta} > \eta E[\phi_0] \quad \text{for some} \quad \eta > 1. \tag{3.12}
$$

We next derive a simple large-deviations type upper bound for the right-most expression in (3.11). Denote $x_0 = \frac{1-\varepsilon}{1+\delta}$. It follows from Chebyshev's inequality that for any $\lambda > 0$,

$$
E\bigg[P_\Phi\bigg(\frac{1}{n}\sum_{i=1}^n B_{0,i} > \frac{1-\varepsilon}{1+\delta}\bigg)\bigg] \leqslant \mathrm{e}^{-n\lambda x_0} E[(1-\phi_0+\phi_0 \mathrm{e}^\lambda)^n].
$$

Thus for all $\lambda > 0$ small enough, namely for all $\lambda > 0$ such that $\mathrm{e}^\lambda < 1 + \eta\lambda$, we have

$$
\begin{aligned}
E\bigg[P_\Phi\bigg(\frac{1}{n}\sum_{i=1}^n B_{0,i} > \frac{1-\varepsilon}{1+\delta}\bigg)\bigg] &\leqslant \mathrm{e}^{-n\lambda x_0} E[(1-\phi_0+\phi_0(1+\eta\lambda))^n] \\
&= \mathrm{e}^{-n\lambda x_0} E[(1+\phi_0\eta\lambda)^n] \leqslant \mathrm{e}^{-n\lambda x_0} E[\mathrm{e}^{\phi_0 \cdot n\eta\lambda}].
\end{aligned}
$$

Therefore, for all $\lambda > 0$ small enough we have

$$
\limsup_{n\to\infty} \frac{1}{n} \log E\bigg[P_\Phi\bigg(\frac{1}{n}\sum_{i=1}^n B_{0,i} > \frac{1-\varepsilon}{1+\delta}\bigg)\bigg] \leqslant -\lambda x_0 + \log E[\mathrm{e}^{\eta\lambda\phi_0}].
$$

Given η, let $f(\lambda) = \log E[\mathrm{e}^{\eta\lambda\phi_0}]$. By the bounded convergence theorem, $f'(0) = \eta E[\phi_0]$. Hence, in view of (3.12),

$$
\limsup_{n\to\infty} \frac{1}{n} \log P\bigg(\frac{1}{n}\sum_{i=1}^n B_{0,i} > \frac{1-\varepsilon}{1+\delta}\bigg) < 0.
$$

Since b_n is a regularly varying sequence, it follows from (3.11) that

$$\lim_{n\to\infty} n\, P(D^{(n,1)}_{x,\delta,\varepsilon}) = 0. \tag{3.13}$$

Therefore, since $\varepsilon > 0$ above can be made arbitrary small (in particular, the left-hand side of (3.12) is an increasing function of ε), combining (3.13) together with (3.10) and (3.11) yields,

$$\limsup_{n\to\infty} P(A^{(n)}_{x,\delta}) \leqslant P(xb_n < K_n \leqslant x(1+\delta)b_n),$$

and hence

$$\begin{aligned}\limsup_{n\to\infty} P(M_n > xb_n) &= \limsup_{n\to\infty} \sum_{k=0}^{\infty} P((1+k\delta)xb_n < M_n \leqslant (1+k\delta+\delta)xb_n) \\ &\leqslant \sum_{k=0}^{\infty} \limsup_{n\to\infty} P((1+k\delta)xb_n < M_n \leqslant (1+k\delta+\delta)xb_n) \\ &\leqslant \sum_{k=0}^{\infty} P((1+k\delta)xb_n < K_n \leqslant (1+k\delta+\delta)xb_n) = P(K_n > xb_n).\end{aligned}$$

The proof of Theorem 2.6 is complete.

3.3 Proof of Theorem 2.11

For $n \in \mathbb{Z}$, let

$$Y_n = \sum_{t=n}^{\infty} X_{n,t} \tag{3.14}$$

be the total number of progeny at all generations of all the immigrants entered at time n, including the immigrants themselves. Then

$$\sum_{k=1}^{n} X_k = \sum_{k=1}^{n}\sum_{t=0}^{k} X_{t,k} = \sum_{t=0}^{n}\sum_{k=t}^{n} X_{t,k} = \sum_{t=0}^{n}\left(\sum_{k=t}^{\infty} X_{t,k} - \sum_{k=n+1}^{\infty} X_{t,k}\right) = \sum_{t=0}^{n} Y_t - \sum_{t=0}^{n}\sum_{k=n+1}^{\infty} X_{t,k}.$$

Notice that

$$\sum_{t=0}^{n}\sum_{k=n+1}^{\infty} X_{t,k} =_D \sum_{t=-n}^{0}\sum_{k=1}^{\infty} X_{t,k} \leqslant \sum_{t=-\infty}^{0}\sum_{k=1}^{\infty} X_{t,k} = \sum_{t=-\nu_{-1}}^{0}\sum_{k=1}^{\infty} X_{t,k} \leqslant \sum_{t=-\nu_{-1}}^{\nu_1} Y_t < \infty.$$

Hence, in order to show that S_n/b_n converges in distribution, it suffices to show that $b_n^{-1}\sum_{k=1}^{n} Y_k$ converges to the same limit. Note that the sequence $(Y_n)_{n\in\mathbb{Z}}$ has the same distribution under P_0 as it has under P.

