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Disjoint long cycles in a graph

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Abstract We prove that if G is a graph of order at least $2k$ with $k \geqslant 9$ and the minimum degree of G is at least $k+1$, then G contains two vertex-disjoint cycles of order at least k. Moreover, the condition on the minimum degree is sharp.

Keywords cycles, disjoint cycles, long cycles

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1 Introduction and terminology

A set of graphs are said to be disjoint if no two of them have any vertex in common. Erdős and Callai [9] showed that if G is a 2-connected graph of order n and every vertex of G, possibly except one, has degree at least k, then G contains a cycle of order at least $\min\{n, 2k\}$. El-Zahar [8] proved that if G is a graph of order $n = n_1 + n_2$ with minimum degree at least $\lceil n_1/2 \rceil + \lceil n_2/2 \rceil$ then G contains two disjoint cycles of order n_1 and n_2, respectively. In [13], we showed that if G is a graph of order $n \geqslant 6$ with minimum degree at least $(n+1)/2$ then for any two integers s and t with $s \geqslant 3$, $t \geqslant 3$ and $s + t \leqslant n$, G contains two disjoint cycles of order s and t, respectively unless s, t and n are odd and $G \cong K_{(n-1)/2,(n-1)/2} + K_1$. We ask the question: given a graph of order at least $2k$, when does G have two disjoint cycles of order at least k? Corrádi and Hajnal [5] proved that a graph G of order at least $3k$ with $\delta(G) \geqslant 2k$ contains k disjoint cycles. In [12], we proved that if G is a graph of order at least rk with $\delta(G) \geqslant (k-1)r$ then G contains r disjoint cycles of order at least k. In terms of the lower bound on the orders of cycles only, this minimum degree condition might be in general far from being sharp with $k \geqslant 4$. In this paper, we prove the following theorem:

Main Theorem. *Let k be an integer with $k \geqslant 9$ and G a graph of order at least $2k$. If the minimum degree of G is at least $k+1$, then G contains two disjoint cycles of order at least k.*

For any integer $k \geqslant 3$ and $m \geqslant 3$, $K_3 + mK_{k-2}$ has minimum degree k but it does not have two disjoint cycles of order at least k. In addition, for any odd integer $k \geqslant 3$, $K_{k,m}$ with $m \geqslant k$ has minimum degree k but it does not have two disjoint cycles of order at least k.

For each integer $k \geqslant 3$, let $\mathcal{G}_k$ be the set of all the graphs G of order at least k such that $V(G)$ has a partition $X \cup Y$ with $|X| = \lceil (k-2)/2 \rceil$ and $N_G(y) = X$ for all $y \in Y$. We use $K_n \cdot K_m$ to denote a graph of order $n + m - 1$ obtained from K_n and K_m by identifying a vertex of K_n with a vertex of K_m. In order to provide a unified proof, we did not include particular details here to show that the theorem is true for $k < 9$ for otherwise we would add some special lengthy details which would interrupt the flow of the proof.

Let G be a graph. A path from u to v is called a u-v path. If P is a path of G and v is an endvertex of P, we use $\alpha(P, v)$ to denote the order of the longest u-v subpath of P with $uv \in E(G)$. Clearly, if $\alpha(P, v) \geqslant 3$ then $P + uv$ has a cycle of order $\alpha(P, v)$. Let $w \in V(G)$. Let $P = w_1 w_2 \cdots w_t$ be a longest path starting at $w = w_1$. We say that P is an optimal path at w in G if $\alpha(P', x_t) \leqslant \alpha(P, w_t)$ for any longest path $P' = x_1 x_2 \cdots x_t$ starting at $w = x_1$ in G. In this case, if P is also a longest path of G, we say that P is an optimal path of G.

Let $x \in V(G)$. Let H be a subset of $V(G)$ or a subgraph of G. We define $N(x, H) = \{u \in N_G(x) | u$ belongs to $H\}$. Let $d(x, H) = |N(x, H)|$. If X is a subset of $V(G)$ or a subgraph of G, define $N(X, H) = \bigcup_x N(x, H)$ and $d(X, H) = \sum_x d(x, H)$, where x runs over X. Clearly, if X and H do not have any common vertex, then $d(X, H)$ is the number of edges of G between X and H. For $x, y \in V(G)$, define $I(xy, H) = N(x, H) \cap N(y, H)$ and let $i(xy, H) = |I(xy, H)|$. We use $e(G)$ to denote $|E(G)|$. The order of G is denoted by $|G|$.

If $C = x_1 \cdots x_t x_1$ is a cycle of G, we assume an orientation of C is given by default such that x_2 is the successor of x_1. Then $C[x_i, x_j]$ is the x_i-x_j path on C along the orientation of C. Define $C[x_i, x_j) = C[x_i, x_j] - x_j$ and $C(x_i, x_j] = C[x_i, x_j] - x_i$. The predecessor and successor of x_i on C are denoted by x_i^- and x_i^+. We will use similar definitions for a path. We use $C_{\geqslant k}$ and P_k to represent a cycle of order at least k and a path of order k, respectively. We use kG to represent a set of disjoint k copies of G. In addition, $rC_{\geqslant k}$ means that a set of r disjoint cycles of order at least k. If S is a set of subgraphs of G, we write $G \supseteq S$.

An *endblock* of G is a block of G *which contains at most one cut-vertex of* G. Thus a 2-connected component of G is an endblock. If each X_i $(1 \leqslant i \leqslant m)$ is a subset of $V(G)$ or a subgraph of G, then $[X_1, \dots, X_m]$ is the subgraph of G induced by the set of all the vertices belonging to at least one of $X_1, \dots, X_m$.

A linear forest of G is a subgraph of G such that each component in this subgraph is a path.

We use "h-cycle", "h-connected" and "h-path" for "hamiltonian cycle", "hamiltonian connected" and "hamiltonian path", respectively.

We use [2] for standard terminology and notation except as indicated above. Readers can refer to references [1–3, 6, 10, 11] on relevant topics.

2 Main ideas in the proof of Main Theorem

Let $k \geqslant 9$ be an integer and $G = (V, E)$ a graph of order $n \geqslant 2k$ with $\delta(G) \geqslant k + 1$. By El-Zahar's result [8], we see that $G \supseteq 2C_{\geqslant k}$ if $n \leqslant 2k + 1$. If G is not 2-connected, we readily see, by observing two endblocks of G, that $G \supseteq 2C_{\geqslant k}$. Therefore we may assume that $n \geqslant 2k + 2$ and G is 2-connected. On the contrary, say $G \not\supseteq 2C_{\geqslant k}$. By Lemma 3.8, $G \supseteq C_{\geqslant 2k+2}$. Therefore G has two subgraphs G_1 and G_2 such that $V(G_1) \cap V(G_2) = \emptyset$, $V(G_1 \cup G_2) = V(G)$, $G_1 \supseteq P_{k-1}$ with $|G_1| \geqslant k$ and $G_2 \supseteq P_k$. We choose G_1 and G_2 such that

$$e(G_1) + e(G_2) \text{ is maximum.} \tag{1}$$

Subject to (1), we choose G_1 and G_2 such that

$$|G_1| \text{ is minimum.} \tag{2}$$

We first show that $|G_1| = k$ and $G_2 \supseteq C_{\geqslant k+1}$. This will be accomplished in Section 4. Thus $G_1 \not\supseteq C_{\geqslant k}$. Let $u_0 \in V(G_1)$ with $d(u_0, G_1)$ minimal such that $G_1 - u_0 \supseteq P_{k-1}$. As $G_1 \not\supseteq C_{\geqslant k}$, $d(u_0, G_1) \leqslant (k-1)/2$. Let $H_1 = G_1 - u_0$ and $H_2 = G_2 + u_0$. Clearly, $e(H_1) + e(H_2) = e(G_1) + e(G_2) + d(u_0, G_2) - d(u_0, G_1) \geqslant e(G_1) + e(G_2) + 2$.

Then we choose an h-path $P = x_1 \cdots x_{k-1}$ of H_1 and a shortest path $L = v_1 \cdots v_q$ of H_2 such that $\{x_1 v_1, x_{k-1} v_q\} \subseteq E$. Set $H = H_2 - V(L)$. Thus $P \cup L + x_1 v_1 + x_{k-1} v_q$ is a cycle of order at least k and so

$H \not\supseteq C_{\geqslant k}$. We carefully choose P and L such that $\delta(H_2) \geqslant (k+3)/2$, $|H| \geqslant k+1$ and $\delta(H) \geqslant (k-1)/2$. This will be accomplished in Section 5. Let $B_1, \ldots, B_t$ be a list of endblocks of H. Ideally, we wish to find two disjoint paths P' and P'' in H such that $[P, P'] \supseteq C_{\geqslant k}$ and $[L, P''] \supseteq C_{\geqslant k}$. Otherwise we will find a subset $X \subseteq V(H)$ such that $H_2 - X \supseteq P_k$ and $e(H_1+X)+e(H_2-X) > e(H_1)+e(H_2)-2 \geqslant e(G_1)+e(G_2)$, contradicting (1). This will be accomplished in Sections 6 and 7. Section 6 proves that $t = 2$ and $|B_1 \cap B_2| = 1$. Let $V(B_1) \cap V(B_2) = \{w_1\}$. Section 7 proves that there exists $v_r \in V(L)$ such that $[x_1, v_1 \cdots v_r, B_1] \supseteq C_{\geqslant k}$ and $[v_{r+1} \cdots v_q, P - x_1, B_2 - w_1] \supseteq C_{\geqslant k}$.

3 Lemmas

Let $G = (V, E)$ be a graph of order $n \geqslant 3$. We will use the following lemmas. Lemma 3.1 is an easy observation.

Lemma 3.1. *Let P be a u-v path of order l in G. Then the following three statements hold*:

(a) *If $x \in V(G) - V(P)$ and $P + x$ does not contain a u-v path of order $< l$, then $d(x, P) \leqslant 3$ and if equality holds then $N(x, P)$ contains three consecutive vertices of P.*

(b) *If xy is an edge of $G - V(P)$ with $d(xy, P) \geqslant 5$ and $P + x + y$ does not contain a u-v path of order $< l$, then $i(xy, P) \geqslant 1$ and if $d(xy, P) = 6$ then $i(xy, P) \geqslant 2$.*

(c) *If P' is an x-y path of order at least r in $G - V(P)$ such that $d(x, P) > 0$, $d(y, P) > 0$, $d(x, P) \geqslant k - r$ and $d(y, P) \geqslant k - r - 1$, then $[P, P']$ contains a cycle of order $\geqslant k$.*

Lemma 3.2 (See [8]). *Let $P = x_1x_2 \cdots x_r$ be a path of G with $r \geqslant 2$ and $y \in V(G) - V(P)$. If $d(y, P) \geqslant r/2$, then $P + y$ has a path P' with $V(P') = V(P) \cup \{y\}$. Furthermore, if $d(y, P) > r/2$ then P' is an x_1-x_r path or r is odd and $N(y, P) = \{x_{2i-1} \mid i = 1, 2, \ldots, (r+1)/2\}$.*

Lemma 3.3 (See [9]). *Let C be a cycle of order k in G. Let $\{x, y\} \subseteq V(C)$ with $x \neq y$. Suppose that $d(x, C) + d(y, C) \geqslant k + 1$. Then $[C]$ has a path P from x^+ to y^+ with $V(P) = V(C)$.*

Lemma 3.4 (See [4,13]). *Suppose that G has an h-path and that for any two endvertices x and y of an h-path of G, $d(x, G) + d(y, G) \geqslant n + r$ holds, where r is a fixed positive integer. Then for any two distinct vertices u and v of G, $d(u, G) + d(v, G) \geqslant n + r$ holds. Moreover, for any linear forest F in G with $e(F) \leqslant r$, G has an h-cycle passing through all the edges of F.*

Lemma 3.5 (See [7]). *Let $d \geqslant 2$ be an integer and let G be a 2-connected graph of order at least 3 such that if $d \geqslant 3$ then the order of G is at least 4. Let x and y be two distinct vertices of G. If every vertex in $V(G) - \{x, y\}$, possibly except one, has degree at least d in G, then G contains an x-y path of order at least $d + 1$.*

Lemma 3.6. *Let P be a path of order r in G with $r < |G|$. If G is connected and $d(x) \geqslant r/2$ for each $x \in V(G) - V(P)$ then G contains a path of order at least $r + 1$.*

Proof. Let Q be a longest u-v path in $G - V(P)$ with $d(u, P) > 0$. It is easy to see that $[P, Q] \supseteq P_{r+1}$. □

Lemma 3.7 (See [9]). *Let $P = x_1x_2 \cdots x_t$ be an optimal path at x_1 in G. Let $r = \alpha(P, x_t)$. Suppose that for each $v \in V(G)$, if there exists a longest path starting at x_1 in G such that the path ends at v then $d(v) > r/2$. Then $N(x_i) \subseteq \{x_{t-r+1}, x_{t-r+2}, \ldots, x_t\}$, $[P]$ has an x_1-x_i h-path and $d(x_i) > r/2$ for all $i \in \{t - r + 2, t - r + 3, \ldots, t\}$. Moreover, if $t > r$ then x_{t-r+1} is a cut-vertex of G.*

Lemma 3.8 (See [9]). *Let $h \geqslant 2$ be an integer. If B is a 2-connected graph such that every vertex, possibly except one, has degree at least $h/2$, then B contains a cycle of order at least* $\min(|B|, 2h)$.

