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Normal edge-transitive Cayley graphs on non-abelian groups of order $4p$, where p is a prime number

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Abstract We determine all connected normal edge-transitive Cayley graphs on non-abelian groups with order $4p$, where p is a prime number. As a consequence we prove if $|G| = 2^\delta p, \delta = 0, 1, 2$ and p prime, then $\Gamma = \mathrm{Cay}(G, S)$ is a connected normal $\frac{1}{2}$ arc-transitive Cayley graph only if $G = F_{4p}$, where S is an inverse closed generating subset of G which does not contain the identity element of G and F_{4p} is a group with presentation $F_{4p} = \langle a, b \mid a^p = b^4 = 1, b^{-1}ab = a^\lambda\rangle$, where $\lambda^2 \equiv -1 \pmod p$.

Keywords Cayley graph, automorphism group, normal edge-transitive graph

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1 Introduction

Let $\Gamma = (V, E)$ be a simple graph, where V is the set of vertices and E is the set of edges of Γ. An edge joining the vertices u and v is denoted by $\{u, v\}$. The group of automorphisms of Γ is denoted by $\mathrm{Aut}(\Gamma)$, which acts on vertices, edges and arcs of Γ. If $\mathrm{Aut}(\Gamma)$ acts transitively on vertices, edges or arcs of Γ, then Γ is called vertx-transitive, edge-transitive or arc-transitive respectively. If Γ is vertex and edge-transitive but not arc-transitive, then Γ is called $\frac{1}{2}$ arc-transitive.

Let G be a finite group and S be an inverse closed subset of G, i.e., $S = S^{-1}$, such that $1 \notin S$, the Cayley graph $\Gamma = \mathrm{Cay}(G, S)$ on G with respect to S is a graph with vertex set G and edge set $\{\{g, sg\} \mid g \in G, s \in S\}$. Γ is connected if and only if $G = \langle S\rangle$. For $g \in G$, define the mapping $\rho_g : G \to G$ by $\rho_g(x) = xg, x \in G$. $\rho_g \in \mathrm{Aut}(\Gamma)$ for every $g \in G$, thus $R(G) = \{\rho_g \mid g \in G\}$ is a regular subgroup of $\mathrm{Aut}(\Gamma)$ isomorphic to G, forcing Γ to be a vertex-transitive graph.

Let $\Gamma = \mathrm{Cay}(G, S)$ be a Cayley graph of a finite group G on S. Let $\mathrm{Aut}(G, S) = \{\alpha \in \mathrm{Aut}(G) \mid S^\alpha = S\}$ and $A = \mathrm{Aut}(\Gamma)$. Then the normalizer of $R(G)$ in A is equal to

$$N_A(R(G)) = R(G) \rtimes \mathrm{Aut}(G, S),$$

where $\rtimes$ denotes the semi-direct product of two groups. In [11] the graph Γ is called normal if $R(G)$ is a normal subgroup of $\mathrm{Aut}(\Gamma)$.

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Therefore, according to [3], $\Gamma = \mathrm{Cay}(G,S)$ is normal if and only if $A := \mathrm{Aut}(\Gamma) = R(G) \rtimes \mathrm{Aut}(G,S)$, and in this case $A_1 = \mathrm{Aut}(G,S)$, where A_1 is the stabilizer of the identity element of G under A. The normality of Cayley graphs has been extensively studied from different points of views by many authors. In [10] all disconnected normal Cayley graphs are obtained. Therefore, it suffices to study the connected Cayley graphs when one investigates the normality of Cayley graphs, which we use in this paper.

Therefore, in this paper when we talk about a Cayley graph Γ, we mean $\Gamma = \mathrm{Cay}(G,S)$, where G is a finite group and S is a non-empty generating subset of G such that $1 \notin S$ and $S = S^{-1}$. We also denote $\mathrm{Aut}(\Gamma)$ by A.

Definition 1.1. A Cayley graph Γ is called normal edge-transitive or normal arc-transitive if $N_A(R(G))$ acts transitively on the set of edges or arcs of Γ, respectively. If Γ is normal edge-transitive, but not normal arc-transitive, then it is called a normal $\frac{1}{2}$ arc-transitive Cayley graph.

