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A remark on Kac's scattering length formula

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Abstract The scattering length formula was formulated and proved in special cases by Kac in 1974 and 1975. It was discussed by a series of authors, including Taylor 1976, Tamura 1992 and Takahashi 1990. The formula was proved by Takeda 2010 in symmetric case and by He 2011 assuming weak duality. In this article, we shall use the powerful tool of Kutznetsov measures to prove this formula in the general framework of right Markov processes without further assumptions.

Keywords scattering length, Kuznetsov measure, capacity

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1 Introduction

The scattering length of a potential $V : \mathbb{R}^3 \to \mathbb{R}_+$ is the quantity

$$\Gamma(V) := \int_{\mathbb{R}^3} \varphi_\infty(x)V(x)\,dx,$$

where

$$\varphi_\infty(x) := \mathbf{P}^x\left[\exp\left(-\int_0^\infty V(B_s)\,ds\right)\right], \quad x \in \mathbb{R}^3,$$

and $\mathbf{P}^x$ is the law of 3-dimensional Brownain motion $(B_s)_{s\geqslant 0}$ started at $B_0 = x$. Kac [11, 12] wrote several papers on Γ in the early 1970s, and proved that

$$\Gamma(V) = \lim_{t\to\infty} t^{-1}\int_{\mathbb{R}^3} \mathbf{P}^x\left[1 - \exp\left(-\int_0^t V(B_s)ds\right)\right]dx. \tag{1.1}$$

In addition, Kac showed that if $V = 1_K$ with K compact and "regular" (in the sense that the Lebesgue penetration time of K by the Brownian motion is the same as the hitting time of K), then

$$\lim_{\lambda\uparrow\infty} \Gamma(\lambda 1_K) = \mathrm{Cap}(K), \tag{1.2}$$

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where $\mathrm{Cap}(K)$ is the Newtonian capacity of K. Kac further conjectured that

$$\lim_{\lambda\uparrow\infty}\Gamma(\lambda V)=\mathrm{Cap}(\mathrm{supp}(V)), \tag{1.3}$$

for integrable V with $\{V>0\}$ regular as above. This conjecture was confirmed by Taylor [19] by a probabilistic method and later by Tamura [18] by an analytic argument. Taylor's work contains interesting applications to the spectral theory of the Neumann Laplacian, including a necessary and sufficient condition for discreteness of the spectrum expressed in terms of Γ. The works cited so far concern Brownian motion in $\mathbb{R}^d$ for $d\geqslant 3$; recent work by Siudeja [15] extended Taylor's study to isotropic stable processes in Euclidean space. Takahashi [16] showed that for general symmetric Markov processes the limit in (1.3) depends on V only through $\{V>0\}$, provided V is continuous and of compact support. Takahashi's paper is notable for an integral formula for $\Gamma(V)$ (see (3.4) below) which makes (1.3) quite transparent. A decisive step was taken by Takeda [17], again in the context of symmetric Markov processes, who proved the analog of (1.3) for general positive continuous additive functionals (PCAFs) of a symmetric Markov process. Takeda seems to be the first author to have recognized the relevance of the fine support of the PCAF associated with V. Takeda's time-change method was shown by He [9] to apply to general PCAFs of a non-symmetric Markov process in possession of a dual process. In this paper we shall prove the analog of (1.3) in the most general framework of right Markov processes. For details on notions of probabilistic potential theory that are used in the sequel (such as energy functional, capacity, Revuz measure, Kuznetsov process) we refer the reader to [7].

We end this introduction by noting that (1.1) and (1.3) (and their generalizations) are true only for transient processes. For example, if X is Brownian motion on $\mathbb{R}$ and if $A_t=\int_0^t V(X_s)\,ds$ with $0\leqslant V\in L^1(\mathbb{R})$, then

$$\lim_{\lambda\to\infty}\lim_{t\to\infty}t^{-1/2}\int_{\mathbb{R}}\mathbf{P}^x[1-\mathrm{e}^{-\frac{\lambda}{\sqrt{t}}A_t}]dx=\frac{4}{\sqrt{2\pi}}\left(=\int_{\mathbb{R}}\mathbf{P}^x[T_0\leqslant 1]dx\right),$$

by a scaling argument. Here T_0 is the hitting time of 0.