The following series of technical lemmas will enable us to apply a general stable limit theorem (namely, [45, Theorem 1.1]; see also [28, Corollary 5.7]) to the partial sums of the sequence Y_n.

Lemma 3.8. *The sequence $(Y_n)_{n\in\mathbb{Z}}$ is strongly mixing, i.e.,* $\lim_{n\to\infty}\chi(n) = 0$, *where*

$$\chi(n) := \sup\{P(A\cap B) - P(A)P(B) : A \in \mathcal{F}^n, B \in \mathcal{F}_0\},$$

and $\mathcal{F}^n := \sigma(Y_i : i \geqslant n)$, $\mathcal{F}_n := \sigma(Y_i : i < n)$.

Proof. This is a variation of [40, Lemma 3.2]. For the sake of completeness, we give here a suitable modification of the argument. For $n \in \mathbb{Z}$, let $\mathcal{Y}_n$ and $\mathcal{Y}^n$ denote, respectively, the sequences $(Y_i)_{i<n}$ and $(Y_i)_{i\geqslant n}$. On one hand, for any $A \in \sigma(Y_i : i > n)$ and $B \in \sigma(Y_i :\leqslant 0)$,

$$P(\mathcal{Y}^n \in A, \mathcal{Y}_0 \in B) \geqslant P(\mathcal{Y}^n \in A, \mathcal{Y}_0 \in B, \nu_1 \leqslant n/2)$$

$$
\begin{aligned}
&= E[P_\Phi(\mathcal{Y}_0 \in B, \nu_1 \leqslant n/2) \cdot P_\Phi(\mathcal{Y}^n \in A)] \\
&\geqslant P(\mathcal{Y}_0 \in B, \nu_1 \leqslant n/2) \cdot P(\mathcal{Y}^n \in A) \\
&\geqslant P(\mathcal{Y}_0 \in B) \cdot P(\mathcal{Y}^n \in A) - P(\nu_1 > n/2).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
P(\mathcal{Y}^n \in A, \mathcal{Y}_0 \in B) &\leqslant P(\mathcal{Y}^n \in A, \mathcal{Y}_0 \in B, \nu_1 \leqslant n/2) + P(\nu_1 > n/2) \\
&= E[P_\Phi(\mathcal{Y}_0 \in B, \nu_1 \leqslant n/2) \cdot P_\Phi(\mathcal{Y}^n \in A)] + P(\nu_1 > n/2) \\
&\leqslant P(\mathcal{Y}_0 \in B, \nu_1 \leqslant n/2) \cdot P(\mathcal{Y}^n \in A) + P(\nu_1 > n/2) \\
&\leqslant P(\mathcal{Y}_0 \in B) \cdot P(\mathcal{Y}^n \in A) + P(\nu_1 > n/2).
\end{aligned}
$$

It thus remains to show that $P(\nu_1 < \infty) = 1$. By Proposition 2.7, we have $P_0(\nu_1 < \infty) = 1$. Since, clearly, $P(\phi_1 \circ X_0 = 0) > 0$, the strong Markov property implies $P(\nu_1 < \infty) > 0$. Since the Markov chain (X_n, Z_n) forms an ergodic process according to Corollary 2.4, it follows from the ergodic theorem that the the two-component Markov chain spends asymptotically a positive proportion of time at the set $\{X_n = Z_n\}$ (one can also appeal directly to the Poincaré recurrence theorem). This completes the proof of the lemma. □

In view of the previous lemma we are seeking to apply to Y_n the following general limit theorem for strongly mixing stationary sequences obtained in [45] (see also a similar [28, Corollary 5.7]).

