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On the family of meromorphic functions whose derivatives omit a holomorphic function

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Abstract In this paper, we continue to study the normality of a family of meromorphic functions without simple zeros and simple poles such that their derivatives omit a given holomorphic function. Such a family in general is not normal at the zeros of the omitted function. Our main result is the characterization of the non-normal sequences, and hence some known results are its corollaries.

Keywords meromorphic functions, normal families, non-normal sequences

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1 Introduction

In this paper, following [4, 6], we study the normality of families of meromorphic functions on plane domains, whose zeros and poles are multiple and whose derivatives omit a given holomorphic function.

Recall that a family $\mathcal{F}$ of functions meromorphic on a plane domain $D \subset \mathbb{C}$ is said to be normal on D, if for each sequence $\{f_n\} \subset \mathcal{F}$, there exists a subsequence $\{f_{n_i}\}$ which converges spherically locally uniformly on D. Also, $\mathcal{F}$ is said to be normal at a point in D, if $\mathcal{F}$ is normal on some neighborhood of this point. A known and useful fact is that $\mathcal{F}$ is normal on D if and only if $\mathcal{F}$ is normal at every point in D. See [3, 7, 8].

The starting point of this paper is the following results.

Theorem 1.1 (See [6, Theorem 1]). *Let h be a given holomorphic function on D without zeros, and let $\mathcal{F}$ be a family of meromorphic functions on D, all of whose zeros and poles are multiple. If for every $f \in \mathcal{F}$, $f'(z) \neq h(z)$ at every $z \in D$, then the family $\mathcal{F}$ is normal on D.*

Theorem 1.2 (See [6, Theorem 2]). *Let h be a given holomorphic function on D satisfying $h \not\equiv 0$, and let $\mathcal{F}$ be a family of meromorphic functions on D, all of whose zeros are at least triple and whose poles are multiple. If for every $f \in \mathcal{F}$, $f'(z) \neq h(z)$ at every $z \in D$, then the family $\mathcal{F}$ is normal on D.*

Theorem 1.3 (See [4, Theorem]). *Let h be a given holomorphic function on D satisfying $h \not\equiv 0$, and let $\mathcal{F}$ be a family of holomorphic functions on D, all of whose zeros are multiple. If for every $f \in \mathcal{F}$, $f'(z) \neq h(z)$ at every $z \in D$, then the family $\mathcal{F}$ is normal on D.*

Theorem 1.3 is not true in general for families of meromorphic function as shown by the following examples.

Example 1.4. Let $p \in \mathbb{N}$ be an integer, $h(z) = z^p$ and for each $n \in \mathbb{N}$,

$$f_n(z) = \frac{(z^{p+1} - \frac{1}{n})^2}{(p+1)z^{p+1}} = \frac{1}{p+1}z^{p+1} - \frac{2}{(p+1)n} + \frac{1}{(p+1)n^2 z^{p+1}}.$$

Then we have $f_n'(z) \neq h(z)$ for $z \in \mathbb{C}$. However, $\{f_n\}$ is not normal at 0.

Example 1.5. Let $p \in \mathbb{N}$ be an integer, $h(z) = \mathrm{e}^z(\mathrm{e}^z - 1)^p$ and for every $n \in \mathbb{N}$,

$$f_n(z) = \frac{((\mathrm{e}^z - 1)^{p+1} - \frac{1}{n})^2}{(p+1)(\mathrm{e}^z - 1)^{p+1}} = \frac{1}{p+1}(\mathrm{e}^z - 1)^{p+1} - \frac{2}{(p+1)n} + \frac{1}{(p+1)n^2(\mathrm{e}^z - 1)^{p+1}}.$$

Then $f'(z) = \mathrm{e}^z(\mathrm{e}^z - 1)^p - \frac{\mathrm{e}^z}{n^2(\mathrm{e}^z - 1)^{p+2}} \neq h(z)$ for every $z \in \mathbb{C}$. However, we see that each subsequence of $\{f_n\}$ is not normal at each zero of $\mathrm{e}^z - 1$.

Here, we prove the following result, which characterizes the non-normal sequences of meromorphic functions whose zeros and poles are multiple and whose derivatives omit a holomorphic function. Theorems 1.1–1.3 are direct corollaries to this theorem.