Lemma 3.9. *Let $k \geqslant 5$ be an integer. Let B be a 2-connected graph of order at least k. Let w be a vertex of B. Suppose that $B \not\supseteq C_{\geqslant k}$ and $d(x, B) \geqslant (k-1)/2$ for all $x \in V(B) - \{w\}$. Then k is odd and B has a cycle C of order $k-1$. Moreover, for some vertex u on C, $d(x, C) = (k-1)/2$ and $N(x, B) \subseteq V(C)$ for each $x \in \{u^-, u^+\}$. In addition, if $w \in V(C)$ then $w = u$.*

Proof. Let $P = x_1x_2 \cdots x_t$ be an optimal path at $w = x_1$. As B has no cut-vertex and by Lemma 3.7, $\alpha(P, x_t) = k-1$. Say $r = t-k+2$. Then $C = x_rx_{r+1} \cdots x_tx_r$ is a cycle of order $k-1$. As B is 2-connected and by the optimality of P, there exists $s \in \{r+2, \dots, t-1\}$ such that $d(x_s, B - V(C)) \geqslant 1$. Let a and b be the smallest and largest numbers in $\{r+2, \dots, t-1\}$, respectively such that $d(x_a, B - V(C)) \geqslant 1$ and $d(x_b, B - V(C)) \geqslant 1$. So $N(x_i, B) \subseteq V(C)$ for all $i \in \{r+1, r+2, \dots, a-1, b+1, b+2, \dots, x_t\}$. By the optimality of P, $[C]$ does not have an x_r-x_a h-path. By Lemma 3.3, $d(x_t x_{a-1}, C) \leqslant k-1$. Thus $k-1$ is even with $d(x_t, C) = d(x_{a-1}, C) = (k-1)/2$. Similarly, $d(x_{r+1}, C) = d(x_{b+1}, C) = (k-1)/2$. Thus the lemma holds with $u = x_r$. □

Lemma 3.10. *Let $k \geqslant 3$ be an integer. Let H be a non-h-graph of order k with $H \supseteq P_{k-1}$. Suppose that $d(x, H) \geqslant (k-1)/2$ for each $x \in V(H)$ with $H - x \supseteq P_{k-1}$. Then k is odd and either $H \in \mathcal{G}_k$ or $H \cong K_{(k+1)/2} \cdot K_{(k+1)/2}$.*

Proof. By Lemma 3.2, $H \supseteq P_k$. First, assume that H has a cycle C of order $k-1$. Then $d(v, C) \geqslant (k-1)/2$ where $\{v\} = V(H) - V(C)$. It follows that k is odd and there exists $X \subseteq V(C)$ with $|X| = (k-1)/2$ such that no two vertices of X are consecutive on C and $N(v, C) = X$. Then $H - u \supseteq P_{k-1}$ and so $d(u, H) \geqslant (k-1)/2$ for each $u \in Y = V(H) - X$. Thus $N(u, H) = X$ for each $u \in Y$ as $H \not\supseteq C_k$, i.e., $H \in \mathcal{G}_k$. If $H \not\supseteq C_{k-1}$, then by Lemma 3.7, H has a cut-vertex and it follows that $H \cong K_{(k+1)/2} \cdot K_{(k+1)/2}$. □

Lemma 3.11. *Let $k \geqslant 10$ be an even integer. Let H be a non-h-graph of order k with $H \supseteq P_{k-1}$ such that $d(x, H) \geqslant (k-2)/2$ for each $x \in V(H)$ with $H - x \supseteq P_{k-1}$. Then one of the following two statements hold*:

(a) *H has an h-path and two endblocks X_1 and X_2 such that $V(H) = V(X_1 \cup X_2)$ and $|X_1 \cap X_2| \leqslant 1$.*

(b) *There is a partition $V(H) = X \cup Y$ with $|X| = (k-2)/2$ and $|Y| = (k+2)/2$ such that Y has two vertices u_1 and u_2 such that $N(y, H) = X$ for all $y \in Y - \{u_1, u_2\}$ and $d(u_i, X \cup \{u_1, u_2\}) \geqslant (k-2)/2$ for each $i \in \{1, 2\}$.*

Proof. First, assume that $H \not\supseteq P_k$. Let $y \in V(H)$ and $P = x_1 \cdots x_{k-1}$ be an h-path of $H - y$. Applying Lemma 3.2 to $H - x_1 - x_{k-1}$, we get $N(y, H) = \{x_2, x_4, \dots, x_{k-2}\}$. As $H \not\supseteq P_k$, $\{y, x_1, x_3, \dots, x_{k-1}\}$ is independent. Clearly, for each $i \in \{1, 3, \dots, k-1\}$, $H - x_i \supseteq P_{k-1}$ and so $d(x_i, H) \geqslant (k-2)/2$. It follows that $H \in \mathcal{G}_k$, i.e., (b) holds. Next, assume that H has an h-path. As $d(x, H) \geqslant (k-2)/2$ for each endvertex x of an h-path of H, we see that if H has a cut-vertex then (a) holds.

We now assume that H is 2-connected, $H \supseteq P_k$ and $H \notin \mathcal{G}_k$. Let P be a u-v h-path of H with $\alpha(P, v)$ maximal. As H is 2-connected and by Lemma 3.7, $\alpha(P, v) \geqslant (k-2)$. First, assume that $H \supseteq C_{k-1}$. Let C be a cycle of order $k-1$. Let x be the vertex not on C. Since $k-1$ is odd, $d(x, C) \geqslant (k-2)/2$ and $H \not\supseteq C_k$, there exists a labelling $C = u_1 u_2 \cdots u_{k-1} u_1$ such that $N(x, C) = \{u_3, u_5, \dots, u_{k-1}\}$. Say $X = N(x, C)$ and $Y' = \{x, u_4, u_6, \dots, u_{k-2}\}$. Since $H \not\supseteq C_k$, $Y' \cup \{u_i\}$ is an independent set of H for $i \in \{1, 2\}$. Clearly, each $y \in Y' \cup \{u_1, u_2\}$ is an endvertex of an h-path of H and so $d(y, H) \geqslant (k-2)/2$. Thus (b) holds with $Y = Y' \cup \{u_1, u_2\}$.

Therefore we may assume that $\alpha(P, v) = k-2$. Say $P = x_1 x_2 u_1 u_2 \cdots u_{k-2}$ with $u_1 u_{k-2} \in E$. Let $C = P - x_1 - x_2$. As H is 2-connected, either $d(x_1, C - u_1) > 0$ or $x_1 u_1 \in E$ and $d(x_2, C - u_1) > 0$. Say w.l.o.g. $d(x_1, C - u_1) > 0$. Then $x_1 u_i \notin E$ for each $i \in \{2, 3, k-3, k-2\}$. As $H \not\supseteq C_{\geqslant (k-1)}$, $d(x, C[u_4, u_{k-4}]) \leqslant (k-6)/2$ by Lemma 3.2. As $d(x_1) \geqslant (k-2)/2$, it follows that $N(x_1) = \{x_2, u_1, u_4, u_6, \dots, u_{k-4}\}$. Let $Y = \{u_5, u_7, \dots, u_{k-5}\}$. As $k \geqslant 10$, $Y \neq \emptyset$. Clearly, each $y \in Y \cup \{x_1, x_2, u_2, u_3, u_{k-3}, u_{k-2}\}$ is an endvertex of an h-path of H. Since $H \not\supseteq C_{\geqslant (k-1)}$, $Y \cup \{u_i\}$ is an independent set of H for each $i \in \{2, 3, k-3, k-2\}$ and $d(u_2 u_3, u_{k-3} u_{k-2}) = 0$. It follows that $N(x_2, C) = N(x_1, C)$. Thus $d(y, H) \leqslant (k-4)/2$ for each $y \in Y$, a contradiction. □

Lemma 3.12. *Let $k \geqslant 5$ be an integer. Let H be a 2-connected graph of order at least k. Suppose that $H \not\supseteq C_{\geqslant k}$ and $\delta(H) \geqslant (k-1)/2$. Then k is odd. Moreover, either $H \in \mathcal{G}_k$ or H has a vertex-cut $\{x, y\}$ such that $H - \{x, y\}$ has at least three components and each of them is isomorphic to $K_{(k-3)/2}$.*

Proof. Let P be an optimal path of H. Say P is an optimal u-v path at u. By Lemma 3.9, we see that k is odd and $\alpha(P, v) = k-1$. Say $P = x_1 x_2 \cdots x_t u_1 u_2 \cdots u_{k-1}$ with $u_1 u_{k-1} \in E$. Let $P' = u_1 x_t x_{t-1} \cdots x_1$ and $C = u_1 u_2 \cdots u_{k-1} u_1$. Then P' is a longest path starting at u_1 in $H - \{u_2, \dots, u_{k-1}\}$.

Let us first assume that for each longest path Q starting at u_1 in $H - \{u_2, \dots, u_{k-1}\}$, if Q ends at w then

$d(w, C-u_1) = 0$. In this situation, we may assume that P' is an optimal path at u_1 in $H - \{u_2, \ldots, u_{k-1}\}$. As H is 2-connected and by Lemma 3.7, we see that $\alpha(P', x_1) = k-1$. Hence $H - \{u_2, \ldots, u_{k-1}\}$ has a cycle C' of order $k-1$. Since H is 2-connected, there exist two disjoint paths from C' to C. This implies $H \supseteq C_{\geqslant k}$, a contradiction.

Therefore we may assume w.l.o.g. that $d(x_1, C-u_1) \geqslant 1$. Say $N(x_1, C-u_1) = \{u_{i_1}, \ldots, u_{i_r}\}$ with $1 < i_1 < \cdots < i_r < k-1$. Since $H \nsupseteq C_{\geqslant k}$ and $d(x_1, H) \geqslant (k-1)/2$, we see that $d(x_1, H) = (k-1)/2$, $\{x_2, \ldots, x_t, u_1\} \subseteq N(x_1, H)$, $i_1 = t+2$, $k-t-1 = i_r$ and $i_{j+1} = i_j + 2$ for $1 \leqslant j \leqslant r-1$. Let $I_1 = \{u_2, \ldots, u_{t+1}\}$, $I_2 = \{u_{k-t}, \ldots, u_{k-1}\}$, $I_3 = \{u_{t+2i+1} \mid i = 1, 2, \ldots, (k-1)/2 - t - 1\}$, $I_4 = \{x_1, \ldots, x_t\}$. As $H \nsupseteq C_{\geqslant k}$, we readily see that $d(I_a, I_b) = 0$ for $1 \leqslant a < b \leqslant 4$ and I_3 is an independent set. It is easy to see that each $y \in I_3 \cup I_4 \cup \{u_2, u_{k-1}\}$ is an endvertex of an h-path of $[P]$ which is a longest path of H and so $N(y, H) \subseteq V(P)$. As $\delta(H) \geqslant (k-1)/2$. It follows that $N(x_i, H) = N(x_1, H)$ for $i = 1, 2, \ldots, t$, $N(u_2, H) = I_1 \cup N(x_1, C) - \{u_2\}$, $N(u_{k-1}, H) = I_2 \cup N(x_1, C) - \{u_{k-1}\}$ and $N(u_i, H) = N(x_1, C)$ for all $u_i \in I_3$. If $I_3 \neq \emptyset$ then $t = 1$ for otherwise $d(u_i, H) < (k-1)/2$ for each $u_i \in I_3$. Consequently, $N(y, H) = \{u_1, u_3, \ldots, u_{k-2}\}$ for each $y \in I_3 \cup I_4$. This argument implies that $N(y, H) = \{u_1, u_3, \ldots, u_{k-2}\}$ for all $y \in V(H) - \{u_1, u_3, \ldots, u_{k-2}\}$ and so $H \in \mathcal{G}_k$. If $I_3 = \emptyset$, then $t = (k-3)/2$ and $i_1 = i_r = (k+1)/2$. Thus $N(u_2, H) = I_1 \cup \{u_1, u_{(k+1)/2}\} - \{u_2\}$ and so each $u_i \in I_1$ is an endvertex of an h-path of $[P]$. As $\delta(H) \geqslant (k-1)/2$, it follows that $N(u_i, H) = I_1 \cup \{u_1, u_{(k+1)/2}\} - \{u_i\}$ for each $u_i \in I_1$. Similarly, $N(u_i, H) = I_2 \cup \{u_1, u_{(k+1)/2}\} - \{u_i\}$ for each $u_i \in I_2$. Thus the three components of $[P] - \{u_1, u_{(k+1)/2}\}$ are isomorphic to $K_{(k-3)/2}$ and they are components of $H - \{u_1, u_{(k+1)/2}\}$. This argument implies that all the other components of $H - \{u_1, u_{(k+1)/2}\}$ are isomorphic to $K_{(k-3)/2}$, too. □

4 Four properties on G_1 and G_2

Let G_1 and G_2 be the two subgraphs satisfying (1). We shall show the following four properties.

Property 1. *For each $x \in V(G_1)$ with $G_1 - x \supseteq P_{k-1} \cup K_1$, $d(x, G_1) \geqslant (k+1)/2$, and for each $y \in V(G_2)$ with $G_2 - y \supseteq P_k$, $d(y, G_2) \geqslant (k+1)/2$. Furthermore, G_1 contains at most two components and G_2 is connected. In addition, if G_1 has a component of order at least k containing P_{k-1} then G_1 is connected.*

Proof. By (1), for each $x \in V(G_1)$ with $G_1 - x \supseteq P_{k-1} \cup K_1$, $e(G_1) + e(G_2) \geqslant e(G_1 - x) + e(G_2 + x)$ which implies $d(x, G_1) \geqslant d(x, G_2)$ and so $d(x, G_1) \geqslant (k+1)/2$. Similarly, for each $y \in V(G_2)$ with $G_2 \supseteq P_k$, $d(y, G_2) \geqslant (k+1)/2$. As G is connected, we see that if G_1 contains a component C with $G_1 - V(C) \supseteq P_{k-1} \cup K_1$ then $e(G_1 - V(C)) + e(G_2 + V(C)) > e(G_1) + e(G_2)$, contradicting (1). Therefore G_1 does not have such a component. Similarly, G_2 shall not have a component C' with $G_2 - V(C') \supseteq P_k$. This proves Property 1. □

Property 2. *For each $i \in \{1, 2\}$, if $G_i \nsupseteq C_{k+1}$, then $|G_i| = k$.*

Proof. We first show that if $G_2 \nsupseteq C_{k+1}$, then $|G_2| = k$. On the contrary, say that $G_2 \nsupseteq C_{\geqslant k+1}$ and $|G_2| > k$. Let $P = x_1 x_2 \cdots x_t$ be an optimal path in G_2 with $\alpha(P, x_t)$ maximal. By Lemma 3.6, $t > k$. Thus for any longest path P' in G_2, if v is an endvertex of P', then $G_2 - v \supseteq P_k$ and so $d(v, G_2) \geqslant (k+1)/2$ by Property 1. Say $\alpha(P, x_t) = r$. Then $x_t x_{t-r+1} \in E$. As $G_2 \nsupseteq C_{\geqslant k+1}$, $r \leqslant k$. Say $B_1 = \{x_{t-r+2}, \ldots, x_t\}$. By Lemma 3.7, $N(x_i, G_2) \subseteq B_1 \cup \{x_{t-r+1}\}$ and $(k+1)/2 \leqslant d(x_i, G_2)$ for all $x_i \in B_1$. So x_{t-r+1} is a cut-vertex of G_2. Let $L = P - B_1$. We may assume that L is an optimal path at x_{t-r+1} in $G_2 - B_1$. Say $\alpha(L, x_1) = s$ and $B_2 = \{x_1, \ldots, x_{s-1}\}$. Similarly, $s \leqslant k$, $N(x_i, G_2) \subseteq B_2 \cup \{x_s\}$ and $(k+1)/2 \leqslant d(x_i, G_2)$ for all $x_i \in B_2$. By the maximality of $\alpha(P, x_t)$, $s \leqslant r$. Let $s - 1 = a + b$ such that if $t - (s-1) \geqslant k$ then $a = 0$ and if $t - (s-1) < k$ then $a = k - t + (s-1)$. Let $X = \{x_1, x_2, \ldots, x_b\}$. Then $X \subseteq B_2$, $G_2 - X \supseteq P_k$, $d(X, G_2 - X) \leqslant b(a+1)$ and $d(X, G_1) \geqslant \sum_{x_i \in X}(k+1-d(x_i, G_2)) \geqslant b(k+1-(s-1))$. This yields

$$\begin{aligned} e(G_2 - X) + e(G_1 + X) &\geqslant e(G_2) + e(G_1) - b(a+1) + b(k-s+2) \\ &= e(G_2) + e(G_1) + b(k-s-a+1) > e(G_2) + e(G_1), \end{aligned}$$

contradicting (1). Therefore if $G_2 \not\supseteq C_{k+1}$, then $|G_2| = k$.