Edge-transitivity of Cayley graphs of small valency have received attention in the literature. A relation between regular maps and edge-transitive Cayley graphs of valency 4 is studied in [7], and in [6] Li et al. characterized edge-transitive Cayley graphs of valency four and odd order. In [5] Houlis classified normal edge-transitive Cayley graphs of groups $\mathbb{Z}_{pq}$, where p and q are distinct primes. In [1] normal edge-transitive Cayley graphs on some abelian groups of valency at most 5 are studied. And in [2] edge-transitive Cayley graphs of valency 4 on non-abelian simple groups are studied.

Motivated by the above results, we consider the normal edge-transitive Cayley graphs of groups with order $4p$, where p is a prime number. Since in [9] Talebi has considered the dihedral group of order $4p$, we will deal with the rest of the non-abelian groups of order $4p$. As a consequence, we will prove if Γ is a normal $\frac{1}{2}$ arc-transitive Cayley graph of order $2^\delta p$, where $0 \leqslant \delta \leqslant 2$ and p is a prime number, then $G \cong F_{4p}$. Moreover, we investigate the normal edge-transitive Cayley graphs on certain groups of order $4p$, namely, Q_{4p}, F_{4p} and A_4, where the groups of Q_{4p} and F_{4p} will be defined in Section 3 and A_4 is the alternating group of order 12.

2 Preliminary results

Keeping fixed terminologies used in Section 1, we mention a few results whose proofs can be found in the literature.

The following result is proved in [11] and [3].

Result 2.1. Let $\Gamma = \mathrm{Cay}(G,S)$, then the following hold:

1. $N_A(R(G)) = R(G) \rtimes \mathrm{Aut}(G,S)$;
2. $R(G) \trianglelefteq A$ if and only if $A = R(G) \rtimes \mathrm{Aut}(G,S)$;
3. Γ is normal if and only if $A_1 = \mathrm{Aut}(G,S)$.

The result that we will use in our investigation of normal edge-transitive Cayley graphs is the following that makes it possible to characterize normal edge-transitivity in terms of the action of $\mathrm{Aut}(G,S)$ on S (see [8]).

Result 2.2. Let $\Gamma = \mathrm{Cay}(G,S)$ be a connected Cayley graph (undirected) on S. Then Γ is normal edge-transitive if and only if $\mathrm{Aut}(G,S)$ is either transitive on S, or has two orbits in S in the form of T and T^{-1}, where T is a non-empty subset of S such that $S = T \cup T^{-1}$.

Since in the action of $\mathrm{Aut}(G,S)$ on S, every element of $s \in S$ has the same order as every element in the orbit of s, we can deduce the following corollary from Result 2.2.

Corollary 2.3. *Let $\Gamma = \mathrm{Cay}(G,S)$ and H be the subset of all involutions of the group G. If $\langle H\rangle \neq G$ and Γ is connected normal edge-transitive, then its valency is even.*

For a general graph $\Gamma = (V,E)$, if v is a vertex in Γ, then $\Gamma(v)$ denotes the set of the so-called neighbors of v, i.e., $\Gamma(v) = \{u \in V \mid \{u,v\} \in E\}$. The following result which can be deduced from a result in [4] characterizes normal arc-transitive Cayley graphs in terms of the action of $\mathrm{Aut}(G,S)$ on S.

Result 2.4. Let $\Gamma = \mathrm{Cay}(G, S)$ be a connected Cayley graph (undirected) on S. Then Γ is normal arc-transitive if and only if $\mathrm{Aut}(G, S)$ acts transitively on S.

We can extract the following corollary from Results 2.2 and 2.4 and the observation that if G is an abelian group, then $\sigma : G \to G$ defined by $\sigma(x) = x^{-1}, \forall\, x \in G$, is an automorphism.

Corollary 2.5. *If Γ is a Cayley graph of an abelian group, then Γ is not a normal $\frac{1}{2}$ arc-transitive Cayley graph.*

The following result is obtained in [9].