Theorem 1.6. *Let $h\,(\not\equiv 0)$ be a given holomorphic function on D, and let $\mathcal{F}$ be a family of meromorphic functions on D, all of whose zeros and poles are multiple, such that for every $f \in \mathcal{F}$, $f'(z) \neq h(z)$ at every $z \in D$. If the family $\mathcal{F}$ is not normal at $z_0 \in D$, then z_0 is a zero of h with multiplicity p and there exists a sequence $\{f_n\} \subset \mathcal{F}$ such that on some neighborhood $U(z_0)$ of z_0,*

$$f_n(z) = \frac{P_n^2(z)}{(z - z_0 - w_n)^{p+1}} f_n^*(z) \quad \text{with} \quad f_n^*(z) \to \frac{1}{(z - z_0)^{p+1}} \int_{z_0}^{z} h(z)dz,$$

where w_n are constants satisfying $w_n/\rho_n \to 0$ and P_n are monic polynomials of degree $p+1$ satisfying $P_n(z) \to (z - z_0)^{p+1}$ and

$$\rho_n^{-p-1} P_n(z_0 + \rho_n z) \to z^{p+1} - c$$

for some sequence of positive numbers $\rho_n \to 0$ and some nonzero constant c.

The proof of Theorem 1.6 closely follows the proofs of Theorems 1.1–1.3. However, some new techniques are adopted to cover some new and old difficulties. For example, our approach of dealing with the case $h(z) = z$ is different from that in [4]. The following example shows that the constants w_n may be nonzero.

Example 1.7. Let $p \in \mathbb{N}$ be an integer, $h(z) = z^p$ and for each $n \in \mathbb{N}$,

$$f_n(z) = \frac{(z^{p+1} - \frac{1}{n})^2}{(p+1)(z - \frac{1}{n^3})^{p+1}}.$$

Then we have

$$f_n'(z) - h(z) = \frac{-\frac{1}{n^2} + \frac{2z^p}{n^4} - \frac{2z^{2p+1}}{n^3} + z^{2p+2} - z^p(z - \frac{1}{n^3})^{p+2}}{(z - \frac{1}{n^3})^{p+2}}.$$

It follows that there exists an integer N such that for $n > N$, $f_n'(z) \neq h(z)$ for $|z| < 1$. However, $\{f_n\}_{n>N}$ is not normal at 0.

We remark that we also have characterized the non-normal sequences of meromorphic functions whose zeros and poles are multiple and whose derivatives minus a given meromorphic function have no zero. This together with the presented Theorem 1.6 has been found some applications in the theory of value distribution of functions meromorphic on the plane. These results will be appeared in our forthcoming papers.

2 Auxiliary results

To prove our results, we require some preliminary results.

Lemma 2.1 (See [5, Lemma 2]). *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, all of whose zeros have multiplicity at least k. Then if $\mathcal{F}$ is not normal at z_0, there exist, for each $-1 < \alpha < k$, points $z_n \in D$ with $z_n \to z_0$, functions $f_n \in \mathcal{F}$ and positive numbers $\rho_n \to 0$ such that $g_n(\zeta) := \rho_n^{-\alpha} f_n(z_n + \rho_n \zeta)$ converges locally uniformly with respect to the spherical metric in $\mathbb{C}$ to a nonconstant meromorphic function g of finite order.*

Lemma 2.2 (See [1, Theorem 1.1]). *Let f be a transcendental meromorphic function, all of whose zeros and poles except finitely many are multiple, and R $(\not\equiv 0)$ be a rational function. Then $f' - R$ has infinitely many zeros.*

Lemma 2.3 (See [6, Lemma 5(i)]). *Let f be a nonconstant rational function, all of whose zeros and poles are multiple. Then $f'(z) - 1$ has at least one zero on $\mathbb{C}$.*

Lemma 2.4 (See [2, Lemma 12]). *Let f be a nonconstant rational function satisfying $f' \neq 0$ on $\mathbb{C}$. Then $f(z) = az + b$ or $f(z) = \frac{a}{(z+c)^n} + b$, where $n \in \mathbb{N}$ and $a\,(\neq 0), b, c \in \mathbb{C}$.*