Next, assume that $G_1 \not\supseteq C_{\geqslant k+1}$ but $|G_1| > k$. Let F be a component of G_1 with $F \supseteq P_{k-1}$. If $|F| = k-1$, then G_1 has another component F' and $d(x, F') \geqslant (k+1)/2$ for all $x \in V(F')$ by Property 1. Let B be an endblock of F'. Then B has a vertex $w \in V(B)$ such that $N(x, F') \subseteq V(B)$ for all $x \in V(B) - \{w\}$. As $G_1 \not\supseteq C_{\geqslant k+1}$ and by Lemma 3.8, $|B| \leqslant k$. Therefore $d(x, G_2) \geqslant 2$ for all $x \in V(B) - \{w\}$. Thus $e(G_1 - V(B-w)) + e(G_2 + V(B-w)) > e(G_1) + e(G_2)$, contradicting (1). Hence $|F| \geqslant k$ and so $G_1 = F$ by Property 1. By Lemma 3.6 and Property 1, $G_1 \supseteq P_{k+1}$. Then a contradiction follows by exchanging the roles of G_1 and G_2 in the above paragraph. □

Subject to (1), we now choose G_1 and G_2 to satisfy (2). By Property 2, we see that either $|G_1| = k$ or $|G_2| = k$. If $|G_2| = k$, then $|G_1| > k$ and $G_1 \supseteq C_{\geqslant k+1}$. As $G_2 \supseteq P_{k-1} \cup K_1$ and $G_1 \supseteq P_k$, we shall have $|G_1| = k$ by (2), a contradiction. Hence $|G_1| = k$ and $|G_2| \geqslant n - k \geqslant k+2$ and so $G_2 \supseteq C_{\geqslant k+1}$. Thus $G_2 - x \supseteq P_k$ for all $x \in V(G_2)$. Subject to (1) and (2), we further choose G_1, G_2 and a vertex $u_0 \in V(G_1)$ with $G_1 - u_0 \supseteq P_{k-1}$ such that $d(u_0, G_1)$ is minimum. If $d(u_0, G_1) \geqslant k/2$ then G_1 has an h-path by Lemma 3.2 and so $d(uv, G_1) \geqslant k$ for any u-v h-path of G_1. Consequently, $G_1 \supseteq C_{\geqslant k}$, a contradiction. Hence $d(u_0, G_1) \leqslant (k-1)/2$.

Property 3. *G_2 is 2-connected with $\delta(G_2) \geqslant (k+2)/2$.*

Proof. First, suppose that $d(x_0, G_2) = (k+1)/2$ for some $x_0 \in V(G_2)$. Then $d(x_0, G_1) \geqslant (k+1)/2$. Thus $e(G_1 + x_0) + e(G_2 - x_0) \geqslant e(G_1) + e(G_2)$ with equality only if $d(x_0, G_1) = (k+1)/2$. With $G_1 + x_0$ and $G_2 - x_0$ in place of G_1 and G_2, we see that $G_1 + x_0 \supseteq C_{\geqslant k+1}$ and $G_2 - x_0 \supseteq C_{\geqslant k+1}$ by Property 2 since $|G_1 + x_0| > k$ and $|G_2 - x_0| > k$, a contradiction. Therefore $\delta(G_2) \geqslant (k+2)/2$. Next, suppose that G_2 has a cut-vertex w. Then $G_2 - w$ has two subgraphs J_1 and J_2 such that $G_2 - w = J_1 \cup J_2$, $J_1 \cap J_2 = \emptyset$ and $J_2 + w \supseteq C_{\geqslant k+1}$. Then $J_1 \not\supseteq C_{\geqslant k}$. Let $L = v_1 \cdots v_p$ be a longest path in J_1. Say $d(v_1, L) \geqslant d(v_p, L)$. Then $k-2 \geqslant d(v_1, L)$ and $d(v_i, G_1 - u_0) \geqslant k+1-2-d(v_i, L) \geqslant k - (d(v_1, L)+1)$ for $i \in \{1, p\}$. Since $G_1 - u_0$ has an h-path and $p \geqslant d(v_1, L)+1$, it follows that $[L, G_1 - u_0] \supseteq C_{\geqslant k}$ by Lemma 3.1(c), a contradiction. □

Property 4. *For each $x \in V(G_2)$, $G_1 + x \not\supseteq C_{\geqslant k}$.*

Proof. Assume by contradiction that $G_1 + x_0 \supseteq C_{\geqslant k}$ for some $x_0 \in V(G_2)$. Say $H = G_2 - x_0$. Then $H \not\supseteq C_{\geqslant k}$ and $\delta(H) \geqslant (k+2)/2 - 1 = k/2$. By Lemma 3.8, H is not 2-connected. Let B_1 and B_2 be two endblocks of H. Say $r = |B_1| \leqslant s = |B_2|$. For each $i \in \{1, 2\}$, let w_i be the cut-vertex of H with $w_i \in V(B_i)$. Say $B_i' = V(B_i) - \{w_i\} (i = 1, 2)$. By Lemma 3.8, $r < k$ and $s < k$. By Lemma 3.7, for each $i \in \{1, 2\}$ and each $x \in B_i'$, B_i has a w_i-x h-path. Let $P = x_1 x_2 \cdots x_t$ be a longest path of H with $x_1 \in B_2'$ and $x_t \in B_1'$. Then $B_2 = [x_1, \ldots, x_s]$, $B_1 = [x_{t-r+1}, \ldots, x_t]$, $w_2 = x_s$ and $w_1 = x_{t-r+1}$. Let $r - 1 = a + b$ with $a = \max\{0, k-1-(t-r+1)\}$. Then $[x_0, x_1, \ldots, x_{t-r+1+a}] \supseteq P_k$. Let $X = \{x_{t-b+1}, x_{t-b+2}, \ldots, x_t\}$. Then we have

$$\begin{aligned} & e(G_1 + X) + e(G_2 - X) \\ & \quad \geqslant e(G_1) + \sum_{x \in X} (k+1-d(x, B_1 + x_0)) + e(G_2) - \sum_{x \in X} d(x, B_1 - X + x_0) \\ & \quad \geqslant e(G_1) + e(G_2) + b(k-r+1) - b(a+2) = e(G_1) + e(G_2) + b(k-r-a-1). \end{aligned}$$

As $k > s \geqslant r$ and $t \geqslant r+s-1$, we see that $k-r-a-1 \geqslant 0$. By (1), it follows that $r = s$ and $k = r+a+1$. Furthermore, $xx_0 \in E$ and $d(x, B_1) = r-1$ for all $x \in X$. Since each $x_i \in B_1'$ can play the role of x_t, this argument implies that $B_1 \cong K_r$ and $d(x_0, B_1') = r-1$. Similarly, $B_2 \cong K_r$ and $d(x_0, B_2') = r-1$. Thus $G_2 - X \supseteq [x_0, x_1, \ldots, x_{t-r+1+a}] \supseteq C_{\geqslant k}$. Then $G_1 + X \not\supseteq C_{\geqslant k}$. Since (1) is maintained with $G_1 + X$ and $G_2 - X$ in place of G_1 and G_2, we obtain $|G_1 + X| = k$ by Property 2, a contradiction. □

5 Properties on $G_1 - u_0$ and $G_2 + u_0$

For convenience, let $H_1 = G - u_0$ and $H_2 = G_2 + u_0$. We will choose an h-path $P = x_1 \cdots x_{k-1}$ of H_1 and a shortest path $L = v_1 \cdots v_q$ in H_2 with $\{x_1 v_1, x_{k-1} v_q\} \subseteq E$. Then we set $H = H_2 - V(L)$. The following cases tell us how to choose P and L so that the properties on H_1, H_2 and H allow us to find $2C_{\geqslant k}$ in G or we find that (1) is violated.

As $d(u_0, G_1) \leqslant \lfloor (k-1)/2 \rfloor$, $d(u_0, G_2) \geqslant \lceil (k+3)/2 \rceil$. For $x \in V(G_1)$ and $y \in V(G_2)$, we define $\xi(x,y) = d(x,G_2) + d(y,G_1) - d(x,G_1) - d(y,G_2) - 2d(x,y)$. Then $e(G_1 - x + y) + e(G_2 - y + x) = e(G_1) + e(G_2) + \xi(x,y)$. Clearly, $G_2 - y \supseteq P_k$ and $\xi(x,y) \geqslant 2(k+1) - 2(d(x,G_1) + d(y,G_2) + d(x,y))$. If $G_1 - x + y \supseteq P_{k-1} \cup K_1$ then

$$\xi(x,y) \leqslant 0 \text{ and so } d(x,G_1) + d(y,G_2) + d(x,y) \geqslant k+1. \tag{3}$$

We consider the following cases.

Case 1. G_1 is 2-connected and $e(u_0, G_1) = \lfloor (k-1)/2 \rfloor = \lceil (k-2)/2 \rceil$.

In this case, by Lemmas 3.10 and 3.11, $V(G_1)$ has a partition $X \cup Y$ with $|X| = \lfloor (k-1)/2 \rfloor$ and $|Y| = \lfloor (k+2)/2 \rfloor$ such that either $N(y, G_1) = X$ for all $y \in Y$, or k is even and $[Y]$ has an edge u_1u_2 such that $N(y, G_1) = X$ for all $y \in Y - \{u_1, u_2\}$ and $d(u_i, G_1) \geqslant (k-2)/2$ for each $i \in \{1, 2\}$. Among all the choices of G_1 and G_2 satisfying (1) and (2) in Case 1, we may assume that G_1 and G_2 have been chosen with $e([Y])$ maximal. Thus $e([Y]) \leqslant 1$ and if equality holds then k is even.

Let $L = v_1 \cdots v_q$ be a shortest path of H_2 such that $\{v_1y, v_qy'\} \subseteq E$ for some vertices y and y' of Y with $y \neq y'$. Moreover, if $e([Y]) = 1$ then $\{y, y'\} \subseteq Y - \{u_1, u_2\}$. Subject to the above assumption on G_1 and G_2, we further choose G_1, G_2 and L with $|L|$ being minimal. As $k \geqslant 9$, we may choose $u_0 \in Y$ such that $N(u_0, G_1) = X$ and $u_0 \notin \{y, y'\}$. Then $P = x_1 \cdots x_{k-1}$ is defined to be an h-path of H_1 from y to y'. Clearly,

$$d(x_1x_{k-1}, H_1) = 2\lfloor (k-1)/2 \rfloor \text{ and so } d(x_1x_{k-1}, H) \geqslant 2(k+1) - 2\lfloor (k-1)/2 \rfloor - 2 \geqslant k+1. \tag{4}$$

We claim that

$$\delta(H_2) \geqslant \lceil (k+3)/2 \rceil \text{ and } d(z, L) = 0 \text{ for each } z \in V(H) \text{ with } d(z, H_2) = \lceil (k+3)/2 \rceil. \tag{5}$$

Proof of (5). By Property 4, for all $z \in V(G_2)$, $G_1 + z \not\supseteq C_{\geqslant k}$ and so $d(z, Y) \leqslant 1$. In particular, $q \geqslant 2$. Then we see that for each $z \in V(G_2)$, there is $y \in Y$ with $d(y, G_1) = \lfloor (k-1)/2 \rfloor$ such that $zy \notin E$. By (3), $d(z, G_2) \geqslant (k+1) - \lfloor (k-1)/2 \rfloor = \lceil (k+3)/2 \rceil$. Hence $\delta(H_2) \geqslant \lceil (k+3)/2 \rceil$. Assume that $d(z, L) > 0$ and $d(z, H_2) = \lceil (k+3)/2 \rceil$ for some $z \in V(H)$. Then $d(z, H_1) \geqslant k + 1 - \lceil (k+3)/2 \rceil = \lfloor (k-1)/2 \rfloor$ and $d(z, Y) \leqslant 1$. If $d(z, Y) = 1$ then $z \neq u_0$, k is even and $e([Y]) = 0$ since $H_1 + z \not\supseteq C_{\geqslant k}$. Furthermore, we may replace G_1 and G_2 by $H_1 + z$ and $H_2 - z$ in Case 1 and obtain $e([Y \cup \{z\} - \{u_0\}]) = 1$, contradicting the maximality of $e([Y])$. Hence $N(z, H_1) = X$. As $d(z, L) > 0$, we see that L has a u-v subpath L' with $|L'| < |L|$ such that $\{uz, vz'\} \subseteq E$ for some $z' \in \{y, y'\}$, contradicting the minimality of $|L|$ if we replace G_1 and G_2 with $H_1 + z$ and $H_2 - z$. Therefore $d(z, L) = 0$. □

Case 2. G_1 is not 2-connected and $d(u_0, G_1) = \lfloor (k-1)/2 \rfloor$.