2 Scattering length

Let

$$X=(\Omega,\mathcal{F},(\mathcal{F}_t),(X_t)_{t\geqslant 0},\mathbf{P}^x)$$

be a right Markov process on a state space $(E,\mathscr{E})$, with transition semigroup (P_t) and resolvent (U^α). Let m be an excessive measure for X. For the sake of simplicity we assume that X is *Borel*, meaning that E is homeomorphic to a Borel subset of a compact metric space, $\mathscr{E}$ is the Borel σ-field on E, P_t maps Borel functions to Borel functions, and X is a strong Markov process. For the matters we study here this entails without loss of generality; see [4,6]. For general theory of Markov processes please refer to [3,8,14].

We fix once and for all a positive continuous additive functional $A=(A_t)$ of X with Revuz measure $\nu=\nu_A^m$ determined by

$$\nu_A^m(f):=\uparrow\lim_{t\downarrow 0}t^{-1}\mathbf{P}^m\int_0^t f(X_s)\,dA_s,\quad f\in p\mathscr{E},$$

where $\mathbf{P}^m=\int_E\mathbf{P}^x m(dx)$. Note that we use $\mathbf{P}^x$ both for the law of X started at x and for the associated expectation. The first-increase time $R:=\inf\{t>0:A_t>0\}$ is a stopping time and

$$F:=\{x\in E:\mathbf{P}^x[R=0]=1\}$$

is the *fine support* of A. The set F is finely perfect and carries all the mass of ν_A. For the details of additive functionals please refer to [2,3].

Define

$$\varphi_t(x)=\varphi_t^A(x):=\mathbf{E}^x[\mathrm{e}^{-A_t}],\quad 0\leqslant t<+\infty,\quad x\in E.$$

Since $t \mapsto \varphi_t(x)$ is decreasing, we can write

$$\varphi_\infty(x) :=\downarrow \lim_{t\uparrow+\infty} \varphi_t(x) = \mathbf{E}^x[\mathrm{e}^{-A_\infty}], \quad x \in E. \tag{2.1}$$

With this notation in hand, and following Kac, we define the *scattering length* of A as

$$\Gamma(A) = \Gamma_m(A) := \int_E \varphi_\infty(x)\, \nu_A^m(dx). \tag{2.2}$$

The excessive measure m can be uniquely decomposed as $m = m_d + m_c$, into dissipative and conservative components (corresponding to the transient and recurrent parts of X respectively), and the objects under study are additive in m,

$$\nu_A^m = \nu_A^{m_d} + \nu_A^{m_c} \quad \text{and} \quad \Gamma_m = \Gamma_{m_d}(A) + \Gamma_{m_c}(A).$$

In view of the lemma to follow, there is no loss of generality in restricting our attention to the dissipative case.

Lemma 2.1. *If m is conservative, then*

$$\Gamma_m(A) = 0.$$

Proof. It is easy to check that $1 - \varphi_\infty$ is a bounded excessive function and in fact

$$\mathrm{e}^{-A_t}\varphi_\infty(X_t) = \mathbf{E}^x[\mathrm{e}^{-A_\infty}|\mathcal{F}_t], \quad t > 0,$$

provide we define $\varphi_\infty(\Delta) := 1$, where Δ is the cemetery state for X. Hence $\{\mathrm{e}^{-A_t}\varphi_\infty(X_t)\}$ is a bounded martingale. When m is conservative, we have by a result of Blumenthal [1],

$$\mathbf{P}^m\{\varphi_\infty(X_t) = \varphi_\infty(X_0),\ \forall\, t > 0\} = 1,$$

and then the martingale $\{\mathrm{e}^{-A_t}\varphi_\infty(X_t)\}$ is continuous and decreasing, $\mathbf{P}^x$-a.s. for m-a.e. $x \in E$. Hence it is independent of t and we have

$$\mathrm{e}^{-A_\infty}\varphi_\infty(X_0) = \mathrm{e}^{-A_t}\varphi_\infty(X_t) = \varphi_\infty(X_0), \quad \mathbf{P}^m\text{-a.s.}$$