Theorem 3.9 (See [45, Theorem 1.1 and Corollary 1.2])**.** *Let $(Y_n)_{n\in\mathbb{N}}$ be a stationary strongly mixing sequence of non-negative random variables. Assume that for some $\alpha \in (0,1)$, there exists $h \in \mathcal{R}_\alpha$ such that $\lim_{t\to\infty} h(t) \cdot P(Y_n > t) = 1$. For $n \in \mathbb{N}$, define a process U_n on the Skorokhod space $D(\mathbb{R}_+, \mathbb{R})$ by setting*

$$
U_n(t) = \frac{1}{b_n} \sum_{k=1}^{\lfloor nt \rfloor} Y_k, \quad t \geqslant 0,
$$

where b_n are defined in (2.6)*. Then U_n converges weakly in $D(\mathbb{R}_+, \mathbb{R})$, as $n \to \infty$, to a Lévy α-stable process if and only if the following local dependence condition holds*:

$$
\text{For any } \varepsilon > 0, \text{ we have: } \lim_{k\to\infty} \limsup_{n\to\infty} n \sum_{j=2}^{\lfloor n/k \rfloor} P(Y_j > \varepsilon b_n, Y_1 > \varepsilon b_n) = 0. \tag{3.15}
$$

We remark that the assumption $P(Y_n \in \mathbb{Z}_+) = 1$ is actually not needed and is not included in the original version of the above theorem, as it is stated in [45]. It is not hard to verify that in our setting the random variable Y_1 has regularly varying distribution tails under the law P_Φ. To transform this statement into a corresponding claim under P we will need the following a-priori bound.

Lemma 3.10. *Let Assumption* 2.1 *hold. Then*

$$
\limsup_{x\to\infty} h(x) \cdot P(Y_1 > x) = C < \infty, \tag{3.16}
$$

where $C \in (0, \infty)$ is a positive constant whose value depends on the distribution of ϕ_0 but not on the distribution of Z_0 (as long as Assumption 2.1 *holds and $h(x)$ is defined as in* (A2)).

Proof. For any $x > 0$ and $\gamma \in (0,1)$,

$$
P(Y_1 > x) = P\bigg(\sum_{n=1}^{\infty} X_{1,n} > x(1-\gamma) \sum_{n=1}^{\infty} \gamma^{n-1}\bigg) \leqslant \sum_{n=1}^{\infty} P(X_{1,n} > x\gamma^{n-1}(1-\gamma)).
$$

Therefore,

$$
\limsup_{x\to\infty} h(x) \cdot P(Y_1 > x) \leqslant \sum_{n=1}^{\infty} \limsup_{x\to\infty} h(x) \cdot P(X_{1,n} > x\gamma^{n-1}(1-\gamma))
$$

$$= \sum_{n=1}^{\infty} \limsup_{x\to\infty} \frac{h(x)}{h(x\gamma^{n-1}(1-\gamma))} \cdot h(x\gamma^{n-1}(1-\gamma)) \cdot P(X_{1,n} > x\gamma^{n-1}(1-\gamma))$$
$$= \sum_{n=1}^{\infty} \gamma^{-\alpha(n-1)}(1-\gamma)^{-\alpha} \cdot \limsup_{x\to\infty} h(x) \cdot P(X_{1,n} > x).$$

Applying Lemma 3.1 to the right-most expression in this inequality, we obtain by virtue of the bounded convergence theorem that

$$\limsup_{x\to\infty} h(x) \cdot P(Y_1 > x) \leqslant \sum_{n=1}^{\infty} \gamma^{-\alpha(n-1)}(1-\gamma)^{-\alpha} \cdot E\Big[\lim_{x\to\infty} h(x) \cdot P_\Phi(X_{1,n} > x)\Big]$$
$$= \sum_{n=1}^{\infty} (\gamma^{-\alpha} \cdot E[\phi_0^\alpha])^{n-1}(1-\gamma)^{-\alpha}.$$

Choose now $\gamma \in (0,1)$ such that $\gamma > E[\phi_0^\alpha]$ concludes the proof of the lemma. To justify the above application of the bounded convergence theorem, observe that $X_{1,n} \leqslant Z_1$ and Z_1 is independent of Φ. □

In order to study the exact asymptotic of the distribution tails of Y_1, it is convenient to approximate Y_1 by $Y_1^{(m)}$, where

$$Y_n^{(m)} := \sum_{k=n}^{n+m} X_{n,k}, \quad n \in \mathbb{Z}.$$

We have the following lemma:

Lemma 3.11. *Let Assumption* 2.1 *hold. Then*

$$\lim_{x\to\infty} h(x) \cdot P(Y_1^{(m)} > x) = E\bigg[\bigg(1 + \sum_{i=1}^{m} \prod_{j=1}^{i} \phi_j\bigg)^\alpha\bigg], \tag{3.17}$$

for any $m \in \mathbb{N}$.