Let c_0 be a cut-vertex of G_1. First, assume that k is odd. By Lemma 3.10, G_1 has two complete subgraphs X_1 and X_2 of order $(k+1)/2$ with $V(X_1) \cap V(X_2) = \{c_0\}$. Let z be an arbitrary vertex of G_2. By Property 4, $N(z, G_1) \subseteq V(X_1)$ or $N(z, G_1) \subseteq V(X_2)$. Say w.l.o.g. $N(z, G_1) \subseteq V(X_2)$. Let $x \in V(X_1) - \{c_0\}$. By (3), $d(z, G_2) \geqslant k + 1 - d(x, G_1) \geqslant (k+3)/2$. If $d(z, G_2) = (k+3)/2$ then $\xi(x, z) \geqslant 0$ and so $\xi(x, z) = 0$, i.e., $e(G_1 - x + z) + e(G_2 - z + x) = e(G_1) + e(G_2)$ and $d(y, G_1 - x + z) = (k-3)/2$ for all $y \in V(X_1 - c_0)$, contradicting the minimality of $d(u_0, G_1)$. Thus $\delta(G_2) \geqslant (k+5)/2$. Let $L = v_1 \cdots v_q$ be a shortest path of G_2 such that $\{v_1y, v_qy'\} \subseteq E$ for some $y \in V(X_1 - c_0)$ and $y' \in V(X_2 - c_0)$. We may choose $u_0 \in V(G_1) - \{y, y', c_0\}$. Let $P = x_1 \cdots x_{k-1}$ be a y-y' h-path of H_1. By the minimality of $|L|$, we conclude that if k is odd then

$$d(x_1x_{k-1}, H_1) = k - 2 \text{ and so } d(x_1x_{k-1}, H) \geqslant k + 2; \tag{6}$$

$$d(u_0, H_2) \geqslant (k+3)/2, \quad \delta(H_2 - u_0) \geqslant (k+5)/2, \quad u_0 \notin V(L), \quad d(u_0, L) \leqslant 1$$
$$\text{and if } d(u_0, L) = 1 \text{ then } d(u_0, v_1v_q) = 1. \tag{7}$$

Next, assume that k is even. By Lemma 3.11, G_1 has an h-path and two endblocks X_1 and X_2 with $V(G_1)=V(X_1\cup X_2)$. Say $|X_1|\leqslant|X_2|$. Then $|X_1|=k/2$ and $|X_2|\leqslant k/2+1$. Let $c_i\in V(X_i)$ be the cut-vertex of G_1 for $i\in\{1,2\}$. As $d(x,G_1)\geqslant(k-2)/2$ for each endvertex x of an h-path of G_1, it follows that $X_1\cong K_{k/2}$. Moreover, we see, by Lemma 3.7, that $d(x,X_2)\geqslant(k-2)/2$ for all $x\in V(X_2-c_2)$. As $k\geqslant 9$, $\delta(X_2-c_2)\geqslant(k-2)/2-1>k/4$ and so X_2-c_2 is h-connected by Lemma 3.4. Let z be an arbitrary vertex of G_2. By Property 4, $N(z,G_1)\subseteq V(X_1)\cup\{c_2\}$ or $N(z,G_1)\subseteq V(X_2)\cup\{c_1\}$. If $N(z,G_1)\not\supseteq V(X_1)-\{c_1\}$, let $x\in V(X_1)-\{c_1\}$ with $xz\notin E$, and by (3), we see that $d(z,G_2)\geqslant k+1-d(x,G_1)\geqslant(k+4)/2$. Moreover, if equality holds then $d(z,X_2-c_2)>0$ and $e(G_1-x+z)+e(G_2-z+x)\geqslant e(G_1)+e(G_2)$. But then we see that $d(y,G_1-x+z)=(k-4)/2$ for each $y\in V(X_1)-\{x,c_1\}$, contradicting the minimality of $d(u_0,G_1)$. Therefore if $N(z,G_1)\not\supseteq V(X_1)-\{c_1\}$ then $d(z,G_2)\geqslant(k+6)/2$. If $N(z,G_1)\supseteq V(X_1)-\{c_1\}$, then $d(z,X_2-c_2)=0$ and by (3), $d(z,G_2)\geqslant k+1-d(w,G_1)\geqslant(k+2)/2$ where $w\in V(X_2)-\{c_2\}$. We conclude that if k is even then for each $x\in V(G_2)$,

$$\text{if } N(x,G_1)\not\supseteq V(X_1-c_1) \text{ then } d(x,G_2)\geqslant(k+6)/2; \tag{8}$$

$$\text{if } N(x,G_1)\supseteq V(X_1-c_1) \text{ then } d(x,G_2)\geqslant(k+2)/2. \tag{9}$$

Let $L=v_1\cdots v_q$ be a shortest path in G_2 such that $\{yv_1,y'v_q\}\subseteq E$ for some $y\in V(X_1-c_1)$ and $y'\in V(X_2-c_2)$. In this Case 2 with k even, we further choose G_1, G_2 and L such that $|L|$ is minimal. Then we choose $u_0\in V(X_1)-\{y,c_1\}$. Let $P=x_1\cdots x_{k-1}$ be a y-y' h-path of H_1. By (8) and (9), we see that $\delta(H_2)\geqslant(k+4)/2$. Moreover, if $d(z,H_2)=(k+4)/2$ with $z\in V(H_2)$, then either $zu_0\in E$ and $\xi(u_0,z)=0$ or $z=u_0$. Consequently, by the assumption on G_1, G_2 and L, we see that if $d(z,H_2)=(k+4)/2$ with $z\in V(H)$, then (1) and (2) are maintained if z and w are exchanged with $w\in V(X_2)-\{c_2,y'\}$ and $wz\notin E$, and so $d(z,L)\leqslant 1$ by the minimality of $|L|$. We conclude that if k is even then

$$d(x_1x_{k-1},H_1)\leqslant k-2 \text{ and so } d(x_1x_{k-1},H)\geqslant k+2; \tag{10}$$

$$u_0\notin V(L),\quad d(u_0,L)\leqslant 1,\quad \delta(H_2)\geqslant(k+4)/2; \tag{11}$$

$$d(x,L)\leqslant 1 \text{ for each } x\in V(H) \text{ with } d(x,H_2)=(k+4)/2. \tag{12}$$

Case 3. $d(u_0,G_1)\leqslant\lfloor(k-1)/2\rfloor-1=\lfloor(k-3)/2\rfloor$.

Then $d(u_0,G_2)\geqslant\lceil(k+5)/2\rceil$. Let z be an arbitrary vertex of G_2 with $d(z,G_2)=\delta(G_2)$. By (3), $\xi(u_0,z)\leqslant 0$ and so $d(z,G_2)\geqslant\lceil(k+3)/2\rceil$. Moreover, if $d(z,G_2)=\lceil(k+3)/2\rceil$ then $u_0z\in E$. Thus $\delta(H_2)\geqslant\lceil(k+5)/2\rceil$.

We claim that H_1 is not h-connected. If this is not true, say H_1 is h-connected. By Property 4, $d(x,H_1)\leqslant 1$ and so $d(x,H_2)\geqslant k$ for all $x\in V(H_2)$. Let $R=u_1\cdots u_q$ be a shortest path of H_2 such that $\{x_1u_1,x_2u_q\}\subseteq E$ for some $\{x_1,x_2\}\subseteq V(H_1)$ with $x_1\neq x_2$. Then $H_1+V(R)\supseteq C_{\geqslant k}$. Say $S=H_2-V(L)$. Then

$$|S|\geqslant\sum_{x\in V(H_1)}d(x,H_2)-2\geqslant(k-1)(k+1-(k-2))-2>2k.$$

By the minimality of $|L|$, we see that $d(x,R)\leqslant 2$ for each $x\in N(H_1,S)$. Therefore $\delta(S)\geqslant k-2$. As $S\not\supseteq C_{\geqslant k}$ and by Lemma 3.8, we see that each end block is a complete graph of order $k-1$. Let B_1 and B_2 be two distinct end blocks of S. Let w be a vertex of B_2 such that if B_2 contains a cut-vertex of S then w is the vertex. Let $\{z_1,z_2\}\subseteq V(B_2)-\{w\}$ with $z_1\neq z_2$. Then $d(z_i,H_1\cup R)\geqslant 3$ for $i\in\{1,2\}$. By the minimality of $|L|$, we readily see that there exists a vertex $v\in I(z_1z_2,R)$. Thus $B_2+v\supseteq C_{\geqslant k}$. Clearly, $[H_1+V(R)-v,B_1-w]\supseteq C_{\geqslant k}$, a contradiction. Hence H_1 is not h-connected.

Let $P=x_1\cdots x_{k-1}$ be an h-path of H_1 with $d(x_1x_{k-1},H_1)$ minimal. By Lemma 3.4, $d(x_1x_{k-1},H_1)\leqslant k-1$. Let $L=v_1\cdots v_q$ be a shortest path of H_2 with $\{x_1v_1,x_{k-1}v_q\}\subseteq E$. We conclude:

$$d(x_1x_{k-1},H_1)\leqslant k-1,\quad d(x_1x_{k-1},H)\geqslant k+1 \quad\text{and}\quad \delta(H_2)\geqslant(k+5)/2. \tag{13}$$

6 Nine propositions on H

The purpose of this section is to prove that H is connected and has exactly two blocks. By (5), (7), (11)–(13) and Lemma 3.1(a), we see that $\delta(H_2) \geqslant (k+3)/2$ and if $x \in V(H)$ then

$$d(x,H) \geqslant d(x,H_2) - d(x,L) \geqslant (k-1)/2 \text{ with the last equality only if } d(x,H_2) = (k+5)/2 \text{ and } d(x,L) = 3. \tag{14}$$

Therefore $\delta(H) \geqslant (k-1)/2$. Let $\tilde{L}$ denote the h-cycle $P \cup L + x_1v_1 + x_{k-1}v_q$ of $[H_1, L]$. Clearly, $|\tilde{L}| \geqslant k+1$ and so $H \not\supseteq C_{\geqslant k}$. Let $B_1, \ldots, B_t$ be a list of endblocks of H. Let w_i be any fixed vertex of B_i if B_i is a component of H. Otherwise let w_i be the cut-vertex of H that contained in B_i. Set $r_i = |B_i|$ and $B_i' = V(B_i) - \{w_i\} (1 \leqslant i \leqslant t)$. As $\delta(H) \geqslant (k-1)/2$, $r_i \geqslant (k+1)/2$ for all $i \in \{1, 2, \ldots, t\}$. By Lemma 3.8, for each $i \in \{1, 2, \ldots, t\}$, if $r_i \leqslant k-1$ then B_i is hamiltonian. As $\delta([B_i']) \geqslant (k-1)/2 - 1 = (k-3)/2$, we also see that if $r_i \leqslant k-2$ then $[B_i']$ is hamiltonian and if $r_i \leqslant k-3$ then $[B_i']$ is h-connected. For each $i \in \{1, 2, \ldots, t\}$, let $B_i^* = \{x \in V(B_i) | d(x,L) = 3, d(x,B_i) = r_i - 1 \text{ and } d(x,H_1) = k - r_i - 1\}$. By the minimality of $|L|$,

$$\text{for each } x \in V(H) \text{ with } d(x,L) = 3,\ N(x,L) \text{ is consecutive on } L; \tag{15}$$

$$\text{for each } xy \in E(H) \text{ with } d(xy,L) \geqslant 5,\ N(x,L) \cap N(y,L) \neq \emptyset; \tag{16}$$

$$\text{for each } x \in N(x_1x_{k-1},H), d(x,L) \leqslant 2 \text{ and so } x \notin B_i^* \text{ for all } 1 \leqslant i \leqslant t. \tag{17}$$

Let $\epsilon = d(u_0,G_2) - d(u_0,G_1)$. For each $X \subseteq V(H_2)$, let $\xi(X) = d(X,H_1) - d(X,H_2 - X)$. Clearly,

$$d(X,H_1) \geqslant \sum_{x \in X} (k+1 - d(x,H_2)) \text{ and so}$$
$$\xi(X) \geqslant (k+1)|X| - d(X,H_2) - d(X,H_2 - X) \text{ for all } X \subseteq V(H_2). \tag{18}$$

If $X \subseteq H_2$, we define $\xi(X) = \xi(V(X))$. Clearly, $\epsilon \geqslant \lceil (k+3)/2 \rceil - \lfloor (k-1)/2 \rfloor \geqslant 2$ and $e(H_1) + e(H_2) = e(G_1) + e(G_2) + \epsilon$. Thus $e(H_1 + X) + e(H_2 - X) = e(G_1) + e(G_2) + \epsilon + \xi(X)$ for all $X \subseteq V(H_2)$. By (1) and Property 2, we obtain

$$\text{For each } \emptyset \neq X \subseteq V(H_2),\ \text{if } H_2 - X \supseteq P_k,\ \text{then } \xi(X) \leqslant -2$$
$$\text{and in addition if } |H_1 + X| > k \text{ and } |H_2 - X| > k \text{ then } \xi(X) < -2. \tag{19}$$

By (4), (6), (10), (13) and Property 4, we have

$$|H| \geqslant |N(x_1x_{k-1},H)| = d(x_1x_{k-1},H) \geqslant k+1. \tag{20}$$

By Lemma 3.5, the following Propositions 1 and 2 hold:

Proposition 1. *In each B_i, any two vertices of B_i are connected by a path of order at least $\lceil (k+1)/2 \rceil$ and therefore $[B_i, B_j, L] \supseteq P_{k+1}$ for all $\{i,j\} \subseteq \{1,2,\ldots,t\}$ with $i \neq j$. Moreover, for any $\{i,j\} \subseteq \{1,\ldots,t\}$ with $i \neq j$, if $d(B_i',H_1) \geqslant 1$ and $d(B_j',H_1) \geqslant 1$ then $[B_i,B_j,H_1] \supseteq P_{k+1}$.*

Proposition 2. *If B_i and B_j are in the same component of H with $i \neq j$, then for each $x \in B_i'$ and $y \in B_j'$, H has an x-y path P' of order at least k and therefore $[B_i,B_j,P',L] \supseteq C_{\geqslant k+1}$. Furthermore, if $d(B_i',H_1) \geqslant 1$ and $d(B_j',H_1) \geqslant 1$, then $[B_i,B_j,P',H_1] \supseteq C_{\geqslant k+1}$.*

Proposition 3. *If $r_i \geqslant k$, then $[B_i',H_1] \supseteq C_{\geqslant k}$ and $[B_i',L] \supseteq C_{\geqslant k}$.*

Proof. As $B_i \not\supseteq C_{\geqslant k}$ and by Lemma 3.9, $[B_i']$ has a path u-v path of order $k-1$ such that $d(u,B_i) = d(v,B_i) = (k-1)/2$. By (14), $d(u,L) = d(v,L) = 3$ and so $d(u,H_1) \geqslant (k-3)/2$ and $d(v,H_1) \geqslant (k-3)/2$. Thus $[B_i',H_1] \supseteq C_{\geqslant k}$ and $[B_i',L] \supseteq C_{\geqslant k}$. □

Proposition 4. *For each $x \in B_i'$, $d(x,H_1) \geqslant k - r_i - 1$ and so $x \in B_i^*$ if and only if $d(x,H_1) \leqslant k - r_i - 1$. In addition, if $B_i^* \supseteq B_i'$ then $B_i \cong K_{r_i}$ and if $B_i^* \supseteq B_i' - \{u\}$ for some $u \in B_i'$ then $B_i + w_iu \cong K_{r_i}$.*