Lemma 2.5. *Let*

$$f(z) = \frac{a}{(z+c)^n} + z^p + b, \tag{2.1}$$

where $a\,(\neq 0), b, c \in \mathbb{C}$ and $n, p \in \mathbb{N}$. Suppose that $n \geqslant 2$ when $c \neq 0$. If all zeros of f are multiple, then $c = 0$, $p = n$ and

$$f(z) = \frac{(z^n + \frac{1}{2}b)^2}{z^n}. \tag{2.2}$$

Proof. Since

$$f(z) = \frac{z^p(z+c)^n + b(z+c)^n + a}{(z+c)^n} \tag{2.3}$$

and all zeros of f are multiple, all zeros of the polynomial

$$P(z) = z^p(z+c)^n + b(z+c)^n + a \tag{2.4}$$

are multiple. Now let z_0 be a zero of P. Then $f(z_0) = 0$ and hence $f'(z_0) = 0$, i.e.,

$$\frac{a}{(z_0+c)^n} + z_0^p + b = 0 \quad \text{and} \quad -\frac{na}{(z_0+c)^{n+1}} + pz_0^{p-1} = 0. \tag{2.5}$$

It follows that $Q_1(z_0) = Q_2(z_0) = 0$, where

$$Q_1(z) = (p+n)z^p + pcz^{p-1} + nb, \quad Q_2(z) = pb(z+c)^{n+1} + a(p+n)z + pac. \tag{2.6}$$

Case 1. Suppose $b = 0$. Then $c \neq 0$ and

$$z_0 = -\frac{pc}{p+n}, \tag{2.7}$$

so that P has a unique (multiple) zero. It is not difficult to see that the multiplicity of this zero is double. Thus

$$P(z) = z^p(z+c)^n + a = \left(z + \frac{pc}{p+n}\right)^2. \tag{2.8}$$

It follows that $p = n = 1$ and $a = c^2/4$. This contradicts that $n \geqslant 2$ when $c \neq 0$.

Case 2. Suppose $b \neq 0$. Next we consider two subcases.

Case 2.1. All zeros of P are double. Then by the above analysis, the two quotients $R_1 = Q_1^2/P$ and $R_2 = Q_2^2/P$ are polynomials. Thus $2\deg(Q_1) \geqslant \deg(P)$ and $2\deg(Q_2) \geqslant \deg(P)$, so $n \leqslant p \leqslant n+2$. Note that $\deg(R_1) = p - n$ and $\deg(R_2) = n + 2 - p$.

If $p = n$, then $R_1 = Q_1^2/P$ is a constant. Thus

$$(2nz^n + ncz^{n-1} + nb)^2 = 4n^2[z^n(z+c)^n + b(z+c)^n + a]. \tag{2.9}$$

It follows that $a = (b-c)^2/4$ and if $n > 1$ then $c = 0$. This together with the assumption that $n \geqslant 2$ when $c \neq 0$ shows that $c = 0$ always holds. Thus f has the form stated in (2.2).

If $p = n+1$, then $\deg(R_1) = \deg(Q_1^2/P) = 1$. Thus

$$[(2n+1)z^{n+1} + (n+1)cz^n + nb]^2 = [(2n+1)^2 z + \alpha][z^{n+1}(z+c)^n + b(z+c)^n + a] \tag{2.10}$$

for some constant α. It follows that $n = 1$, $a = -\frac{8}{27}c^3$ and $b = \frac{1}{3}c^2$. This contradicts that $n \geqslant 2$ when $c \neq 0$.

If $p = n+2$, then $R_2 = Q_2^2/P$ is a constant. Thus

$$[(n+2)b(z+c)^{n+1} + (2n+2)az + (n+2)ac]^2 = (n+2)^2 b^2 [z^{n+2}(z+c)^n + b(z+c)^n + a]. \tag{2.11}$$

It follows that $n = 1,\ a = b = 64/729,\ c = -8/9$. This contradicts that $n \geqslant 2$ when $c \neq 0$.

Case 2.2. P has a zero z_0 which is at least triple. Then we also have

$$f''(z_0) = \frac{n(n+1)a}{(z_0+c)^{n+2}} + p(p-1)z_0^{p-2} = 0. \tag{2.12}$$

This together with (2.5) gives that $p \geqslant 2$,

$$z_0 = -\frac{(p-1)c}{n+p} \tag{2.13}$$

and

$$a = (-1)^{p-1}\frac{p(p-1)^{p-1}(n+1)^{n+1}}{n(n+p)^{n+p}}c^{n+p}, \quad b = (-1)^p \frac{(p-1)^{p-1}}{n(n+p)^{p-1}}c^p. \tag{2.14}$$

Thus $c \neq 0$ and hence $n \geqslant 2$. This shows that all zeros of P are double with one possible exception which is triple. Thus by the above analysis, the two quotients $R_1 = (z-z_0)Q_1^2/P$ and $R_2 = (z-z_0)Q_2^2/P$ are polynomials, where z_0 is given by (2.13). Thus $2\deg(Q_1) + 1 \geqslant \deg(P)$ and $2\deg(Q_2) + 1 \geqslant \deg(P)$, so that $n-1 \leqslant p \leqslant n+3$. Note that $\deg(R_1) = p-n+1$ and $\deg(R_2) = n+3-p$.