Proof. Note that

$$Y_n^{(m)} = \sum_{k=1}^{Z_n} \bigg(1 + \sum_{i=1}^{m} B_{n,k}^{(i)}\bigg),$$

where $B_{n,k}^{(i)}$ is the number of progeny (either zero or one) of the k-th immigrant at generation n, who is present (or not) at the system at generation $n+i$. Then an argument similar to the one which we have employed in order to prove Lemma 3.1 (see also Remark 3.2) along with (3.1) ensure that

$$\lim_{x\to\infty} h(x) \cdot P_\Phi(Y_0^{(m)} > x) = \bigg(1 + \sum_{i=1}^{m} E_\Phi[B_{0,1}^{(i)}]\bigg)^\alpha = \bigg(1 + \sum_{i=1}^{m} \prod_{j=1}^{i} \phi_j\bigg)^\alpha.$$

Since $Y_0^{(m)} \leqslant mZ_0$ and Z_0 is independent of Φ, the bounded convergence theorem yields

$$\lim_{x\to\infty} h(x) \cdot P(Y_0^{(m)} > x) = E\Big[\lim_{x\to\infty} h(x) \cdot P_\Phi(Y_0^{(m)} > x)\Big] = E\bigg[\bigg(1 + \sum_{i=1}^{m} \prod_{j=1}^{i} \phi_j\bigg)^\alpha\bigg],$$

which completes the proof of the lemma. □

Combining together the results of Lemmas 3.10 and 3.11, we can deduce the following:

Lemma 3.12. *Let Assumption* 2.1 *hold. Then*

$$\lim_{x\to\infty} h(x) \cdot P(Y_1 > x) = E\bigg[\bigg(1 + \sum_{i=1}^{\infty} \prod_{j=1}^{i} \phi_j\bigg)^\alpha\bigg] < \infty.$$

Proof. First, observe that the lower bound

$$\lim_{x\to\infty} h(x)\cdot P(Y_1>x) \geqslant \lim_{m\to\infty}\lim_{x\to\infty} h(x)\cdot P(Y_1^{(m)}>x) = E\left[\left(1+\sum_{i=1}^{\infty}\prod_{j=1}^{i}\phi_j\right)^{\alpha}\right]$$

holds by virtue of Lemma 3.11 and the monotone convergence theorem.

To prove the matching upper bound, notice that the difference $Y_1 - Y_1^{(m)}$ is distributed under the law P as Y_1 is distributed under the law Q, where Q is defined in the same way as P with the only exception that in the former case the distribution of Z_n is assumed to be that of $\Pi_{m+1}\circ Z_0$ under P. Furthermore, since $\Pi_{m+1}\circ Z_0 \leqslant Z_0$, Lemma 3.1 and the bounded convergence theorem imply that

$$\lim_{x\to\infty} h(x)\cdot P(\Pi_{m+1}\circ Z_0 > x) = (E[\phi_0^{\alpha}])^{m+1}.$$

It follows then from (3.16) with the probability measure P replaced by Q, that

$$\lim_{m\to\infty}\limsup_{x\to\infty} h(x)\cdot P(Y_1 - Y_1^{(m)} > x) = 0.$$

Thus, using again Lemma 3.11 and the monotone convergence theorem, we obtain that the following holds for any $\varepsilon > 0$:

$$\begin{aligned}
&\limsup_{x\to\infty} h(x)\cdot P(Y_1 > x)\\
&\quad\leqslant \lim_{m\to\infty}\left\{\lim_{x\to\infty} h(x)\cdot P(Y_1^{(m)} > x(1-\varepsilon)) + \limsup_{x\to\infty} h(x)\cdot P(Y_1 - Y_1^{(m)} > x\varepsilon)\right\}\\
&\quad= \lim_{m\to\infty}\lim_{x\to\infty} h(x)\cdot P(Y_1^{(m)} > x(1-\varepsilon)) = E\left[\left(1+\sum_{i=1}^{\infty}\prod_{j=1}^{i}\phi_j\right)^{\alpha}\right]\cdot(1-\varepsilon)^{-\alpha}.
\end{aligned}$$

Taking $\varepsilon \to 0$ yields the desired upper bound. To conclude the proof of the lemma, it remains to note that by Jensen's inequality,

$$E\left[\left(1+\sum_{i=1}^{\infty}\prod_{j=1}^{i}\phi_j\right)^{\alpha}\right] \leqslant \left(E\left[1+\sum_{i=1}^{\infty}\prod_{j=1}^{i}\phi_j\right]\right)^{\alpha} = (1-E[\phi_0])^{-\alpha} < \infty,$$

where we used the assumption $\alpha \in (0,1)$. □

We are now in a position to complete the proof of Theorem 2.11. It suffices to verify that the conditions of Theorem 3.9 hold for the sequence $(Y_n)_{n\geqslant 1}$. In view of Lemmas 3.8 and 3.12, we only need to check the validity of the "local dependence" condition (3.15). To this end, observe that for any $j \geqslant 2$, Y_j and Y_1 are independent under the law P_ϕ, and hence Cauchy-Schwarz inequality yields

$$P(Y_j > \varepsilon b_n, Y_1 > \varepsilon b_n) = E[P_\Phi(Y_j > \varepsilon b_n)\cdot P_\Phi(Y_1 > \varepsilon b_n)] \leqslant E[P_\Phi^2(Y_1 > \varepsilon b_n)].$$