Consequently there is an m-exceptional set N such that for $x \notin N$ either $\varphi_\infty(x) = 0$ (in which case $\mathbf{P}^x(A_\infty = \infty) = 1$) or $\varphi_\infty(x) > 0$ and $\mathbf{P}^x[A_\infty = 0] = 1$ (in which case $\varphi_\infty(x) = 1$). Therefore,

$$\Gamma_m(A) = \nu_A^m(\varphi_\infty) = \nu_A^m(\varphi_\infty = 1) = 0.$$

The final equality above follows because m is invariant (being conservative), so that

$$\begin{aligned}\nu_A^m(\varphi_\infty = 1) &= \mathbf{P}^m \int_0^1 1_{\{\varphi_\infty(X_t)=1\}} dA_t \\ &= \mathbf{P}^m \int_0^1 \mathbf{P}^{X_t}(A_\infty = 0) dA_t \\ &= \mathbf{P}^m \int_0^1 1_{\{A_\infty = A_t\}} dA_t = 0.\end{aligned}$$

□

For the remainder of the paper, with the exception of the very last result (Theorem 3.6), we assume that the excessive measure m is dissipative.

We end this section with the statement and proof of (1.1) in our context, under the additional condition that m is invariant. The general case must wait until the next section and the introduction of the Kuznetsov measure associated with X and m.

Theorem 2.2. *If m is invariant and $\nu_A^m(E) < \infty$, then*

$$\Gamma(A) = \lim_{t\uparrow\infty} t^{-1}\mathbf{E}^m[1 - \mathrm{e}^{-A_t}]. \tag{2.3}$$

Proof. Recall that θ_t is the shift operator for X. We have

$$1 - \exp(-A_t) = \exp(-A_t) \int_0^t \exp(A_s) dA_s$$
$$= \int_0^t \exp(-A_{t-s} \circ \theta_s) dA_s,$$

from which it follows that

$$\mathbf{P}^x [1 - \exp(-A_t)] = \mathbf{P}^x \int_0^t \exp(-A_{t-s} \circ \theta_s) dA_s$$
$$= \mathbf{P}^x \int_0^t \mathbf{P}^{X_s} [\exp(-A_{t-s})] dA_s$$
$$= \mathbf{P}^x \int_0^t \varphi_{t-s}(X_s) dA_s.$$

Hence, because m is invariant,

$$\mathbf{P}^m [1 - \exp(-A_t)] = \mathbf{P}^m \int_0^t \varphi_{t-s}(X_s) dA_s$$
$$= \int_0^t ds \int_E \varphi_{t-s}(x) \nu_A^m(dx)$$
$$= \int_0^t du \int_E \varphi_u(x) \nu_A^m(dx).$$

Consequently,

$$\frac{1}{t}\mathbf{P}^m [1 - \exp(-A_t)] - \int_E \varphi_\infty(x) \nu_A^m(dx) = \frac{1}{t} \int_0^t du \int_E [\varphi_u(x) - \varphi_\infty(x)] \nu_A^m(dx).$$

Because $\nu_A^m(E) < \infty$, the monotonicity in u of $\varphi_u(x) - \varphi_\infty(x)$ (for each x) implies that

$$\int_E [\varphi_u(x) - \varphi_\infty(x)] \nu_A^m(dx)$$

decreases to zero as $u \uparrow +\infty$, and the conclusion follows. □

3 Scattering length and capacity

We will prove in this section that Kac's conjecture holds in complete generality. Recall that F is the fine support of the PCAF A. We shall write Cap(F) for the Getoor-Steffens *capacity* of F relative to m, as discussed in [7]; see [7, (10.12)]. We are still assuming that m is dissipative.