For $p = n-1$, we have $n \geqslant 3$ and R_1 is constant. Thus

$$(z-z_0)[(2n-1)z^{n-1} + (n-1)cz^{n-2} + nb]^2 = (2n-1)^2[z^{n-1}(z+c)^n + b(z+c)^n + a]. \tag{2.15}$$

Comparing the coefficients of the term z^{2n-2} yields a contradiction.

For $p = n$, we have $\deg(R_1) = 1$. Thus

$$(z-z_0)(2nz^n + ncz^{n-1} + nb)^2 = 4n^2(z+\alpha)[z^n(z+c)^n + b(z+c)^n + a] \tag{2.16}$$

for some constant α. If $n > 2$, then comparing the coefficients of the terms z^{2n} and z^{2n-1} yields a contradiction. If $n = 2$, then $a = -\frac{27}{256}c^4, b = \frac{1}{8}c^2$, therefore $P(z) = z^n(z+c)^n + b(z+c)^n + a = (4z+c)^3(4z+5c)/256$. This contradicts that all zeros of P are multiple.

For $p = n+1$, we have $\deg(R_1) = 2$. Thus

$$\begin{aligned}(z-z_0)&[(2n+1)z^{n+1} + (n+1)cz^n + nb]^2 \\ &= (2n+1)^2(z^2+\alpha z+\beta)[z^{n+1}(z+c)^n + b(z+c)^n + a].\end{aligned} \tag{2.17}$$

Again, for $n > 3$, a contradiction follows from comparing the coefficients of the terms z^{2n+2}, z^{2n+1} and z^{2n}. For $n = 3$ and $n = 2$, we can compute a and b and directly yield that the polynomial $P(z) = z^{n+1}(z+c)^n + b(z+c)^n + a$ must have simple zero. This contradicts that all zeros of P are multiple.

For $p = n+2$, we have $\deg(R_2) = 1$. Thus

$$\begin{aligned}(z-z_0)&[(n+2)b(z+c)^{n+1} + (2n+2)az + (n+2)ac]^2 \\ &= (n+2)^2 b^2 (z+\alpha)[z^{n+2}(z+c)^n + b(z+c)^n + a].\end{aligned} \tag{2.18}$$

Again, for $n > 2$, a contradiction follows from comparing the coefficients of the terms z^{2n+2} and z^{2n+1}. For $n = 2$, we can directly yield that the polynomial P must have simple zero to obtain a contradiction.

For $p = n + 3$, R_2 is a constant. Thus

$$\begin{aligned}(z - z_0)[(n+3)b(z+c)^{n+1} + (2n+3)az + (n+3)ac]^2 \\ = (n+3)^2 b^2 [z^{n+3}(z+c)^n + b(z+c)^n + a].\end{aligned} \tag{2.19}$$

Again, by comparing the coefficients of the term z^{2n+2}, a contradiction follows. □

3 Proof of Theorem 1.6

By Theorem 1.1, z_0 must be a zero of h. So by making normalization, we may assume that

$$h(z) = z^p \widehat{h}(z) \tag{3.1}$$

for some $p \in \mathbb{N}$ and zero-free holomorphic function $\widehat{h}$ on $\Delta(0,1)$ such that $\widehat{h}(0) = 1$. Here and in the sequel, $\Delta(0,r) = \{z : |z| < r\}$ and $\Delta^\circ(0,r) = \{z : 0 < |z| < r\}$. Thus by Theorem 1.1, $\mathcal{F}$ is normal on $\Delta^\circ(0,1)$. Now let

$$\mathcal{G} = \{g_f(z) = z^{-p} f(z) : \ f \in \mathcal{F}\}. \tag{3.2}$$

Since $f' \neq h$ and the zeros of f are multiple, we see that $f(0) \neq 0$, and hence 0 is a pole of $g_f(0)$ with multiplicity at least p, and the other poles and all the zeros of g_f are multiple. Furthermore, $\mathcal{G}$ is also normal on $\Delta^\circ(0,1)$.

We claim that $\mathcal{G}$ is not normal at 0. If this is not true, then $\mathcal{G}$ would be normal on the whole disk $\Delta(0,1)$, and hence, by the same argument in [4, p. 105] or [6, p. 8], $\mathcal{F}$ would be normal on the whole disk $\Delta(0,1)$. This is a contradiction.