An argument similar to the one we employed to prove Lemma 3.12 shows then that the following limit exists and the identity holds:

$$\lim_{x\to\infty} h(x)^2\cdot E[P_\Phi^2(Y_1 > x)] = E\left[\left(1+\sum_{i=1}^{\infty}\prod_{j=1}^{i}\phi_j\right)^{2\alpha}\right] < \infty.$$

Thus

$$\lim_{n\to\infty} n^2\cdot E[P_\Phi^2(Y_1 > \varepsilon b_n)] = \varepsilon^{-2\alpha}\cdot E\left[\left(1+\sum_{i=1}^{\infty}\prod_{j=1}^{i}\phi_j\right)^{2\alpha}\right] < \infty,$$

and

$$\lim_{k\to\infty}\limsup_{n\to\infty} n\sum_{j=2}^{\lfloor n/k\rfloor} P(Y_j > \varepsilon b_n, Y_1 > \varepsilon b_n)$$

$$\leqslant \lim_{k\to\infty} \limsup_{n\to\infty} \frac{n^2}{k} \cdot E[P_\Phi^2(Y_1 > \varepsilon b_n)]$$

$$= \lim_{k\to\infty} \limsup_{n\to\infty} \frac{n^2}{k} \cdot n^{-2}\varepsilon^{-2\alpha} \cdot E\left[\left(1 + \sum_{i=1}^{\infty}\prod_{j=1}^{i}\phi_j\right)^{2\alpha}\right] = 0,$$

as desired. The proof of Theorem 2.11 is completed.

3.4 Proof of Lemma 2.13

Recall Y_n from (3.14). Define

$$Q_n = X_n + \text{total progeny of the } X_n \text{ particles present at generation } n.$$

For all $A > 0$ define its stopping time $\varsigma_A = \inf\{n : X_n > A\}$. The random variable W_1 can be represented on the event $\{\varsigma_A < \nu_1\}$ in the following form:

$$W_1 = \sum_{n=0}^{\varsigma_A - 1} X_n + Q_{\varsigma_A} + \sum_{\varsigma_A < n < \nu_1} Y_n. \tag{3.18}$$

The three terms in the right-hand side of (3.18) are evaluated in the following series of lemmas. It will turn out that for large A, the main contribution to W_1 in (3.18) comes from the second term. Fix any $\delta > 0$. It follows from (2.8) that for any $A > 0$,

$$P_0\left(\sum_{n=0}^{\min\{\varsigma_A,\nu_1\}-1} X_n \geqslant \delta t\right) \leqslant P_0(A\nu_1 \geqslant \delta t) \leqslant K_1 \mathrm{e}^{-K_2\delta t/A},$$

and hence

$$P_0(W_1 \geqslant \delta t, \varsigma_A \geqslant \nu_1) \leqslant P(A\nu_1 \geqslant \delta t) \leqslant K_1 \mathrm{e}^{-K_2\delta t/A}, \tag{3.19}$$

$$P_0\left(\sum_{n=0}^{\varsigma_A-1} X_n \geqslant \delta t, \varsigma_A < \nu_1\right) \leqslant P_0(A\nu_1 \geqslant \delta t) \leqslant K_1 \mathrm{e}^{-K_2\delta t/A}. \tag{3.20}$$

Lemma 3.13. *For all $\delta > 0$ there exists an $A_0 = A_0(\delta) < \infty$ such that*

$$h(t) \cdot P_0\left(\sum_{\varsigma_A < n < \nu_1} Y_n \geqslant \delta t\right) \leqslant \delta, \quad \text{for all } A \geqslant A_0 \text{ and } t > 0. \tag{3.21}$$

Proof. Using the identity $\sum_{n=1}^{\infty} n^{-2} = \pi^2/6 < 2$ and the fact that Y_n is independent of $\mathbf{1}_{\{\varsigma_A < n < \nu_1\}}$ under the law P_0, we obtain that the following holds for all $t > 0$:

$$\begin{aligned} h(t) \cdot P_0\left(\sum_{\varsigma_A < n < \nu_1} Y_n \geqslant \delta t\right) &= h(t) \cdot P_0\left(\sum_{n=1}^{\infty} Y_n \mathbf{1}_{\{\varsigma_A < n < \nu_1\}} \geqslant 6\delta t \pi^{-2} \sum_{n=1}^{\infty} n^{-2}\right) \\ &\leqslant \sum_{n=1}^{\infty} P_0(\varsigma_A < n < \nu_1) \cdot h(t) \cdot P_0(Y_n \geqslant 1/2 \cdot \delta t n^{-2}) \\ &\leqslant \sum_{n=1}^{\infty} P_0(\varsigma_A < n < \nu_1) \cdot \frac{h(t)}{h(1/2 \cdot \delta t n^{-2})} \cdot h(1/2 \cdot \delta t n^{-2}) \\ &\quad \cdot P_0(Y_n \geqslant 1/2 \cdot \delta t n^{-2}). \end{aligned} \tag{3.22}$$