Theorem 3.1. *If A is a* PCAF *of X with fine support F, then*

$$\uparrow \lim_{\alpha \uparrow \infty} \Gamma(\alpha A) = \mathrm{Cap}(F). \tag{3.1}$$

Our proof of this result relies on an expression for $\Gamma(A)$ in terms of the Kuznetsov process

$$Y = (W, \mathcal{G}_t, Y_t, \sigma_t, \mathbb{Q}_m)$$

associated with X and m. Here $(Y_t)_{t \in \mathbb{R}}$ is the coordinate process on the space W of paths $w : \mathbb{R} \to E \cup \{\Delta\}$ with birth time $\alpha(w)$ and death time $\beta(w)$ ($w(t) = \Delta$ for $t \notin [\alpha(w), \beta(w)]$). $\mathbb{Q}_m$ is a σ-finite measure on $(W, \mathfrak{G})$ (where $\mathfrak{G} := \sigma\{Y_t; t \in \mathbb{R}\}$) such that

$$\mathbb{Q}_m[Y_t \in B; \alpha < t < \beta] = m(B), \quad \forall\, B \in \mathscr{E},$$

and

$$(Y_s)_{s>t} \text{ is } (P_s)\text{-Markov on } \{\alpha < t < \beta\}, \quad \forall t \in \mathbb{R}.$$

In particular, Y is a stationary process; that is, $\mathbb{Q}_m$ is invariant with respect to the shift operators σ_t, $t \in \mathbb{R}$, defined by

$$[\sigma_t w](s) := w(s+t), \quad s, t \in \mathbb{R}.$$

Because m is dissipative, there is a random time $S : W \to [-\infty, \infty]$ such that $\mathbb{Q}_m[S \notin \mathbb{R}] = 0$ and $S(\sigma_t w) = S(w) - t$ for all $w \in W$ and $t \in \mathbb{R}$. Writing $\mathring{A}$ for the class of (σ_t)-invariant elements of $\mathfrak{G}$, the formula

$$\mathbb{P}_m[F] := \mathbb{Q}_m[F; 0 < S < 1], \quad F \in \mathring{A},$$

then defines a σ-finite measure on $\mathring{A}$. (This measure corresponds to the *quasi-process* of Hunt [10] and Weil [20]; see [5].) Because of the stationarity of $\mathbb{Q}_m$, the measure $\mathbb{P}_m$ does not depend on the specific choice of S subject to the two conditions imposed above.

The relevance of $\mathbb{P}_m$ lies in the following formula developed in [5]. Let $H : \Omega \to [0, \infty]$ be "excessive" in the sense that $t \mapsto H(\theta_t \omega)$ is decreasing and right-continuous on $[0, \infty[$ for each $\omega \in \Omega$. The function $h(x) := \mathbf{P}^x[H]$, $x \in E$, is then an excessive function of X, and the formula

$$H^*(w) :=\uparrow \lim_{t \downarrow \alpha(w)} H(\theta_t w), \quad w \in W,$$

defines a (σ_t)-invariant function on W for which

$$\mathbb{P}_m[H^*] = L(m, h),$$

where L is the energy functional associated with X. Recall from [7] that

$$L(m, h) := \sup\{\mu(h) : \mu U \leqslant m\}.$$

For example, if B is a nearly Borel subset of E and $T_B := \inf\{t > 0 : X_t \in B\}$, then $H := 1_{\{T_B < \infty\}}$ is excessive in the above sense,

$$h(x) = P_B 1(x) = \mathbf{P}^x[T_B < \infty]$$

is the hitting probability of B, and $H^* = 1_{\{\tau_B < \infty\}}$, where $\tau_B := \inf\{t > \alpha : Y_t \in B\}$. In this case,

$$\mathbb{P}_m[\tau_B < \infty] = \mathbb{P}_m[Y \text{ hits } B] = L(m, P_B 1) = \mathrm{Cap}(B). \tag{3.2}$$