Thus $\mathcal{G}$ is not normal at 0. Hence, by Lemma 2.1, there exist points $z_n \to 0$, functions $\{f_n\}$ and positive numbers $\rho_n \to 0$ such that

$$G_n(\zeta) := \rho_n^{-1}(z_n + \rho_n\zeta)^{-p} f_n(z_n + \rho_n\zeta) \to G(\zeta) \tag{3.3}$$

spherically locally uniformly on $\mathbb{C}$, where G is a nonconstant meromorphic function on the plane, all of whose zeros on $\mathbb{C}$ and poles on $\mathbb{C}^*$ are multiple.

Now the case $z_n/\rho_n \to \infty$ can be ruled out as in [4, pp. 103–104]. So by taking subsequence, we can assume that $z_n/\rho_n \to \alpha \in \mathbb{C}$. Thus, by (3.3),

$$\Phi_n(\zeta) := \rho_n^{-p-1}\zeta^{-p} f_n(\rho_n\zeta) = G_n(\zeta - z_n/\rho_n) \to \Phi(\zeta) := G(\zeta - \alpha) \tag{3.4}$$

on $\mathbb{C}$, so all of the zeros on $\mathbb{C}$ and poles on $\mathbb{C}^*$ of Φ are multiple, while 0 is a pole of Φ with multiplicity at least p.

It follows from (3.4) that

$$\Psi_n(\zeta) := \rho_n^{-p-1} f_n(\rho_n\zeta) = \zeta^p \Phi_n(\zeta) \to \Psi(\zeta) := \zeta^p \Phi(\zeta) \tag{3.5}$$

locally uniformly on $\mathbb{C} \setminus \Phi^{-1}(\infty)$. Since 0 is a pole of Φ with multiplicity at least p, we have $\Psi(0) \neq 0$. Furthermore, all zeros and poles on $\mathbb{C}^*$ of Ψ are multiple.

If $\Psi'(\zeta) - \zeta^p \equiv 0$, then $\Psi = \frac{1}{p+1}\zeta^{p+1} + c$ for some constant. As the zeros of Ψ on $\mathbb{C}^*$ are multiple, we get $c = 0$, and hence $\Phi(\zeta) = \zeta^{-p}\Psi(\zeta) = \zeta/(p+1)$, which contradicts that $\Phi(0) = \infty$.

Thus $\Psi'(\zeta) - \zeta^p \not\equiv 0$. Since $\Psi_n'(\zeta) - \zeta^p \widehat{h}(\rho_n\zeta) \to \Psi'(\zeta) - \zeta^p$ on $\mathbb{C} \setminus \Phi^{-1}(\infty)$ and

$$\Psi_n'(\zeta) - \zeta^p \widehat{h}(\rho_n\zeta) = \rho_n^{-p}[f_n'(\rho_n\zeta) - h(\rho_n\zeta)] \neq 0, \tag{3.6}$$

by maximum modulus principle, we see that $\Psi_n'(\zeta) - \zeta^p \widehat{h}(\rho_n\zeta) \to \Psi'(\zeta) - \zeta^p$ spherically locally uniformly on $\mathbb{C}$, and hence by Hurwitz's theorem, $\Psi'(\zeta) - \zeta^p \neq 0$. It then follows from Lemma 2.2 that Ψ is a rational function.

We prove first that Ψ is not a polynomial. Suppose now that Ψ is a polynomial. Then $\Psi'(\zeta) = \zeta^p - c$ for some constant $c \neq 0$. Thus

$$\Psi(\zeta) = \frac{1}{p+1}\zeta^{p+1} - c\zeta + c_1 \tag{3.7}$$

for some constant c_1. Since $\Psi(0) \neq 0$, $c_1 \neq 0$. Thus by all zeros of Ψ being multiple, we get $p = 1$ and $\Psi(\zeta) = (\zeta - c)^2/2$. Hence

$$\Phi(\zeta) = \frac{(\zeta - c)^2}{2\zeta}. \tag{3.8}$$

Thus by (3.4), $f_n(0) \neq \infty$ and f_n has a double zero z_n such that $z_n/\rho_n \to c$. The poles and the other zeros z_n^* of f_n satisfy $z_n^*/\rho_n \to \infty$.