Result 2.6. Let $\Gamma = \mathrm{Cay}(G, S)$ be a connected normal edge-transitive Cayley graph of the dihedral group D_{2n}. Then $\mathrm{Aut}(D_{2n}, S)$ is transitive on S.

Corollary 2.7. *If Γ is a Cayley graph of a dihedral group D_{2n}, then Γ is not a normal $\frac{1}{2}$ arc-transitive Cayley graph.*

The following result is mentioned in [8].

Result 2.8. Let Γ be a connected Cayley graph of a non-abelian simple group with valency 3. If Γ is normal edge-transitive, then it is normal.

3 Classifications of Cayley graphs of order $4p$

It is easy to prove that any non-abelian group of order $4p$ is isomorphic to one of the following groups which are given by generators and relations:

$F_{4p} = \langle a, b \mid a^p = b^4 = 1, b^{-1}ab = a^{\lambda}\rangle$, where $\lambda^2 \equiv -1 \pmod p$,

$Q_{4p} = \langle a, b \mid a^{2p} = 1, a^p = b^2, b^{-1}ab = a^{-1}\rangle$,

$A_4 = \langle a, b \mid a^3 = b^3 = (ab)^2 = 1\rangle$ $(p = 3)$,

$D_{4p} = \langle a, b \mid a^{2p} = b^2 = 1, b^{-1}ab = a^{-1}\rangle$.

If $p = 2$, then F_8 is abelian and Q_8 is the quaternion group of order 8, which are not of interest to us. Also the dihedral group D_{4p} is studied in [9], hence we study the groups Q_{4p} and F_{4p}, where p is an odd prime, and the alternating group of order 12.

Elements of Q_{4p} can be written uniquely in the form $a^ib^j, 0 \leqslant i < 2p, j = 0, 1$. Element orders in Q_{4p} are as follows:

$$O(a^i) = \frac{2p}{(i, 2p)} = \begin{cases} 2p, & \text{if } (i, 2p) = 1, \\ p, & \text{if } (i, 2p) = 2, \\ 2, & \text{if } i = p, \end{cases}$$

where $1 \leqslant i \leqslant 2p - 1$. We have $O(a^ib) = 4$, $0 \leqslant i \leqslant 2p - 1$. Using the above facts we can find $\mathrm{Aut}(Q_{4p})$.

Lemma 3.1. *For odd prime p, $\mathrm{Aut}(Q_{4p}) \cong \mathbb{Z}_{2p} \rtimes \mathbb{Z}_{p-1}$, and it has the following orbits on Q_{4p} : $\{1\}$, $\{a^i \mid 1 \leqslant i < 2p,\ (i, 2p) = 1\}, \{a^p\}, \{a^{2i} \mid 1 \leqslant i \leqslant p - 1\}$ and $\{a^ib \mid 0 \leqslant i \leqslant 2p - 1\}$.*

Proof. Any $\sigma \in \mathrm{Aut}(Q_{4p})$ is determined by its effect on a and b. Taking orders into account we have $\sigma(a) = a^i$, where $1 \leqslant i < 2p$, $(i, 2p) = 1$ and $\sigma(b) = a^jb$, $0 \leqslant j \leqslant 2p - 1$. It can be verified that $\sigma = f_{i,j}$ defined as above can be extended to an automorphism of Q_{4p}. Therefore, $\mathrm{Aut}(Q_{4p}) = \{f_{i,j} \mid 1 \leqslant i < 2p, (i, 2p) = 1, 0 \leqslant j \leqslant 2p - 1\}$ is a group of order $2p\varphi(2p) = 2p(p - 1)$.