To bound the term $\frac{h(t)}{h(1/2 \cdot \delta t n^{-2})}$, we apply the following simplified version of [19, Lemma 1]:

$$\text{There exists } K > 1 \text{ such that } \frac{h(\lambda t)}{h(t)} \leqslant K(\lambda^{-\alpha} + \lambda^{\alpha}) \text{ for all } \lambda > 0,\ t > 0. \tag{3.23}$$

It follows that

$$\frac{h(t)}{h(1/2\cdot\delta t n^{-2})}\leqslant K2^{\alpha}n^{2\alpha}(\delta^{-\alpha}+\delta^{\alpha}),\quad t>0.$$

Using Lemma 3.12 and (3.23), we obtain from (3.22) and the above bound that the following holds for all $t>0$ with a suitable constant $C>0$ independent of n,δ,A and t :

$$\begin{aligned}h(t)\cdot P_0\Bigg(\sum_{\varsigma_A<n<\nu_1}Y_n\geqslant\delta t\Bigg)&\leqslant C2^{\alpha}t^{-\alpha}(\delta^{-\alpha}+\delta^{\alpha})E_0[\nu_1^{2\alpha+1};\varsigma_A<\nu_1]\\&\leqslant C2^{\alpha}t^{-\alpha}(\delta^{-\alpha}+\delta^{\alpha})\sqrt{E_0(\nu_1^{4\alpha+2})}\cdot\sqrt{P_0(\varsigma_A<\nu_1)}.\end{aligned}$$

The claim follows now from (2.8), the first square root being bounded and the second one going to zero as $A\to\infty$. □

It follows from (3.18), taking estimates (3.19)–(3.21) into account, that for any $A>A_0(\delta)$ (where A_0 is given by (3.21)) there exists $t_A>0$ such that

$$\begin{aligned}h(t)\cdot P_0(\varsigma_A<\nu_1,Q_{\varsigma_A}\geqslant t)&\leqslant h(t)\cdot P_0(W_1\geqslant t)\\&\leqslant h(t)\cdot P_0(\varsigma_A<\nu_1,Q_{\varsigma_A}\geqslant t(1-2\delta))+3\delta,\end{aligned}\tag{3.24}$$

for all $t>t_A$. Thus, W_1 can be approximated by Q_{ς_A}. The following lemma deals with the distribution tails of the latter.

Lemma 3.14. *Let Assumption* 2.1 *hold. Then,*

(a) *We have*

$$\limsup_{A\to\infty}\limsup_{t\to\infty}h(t)\cdot P_0(X_{\varsigma_A}\geqslant t,\varsigma_A<\nu_1)<\infty.\tag{3.25}$$

(b) *The following limit exists and is finite for any given* $A>0$:

$$\lim_{t\to\infty}h(t)\cdot P_0(X_{\varsigma_A}\geqslant t,\varsigma_A<\nu_1).$$

(c) *The following limit exists and is finite for any given* $A>0$:

$$\lim_{t\to\infty}h(t)\cdot P_0(Q_{\varsigma_A}\geqslant t,\varsigma_A<\nu_1).$$

Proof. (a) Recall M_n from (2.5). For $t>A$ we have

$$\begin{aligned}P_0(X_{\varsigma_A}>t;\varsigma_A<\nu_1)&=\sum_{n\geqslant1}\sum_{a=0}^{A-1}P_0(X_{\varsigma_A}>t,\varsigma_A=n,X_n>A,X_{n-1}=a,\nu_1>n)\\&=\sum_{n\geqslant1}\sum_{a=0}^{A-1}P_0(X_n>t,M_{n-1}<A,X_{n-1}=a,\nu_1>n)\\&\leqslant\sum_{n\geqslant1}P(Z_n>t-A,M_{n-1}<A,\nu_1>n)\\&=\sum_{n\geqslant1}P(Z_n>t-A)\cdot P(M_{n-1}<A,\nu_1>n)\leqslant P(Z_0>t-A)\cdot E_0[\nu_1].\end{aligned}$$

In view of Assumption 2.1 and (2.8), this completes the proof of part (a).