Now write

$$f_n(z) = (z - z_n)^2 f_n^*(z). \tag{3.9}$$

Then by (3.4), we get

$$F_n^*(\zeta) := f_n^*(\rho_n\zeta) \to \frac{1}{2} \tag{3.10}$$

on $\mathbb{C}^*$, and hence on $\mathbb{C}$ as F_n^* are uniformly locally holomorphic on $\mathbb{C}$.[1)]

We claim that $f_n^*(z) \neq 0$ on some disk $\Delta(0,\delta)$. If this is not true, then f_n^* has zeros tending to 0. Let $z_n^* \to 0$ be the zero with smallest modulus. Then by (3.10), $z_n^*/\rho_n \to \infty$. Let

$$\widehat{f_n^*}(z) = f_n^*(z_n^* z). \tag{3.11}$$

Then $\widehat{f_n^*}$ are well defined on $\mathbb{C}$ with $\widehat{f_n^*}(z) \neq 0$ for $z \in \Delta(0,1)$ and $\widehat{f_n^*}(1) = 0$. Since $f_n'(z) \neq h(z) = z\widehat{h}(z)$, we have

$$T_n(z) := \left[\left(z - \frac{z_n}{z_n^*}\right)^2 \widehat{f_n^*}(z)\right]' - z\widehat{h}(z_n^* z) = (z_n^*)^{-1}[f_n'(z_n^* z) - h(z_n^* z)] \neq 0. \tag{3.12}$$

Note that $z_n/z_n^* = z_n/\rho_n \cdot \rho_n/z_n^* \to 0$. Now, by (3.12), using Lemma 2.1 with Lemmas 2.2 and 2.3, it can be seen that $\{\widehat{f_n^*}\}$ is normal on $\mathbb{C}^*$. Thus, as $\widehat{f_n^*}(1) = 0$, by taking a subsequence, we can assume that $\{\widehat{f_n^*}\}$ converges spherically locally uniformly on $\mathbb{C}^*$ to a function $\widehat{f}^*$ which is meromorphic on $\mathbb{C}^*$ and satisfies $\widehat{f}^*(1) = 0$. Note that 1 is a multiple zero of $\widehat{f}^*$.

We claim that $\widehat{f}^* \not\equiv 0$. If this is not true, then we have $\widehat{f_n^*} \to 0$, $\widehat{f_n^*}' \to 0$ and $\widehat{f_n^*}'' \to 0$ on $\mathbb{C}^*$. Hence $T_n \to -z$ and $T_n' \to -1$ on $\mathbb{C}^*$. It follows that

$$\left| n(1, T_n) - n\left(1, \frac{1}{T_n}\right) \right| = \frac{1}{2\pi}\left| \int_{|z|=1} \frac{T_n'}{T_n} dz \right| \to \frac{1}{2\pi}\left| \int_{|z|=1} \frac{1}{z} dz \right| = 1, \tag{3.13}$$

where the notation $n(r, f)$ denotes the number of poles of f in $\Delta(0, r)$, counting multiplicity. Thus by (3.12) and (3.13), we get $n(1, T_n) = 1$. It follows that $[(z - \frac{z_n}{z_n^*})^2 \widehat{f_n^*}(z)]'$ has one simple pole. This is impossible.

Thus $\widehat{f}^* \not\equiv 0$, and hence $1/\widehat{f_n^*} \to 1/\widehat{f}^* \not\equiv \infty$ on $\mathbb{C}^*$. Since $1/\widehat{f_n^*}$ are holomorphic on $\Delta(0,1)$, by the maximum modulus principle, we get $1/\widehat{f_n^*} \to 1/\widehat{f}^*$ and hence $\widehat{f_n^*} \to \widehat{f}^*$ on $\Delta(0,1)$. Thus $\widehat{f_n^*} \to \widehat{f}^*$ on $\mathbb{C}$.

Since $\widehat{f_n^*}(0) = f_n^*(0) \to 1/2$, we have $\widehat{f}^*(0) = 1/2$ and hence $\widehat{f}^*$ is nonconstant as $\widehat{f}^*(1) = 0$. Furthermore, by (3.12), $T_n(z) \to T(z) := [z^2 \widehat{f}^*(z)]' - z$ on $\mathbb{C}^*$. Since $T(1) = -1$, $T(z) \not\equiv 0$. Thus by $T_n \neq 0$ and the maximum modulus principle, $T_n \to T$ on $\mathbb{C}$. Now by Hurwitz's theorem and $T_n \neq 0$, we get $T(z) \neq 0$ on $\mathbb{C}$. This is impossible as $T(0) = 0$.

This contradiction shows that $f_n^* \neq 0$ on some disk $\Delta(0,\delta)$.