Proof. For each $x \in B_i'$, $d(x,H_1) \geqslant k+1 - d(x,B_i) - d(x,L) \geqslant k+1 - (r_i - 1) - 3 = k - r_i - 1$, and then the proposition follows. □

Proposition 5. *Let $i \in \{1, 2, \ldots, t\}$. The following two statements hold*:

(a) *If a is the minimal number in $\{1, 2, \ldots, q\}$ and b is the maximal number in $\{1, 2, \ldots, q\}$ such that $d(v_a, B_i') \geqslant 1$ and $d(v_b, B_i') \geqslant 1$. Then $[\tilde{L} - \{v_1, \ldots, v_a\}, B_i'] \supseteq C_{\geqslant k}$ and $[\tilde{L} - \{v_b, \ldots, v_q\}, B_i'] \supseteq C_{\geqslant k}$*

(b) *If $[B_i, H_1] \not\supseteq C_{\geqslant k}$, then $r_i \leqslant k-1$ and for some $u \in V(B_i)$, $B_i' - \{u\} \subseteq B_i^*$ and if $r_i \leqslant k-2$ then $u = w_i$. In addition, if B_i is a component of H then $|B_i^*| \geqslant k-2$ if $r_i = k-1$ and $B_i^* = V(B_i)$ if $r_i \leqslant k-2$.*

Proof. If $r_i \geqslant k$, $C_{\geqslant k} \subseteq [H_1, B_i'] \subseteq [H_1, B_i]$ by Proposition 3, and so Proposition 5 holds. We now assume $r_i \leqslant k-1$. Then B_i has an h-cycle $C = y_1 \cdots y_{r_i} y_1$ with $y_1 = w_i$. Clearly, $d(y_j, \tilde{L} - \{v_1, \ldots, v_a\}) \geqslant k+1-(r_i-1)-1 = k-(r_i-1)$ for $j \in \{2, r_i\}$. By Lemma 3.1(c), $[B_i', \tilde{L} - \{v_1, \ldots, v_a\}] \supseteq C_{\geqslant k}$. Similarly, $[B_i', \tilde{L} - \{v_b, \ldots, v_q\}] \supseteq C_{\geqslant k}$. Thus (a) holds. To show (b), we have that $d(y, H_1) \geqslant k+1-d(y, B_i)-d(y, L) \geqslant k-r_i-1$ for all $y \in V(B_i)$ except possibly $y = w_i$ with w_i being a cut-vertex of B_i. By Proposition 4, we see that if (b) fails, $d(y_c, H_1) \geqslant k-r_i$ for some y_c. As either $y_1 \neq y_{c-1}$ or $y_1 \neq y_{c+1}$, say w.l.o.g. that $y_1 \neq y_{c-1}$. As $[B_i, H_1] \not\supseteq C_{\geqslant k}$ and by Lemma 3.1(c), we must have that $d(y_{c-1}, H_1) = k-r_i-1 = 0$ and so $y_{c-1} \in B_i^*$ with $r_i = k-1$. It follows that for each $y_s \in B_i' - \{y_c, y_1\}$, B_i has a y_c-y_s h-path and so $d(y_s, H_1) = 0$ as $[B_i, H_1] \not\supseteq C_{\geqslant k}$ and so $y_s \in B_i^*$. Thus $B_i^* \supseteq B_i' - \{y_c\}$. If B_i is a component, then y_s can take on y_1 as well. Thus (b) holds. □

Proposition 6. *Let $i \in \{1, 2, \ldots, t\}$. The following two statements hold*:

(a) *If $[B_i', H_1] \not\supseteq C_{\geqslant k}$ and $[B_i', L] \not\supseteq C_{\geqslant k}$, then $r_i \leqslant k-2$ and if $r_i = k-2$ then $B_i \cong K_{k-2}$ and for each $x \in B_i'$, $d(x, H_1) = d(x, L) = 2$. Moreover, if $r_i \leqslant k-3$ then either $B_i' - \{u\} \subseteq B_i^*$ for some $u \in B_i'$ or $d(x, H_1) \leqslant k-r_i$ and so $d(x, L) \geqslant 2$ for all $x \in B_i'$.*

(b) *If $[B_i, H_1] \not\supseteq C_{\geqslant k}$ and $[B_i, L] \not\supseteq C_{\geqslant k}$, then $r_i \leqslant k-4$ and $B_i' \subseteq B_i^*$.*

Proof. By Proposition 3, we may assume $r_i \leqslant k-1$. Then B_i has an h-cycle. We show (a) first. Let $u_2 \cdots u_{r_i}$ be an h-path of $[B_i']$ with $d(u_2, H_1)$ maximal. First, assume that $d(u_2, H_1) \geqslant k-r_i+1$. As $[B_i', H_1] \not\supseteq C_{\geqslant k}$ and by Lemma 3.1(c), $d(u_{r_i}, H_1) \leqslant k-r_i-1$, i.e., $u_{r_i} \in B_i^*$ by Proposition 4. Thus for each $u_j \in B_i' - \{u_2\}$, $[B_i']$ has a u_2-u_j h-path and consequently, $u_j \in B_i^*$. As $[B_i', L] \not\supseteq C_{\geqslant k}$, this yields $r_i \leqslant k-3$ and so (a) holds. Next, assume $d(u_2, H_1) \leqslant k-r_i$. Then $d(u_2, L) \geqslant k+1-(k-r_i)-(r_i-1) = 2$. Similarly, $d(u_{r_i}, H_1) \leqslant k-r_i$ and $d(u_{r_i}, L) \geqslant 2$. These two inequalities will hold for each $x \in B_i'$ if $[B_i']$ is h-connected. Hence (a) holds if $r_i \leqslant k-3$. So assume that $r_i \geqslant k-2$. As $[B_i', L] \not\supseteq C_{\geqslant k}$, it follows that $r_i = k-2$ then $d(u_2, L) = d(u_{k-2}, L) = 2$ and so $d(u_2, B_i) = d(u_{k-2}, B_i) = k-3$. Thus for each $x \in B_i' - \{u_2\}$, $[B_i']$ has a u_2-x h-path and so $d(x, B_i) = k-3$ and $d(x, L) = 2$, i.e., (a) holds. To prove (b), we see that $r_i \leqslant k-2$ by (a) as $[B_i'] \subseteq B_i$. As $[B_i, H_1] \not\supseteq C_{\geqslant k}$ and by Proposition 5(b), $B_i' \subseteq B_i^*$. Thus $r_i \leqslant k-4$ as $[B_i, L] \not\supseteq C_{\geqslant k}$. □

Proposition 7. *It holds that $t \geqslant 2$ and the following two statements hold*:

(a) *For each $i \in \{1, 2, \ldots, t\}$, either $[B_i', H_1] \not\supseteq C_{\geqslant k}$ or $[B_i', L] \not\supseteq C_{\geqslant k}$ and if B_i is a component of H or $d(w_i, H - V(B_i)) = 1$ then $[B_i, H_1] \not\supseteq C_{\geqslant k}$ or $[B_i, L] \not\supseteq C_{\geqslant k}$.*

(b) *For all $i \in \{1, 2, \ldots, t\}$ and $v \in V(\tilde{L})$ and $uv \in E(\tilde{L})$, we have that $r_i \leqslant k-1$, $[\tilde{L} - v, B_i'] \supseteq C_{\geqslant k}$, $[\tilde{L} - u - v, B_i] \supseteq C_{\geqslant k}$ and $d(B_i', H_1) > 0$. Moreover, if $q \leqslant 2k-9$ then $r_i \leqslant k-2$ for all $i \in \{1, 2, \ldots, t\}$.*

Proof. First, we show that $t \geqslant 2$. On the contrary, say $t = 1$. Then H is 2-connected. Let $Y = \{x \in V(H) \mid d(x, H) = (k-1)/2\}$. By Lemma 3.12, we see that $|H| - |Y| = 2$ or $(k-1)/2$. By (14), we see that $d(x, L) = 3$ for all $x \in Y$. By (17), $d(x_1 x_{k-1}, Y) = 0$. By (20), $|H| - |Y| \geqslant k+1$, a contradiction. Hence $t \geqslant 2$.

Next, we show (a). With B_i in place of B_i', the proof of the conclusion with respect to B_i is the same as (somehow simpler than) the proof of the conclusion with respect to B_i' since we have no concern with w_i. So we provide the proof of the conclusion with respect to B_i'. On the contrary, say $[B_i', H_1] \supseteq C_{\geqslant k}$ and $[B_i', L] \supseteq C_{\geqslant k}$. Let $j \in \{1, 2, \ldots, t\} - \{i\}$. Then $[B_j, H_1] \not\supseteq C_{\geqslant k}$ and $[B_j, L] \not\supseteq C_{\geqslant k}$. By Proposition 6(b), $r_j \leqslant k-4$ and $B_j' \subseteq B_j^*$. By (17) $d(x_1 x_{k-1}, B_j') = 0$. If $t \geqslant 3$, let $l \in \{1, 2, \ldots, t\} - \{i, j\}$. Then we also have that $r_l \leqslant k-4$ and $B_l' \subseteq B_l^*$. Thus B_j and B_l are not in the same component of H for otherwise $[H - B_i', L] \supseteq C_{\geqslant k+1}$ by Proposition 2. It follows that H has a component F with $B_i \not\subseteq F$ such that only one of B_j and B_l, say B_l, is in F. As $[F, L] \not\supseteq C_{\geqslant k}$ and by Proposition 2, we see that $F = B_l$. As

$r_l \leqslant k-4$, $d(x, H_1) \geqslant k+1-(r_l-1)-3 \geqslant 3$ for all $x \in V(B_l)$ and so $\xi(B_l) \geqslant 0$. By Proposition 1, $H_2 - V(B_l) \supseteq P_{k+1}$. By (19), $\xi(B_l) \leqslant -2$, a contradiction. Hence $t=2$.

We claim that $V(H) = V(B_1 \cup B_2)$. If this is not true, then H must be connected. As $\delta(H) \geqslant (k-1)/2$, H has another block B with $|B| \geqslant \delta(H)+1 \geqslant (k+1)/2$ such that B contains exactly two cut-vertices, say c_1 and c_2, of H. As $B \not\supseteq C_{\geqslant k}$, we readily see that $d(w, B) < k$ for some $w \in V(B) - \{c_1, c_2\}$. Thus $d(w, L) > 0$ or $d(w, H_1) > 0$. By Lemma 3.5, w is connected to c_2 in B by a path of order at least $(k+1)/2$. Let P' be a w_2-c_2 path of H. By Proposition 2, $[B, P', B_2, L] \supseteq C_{\geqslant k}$ or $[B, P', B_2, H_1] \supseteq C_{\geqslant k}$, and so $G \supseteq 2C_{\geqslant k}$, a contradiction. Hence the claim holds.

Recall that $r_2 \leqslant k-4$, $B_2' \subseteq B_2^*$ and $d(x_1x_{k-1}, B_2') = 0$. By (20), $r_1 + 1 \geqslant |H - B_2'| \geqslant k+1$. Therefore $r_1 \geqslant k$. By Lemma 3.9, B_1 has a cycle $C = u_1 \cdots u_{k-1}u_1$ such that $N(u_2u_{k-1}, B_1) \subseteq V(C)$, $d(u_2, B_1) = d(u_{k-1}, B_1) = (k-1)/2$ and $w_1 \notin V(C - u_1)$. By (14), $d(u_2, L) = d(u_{k-1}, L) = 3$. Let $z \in B_2'$. Say $N(z, L) = \{v_s, v_{s+1}, v_{s+2}\}$. Let v_a be the first vertex and v_b be the last vertex on L such that $d(v_a, u_2u_{k-1}) > 0$ and $d(v_b, u_2u_{k-1}) > 0$. Clearly, $[L[v_1, v_s], H_1, B_2] \supseteq C_{\geqslant k}$. So $[C - u_1, L[v_{s+1}, v_q]] \not\supseteq C_{\geqslant k}$. This implies that $a < s$. Say w.l.o.g. $u_2v_a \in E$. Similarly, $b > s+1$. Then $v_bu_{k-1} \in E$. As $v_au_2u_1u_{k-1}v_b$ is a path and by the minimality of $|L|$, $a = s-1$ and $b = s+2$. Thus $[C - u_1, L[v_{s+1}, v_q]] \supseteq C_{\geqslant k}$, a contradiction.

To prove (b), we see, by (a) and Proposition 3, that $r_i \leqslant k-1$ for all $i \in \{1, 2, \ldots, t\}$. Thus B_i is hamiltonian and $[B_i']$ has an h-path for all $i \in \{1, 2, \ldots, t\}$. As $d(x, \tilde{L}) \geqslant k+1-(r_i-1) = k - r_i + 2$ for all $x \in B_i'$ and $i \in \{1, 2, \ldots, t\}$ and by Lemma 3.1(c), $[\tilde{L} - v, B_i'] \supseteq C_{\geqslant k}$ and $[\tilde{L} - u - v, B_i] \supseteq C_{\geqslant k}$ for all $i \in \{1, 2, \ldots, t\}$, $v \in V(L)$ and $uv \in E(\tilde{L})$. If $d(B_i', H_1) = 0$ for some $i \in \{1, 2, \ldots, t\}$, then $B_i' = B_i^*$ and $r_i = k-1$ as $\delta(G) \geqslant k+1$. Thus $B_i + v \supseteq C_{\geqslant k}$ for some $v \in V(L)$. Consequently, $G \supseteq 2C_{\geqslant k}$ as $[\tilde{L} - v, B_j'] \supseteq C_{\geqslant k}$ for $j \neq i$, a contradiction.