(b) The computation is quite similar to the one in part (a). Namely, for $t>A$ we have

$$P_0(X_{\varsigma_A}>t;\varsigma_A<\nu_1)=\sum_{n\geqslant1}\sum_{a=1}^{A-1}P_0(Z_n>t-a,\phi_n\circ X_{n-1}=a,M_{n-1}<A,\nu_1>n)$$

$$= \sum_{n \geqslant 1} \sum_{a=1}^{A-1} P(Z_0 > t - a) \cdot P_0(\phi_n \circ X_{n-1} = a, M_{n-1} < A, \nu_1 > n)$$
$$= \sum_{a=1}^{A-1} P(Z_0 > t - a) \cdot \sum_{n \geqslant 1} P_0(\phi_n \circ X_{n-1} = a, M_{n-1} < A, \nu_1 > n).$$

As before,

$$\sum_{n \geqslant 1} P_0(\phi_n \circ X_{n-1} = a, M_{n-1} < A, \nu > n) \leqslant \sum_{n \geqslant 1} P_0(\nu_1 > n) = E[\nu_1] < \infty,$$

from which the claim of part (b) follows in view of Assumption 2.1.

(c) This is merely Lemma 3.12 applied to X_{ς_A} under the conditional law $P(\,\cdot \mid \varsigma_A < \nu_1)$ rather than to Z_1 under the regular measure P. □

We are now in a position to conclude the proof of Lemma 2.13. It follows from (3.24), (3.23), and Lemma 3.14 that

$$\lim_{t \to \infty} h(t) \cdot P_0(W_1 > t) = \lim_{A \to \infty} \lim_{t \to \infty} h(t) \cdot P_0(Q_{\varsigma_A} > t; \varsigma_A < \nu_1) < \infty.$$

The second limit, taken as $A \to \infty$, in the right-hand side exists since the limit in the left-hand side does not depend on A. Furthermore, by Assumption 2.1,

$$\lim_{t \to \infty} h(t) \cdot P_0(W_1 > t) \geqslant \lim_{t \to \infty} h(t) \cdot P(Z_1 > t) > 0,$$

concluding the proof of Lemma 2.13.

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Appendix: Proof of two auxiliary propositions

Proof of Proposition 2.3. (a) By Jensen's inequality, if $E[Z_0^\beta] < \infty$ for $\beta > 0$, then $E[Z_0^{\beta/m}] < \infty$ for any $m \in \mathbb{N}$. Therefore, without loss of generality we can assume that $\beta \in (0,1)$ in Assumption 2.2. Assuming from now on and throughout the proof of part (a) of Proposition 2.3 that $\beta \in (0,1)$, we obtain by virtue of Jensen's inequality for conditional expectations that

$$\begin{aligned} E_0[(\Pi_k \circ Z_k)^\beta] &= E_0[E_0[(\Pi_k \circ Z_k)^\beta | \Phi, \mathcal{Z}]] \leqslant E_0[(E_0[\Pi_k \circ Z_k | \Phi, \mathcal{Z}])^\beta] \\ &= E\bigg[\bigg(\prod_{j=1}^k \phi_j \cdot Z_k\bigg)^\beta\bigg] = E[Z_0^\beta] \cdot (E[\phi_0^\beta])^k. \end{aligned} \tag{A.1}$$

Hence

$$E[X_\infty^\beta] = E\bigg[\bigg(\sum_{k=0}^\infty X_{0,k}\bigg)^\beta\bigg] \leqslant \sum_{k=0}^\infty E[X_{0,k}^\beta] \leqslant E[Z_0^\beta] \cdot \sum_{k=0}^\infty (E[\phi_0^\beta])^k < \infty.$$

In particular, X_∞ is P-a.s. finite.

(b) For $n \in \mathbb{N}$, we have

$$X_n = \sum_{k=1}^n X_{k,n} + X^{(0,n)},$$

where $X^{(0,n)} =_P \Pi_n \circ X_0$. Since $P(\lim_{n\to\infty} \Pi_n \circ X_0 = 0) = 1$ for any $X_0 \in \mathcal{N}_+$, the limiting distribution of X_n, if exists, is independent of X_0. Furthermore, if $X_0 = 0$, the i.i.d. structure of $(\Phi, \mathcal{Z})$ yields,

$$X_n =_P \sum_{k=-n+1}^0 X_{k,0} =_P Z_0 + \sum_{k=1}^{n-1} \Pi_k \circ Z_k,$$

The claim of part (b) follows now from the almost sure convergence of the series on the right-hand side of the above identity to X_∞.