Since $\{f_n\}$ and hence $\{f_n^*\}$ is normal on $\Delta^\circ(0,1)$, we can assume that $f_n^* \to f^*$ on $\Delta^\circ(0,1)$, where f^* may be ∞ identically. By (3.9), we also have $f_n(z) \to z^2 f^*(z)$ on $\Delta^\circ(0,1)$.

1) We say the functions f_n ($n \in \mathbb{N}$) are uniformly locally holomorphic on $\mathbb{C}$, if for each $R > 0$ there exists an $N \in \mathbb{N}$ such that for $n > N$, f_n are holomorphic on $\Delta(0, R)$.

We claim that $f^* \not\equiv 0$ on $\Delta^\circ(0,1)$. Suppose $f^* \equiv 0$, then we have $f_n' \to 0$ and $f_n'' \to 0$ on $\Delta^\circ(0,1)$, and hence

$$\left|n\left(\frac{1}{2}, f_n' - h\right) - n\left(\frac{1}{2}, \frac{1}{f_n' - h}\right)\right| = \frac{1}{2\pi}\left|\int_{|z|=\frac{1}{2}} \frac{f_n'' - h'}{f_n' - h}dz\right| \to \frac{1}{2\pi}\left|\int_{|z|=\frac{1}{2}} \frac{h'}{h}dz\right| = p = 1. \tag{3.14}$$

Thus by $f_n' - h \neq 0$,

$$n\left(\frac{1}{2}, f_n'\right) = n\left(\frac{1}{2}, f_n' - h\right) = 1. \tag{3.15}$$

This is impossible.

Thus $f^* \not\equiv 0$ on $\Delta^\circ(0,1)$. Then $1/f^* \not\equiv \infty$. As $1/f_n^*$ are holomorphic on $\Delta(0,\delta)$, and $1/f_n^* \to 1/f^*$ on $\Delta^\circ(0,1)$, it can be seen by maximum modulus principle that $1/f_n^* \to 1/f^*$ and hence $f_n^* \to f^*$ on $\Delta(0,1)$.

Since $f_n^*(0) \to 1/2$, we get $f^*(0) = 1/2$ and hence f^* is holomorphic at 0. By (3.9), we see that f_n is normal at 0, which is a contradiction.

Thus Ψ is not a polynomial. Hence by $[\Psi(\zeta) - \frac{1}{p+1}\zeta^{p+1}]' = \Psi'(\zeta) - \zeta^p \neq 0$ and Lemma 2.4, we get

$$\Psi(\zeta) = \frac{1}{p+1}\zeta^{p+1} + c_1 + \frac{c_2}{(\zeta - \zeta_0)^m} \tag{3.16}$$

for some constants c_1, c_2, ζ_0 with $c_2 \neq 0$ and $m \in \mathbb{N}$. Thus by Lemma 2.5,

$$\Psi(\zeta) = \frac{(\zeta^{p+1} - c)^2}{(p+1)\zeta^{p+1}} \tag{3.17}$$

for some constant $c \neq 0$. Hence

$$\Phi(\zeta) = \frac{(\zeta^{p+1} - c)^2}{(p+1)\zeta^{2p+1}}. \tag{3.18}$$

Thus by (3.4), f_n has $p+1$ double zeros $z_{n,i}$ such that $\zeta_{n,i} = z_{n,i}/\rho_n \to \zeta_i$ with $\zeta_i^{p+1} = c$ and the sum of the multiplicities of the poles w_n of f_n that satisfy $w_n/\rho_n \to 0$ is equal to $p+1$.

We claim that f_n has exactly one pole w_n satisfying $w_n/\rho_n \to 0$ (and hence it has exact multiplicity $p+1$). If this is not true, then by (3.4), we see that f_n has $2 \leqslant s \leqslant p+1$ poles $w_{n,i}$ with multiplicity s_i such that $\xi_{n,i} = w_{n,i}/\rho_n \to 0$ and $\sum_{i=1}^s s_i = p+1$. By choosing subsequence, we may assume that the number s and the multiplicities s_i of distinct poles of f_n are independent of f_n.