We have $f_{i,j} \circ f_{i',j'} = f_{ii',ij'+j}$, hence if we define a group of 2×2 matrices

$$G = \left\{ \begin{bmatrix} i & j \\ 0 & 1 \end{bmatrix} \,\middle|\, 1 \leqslant i < 2p, (i, 2p) = 1, 0 \leqslant j \leqslant 2p - 1 \right\},$$

then $\mathrm{Aut}(Q_{4p}) \cong G$. But it is easy to prove that $G \cong \mathbb{Z}_{2p} \rtimes \mathbb{Z}_{p-1}$ and the lemma is proved. □

Lemma 3.2. *If $\mathrm{Cay}(Q_{4p}, S)$ is a connected normal edge-transitive Cayley graph on S, then S consists of elements of order 4. Moreover, $|S| > 2$ is even.*

Proof. By Result 2.2 elements in S have the same order. Since $\langle S\rangle = Q_{4p}$, the set S cannot contain elements of order p or $2p$, and should contain elements of order 4 only. Being of even valency arises from Corollary 2.3. □

Lemma 3.3. *Let $i \geqslant j$. $S = \{a^i b, a^{i+p}b, a^j b, a^{j+p}b\}$ generates Q_{4p} if and only if $i \neq j$ and $i \neq j+p$, $0 \leqslant i, j \leqslant 2p-1$. Moreover, in this case $\mathrm{Aut}(Q_{4p}, S) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof. Generating condition of S comes from the relations $(a^k b)^{-1} = a^{k+p}b$ for $k = i, j$, $a^i b a^{j+p} b = a^{i-j}$ and $a^i b a^j b = a^{i-j+p}$.

Now let $S = \{x, x^{-1}, y, y^{-1}\}$ and $G = \mathrm{Aut}(Q_{4p}, S)$. Then G acts on S faithfully, and so is a subgroup of $\mathbb{S}_4$. But G does not have elements of order 3 or 4, because if $f \in G$ has order 3, then it should fix an element on S such as s, thus $f(s^{-1}) = s^{-1}$, contradicting the order of f. Also if f is an element of order 4, then its cycle structure on S have the form $(x \;\; y \;\; x^{-1} \;\; y^{-1})$ or $(x \;\; y^{-1} \;\; x^{-1} \;\; y)$, where $x = a^i b, y = a^j b$ and $f = f_{r,s}$ (as mentioned in Lemma 3.1, $(r, 2p) = 1, 0 \leqslant s \leqslant 2p-1)$) and we may assume $i > j$, thus $ri + s \equiv j \pmod{2p}$ and $rj + s \equiv i + p \pmod{2p}$, or for the latter case $ri + s \equiv j + p \pmod{2p}$ and $rj + s \equiv i \pmod{2p}$. But in each case we obtain $p \mid (r+1)(i-j)$. But $p \mid (r+1)$ implies either $r = p-1$, i.e., r is even, or $r = 2p-1$. If r is even, then it contradicts $(r, 2p) = 1$. If $r = 2p-1$, then substituting it into the congruences $ri + s \equiv j \pmod{2p}$ and $rj + s \equiv i + p \pmod{2p}$, we obtain the new congruences $-(i+j) + s \equiv 0 \pmod{2p}$ and $-(i+j) + s \equiv p \pmod{2p}$ respectively which are absurd. $p \mid (i-j)$ implies $i = j + p$ which is also a contradiction.

Therefore, G is a subgroup of $\mathbb{S}_4$ which does not have any element of order 3 or 4, but at least it has two elements of order 2 such as $f_{1,p}$ and $f_{-1,i+j}$, where $x = a^i b$ and $y = a^j b$, implying $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. □

Next, we can express the main condition under which the Cayley graph of group Q_{4p} becomes connected normal edge-transitive.

Theorem 3.4. *$\Gamma = \mathrm{Cay}(Q_{4p}, S)$ is a connected normal edge-transitive Cayley graph if and only if its valency is even, greater than two, $S \subseteq \{a^i b, a^{i+p}b \mid 0 \leqslant i < p\}$, $S = S^{-1}$ and $\mathrm{Aut}(Q_{4p}, S)$ acts transitively on S.*