The choice $H = 1 - \exp(-A_\infty)$ yields $H^* = 1 - \exp(-\kappa(\mathbb{R}))$, where κ is the diffuse homogeneous random measure (HRM) over Y that extends A; see [7, pp. 89–90]. This choice leads to the key result of this section.

Proposition 3.2. *If κ is the* HRM *associated with A, then*

$$\Gamma(A) = \mathbb{P}_m[1 - \exp(-\kappa(\mathbb{R}))]. \tag{3.3}$$

Proof. In view of the preceding discussion, we need only show that $L(m, h) = \Gamma(A)$ when $h = 1 - \varphi_\infty$. But arguing as in the proof of Theorem 2.2 we find that

$$h(x) = 1 - \varphi_\infty(x) = \mathbf{P}^x \int_0^\infty \varphi_\infty(X_t)\, dA_t = U_A \varphi_\infty(x),$$

where U_A is the potential kernel associated with A. Consequently,

$$L(m, 1 - \varphi_\infty) = L(m, U_A \varphi_\infty) = \nu_A^m(\varphi_\infty) = \Gamma(A),$$

where is called the Meyer's formula for which the interested readers may refer to [7, (8.13)] and [13] the second equality. □

Proposition 3.2 was motivated by a result of Takahashi, which was proved in [16] for symmetric Markov processes and A of the form

$$A_t = \int_0^t V(X_s)ds.$$

The corollary to follow is essentially Takahashi's formula expressed in full generality using the Kuznetsov process.

Corollary 3.3. *If κ is the* HRM *associated with A, then*

$$\Gamma(A) = \mathbb{Q}_m\left[\frac{1-\exp(-\kappa(\mathbb{R}))}{\kappa(\mathbb{R})}\kappa[0,1];\kappa(\mathbb{R})>0\right]. \tag{3.4}$$

Proof. The (σ_t)-invariance of $\mathbb{Q}_m$ leads easily to the identity

$$\mathbb{Q}_m[\overline{F}\cdot G\cdot A] = \mathbb{Q}_m[F\cdot\overline{G}\cdot A], \tag{3.5}$$

in which F and G are positive and $\mathfrak{G}$-measurable, A is positive and $\mathring{A}$-measurable, and $\overline{F}(w) := \int_{\mathbb{R}} F(\sigma_t w)dt$, etc. Taking

$$A = 1_{\{\kappa(\mathbb{R})>0\}}(1-\exp(-\kappa(\mathbb{R})))/\kappa(\mathbb{R}), \quad F=\kappa[0,1],$$

and $G = 1_{\{0<S<1\}}$ (so that $\overline{F}=\kappa(\mathbb{R})$ and $\overline{G}=1_{\{S\in\mathbb{R}\}}$), we see that (3.4) follows from (3.5) and Proposition 3.2. □

The interpretation (3.3) of $\Gamma(A)$ yields Kac's conjecture immediately.

Theorem 3.4. *Let A be a* PCAF *of X with fine support F. Then*

$$\uparrow\lim_{\lambda\uparrow\infty}\Gamma(\lambda A) = \mathrm{Cap}(F).$$

Proof. By Proposition 3.2, with A replaced by λA, we have

$$\Gamma(\lambda A) = \mathbb{P}_m\left[1-\exp(-\lambda\kappa(\mathbb{R}))\right].$$

As λ increases to $+\infty$, the integrand $1-\exp(-\lambda\kappa(\mathbb{R}))$ increases to $1_{\{\kappa(\mathbb{R})>0\}}$. But κ has positive mass if and only if Y hits F. The result now follows from the discussion of hitting times before the statement of Proposition 3.2. □

A second corollary of Proposition 3.2 follows from the monotone convergence theorem.