If $q \leqslant 2k-9$ and $r_i \not\leqslant k-2$ for some $i \in \{1, 2, \ldots, t\}$, let $C = u_1 \cdots u_{k-1}u_1$ be an h-cycle of B_i with $w_i = u_1$. As $e(C - u_1 - u_2, \tilde{L}) \geqslant \sum_{3 \leqslant l \leqslant k-1}(k+1-d(u_l, B_i)) \geqslant 3(k-3) \geqslant |\tilde{L}|+1$. This implies that there exists $v \in I(u_au_b, \tilde{L}) \neq \emptyset$ for some $3 \leqslant a < b \leqslant k-1$. Let $j \in \{1, 2, \ldots, t\} - \{i\}$. Since $[\tilde{L} - v, B_j'] \supseteq C_{\geqslant k}$, $B_i + v \not\supseteq C_{\geqslant k}$ and so B_i does not have a u_a-u_b h-path. By Lemma 3.3, $d(u_{a-1}u_{b-1}, C) \leqslant k-1$. As $\delta(H) \geqslant (k-1)/2$, it follows that k is odd and $d(u_{a-1}, B_i) = d(u_{b-1}, B_i) = (k-1)/2$. By (14), $d(u_{a-1}u_{b-1}, L) = 6$. Thus $I(u_{a-1}u_{b-1}, L) \neq \emptyset$. Similarly, we obtain $d(u_au_b, B_i) = 6$. Thus $I(u_{a-1}u_a, L) \neq \emptyset$ and so $B_i + v' \supseteq C_{\geqslant k}$ for some $v' \in V(L)$, a contradiction. This proves (b). □

Proposition 8. *For each $i \in \{1, 2, \ldots, t\}$, $d(w_i, H - V(B_i)) \geqslant 2$. In addition, if $t = 2$ then $w_1 = w_2$.*

Proof. On the contrary, say w.l.o.g. that $d(w_t, H - V(B_t)) \leqslant 1$ and $d(w_t, H - V(B_t)) \leqslant d(w_i, H - V(B_i))$ for all B_i. First, assume that $t \geqslant 3$. We claim that for all $1 \leqslant i < j \leqslant t-1$, B_i and B_j are not in the same component of H. If this is not true, say for $i = 1$ and $j = 2$. Then $H - V(B_t)$ has an w_1-w_2 path P' with $w_t \notin V(P')$. By Propositions 2 and 7(b), $[B_1, B_2, P', L] \supseteq C_{\geqslant k+1}$ and $[B_1, B_2, P', H_1] \supseteq C_{\geqslant k+1}$. By Proposition 6(b), $r_t \leqslant k-4$ and $B_t' \subseteq B_t^*$. By (19), $\xi(B_t) < -2$. As $e(B_t, L) \leqslant 3r_t$, $e(B_t, H_2 - V(B_t)) \leqslant 3r_t + 1$. By (18), $\xi(B_t) \geqslant r_t(k+1-(r_t-1)-3-3) - 2 \geqslant -2$, a contradiction.

Therefore B_i is a component of H for each $i \in \{1, 2, \ldots, t-1\}$ since $d(w_t, H - V(B_t)) \leqslant d(w_i, H - V(B_i))$ for all B_i. Thus B_t is a component of H. As $[B_i, B_j, L] \supseteq P_{k+1}$ for all $1 \leqslant i < j \leqslant k$ and by (19), $\xi(B_i) < -2$ and so $r_i \geqslant k-3$ for all $i \in \{1, 2, \ldots, t\}$. We claim that $[B_i, L] \not\supseteq C_{\geqslant k}$ for all $i \in \{1, 2, \ldots, t\}$. If this is false, say w.l.o.g. that $[B_t, L] \supseteq C_{\geqslant k}$. Then $[B_i, H_1] \not\supseteq C_{\geqslant k}$ for all $i \in \{1, 2, \ldots, t-1\}$. Let $i \in \{1, 2, \ldots, t-1\}$. By Proposition 5(b), for all $i \in \{1, 2, \ldots, t-1\}$, $|B_i^*| \geqslant k-2$ if $r_i = k-1$ and $B_i^* = V(B_i)$ if $r_i \leqslant k-2$. It follows that $[B_1, L] \supseteq C_{\geqslant k}$ as $r_1 \geqslant k-3$. Similarly, we must have that $[B_t, H_1] \not\supseteq C_{\geqslant k}$, $|B_t^*| \geqslant k-2$ if $r_t = k-1$ and $B_t^* = V(B_t)$ if $r_t \leqslant k-2$. By Proposition 7(b), $[\tilde{L} - u - v, B_j] \supseteq C_{\geqslant k}$ and so $[uv, B_i] \not\supseteq C_k$ for all $uv \in E(L)$ and $\{i, j\} \subseteq \{1, 2, \ldots, t\}$ with $i \neq j$. This implies that $r_i = k-3$ for all $i \in \{1, 2, \ldots, t\}$. Thus $B_i^* = B_i$ and so $d(x_1x_{k-1}, B_i) = 0$ by (17) for all $i \in \{1, 2, \ldots, t\}$, i.e., $d(x_1x_{k-1}, H) = 0$, a contradiction. Therefore $[B_i, L] \not\supseteq C_{\geqslant k}$ for all B_i. Let i be arbitrary in $\{1, 2, \ldots, t\}$ and $u_1 \cdots u_{r_i}u_1$ be an h-cycle of B_i. As H_2 is 2-connected, there are two independent edges u_jv and u_lv' between B_i and L. As $\delta(H_2) \geqslant (k+3)/2$, either $d(u_{j-1}, L) \geqslant 2$ or $d(u_{j-1}, B_i) \geqslant (k+1)/2$. If the latter holds then $d(u_{j-1}u_{l-1}, B_i) \geqslant (k+1)/2 + (k-1)/2 = (k-1)+1$ and by Lemma 3.3, B_i has a u_j-u_l h-path. In either situation, we see that $[B_i, L] \supseteq C_{\geqslant r_i+2}$. Thus $r_i = k-3$ for all $i \in \{1, 2, \dots, t\}$. Let C be an h-cycle of B_t. As $[B_t, L] \not\supseteq C_{\geqslant k}$, $d(xx^+, L) \leqslant 4$ for all $x \in V(C)$. Thus $e(B_t, L) \leqslant 2r_t$. By (18), $\xi(B_t) > 0$, a contradiction.

Therefore $t = 2$. Then either B_1 and B_2 are two components of H or H has a sequence $D_1, \dots, D_m$ of blocks with $|D_m| = 2$ such that a w_1-w_2 path P' passes through $D_1, \dots, D_m$ successively. We claim that there is no D_i with $|D_i| \geqslant 3$. If this is false, let i be the largest index with $|D_i| \geqslant 3$. Let c_1 and c_2 be the two cut-vertices of H that are contained in D_i with c_2 behind c_1 on P'. By Lemma 3.5, each vertex of $D_i - c_1$ is connected to c_1 by a path of order at least $(k+1)/2$ in D_i. Consequently, $H - V(B_2) \supseteq P_{k+1}$. If $r_2 \leqslant k-4$, then by (18), $\xi(B_2) \geqslant -2$, contradicting (19). Hence $r_2 \geqslant k-3$. If $d(x, H_1) = 0$ for all $x \in V(D_i) - \{c_1\}$ then $d(x, D_i) \geqslant k-2$ for all $x \in V(D_i) - \{c_1, c_2\}$ and $d(c_2, D_i) \geqslant k-3$. As $D_i \not\supseteq C_{\geqslant k}$, $|D_i| \leqslant k-1$ by Lemma 3.8. It follows that $|D_i| = k-1$, $d(D-c_1-c_2, L) = 3(k-3)$ and $D_i + c_1c_2 \cong K_{k-1}$. Then $[D_i, v] \supseteq C_{\geqslant k}$ for some $v \in V(L)$. By Proposition 7(b), $[B_2, \tilde{L} - v] \supseteq C_{\geqslant k}$, a contradiction. Hence $d(D_i - c_1, H_1) > 0$. As $d(B_1', H_1) > 0$ by Proposition 7(b), we see that $[H - V(B_2), H_1] \supseteq C_{\geqslant k}$. Thus $[B_2, L] \not\supseteq C_{\geqslant k}$. Then $[B_2, H_1] \supseteq C_{\geqslant k}$ for otherwise $r_2 \leqslant k-4$ by Proposition 6(b). Hence $[H - V(B_2), L] \not\supseteq C_{\geqslant k}$. As $d(B_1', L) > 0$, it follows that $d(D_i - c_1, L) = 0$. As $\delta(H_2) \geqslant (k+3)/2$, $d(x, D) \geqslant (k+3)/2 - 1 = (k+1)/2$ for all $x \in V(D_i) - \{c_1\}$. As $D_i \not\supseteq C_{\geqslant k}$ and by Lemma 3.8, it follows that $|D_i| \leqslant k-1$ and so $\xi(D - c_1) > 0$ by (18) . By Proposition 2, $[B_1, B_2, L] \supseteq P_{k+1}$ and so $\xi(D - c_1) < -2$ by (19), a contradiction. Therefore the claim holds.

As $\delta(H) \geqslant (k-1)/2$, it follows that either $m = 1$ with $w_1w_2 \in E$ or B_1 and B_2 are two components of H. We claim that $q \leqslant 7$. If this is not true, then $I(xy, H) = \emptyset$ for each $\{x, y\} \subseteq \{x_1, x_{k-1}, v_3, v_6\}$ with $x \neq y$ by the minimality of q. As $\delta(H_2) \geqslant (k+3)/2$, $d(v_i, H) \geqslant (k+3)/2 - 2$ for each $v_i \in V(L)$, we see that $2(k-1) \geqslant |H| \geqslant d(x_1x_{k-1}, H) + d(v_3v_6, H) \geqslant k+1+(k-1) \geqslant 2k$, a contradiction. Hence $q \leqslant 7$. By Proposition 7(b), $r_1 \leqslant k-2$ and $r_2 \leqslant k-2$. So by Lemma 3.7, for each $i \in \{1, 2\}$ and $x \in B_i'$, B_i has a w_i-x h-path. We shall find $X \subseteq V(B_2)$ such that (19) is violated.

Let L' be a longest u-v subpath of L with $d(u, B_1') > 0$ such that if B_1 and B_2 are two components of H then $d(v, B_2') > 0$. Set $q' = |L'|$. Let $r_2 = a + b$ with $a = \max\{0, k - r_1 - q'\}$. As $q \geqslant 2$ and H_2 is 2-connected, $q' \geqslant 2$. Let $z_1 \cdots z_{r_2} z_1$ be an h-cycle of B_2 such that if $w_1w_2 \in E$ then $z_1 = w_2$ and if $w_1w_2 \notin E$ then $z_1v \in E$. Clearly, $[L', B_1, z_1 \cdots z_a]$ has an h-path P' of order $r_1 + q' + a \geqslant k$. Let $X = \{z_{a+1}, \dots, z_{r_2}\}$. By (19), $\xi(X) \leqslant -2$.

We now divide the remaining proof into two cases.

Case 1. $r_1 \geqslant k-3$ and $r_2 \geqslant k-3$.

By Propositions 6–7, for each $i \in \{1, 2\}$, either $[B_i, H_1] \supseteq C_{\geqslant k}$ and $[B_i, L] \not\supseteq C_{\geqslant k}$, or $[B_i, H_1] \not\supseteq C_{\geqslant k}$ and $[B_i, L] \supseteq C_{\geqslant k}$. First, assume that $[B_1, H_1] \not\supseteq C_{\geqslant k}$ and $[B_1, L] \supseteq C_{\geqslant k}$. Then $[B_2, H_1] \not\supseteq C_{\geqslant k}$. By Proposition 5(b), for each $i \in \{1, 2\}$, $B_i' \subseteq B_i^*$ as $r_i \leqslant k-2$. By (17), $d(x_1x_{k-1}, H) \leqslant 2$, a contradiction. Therefore $[B_1, H_1] \supseteq C_{\geqslant k}$ and $[B_1, L] \not\supseteq C_{\geqslant k}$. Similarly, $[B_2, H_1] \supseteq C_{\geqslant k}$ and $[B_2, L] \not\supseteq C_{\geqslant k}$. Say w.l.o.g. $r_1 \geqslant r_2$.

Let $\tau = k - 2 - r_2$. Then $\tau \in \{0, 1\}$. Clearly, $1 \geqslant a$ and if $a = 1$ then $q' = 2$ and $r_1 = k-3$. Thus if $a = 1$ then $r_1 = r_2 = k-3$ and so $\tau = 1$. As $[B_2, L] \not\supseteq C_{\geqslant k}$, $d(z_iz_{i+1}, L) \leqslant 3 + \tau$ for all $i \in \{1, \dots, r_2 - 1\}$. Thus if b is even, then $d(X, L) \leqslant b(3+\tau)/2$. If b is odd, then $d(z_{r_2}, L) \leqslant 3$ and $d(X, L) \leqslant (b-1)(3+\tau)/2 + d(w_1, X) + 3 \leqslant b(3+\tau)/2 + d(w_1, X) + (3-\tau)/2$. Obviously, $d(w_1, X) = 0$ if $a > 0$ and otherwise $d(w_1, X) \leqslant 1$. Clearly, $d(X, H - X) \leqslant ba + d(w_1, X)$. Then $d(X, H_1) \geqslant \sum_{z \in X}(k+1-(r_2-1)-d(z, L)) - d(w_1, X) \geqslant b(k+1-(r_2-1)) - b(3+\tau)/2 - d(w_1, X) - \theta$, where $\theta = (3-\tau)/2$ if b is odd and otherwise $\theta = 0$. Thus $-2 \geqslant \xi(X) \geqslant b(k - r_2 - 1 - \tau - a) - 2d(w_1, X) - 2\theta = b(1-a) - 2d(w_1, X) - 2\theta$. As $r_2 \geqslant k-3 \geqslant 6$, this implies that $a = 1$. Thus $\tau = 1$ and $-2 \geqslant \xi(X) \geqslant -2\theta = -2$. It follows that $d(z_{r_2}, L) = 3$. As $r_1 = r_2$, this argument implies $d(y, L) = 3$ for some $y \in B_1'$. Thus $q' = 3$, a contradiction.

Case 2. Either $r_1 \leqslant k-4$ or $r_2 \leqslant k-4$.

For the proof, say $r_1 \geqslant r_2$ and $r_2 \leqslant k-4$. As $d(x_1x_{k-1}, H) \geqslant k+1$, $d(x_1x_{k-1}, B_2') \geqslant 2$. As $r_1 \geqslant (k+1)/2$, $a \leqslant k - (k+1)/2 - 2$ and so $b = r_2 - a \geqslant 3$. Let $\lambda = \max_{x \in X} d(x, L)$. Then $d(X, H_1) \geqslant \sum_{x \in X}(k+1-d(x, H_2)) \geqslant b(k+2-r_2-\lambda) - d(w_1, X)$ and $d(X, H_2 - X) = \sum_{x \in X} d(x, H_2 - X) \leqslant$

$b(a+\lambda)+d(w_1,X)$. Thus $\xi(X) \geqslant b(k+2-r_2-a-2\lambda)-2d(w_1,X)$.

First, assume $\lambda \leqslant 2$. Since $\xi(X) \leqslant -2$, $a>0$ and so $d(w_1,X)=0$. Then $\xi(X) \geqslant b(k-r_2-a-2) = b(r_1-r_2+q'-2) \geqslant 0$, a contradiction.