Proof. If Γ is a connected normal edge-transitive Cayley graph, then by Corollary 2.3 its valency should be even. By Lemma 3.2, $S \subseteq \{a^i b, a^{i+p}b \mid 0 \leqslant i < p\}$, since the graph is undirected, $S = S^{-1}$, and by Lemma 3.3, if $|S| > 2$, then $\langle S\rangle = Q_{4p}$. Hence $S = \{a^i b, a^{i+p}b \mid \text{for some } 0 \leqslant i < p\}$. From Result 2.2, either $\mathrm{Aut}(Q_{4p}, S)$ acts on S transitively, or $S = T \cup T^{-1}$, where T and T^{-1} are orbits of the action of $\mathrm{Aut}(Q_{4p}, S)$ on S. But we observe $f_{1,p} \in \mathrm{Aut}(Q_{4p}, S)$, which implies both of $a^i b$ and $(a^i b)^{-1} = a^{i+p}b$ belong to the same orbit for $0 \leqslant i < p$ in which $a^i b \in S$, and that contradicts the assumption $S = T \cup T^{-1}$, forcing $\mathrm{Aut}(Q_{4p}, S)$ to act transitively on S. □

Corollary 3.5. *If Γ is a connected Cayley graph of the group Q_{4p}, then Γ is not normal $\frac{1}{2}$ arc-transitive.*

Example 3.6. If $S = \{a^i b \mid 0 \leqslant i \leqslant 2p-1\}$, then $\mathrm{Cay}(Q_{4p}, S)$ is a connected normal edge-transitive Cayley graph of valency $2p$.

Recall that Result 2.8 expresses some conditions in which a connected normal edge-transitive Cayley graph becomes a normal Cayley graph. Here we provide a counter example when the conditions are omitted, this shows somehow the necessity of the conditions.

Example 3.7. Let $\Gamma = \mathrm{Cay}(Q_{4p}, S)$ be a Cayley graph of valency 4. Γ is a connected normal edge-transitive Cayley graph if and only if $S = \{a^i b, a^{i+p}b, a^j b, a^{j+p}b\}$, where $i \neq j$ and $i \neq j+p$. Moreover, in this case Γ is not a normal Cayley graph, i.e., there is a connected normal edge-transitive Cayley graph which is not normal Cayley graph.

Proof. By Theorem 3.4 it is enough to show $\mathrm{Aut}(Q_{4p}, S)$ acts transitively on S. S is equivalent to $S' = \{b, b^{-1}, ab, (ab)^{-1}\}$, since $(S')^{f_{j-i,i}} = S$. Therefore, it is sufficient to check it for S'. But $f_{1,p}, f_{-1,1}, f_{-1,p+1}$ are all in $\mathrm{Aut}(Q_{4p}, S)$ and send b to $b^{-1}, ab, (ab)^{-1}$, respectively.

For the second part, it is enough to check the case $S' = \{b, b^{-1}, ab, (ab)^{-1}\} = \{b, a^p b, ab, a^{p+1}b\}$. We have $\Gamma(ab) = \{a^{p-1}, a^{-1}, a^p, 1\} = \Gamma(a^{p+1}b)$, thus $\sigma = (ab \quad (ab)^{-1}) \in (\mathrm{Aut}\Gamma)_1$, but $f_{1,p}, f_{-1,1}, f_{-1,p+1} \in$

$\mathrm{Aut}(Q_{4p}, S)$ and Lemma 3.3 show that $\sigma \notin \mathrm{Aut}(Q_{4p}, S)$, i.e., $(\mathrm{Aut}\Gamma)_1 \neq \mathrm{Aut}(Q_{4p}, S)$ and by Result 2.1, Γ is not a normal Cayley graph. □

Theorem 3.8. *Let $\Gamma = \mathrm{Cay}(Q_{4p}, S)$ be a normal edge-transitive Cayley graph of valency $2d$. Then either $d = p$ or $d \mid (p-1)$. Moreover, for each of the above numbers, there is, up to isomorphism, one normal edge-transitive Cayley graph of valency $2d$.*