Corollary 3.5. *Let A be a* PCAF *of X with Revuz measure ν_A^m. Then*

$$\uparrow\lim_{\lambda\downarrow 0}\frac{\Gamma(\lambda A)}{\lambda} = \nu_A^m(E).$$

The fine support F of A is an *equilibrium set* if the balayage $R_F m$ of m on F, defined by

$$R_F m(g) = L(m, P_F g), \quad g\in p\mathscr{E},$$

is a potential, say $R_F m = \pi_F U$. (Here $P_F g(x) := \mathbf{P}^x[g(X_{T_F}); T_F<\infty]$ is the hitting kernel associated with F). The measure π_F is the capacitory measure of F, and the total mass of π_F is equal to $\mathrm{Cap}(F)$. The switching identity

$$L(R_F m, h) = L(m, P_F h),$$

valid for excessive h, then implies

$$\begin{aligned}\Gamma(A) &= \nu_A^m(\varphi_\infty) = L(m, U_A\varphi_\infty)\\ &= L(m, P_F U_A\varphi_\infty) = L(R_F m, U_A\varphi_\infty)\\ &= L(\pi_F U, U_A\varphi_\infty) = \pi_F U_A\varphi_\infty\end{aligned}$$

$$= \pi_F(1 - \varphi_\infty).$$

That is,

$$\Gamma(A) = \mathbf{P}^{\pi_F}[1 - \mathrm{e}^{-A_\infty}],$$

when F is an equilibrium set with capacitory measure π_F. In particular, $\Gamma(A) \leqslant \mathrm{Cap}(F)$.

Finally, the Kuznetsov process allows us to extend Kac's asymptotic formula (2.3) to general m.

Theorem 3.6. *Let A be a* PCAF *with associated* HRM κ *and scattering length* $\Gamma(A)$. *If* $\nu_A^m(E) < \infty$, *then*

$$\Gamma(A) = \lim_{t\to\infty} t^{-1}\mathbb{Q}_m[1 - \exp(-\kappa]0,t])]. \tag{3.6}$$

Proof. As in the proof of Theorem 2.2 we may write

$$1 - \exp(-\kappa]0,t]) = \int_0^t \exp(-A_{t-s}\circ\theta_s)\,\kappa(ds), \quad t > 0.$$

Integrating with respect to $\mathbb{Q}_m$ and using the Revuz formula (see [7, (8.25)]) we obtain

$$\begin{aligned}\mathbb{Q}_m[1 - \exp(-\kappa]0,t])] &= \mathbb{Q}_m \int_0^t \varphi_{t-s}(Y_s)\,\kappa(ds) \\ &= \int_0^t ds \int_E \varphi_{t-s}(x)\nu_A^m(dx) \\ &= t\Gamma(A) + \int_0^t \nu_A^m(\varphi_u - \varphi_\infty)du.\end{aligned}$$

The assertion follows as before because $\int_0^t \nu_A^m(\varphi_u - \varphi_\infty)\,du$ is $o(t)$ as $t \to \infty$ when $\nu_A^m(E)$ is finite. □

It is known from [7] that the measure m is invariant if and only if $\mathbb{Q}_m[\alpha > -\infty] = 0$. Therefore, for invariant m,

$$\begin{aligned}\mathbb{Q}_m[1 - \exp(-\kappa]0,t])] &= \mathbb{Q}_m[1 - \exp(-\kappa]0,t]); \alpha < 0 < \beta] \\ &= \mathbf{P}^m[1 - \exp(-A_t)],\end{aligned}$$

because $\kappa]0,t] = 0$ on $\{\beta \leqslant 0\}$. This means that (2.3) is a special case of (3.6).

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