(c) To see that the stationary distribution is unique, consider two stationary solutions $(X_n^{(1)})_{n\in\mathbb{Z}_+}$ and $(X_n^{(2)})_{n\in\mathbb{Z}_+}$ to (1.1) corresponding to different initial values, $X_n^{(1)}$ and $X_n^{(2)}$, respectively. Then, since Π_n are "thinning" operators,

$$|X_n^{(1)} - X_n^{(2)}| \leqslant \Pi_{n+1} \circ |X_0^{(1)} - X_0^{(2)}|,$$

and hence

$$\lim_{n\to\infty} (X_n^{(1)} - X_n^{(2)}) = 0, \quad P\text{-a.s.}$$

The proof of the proposition is complete. □

Proof of Proposition 2.7. (a) By Corollary 2.4, $P_0(X_n = k_{\min} \text{ i.o.}) = 1$, and hence $P_0(\nu_n < \infty) = 1$ for all $n \in \mathbb{N}$. The argument showing that the pairs $(\sigma_n, W_n)_{n\in\mathbb{N}}$ form an i.i.d. sequence is standard (cf. [2]) and is based on the following two observation along with the use of the strong Markov property:

(i) The random times ν_n are times of the successive visits to the set $\{(x, y) \in \mathbb{Z}^2 : x = y\}$ by the two-component Markov chain $(X_n, Z_n)_{n\in\mathbb{N}}$. Furthermore, $X_{\nu_n} = Z_{\nu_n} =_P Z_0$.

(ii) Transition kernel of the Markov chain (X_n, Z_n) depends only on the current value of the first component, but not on the value of the second.

(b) In order to prove part (b) of the proposition, it suffices to show that the following power series has a radius of convergence greater than 1:

$$V(z) = \sum_{t=0}^{\infty} P_0(\sigma_1 > t)z^t.$$

Let us introduce some notation. Let $v(t) = P_0(\sigma_1 > t)$,

$$h(r, t) = P\Bigg(X_{r,t} \neq 0, \sum_{j=r+1}^{t-1} X_{j,t} = 0\Bigg) \quad \text{and} \quad g(r, t) = P\Bigg(\sum_{j=r}^{t-1} X_{j,t} = 0\Bigg).$$

Then

$$\begin{aligned} v(t) &= P_0(\sigma_1 > t, X_t \neq 0) = P_0\Bigg(\sigma_1 > t, \sum_{k=0}^{t-1} X_{k,t} \neq 0\Bigg) \\ &= \sum_{k=0}^{t-2} P_0\Bigg(\sigma_1 > t, X_{k,t} \neq 0, \sum_{j=k+1}^{t-1} X_{k,t} = 0\Bigg) + P_0(\sigma_1 > t, X_{t-1,t} \neq 0). \end{aligned} \tag{A.2}$$

Using the i.i.d. structure of the sequence of random coefficients (Φ, Z), we obtain

$$\begin{aligned} P_0\Bigg(\sigma_1 > t, X_{k,t} \neq 0, \sum_{j=k+1}^{t-1} X_{k,t} = 0\Bigg) &= P_0\Bigg(\sigma_1 > k, X_{k,t} \neq 0, \sum_{j=k+1}^{t-1} X_{k,t} = 0\Bigg) \\ &= P_0(\sigma_1 > k)P\Bigg(X_{k,t} \neq 0, \sum_{j=k+1}^{t-1} X_{k,t} = 0\Bigg) = v(k)h(k, t) \end{aligned} \tag{A.3}$$

and

$$g(k, t) = P(X_{t-k} = 0). \tag{A.4}$$

Let $g(t) = P_0(X_t = 0)$. It follows from (A.4) that

$$g(k, t) = g(t - k). \tag{A.5}$$

Next, let $h(t) = g(t - 1) - g(t)$. Since $h(k, t) + g(k, t) = g(k + 1, t)$, then (A.5) implies that

$$h(k, t) = h(t - k). \tag{A.6}$$

Substituting (A.3) into (A.2) and then using (A.6) gives

$$v(t) = \sum_{k=0}^{t-1} v(k)h(t - k).$$

In addition, we have $v(0) = 1$, $h(k) > 0$ for all $k > 0$, and

$$\sum_{k=1}^{\infty} h(k) = 1 - \lim_{t\to\infty} g(t) = 1 - \lim_{t\to\infty} P_0(X_t = 0) < 1,$$

where for the last inequality we used Theorem 3.3. Therefore, $\{v(k) : k = 0, 1, 2, \dots\}$ is a renewal sequence. Therefore (see [15, Section XIII.3]), $V(z) = (1 - H(z))^{-1}$, where

$$H(z) := \sum_{t=0}^{\infty} h(t) z^t.$$

To conclude the proof of the proposition, observe that, using (A.1) and Chebyshev's inequality,

$$h(t) = h(0, t) < P(X_{0,t} \neq 0) \leqslant E[X_{0,t}] = E[Z_0^{\beta}] \cdot (E[\phi_0^{\beta}])^t,$$

and hence the radius of convergence of $H(z)$ is greater than 1. $\square$