Proof. By Theorem 3.4, $S \subseteq \{a^i b, a^{i+p} b \mid 0 \leqslant i < p\}$ and by Example 3.6, $\mathrm{Cay}(Q_{4p}, U)$ is a connected normal edge-transitive graph of valency $2p$, where $U = \{a^i b, a^{i+p} b \mid 0 \leqslant i < p\}$. Now suppose $S \subseteq \{a^i b, a^{i+p} b \mid 0 \leqslant i < p\}, \langle S \rangle = Q_{4p}$ and Γ is a Cayley graph of valency $2d$. Since $\mathrm{Aut}(Q_{4p}, S) \leqslant \mathrm{Aut}(Q_{4p})$ and $\mathrm{Aut}(Q_{4p}, S)$ is transitive on S (Theorem 3.4), we have $|S| = 2d \mid |\mathrm{Aut}(Q_{4p}, S)| \mid |\mathrm{Aut}(Q_{4p})| = 2p(p-1)$, implying $d \mid p(p-1)$. On the other hand, we have $d \leqslant p$, hence either $d = p$ or $d \mid (p-1)$ proving the first assertion of the theorem.

To prove the existence and uniqueness part in the theorem, if $d = p$, then as mentioned above, $\mathrm{Cay}(Q_{4p}, U)$ is the unique normal maximal edge-transitive Cayey graph of valency $2p$. Now suppose $d \mid (p-1), d > 1$. The stabilizer of b under $A = \mathrm{Aut}(Q_{4p})$ is the group $A_b = \{f_{i,0} \mid 1 \leqslant i < 2p, i \text{ odd}, i \neq p\} \cong \mathbb{Z}_{p-1}$, where $\mathbb{Z}_{p-1}$ is the cyclic group with multiplicative low of composition. Let t be a generator of $\mathbb{Z}_{p-1}$, so that $A_b = \langle f_{t,0} \rangle$. Since $d \mid (p-1)$, the group $\mathbb{Z}_{p-1}$ contains a unique subgroup of order d, and if we set $u = t^{\frac{p-1}{d}}$, then $\langle f_{u,0} \rangle$ is a subgroup of A_b with order d. Now consecutive effects of $f_{u,0}$ on ab yields the set $T = \{ab, a^u b, \ldots, a^{u^{d-1}}\}$ whose size is d and is invariant under $f_{u,0}$. Let us set $T^{-1} = \{x^{-1} \mid x \in T\} = \{a^p b, a^{u+p} b, \ldots, a^{u^{d-1}+p} b\}$ and $S = T \cup T^{-1}$. We claim that $\mathrm{Cay}(Q_{4p}, S)$ is a connected normal edge-transitive Cayley graph. By the argument used in Lemma 3.3, we have $\langle S \rangle = Q_{4p}$. It is easy to see that $f_{u,p}$ interchanges elements of T and T^{-1}, also the automorphism group of $\mathrm{Cay}(Q_{4p}, S)$ is $\langle f_{u,0}, f_{u,p} \rangle$, implying $\mathrm{Cay}(Q_{4p}, S)$ is connected normal edge-transitive of valency $2d$. □

Next, we consider the group F_{4p}, which is defined as follows and we will prove its Cayley graph on some set can be connected $\frac{1}{2}$ arc-transitive Cayley graph,

$$F_{4p} = \langle a, b \mid a^p = b^4 = 1, b^{-1}ab = a^{\lambda} \rangle, \quad \text{where } \lambda^2 \equiv -1 \pmod{p}.$$

Recall that we assume p is an odd prime. The existence of λ satisfying the condition $\lambda^2 \equiv -1 \pmod{p}$ implies that $4 \mid (p-1)$, hence p must be a prime of the form $p = 1 + 4k$.

The orders of non-identity elements of F_{4p} are as follows:

$O(a^i) = p$, $1 \leqslant i \leqslant p-1$;

$O(a^i b) = O(a^i b^3) = 4$, $0 \leqslant i \leqslant p-1$;

$O(a^i b^2) = 2$, $0 \leqslant i \leqslant p-1$.

Thus if $\sigma \in \mathrm{Aut}(F_{4p})$, then $\sigma(a) = a^i$ and either $\sigma(b) = a^j b$ or $\sigma(b) = a^j b^3$ for $1 \leqslant i \leqslant p-1, 0 \leqslant j \leqslant p-1$, but we also have $\sigma(b^{-1}ab) = \sigma(a^{\lambda})$, thus in the latter case we obtain a contradiction. Therefore, we have

$$\mathrm{Aut}(F_{4p}) = \{g_{i,j} \mid g_{i,j}(a) = a^i, g_{i,j}(b) = a^j b, 1 \leqslant i \leqslant p-1, 0 \leqslant j \leqslant p-1\}$$

is a group of order $p(p-1)$.