Therefore $\lambda=3$, i.e., $d(x_0,L)=3$ for some $x_0 \in X$, and so $\xi(X) \geqslant b(k-r_2-a-4)-2d(w_1,X)$. First, assume that $a=0$. By (17), $d(x,L) \leqslant 2$ and so $d(x,H_1) \geqslant k-r_2$ for each $x \in N(x_1x_{k-1},B_2')$. It follows that $\xi(X) \geqslant b(k-r_2-4)-2d(w_1,X)+2d(x_1x_{k-1},B_2') > 0$, a contradiction. Hence $a>0$ and so $d(w_1,X)=0$.

Assume $r_1=r_2$. Similarly, $d(y_0,L)=3$ for some $y_0 \in V(B_1)$ with $d(y_0,B_2)=0$. Thus $q' \geqslant 3$. Say w.l.o.g. $d(x_1x_{k-1},B_2) \geqslant d(x_1x_{k-1},B_1)$. Let $S=N(x_1x_{k-1},X)$. As $d(x_1x_{k-1},H) \geqslant k+1$, $d(x_1x_{k-1},B_2) \geqslant (k+1)/2$ and so $|S| \geqslant (k+1)/2-a$. As $b=r_2-a$, $2|S|-b \geqslant k+1-r_2-a = q'+1>0$. Thus $d(X,H_2-X)=d(X,H-X)+d(X,L) \leqslant ba+2|S|+3(b-|S|)$ and $d(X,H_1) \geqslant |S|(k-r_2)+(b-|S|)(k-r_2-1)$. Then $\xi(X) \geqslant b(k-r_2-a-3)+2|S|-b \geqslant b(q'-3)+q'+1>0$, a contradiction.

Therefore $r_1>r_2$. If $q' \geqslant 3$ or $r_1 \geqslant r_2+2$ then $\xi(X) \geqslant b(k+2-r_2-a-2\lambda) = b(r_1-r_2+q'-4) \geqslant 0$, a contradiction. Hence $q'=2$ and $r_1=r_2+1$. Say $N(x_0,L)=\{v_c,v_{c+1},v_{c+2}\}$. As $q'=2$ and H_2 is 2-connected, $N(B_1',L) \subseteq \{v_{c+1}\}$ and $w_1w_2 \in E$. Let $r_1=d+l$ with $d=k-r_2-3$ and $u_1u_2\cdots u_{r_1}$ be an h-path of B_1 with $u_1=w_1$. Set $Y=\{u_{d+1},\ldots,u_{r_1}\}$. Then $[L,B_2,u_1\cdots u_d] \supseteq P_k$. Clearly, $\xi(Y) \geqslant l(k-r_1+1)-l(d+1)>0$, a contradiction. □

Proposition 9. $t=2$.

Proof. On the contrary, say $t \geqslant 3$. First, assume that H is disconnected. By Proposition 8, each component contains at least two end blocks. Thus if D_1 and D_2 are two components then $[D_1,L] \supseteq C_{\geqslant k+1}$ by Proposition 2 and $[D_2,H_1] \supseteq C_{\geqslant k+1}$ by Proposition 2 and Proposition 7(b), a contradiction.

Hence H is connected. Let v_a and v_b be the first two vertices on L such that $d(v_a,B_i')>0$ and $d(v_b,B_j')>0$ for some $\{i,j\} \subseteq \{1,2,\ldots,t\}$ with $i \neq j$. Say $d(v_a,B_1')>0$ and $d(v_b,B_2')>0$. Then $[v_a\cdots v_b, H-B_3'] \supseteq C_{\geqslant k+1}$ by Proposition 2. Clearly, $d(x,v_a\cdots v_b) \leqslant 1$ for all $x \in B_3'$. Thus $d(x,\tilde{L}-\{v_1,\ldots,v_b\}) \geqslant k-(r_3-1)$ for all $x \in B_3'$. As $[B_3']$ has an h-path, $[B_3',\tilde{L}-\{v_a,\ldots,v_b\}] \supseteq C_{\geqslant k}$ by Lemma 3.1(c), a contradiction. □

7 Proof of Main Theorem

We now have that $t=2$, $w_1=w_2$ and $r_i \leqslant k-1$ $(i=1,2)$. As $\delta(G) \geqslant k+1$, $d(x_i,H) \geqslant 2$ for $i \in \{1,k-1\}$. As $d(x_1x_{k-1},H) \geqslant k+1$, we may assume w.l.o.g. that $d(x_1,B_1') \geqslant 1$ and $d(x_{k-1},B_2') \geqslant 1$. As $\delta(H) \geqslant (k-1)/2$, we see that the distance of any two vertices of H is at most 4 in H. Thus $q \leqslant 5$. By Proposition 7(b), $r_1 \leqslant k-2$ and $r_2 \leqslant k-2$. As $\delta(H) \geqslant (k-1)/2$ and by Lemma 3.7, there is a w_i-x h-path in B_i for each $i \in \{1,2\}$ and $x \in B_i'$. Set $\lambda = \max_{x \in B_2'} d(x,L)$. The proof consists of the following six claims.

Claim a. *For each* $i \in \{1,2\}$, $[B_i',L] \not\supseteq C_{\geqslant k}$.

Proof. On the contrary, say w.l.o.g. that $[B_1',L] \supseteq C_{\geqslant k}$. By Proposition 5(b), $B_2' \subseteq B_2^*$. By (17), $d(x_1x_{k-1},B_2^*)=0$. Thus $r_1 \geqslant d(x_1x_{k-1},H) \geqslant k+1$, a contradiction. □

Claim b. *Let* $\{i,j\}=\{1,2\}$. *If* $[B_i,L] \supseteq P_k$ *then* $r_j=k-2$ *if* $\max_{x \in B_j'} d(x,L) \leqslant 2$ *and* $r_j \geqslant k-4$ *if* $\max_{x \in B_j'} d(x,L)=3$.

Proof. On the contrary, say w.l.o.g. that $[B_1,L] \supseteq P_k$ such that $r_2 \leqslant k-3$ if $\lambda \leqslant 2$ and $r_2 \leqslant k-5$ if $\lambda=3$. Clearly, $d(B_2',H_2-B_2') \leqslant (r_2-1)(\lambda+1)$, $d(B_2',H_1) \geqslant (r_2-1)(k+1-(r_2-1)-\lambda)$. Then $\xi(B_2') \geqslant (r_2-1)(k+1-r_2-2\lambda) \geqslant 0$, contradicting (19). □

Claim c. *For each* $i \in \{1,2\}$, $r_i \leqslant k-3$.

Proof. On the contrary, say $r_1=k-2$. Let u and v be the two end vertices of an arbitrary h-path of $[B_1']$. As $[B_1',L] \not\supseteq C_{\geqslant k}$ by Claim a, $d(uv,L) \leqslant 4$. Moreover, we see that if $d(uv,L)=4$ with $d(u,L)=1$ then $d(u,v_1v_q)=0$. By (5), (7), (11)–(13), $d(uv,B_1) \geqslant d(uv,H_2)-d(uv,L) \geqslant k+1$. Consequently, $d(uv,B_1') \geqslant k+1-2 = |B_1'|+2$. By Lemma 3.4, we see that $d(xy,B_1') \geqslant |B_1'|+2$ for all $\{x,y\} \subseteq B_1'$

with $x \neq y$. Let $u_1 \cdots u_{k-3}u_1$ be an h-cycle of $[B_1']$ with $d(u_1, L)$ maximal. We break into two cases.

Case 1. Either $d(u_1, L) = 3$ or $d(u_i, L) \leqslant 1$ for all $i \in \{2, \ldots, r_1 - 1\}$.

Set $B_1'' = B_1' - \{u_1\}$. Since $[B_1', L] \not\supseteq C_{\geqslant k}$ and $[B_1']$ is h-connected, we see that if $d(u_1, L) = 3$ then $d(x, L) \leqslant 1$ for all $x \in B_1''$ by Lemma 3.1. In either situation, we have that $d(B_1'', H_2 - B_1'') \leqslant 3(k-4)$ and $d(B_1'', H_1) \geqslant (k-4)(k+1-(k-3)-1) = 3(k-4)$. Thus $\xi(B_1'') \geqslant 0$. By (19), $[B_2, L, u_1] \not\supseteq P_k$. Thus $r_2 \leqslant k-3$. As $[B_1, L] \supseteq P_k$ and by Claim b, $\lambda = 3$ and $r_2 \geqslant k-4$. Moreover, we see that $d(u_1, L) = 1$ and $d(u_1, v_1v_q) = 0$ as $[B_2, L, u_1] \not\supseteq P_k$. Hence $d(v_1v_q, B_1') = 0$ for otherwise we may choose $u \in N(v_1v_q, B_1')$ to replace u_1 in the above argument and a contradiction follows. Thus $d(v_1v_q, B_2) \geqslant 2\delta(H_2) - 2 \geqslant k+1$ and so $[B_2, L]$ has an h-cycle. Consequently, $[B_2, L, u_1] \supseteq P_k$, a contradiction.

Case 2. For some $u_m \in B_1' - \{u_1\}$, $d(u_m, L) = d(u_1, L) = 2$.

Since $[B_1']$ is h-connected and $[B_1', L] \not\supseteq C_{\geqslant k}$ by Claim a, we see that $N(B_1', L) = \{v_b, v_{b+1}\}$ for some $1 \leqslant b \leqslant q-1$. Clearly, $d(u, H_1) \geqslant k+1-(k-3)-2 = 2$ for $u \in \{u_1, u_m\}$ and $d(u_i, H_1) \geqslant 1$ for all u_i. Thus $[B_1, H_1] \supseteq C_{\geqslant k}$ by Lemma 3.1. Say $Z = \{v_b, v_{b+1}\}$.

First, assume that $[B_1, Z] \supseteq C_{\geqslant k}$. Let s and t be the two end vertices of an arbitrary h-path of $[B_2']$. Then $d(z, \tilde{L} - Z) \geqslant k+1-(r_2-1)-2 = k-1-(r_2-1)$ for each $z \in \{s, t\}$. As $[B_2', \tilde{L} - Z] \not\supseteq C_{\geqslant k}$, it follows that $d(s, \tilde{L} - Z) = d(t, \tilde{L} - Z) = k-1-(r_2-1)$, $N(s, \tilde{L} - Z) = N(t, \tilde{L} - Z)$, $Z \subseteq I(st, L)$, and $d(st, B_1) = 2(r_2 - 1)$. Moreover, the vertices of $N(s, \tilde{L} - Z)$ are consecutive on $\tilde{L}$. Thus s and t can be any two distinct vertices of B_2' in this argument and so these equalities hold for all $\{s, t\} \subseteq B_2'$ with $s \neq t$. Choose $s \in N(x_{k-1}, B_2') > 0$. By the minimality of q, $v_{b+1} = v_q$. Then we see that $[x_{r_2}x_{r_2+1} \cdots x_{k-1}, B_2] \supseteq C_{\geqslant k}$. Since $d(x_1, B_1') > 0$ and $[B_1']$ is h-connected, we see that $[x_1, L, B_1'] \supseteq C_{\geqslant k}$, a contradiction.

Therefore $[B_1, Z] \not\supseteq C_{\geqslant k}$. If $N(w_1, B_1) \neq \{u_1, u_m\}$ or $|N(v_bv_{b+1}, B_1')| \neq \{u_1, u_m\}$, we can readily choose two pairs (u_i, u_j) and (u_r, u_l) of vertices of B_1' such that $u_i \neq u_j$, $u_r \neq u_l$, $|\{u_i, u_j, u_r, u_l\}| \geqslant 3$, $d(u_i, Z) \geqslant 1$, $d(u_j, Z) = 2$ and $\{u_r, u_l\} \subseteq N(w_1)$. By Lemma 3.4, $[B_1'] + u_iu_j + u_ru_l$ has an h-cycle passing through u_iu_j and u_ru_l. Thus $[B_1, Z]$ is hamiltonian, a contradiction. Therefore $d(u_i, L) = 0$ for all $u_i \in V(B_1') - \{u_1, u_m\}$ and $N(w_1, B_1) = \{u_1, u_m\}$. Say $X = B_1' - \{u_1, u_m\}$. By (18), $\xi(X) \geqslant |X|(k+1-(r_1-2)) - 2|X| > 0$. By (19), $[L, B_2, u_1, u_m] \not\supseteq P_k$. This implies $r_2 \leqslant k-5$, contradicting Claim b as $[B_1, L] \supseteq P_k$. □

Claim d. $|r_1 - r_2| \leqslant 1$.

Proof. On the contrary, say w.l.o.g. $r_1 \geqslant r_2 + 2$. Then $r_2 \leqslant k-5$. Let $P = y_1 \cdots y_{r_2}$ be an h-path of B_2 with $y_1 = w_1$ and let P' be a longest u-v path on L with $d(v, B_1') \geqslant 1$. Say $q' = |P'|$. Then $q' \geqslant 2$. Let $r_2 - 1 = a + b$ with $a = \max\{0, k - r_1 - q'\}$ and $X = \{y_{r_2-b+1}, \ldots, y_{r_2}\}$. Then $[B_1, L', y_1 \cdots y_{a+1}] \supseteq P_k$ and $\xi(X) \geqslant b(k+1-(r_2-1)-\lambda) - b(a+1+\lambda) = b(k+1-r_2-a-2\lambda)$. By (19), $\xi(X) \leqslant -2$. Thus $a > 0$ and so $a = k - r_1 - q'$. Hence $k+1-r_2-a-2\lambda = r_1 - r_2 + 1 + q' - 2\lambda$. It follows that $\lambda = 3$, $q' = 2$ and $r_1 = r_2 + 2$. As $q' = 2$, we obtain that $q = 3$ and $N(B_1') = \{v_2\}$.

As $r_2 \geqslant (k+1)/2$, $b = r_2 - 1 - a = q' + r_1 + r_2 - 1 - k \geqslant 4$. Assume that $d(x, L) = 3$ for at most two vertices $x \in X$. Then $\xi(X) \geqslant (b-2)(r_1 - r_2 + 1 + q' - 4) + 2(r_1 - r_2 + 1 + q' - 6) \geqslant 0$, a contradiction. Therefore there exist two vertices z_1 and z_2 in X such that $d(z_1z_2, L) = 6$ and $d(w_1, B_2' - \{z_1, z_2\}) \geqslant 1$. Clearly, $[z_1, \tilde{L} - v_2] \supseteq C_{\geqslant k}$ and $\delta([B_2' - \{z_1\}]) \geqslant (k-1)/2 - 2 = (k-5)/2$. As $|B_2'| - 1 \leqslant (k-5) - 1$ and by Lemma 3.4, $[B_2' - \{z_1\}]$ is h-connected and it follows that $[B_1, B_2 - \{z_1\}, v_2] \supseteq C_{\geqslant k}$, a contradiction. □

Let $v_0 = x_1$ and $v_{q+1} = x_{k-1}$. Set $L^* = v_0Lv_{q+1}$. By (5), (7), (11)–(13) and (17), for each $x \in N(x_1x_{k-1}, H - w_1)$, $d(x, H) \geqslant (k+1)/2$. Thus $r_1 \geqslant (k+3)/2$ and $r_2 \geqslant (k+3)/2$.