Let

$$f_n(z) = \frac{P_n^2}{Q_n}f_n^*(z), \tag{3.19}$$

where

$$P_n(z) = \prod_{i=1}^{p+1}(z - z_{n,i}), \quad Q_n(z) = \prod_{i=1}^{s}(z - w_{n,i})^{s_i}, \quad \sum_{i=1}^{s} s_i = p+1. \tag{3.20}$$

Then by (3.4),

$$F_n^*(\zeta) := f_n^*(\rho_n\zeta) \to \frac{1}{p+1} \tag{3.21}$$

on $\mathbb{C}^*$, and hence on $\mathbb{C}$ by maximum modulus principle, as F_n^* are uniformly locally holomorphic on $\mathbb{C}$. By (3.19), we have

$$\Psi_n(\zeta) = \rho_n^{-p-1}f_n(\rho_n\zeta) = \frac{\widehat{P}_n^2(\zeta)}{\widehat{Q}_n(\zeta)}F_n^*(\zeta), \tag{3.22}$$

where

$$\widehat{P}_n(\zeta) = \prod_{i=1}^{p+1}(\zeta - \zeta_{n,i}) \to \zeta^{p+1} - c, \quad \widehat{Q}_n(\zeta) = \prod_{i=1}^{s}(\zeta - \xi_{n,i})^{s_i} \to \zeta^{p+1}. \tag{3.23}$$

Thus

$$\Psi_n' = \frac{(\widehat{P}_n^2F_n^*)'}{\widehat{Q}_n} - \frac{\sum_{i=1}^s s_i\prod_{j\neq i}(\zeta - \xi_{n,j})}{\prod_{i=1}^s(\zeta - \xi_{n,i})^{s_i+1}}\widehat{P}_n^2F_n^* = \frac{L_n(\zeta)}{\prod_{i=1}^s(\zeta - \xi_{n,i})^{s_i+1}}, \tag{3.24}$$

where

$$L_n(\zeta) := (\widehat{P}_n^2 F_n^*)' \prod_{i=1}^{s} (\zeta - \xi_{n,i}) - \widehat{P}_n^2 F_n^* \sum_{i=1}^{s} s_i \prod_{j \neq i} (\zeta - \xi_{n,j}) \to \zeta^{s-1}(\zeta^{2p+2} - c^2). \tag{3.25}$$

Since $\Psi_n'(\zeta) - \zeta^p \widehat{h}(\rho_n \zeta) \neq 0$, we get

$$M_n(\zeta) := L_n(\zeta) - \zeta^p \widehat{h}(\rho_n \zeta) \prod_{i=1}^{s} (\zeta - \xi_{n,i})^{s_i+1} \neq 0. \tag{3.26}$$

We have

$$M_n(\zeta) \to M(\zeta) := \zeta^{s-1}(\zeta^{2p+2} - c^2) - \zeta^p \cdot \zeta^{p+1+s} = -c^2 \zeta^{s-1}. \tag{3.27}$$

Hence, by $M_n \neq 0$ and Hurwitz's theorem, we have either $M \equiv 0$ or $M \neq 0$. This is impossible, since $s \geqslant 2$.

Thus f_n has exactly one pole w_n satisfying $w_n/\rho_n \to 0$ and with exact multiplicity $p+1$. Hence by letting

$$f_n(z) = R_n(z) f_n^*(z), \tag{3.28}$$

where

$$R_n(z) = \frac{[\prod_{i=1}^{p+1}(z - z_{n,i})]^2}{(z - w_n)^{p+1}}, \tag{3.29}$$

we get by (3.4)

$$F_n^*(\zeta) := f_n^*(\rho_n \zeta) \to \frac{1}{p+1} \tag{3.30}$$

on $\mathbb{C}$ as showed above.

Next, by a similar argument to that showed above, we can see that f_n^* has a subsequence which converges to a meromorphic function f^* on $\Delta(0,1)$ such that $f^*(0) = 1/(p+1)$.

By (3.28) and (3.29), we see that $f_n \to z^{p+1} f^*$ on $\Delta^\circ(0,1)$. Thus $f_n' - h \to [z^{p+1} f^*]' - h$ on $\Delta^\circ(0,1) \setminus (f^*)^{-1}(\infty)$. If $[z^{p+1} f^*]' - h \not\equiv 0$, then since $f_n' - h \neq 0$, by the maximum modulus principle, we obtain $f_n' - h \to [z^{p+1} f^*]' - h$ on the whole disk $\Delta(0,1)$, and by Hurwitz's theorem, we get $[z^{p+1} f^*]' - h \neq 0$. Letting $z = 0$, we see that this case cannot occur. Thus $[z^{p+1} f^*]' \equiv h$, and hence

$$f^*(z) = \frac{1}{z^{p+1}} \int_0^z h(z) dz. \tag{3.31}$$

The proof of Theorem 1.6 is completed.

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