Theorem 3.9. *$\Gamma = \mathrm{Cay}(F_{4p}, S)$ is a connected normal edge-transitive Cayley graph if and only if it has even valency, $S = T \cup T^{-1}$, where $T \subseteq \{a^i b \mid 0 \leqslant i \leqslant p-1\}$ and $\mathrm{Aut}(F_{4p}, S)$ acts transitively on T. Moreover, if $\Gamma = \mathrm{Cay}(F_{4p}, S)$ is a normal edge-transitive Cayley graph of valency $2d$, then either $d = p$ or $d \mid (p-1)$. For each of the above numbers, there is, up to isomorphism, one normal edge-transitive Cayley graph of valency $2d$.*

Proof. First assume Γ is a connected normal edge-transitive Cayley graph.

The fact that Γ has even valency follows from Corollary 2.3. By Result 2.2, in the action of $\mathrm{Aut}(F_{4p}, S)$ on S, we can deduce either S is an orbit or $S = T \cup T^{-1}$, where T is an orbit.

We have $(a^i b^2)^{-1} = a^i b^2$, thus if $a^i b^2 \in S$ for some $0 \leqslant i \leqslant p-1$, the case $S = T \cup T^{-1}$ cannot occur, i.e., $\mathrm{Aut}(F_{4p}, S)$ acts transitively on S, but Γ is connected, i.e., $\langle S \rangle = G$, therefore S should contain some element other than $a^j b^2, 0 \leqslant j < p$, say x, such that its order is not 2. Hence, there is no $g_{r,s} \in \mathrm{Aut}(F_{4p}, S) \subseteq \mathrm{Aut}(F_{4p})$ such that $g_{r,s}(x) = a^i b^2$, a contradiction.

Suppose $y = a^i \in S$ for some $1 \leqslant i \leqslant p-1$. Since Γ is connected, i.e., $\langle S \rangle = G$, S should contain an element x, where $x = a^j b$ or $x = a^j b^3$ for some $0 \leqslant j \leqslant p-1$. But since $(x)^{-1} \neq y$, without loss of generality, we can assume x and y are contained in the same orbit. But there is no $g_{r,s} \in \mathrm{Aut}(F_{4p}, S) \subseteq \mathrm{Aut}(F_{4p})$ such that $g_{r,s}(x) = y$, a contradiction.

Therefore, S contains only elements of types $a^i b$ and $a^j b^3$ for $0 \leqslant i, j \leqslant p-1$. But $S = S^{-1}$ and for each $0 \leqslant j \leqslant p-1$, there is some $0 \leqslant i \leqslant p-1$, where $(a^j b)^{-1} = a^i b^3$, hence S contains not only $a^i b$ but also $a^j b^3$ for $0 \leqslant i, j \leqslant p-1$. Therefore, $\mathrm{Aut}(F_{4p})$ and consequently $\mathrm{Aut}(F_{4p}, S)$ is not transitive on S, hence $S = T \cup T^{-1}$, where $T \subseteq \{a^i b \mid 0 \leqslant i \leqslant p-1\}$, and $\mathrm{Aut}(F_{4p}, S)$ acts transitively on T.

The converse is obvious.

The second part of the theorem is similar to the proof of Theorem 3.8 by replacing $U = \{a^i b, a^i b^3 \mid 0 \leqslant i \leqslant p-1\}$ for the case $d = p$ and the same argument in Theorem 3.8 for the case $d \mid (p-1)$ can be repeated here and the assumption that T is the orbit of the action of $\mathrm{Aut}(F_{4p}, S)$ on S. □

Example 3.10. Let $\Gamma = \mathrm{Cay}(F_{4p}, S)$ be a Cayley graph of valency 4. Γ is a connected normal edge-transitive Cayley graph if and only if $S = \{a^i b, a^j b, (a^i b)^{-1}, (a^j b)^{-1} \mid \text{for some } 0 \leqslant i < j \leqslant p-1\}$.