Claim e. *There exists v_m on L such that $N(B_1', L^*) \subseteq \{v_0, v_1, \ldots, v_m\}$ and $N(B_2', L^*) \subseteq \{v_m, \ldots, v_{q+1}\}$.*

Proof. On the contrary, say that the claim is false. Since $d(v_0, B_1') > 0$, $d(v_{q+1}, B_2') > 0$, $d(B_1', L) > 0$ and $d(B_2', L) > 0$, we see that there exists $v_c \in V(L)$ such that either $d(L[v_1, v_c], B_2') \geqslant 1$ and $d(L^*[v_{c+1}, v_{q+1}], B_1') \geqslant 1$ or $d(L^*[v_0, v_{c-1}], B_2') \geqslant 1$ and $d(L[v_c, v_q], B_1') \geqslant 1$. Say that $d(L[v_1, v_c], B_2') \geqslant 1$ and $d(L^*[v_{c+1}, v_{q+1}], B_1') \geqslant 1$. Choose v_c with c maximal. Then $d(B_1', L^*(v_{c+1}, v_{q+1}]) = 0$ and so $N(B_1', L^*) \subseteq V(L^*[v_0, v_{c+1}])$ with $d(v_{c+1}, B_1') > 0$. Note that if $d(x_{k-1}, B_1') > 0$ then $v_{c+1} = v_{q+1} = x_{k-1}$.

Let $\{z_1, z_2\} \subseteq B_1'$ with $\{z_1x_1, z_2v_{c+1}\} \subseteq E$. Since $d(x_1x_{k-1}, H) \geqslant k+1$, $i(x_1x_{k-1}, H) = 0$ and

$r_2 \leqslant k-3$, we get that $d(x_1x_{k-1}, B_1') \geqslant 4$. Thus we may choose z_1 and z_2 such that $z_1 \neq z_2$ and $d(w_1, B_1' - \{z_1, z_2\}) \geqslant 1$. Subject to this, we choose z_1 and z_2 with the distance between z_1 and z_2 minimized in $[B_1']$. If $z_1z_2 \notin E$, then $i(z_1z_2, B_1) \geqslant 2\delta(H) - (r_1 - 2) \geqslant (k-1) - (k-5) = 4$ and we choose $z_0 \in I(z_1z_2, B_1')$ such that $d(w_1, B_1' - \{z_1, z_2, z_0\}) \geqslant 1$. For convenience, we define $z_0 = z_2$ if $z_1z_2 \in E$. Then $[H_1, L^*[v_{c+1}, v_{q+1}], z_1z_2z_0] \supseteq C_{\geqslant k}$ and so $F \not\supseteq C_{\geqslant k}$, where $F = [B_1 - \{z_1, z_2, z_0\}, L[v_1, v_c], B_2]$. Let $B_1'' = B_1 - \{z_1, z_2, z_0\}$ and $M = u_1 \cdots u_t$ an arbitrary longest path at $w_1 = u_1$ in B_1''. By (14), we see that for each $x \in V(B_1'') - \{u_1\}$, $d(x, B_1'') \geqslant d(x, H_2) - d(x, L) - d(x, z_1z_0z_2) \geqslant (k-7)/2$ and if equality holds then $d(x, H_2) = (k+5)/2$, $d(x, L) = 3$ and $d(x, z_1z_0z_2) = 3$. Thus $t \geqslant (k-7)/2 + 1 = (k-5)/2$.

First, assume that $u_tv_i \in E$ for some $v_i \in \{v_1, \ldots, v_c\}$. Let $v_j \in \{v_1, \ldots, v_c\}$ and $z \in B_2'$ with $v_jz \in E$. Choose v_i and v_j with $|j-i|$ maximal. Let P' be a w_1-z h-path of B_2. Then $[M, P', L[v_1, v_c]]$ has a cycle C with $|C| \geqslant r_2 + t + |j-i|$. Since $k-1 \geqslant |C|$, $r_2 \geqslant (k+3)/2$ and $t \geqslant (k-5)/2$, we obtain that $k-1 \geqslant |C| \geqslant (k-5)/2 + (k+3)/2 + |j-i| = k-1+|j-i|$. Thus $i=j$, $r_2 = (k+3)/2$, $t = (k-5)/2$ and $d(u_t, B_1'') = (k-7)/2$. Consequently, $d(u_t, L) = 3$ and $d(u_t, L[v_1, v_c]) \geqslant 2$. Thus $|i-j| \geqslant 1$, a contradiction.

We conclude that $d(u_t, L[v_1, v_c]) = 0$. Thus $N(u_t, L) \subseteq \{v_{c+1}\}$. As $r_1 \leqslant k-3$ and by (5), (7), (11)–(13), we see that $d(u_t, M) \geqslant \lceil (k+3)/2 \rceil - d(u_t, v_{c+1}) - d(u_t, z_1z_2z_0) \geqslant \lceil (k+1-2s)/2 \rceil \geqslant (|B_1''|+1)/2$ where $s = |\{z_1, z_2, z_0\}|$ and $|B_1''| = r_1 - s$. Let M be optimal at w_1 in $[B_1'']$ and set $r = \alpha(N, u_t)$, $D = [u_{t-r+1}, \ldots, u_t]$ and $D' = V(D) - \{u_{t-r+1}\}$. By Lemma 3.7, for each $u_i \in D'$, $d(u_i, D) \geqslant (|B_1''|+1)/2$, $N(u_i, B_1'') \subseteq V(D)$ and $[M]$ has a u_1-u_i h-path. This argument implies that $N(D', L) \subseteq \{v_{c+1}\}$. Since $k-3 \geqslant r_1$ and $\delta(H) \geqslant (k-1)/2$, $d(x, D') \geqslant 1$ for all $x \in V(B_1')$. Thus $B_1'' - \{u_1\} \subseteq V(D)$ and $r \in \{t-1, t\}$.

By (5), (7), (11)–(13), D' contains a vertex x with $d(x, H) \geqslant (k+4)/2 - 1 = (k+2)/2$ and so $r \geqslant d(x, D) + 1 \geqslant (k+2)/2 - d(x, z_1z_0z_2) + 1 \geqslant (k-2)/2$.

Suppose that $d(z_1z_2z_0, L) \geqslant 1$. Let L' be a longest path starting at u_{t-r+1} in

$$[u_{t-r+1}u_{t-r} \cdots u_1, B_2, L, z_1z_2, z_0].$$

As $d(L[v_1, v_c], B_2') > 0$, we see that $|L'| = r_2 + \sigma$ with $\sigma \geqslant 3$ and if $\sigma = 3$ then $t = r$, $v_{c+1} = v_{q+1} = x_{k-1}$ and $N(B_2' \cup \{z_1, z_0, z_2\}) = \{v_l\}$ for some $l \in \{1, \ldots, c\}$. Let $r - 1 = a + b$ with $a = \max\{0, k - r_2 - \sigma\}$ and $Y = \{u_{t-b+1}, \ldots, u_t\}$. Then $[L', u_{t-r+2} \cdots u_{t-r+a+1}\} \supseteq P_k$. As $r \geqslant (k-2)/2$ and $r_2 \geqslant (k+3)/2$, we see that $Y \neq \emptyset$.

Let $y \in Y$. Clearly, $d(y, B_1 \cup L - Y) \leqslant a + 1 + d(y, v_{c+1}z_1z_2z_0)$ and $d(y, H_1) \geqslant k + 1 - (r-1) - d(y, v_{c+1}z_1z_2z_0)$. If $|\{z_0, z_1, z_2\}| = 3$, then by the minimality of the distance between z_1 and z_2, $d(y, z_1v_{c+1}) \leqslant 1$. Thus $\xi(Y) \geqslant \sum_{y \in Y}(k+1-r-a-2d(y, v_{c+1}z_1z_2z_0)) \geqslant b(k+1-r-a-6) = b(k-r-a-5)$. By (19), $\xi(Y) \leqslant -2$. As $r \leqslant r_1 - |\{z_0, z_1, z_2\}| \leqslant k-5$, we see that $a > 0$ and so $a = k - r_2 - \sigma$. Therefore $k - r - a - 5 = r_2 + \sigma - r - 5$. As $|r_1 - r_2| \leqslant 1$ by Claim d, we obtain that $r_2 + \sigma - r - 5 \leqslant 0$ implies that $\sigma = 3$ and $|\{z_1, z_0, z_2\}| = 2$. Thus $v_{c+1} = v_{q+1} = x_{k-1}$. As $N(D', L^*) \subseteq \{v_{c+1}\}$, we obtain $d(Y, L) = 0$. Thus $\xi(Y) \geqslant b(k-r-a-3) = b(r_2 - r + \sigma - 3) \geqslant 0$, a contradiction.

Therefore $d(z_1z_0z_2, L) = 0$. Let $r_1 - 1 = d + l$ with $d = k - r_2 - 2$ and $Z = \{u_{d+2}, \ldots, u_t\}$. Then $[L, B_2, u_1u_2 \cdots u_{d+1}] \supseteq P_k$. As $r \in \{t-1, t\}$, $\{u_2, \ldots, u_t\} \subseteq V(D)$. Set $Z' = Z \cup \{z_1, z_0, z_2\}$. Since $N(D', L) \subseteq \{v_{c+1}\}$, we see that $d(Z', H_2 - Z) \leqslant l(d+2)$ and $d(Z', H_1) \geqslant l(k+1-(r_1-1)-1)$. Thus $\xi(Z') \geqslant l(k - r_1 - d - 1) \geqslant 0$ as $r_1 \leqslant r_2 + 1$, a contradiction. □

By Claim e, for some $v_m \in V(L)$, $N(B_1', L^*) \subseteq \{v_0, v_1, \ldots, v_m\}$ and $N(B_2', L^*) \subseteq \{v_m, \ldots, v_q, v_{q+1}\}$. In particular, $d(v_1, B_1') > 0$ and $d(v_q, B_2') > 0$. Let $\mu = \max_{x \in B_1'} d(x, L)$. Recall $\lambda = \max_{x \in B_2'} d(x, L)$. Thus $q \geqslant \mu + \lambda - 1$.

Claim f. $\mu = 3$ *and* $\lambda = 3$.

Proof. On the contrary, say that it is false. Say w.l.o.g. that $r_1 \geqslant r_2$. First, assume $\lambda \leqslant 2$. Let $u_1 \cdots u_{r_2}$ be an h-path of B_2 with $u_1 = w_1$. Let $r_2 - 1 = a + b$ with $a = \max\{0, k - r_1 - q\}$ and $X = \{u_{r_2-b+1}, \ldots, u_{r_2}\}$. Then $[L, B_1, u_1 \cdots u_{a+1}] \supseteq P_k$, $d(X, H_2 - X) \leqslant b(a+1+\lambda)$ and $d(X, H_1) \geqslant b(k+1-(r_2-1)-\lambda)$. Thus $\xi(X) \geqslant b(k+1-r_2-a-2\lambda)$. As $\xi(X) \leqslant -2$ by (19) and $r_2 \leqslant k-3$, we see that $a > 0$ and so

$a = k - r_1 - q$. Thus $\xi(X) \geqslant b(r_1 - r_2 + q + 1 - 2\lambda)$. It follows that $r_1 = r_2$, $q = 2$ and $\lambda = 2$. Exchanging the roles of B_1 and B_2 in the above argument, we see that $\mu \nleqslant 2$. Thus $q \geqslant 3$, a contradiction.

Therefore $\lambda = 3$ and so $\mu \leqslant 2$. By the above argument, we see that $r_1 \nleqslant r_2$. So $r_1 = r_2 + 1$ by Claim d. Let $y_1 \cdots y_{r_1}$ be an h-path of B_1 with $y_1 = w_1$. Let $r_1 - 1 = c + l$ with $c = \max\{0, k - r_2 - q\}$ and $Y = \{y_{r_1 - l + 1}, \ldots, y_{r_1}\}$. Then $[L, B_2, y_1 \cdots y_{c+1}] \supseteq P_k$ and $-2 \geqslant \xi(Y) \geqslant l(k + 1 - r_1 - c - 2\mu)$. Thus $c > 0$ and so $c = k - r_2 - q \leqslant k - r_2 - (\mu + 3 - 1)$. Then $\xi(Y) \geqslant l(r_2 - r_1 + 3 - \mu) \geqslant 0$, a contradiction. $\square$

By Claim f, $q \geqslant 5$. We claim that $r_i \geqslant k - 4$ $(i = 1, 2)$. If this is not true, say $r_1 \geqslant r_2$ and $r_2 \leqslant k - 5$. Let $u_1 \cdots u_{r_2}$ be an h-path with $u_1 = w_1$ Let $r_2 - 1 = a + b$ with $a = \max\{0, k - r_1 - 5\}$ and $X = \{u_{r_2 - b + 1}, \ldots, u_{r_2}\}$. Then $[L, B_1, u_1 \cdots u_{a+1}] \supseteq P_k$ and $\xi(X) \geqslant b(k + 1 - (r_2 - 1) - \lambda) - b(a + 1 + \lambda) = b(k - r_2 - a - 5) \geqslant 0$. By (19), $\xi(X) \leqslant -2$, a contradiction. Hence $r_i \geqslant k - 4$ $(i = 1, 2)$. Let r be maximal with $v_r z \in E$ for some $z \in B_1'$. Clearly, $d(x, \tilde{L} - \{v_0, \ldots, v_r\}) \geqslant k + 1 - (r_2 - 1) - 1 = k - (r_2 - 1)$ for all $x \in B_2'$. By Lemma 3.1(c), $[B_1', \tilde{L} - \{v_0, \ldots, v_r\}] \supseteq C_{\geqslant k}$. As $d(x_1 x_{k-1}, B_1') \geqslant k + 1 - r_2 \geqslant 4$, $d(x_1, B_1') \geqslant 4$. We can choose an h-cycle C of B_1 and a vertex $y \in B_1'$ such that $\{yx_1, zv_r\}$ and $w_1 \notin \{y^-, z^-\}$. Since $\delta(H) \geqslant (k - 1)/2$ and by Lemma 3.3, B_1 has a y-z h-path and so $[B_1, x_1 v_1 \cdots v_r] \supseteq C_{\geqslant k}$. This proves Main Theorem.

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