Proof. By Theorem 3.9, it is sufficient to put $T = \{a^i b, a^j b\}$ and consider $g_{-1,i+j} \in \mathrm{Aut}(F_{4p}, S)$. □

The above example ensures that we may have some connected normal edge-transitive Cayley graph of the group F_{4p}, and by Theorem 3.9, we see that there is some connected normal $\frac{1}{2}$ arc-transitive Cayley graph of order $4p$. The next result shows that normal $\frac{1}{2}$ arc-transitivity among Cayley graphs of order $2^\delta p$, $0 \leqslant \delta \leqslant 2$, p prime, appears only among group F_{4p}.

In the next theorem we investigate the alternating group of order 12 which shows no $\frac{1}{2}$ arc-transitive Cayley graph arise from this group.

Theorem 3.11. *Let* $\Gamma = \mathrm{Cay}(A_4, S)$ *be a connected Cayley graph. The following are equivalent*:

(1) Γ *is normal edge-transitive.*

(2) *The valency of* Γ *is even,* $|S| > 2$ *and* S *consists of entirely* 3*-cycles.*

(3) Γ *is normal arc-transitive.*

Proof. Let $a = (1\ 2\ 3)$, $b = (1\ 2\ 4)$, $c = (1\ 3\ 4)$, $d = (2\ 3\ 4)$ be all of 3-cycles in A_4 so that none of them are the inverse of the others.

(1) ⇒ (2) Assume Γ is normal edge-transitive. Hence by Corollary 2.3, its valency is even. $\langle S \rangle = A_4$ and $A_4 = \langle a, b \rangle$ imply $|S| > 2$ and S contains at least one 3-cycle. Result 2.2 shows that all elements of S should have the same order, hence S consists of entirely 3-cycles.

(2) ⇒ (3) Under the conditions which are mentioned in (2), $|S| = 4, 6$ or 8, S contains only 3-cycles and $S = S^{-1}$.

If $|S| = 4$, then S can be one of the following types: $S_1 = \{a, b, a^{-1}, b^{-1}\}$, $S_2 = \{a, c, a^{-1}, c^{-1}\}$, $S_3 = \{a, d, a^{-1}, d^{-1}\}$, $S_4 = \{b, c, b^{-1}, c^{-1}\}$, $S_5 = \{b, d, b^{-1}, d^{-1}\}$, $S_6 = \{c, d, c^{-1}, d^{-1}\}$. If we define $\alpha := (1\ 2\ 3\ 4)$, $\beta := (3\ 4\ 2)$, $\gamma := (1\ 4)$, $\delta := (1\ 4\ 2)$, $\zeta := (1\ 3)(2\ 4)$ and $\Gamma_i := \mathrm{Cay}(A_4, S_i)$ for $i = 1, \ldots, 6$, then we can see $\sigma_\alpha, \sigma_\beta, \sigma_\gamma, \sigma_\delta$ and σ_ζ send S_1 to S_2, S_3, S_4, S_5 and S_6 respectively, showing that Γ_1 is equivalent to $\Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5$ and Γ_6, respectively. Hence in this case it is enough to verify the condition in Result 2.4 for Γ_1. One can easily check σ_η, σ_θ and σ_ι are all in $\mathrm{Aut}(A_4, S_1)$ and send a to a^{-1}, b and c respectively, where $\eta = (1\ 2)$, $\theta = (3\ 4)$ and $\iota = (1\ 2)(3\ 4)$, proving that $\mathrm{Aut}(A_4, S_1)$ acts transitively on S_1, i.e., Γ_1 and as a consequence $\Gamma_i, 1 \leqslant i \leqslant 6$ are all normal arc-transitive.

If $|S| = 6$, then S can be one of the following types: $S_7 = \{a, b, c, a^{-1}, b^{-1}, c^{-1}\}$, $S_8 = \{a, b, d, a^{-1}, b^{-1}, d^{-1}\}$, $S_9 = \{a, c, d, a^{-1}, c^{-1}, d^{-1}\}$ and $S_{10} = \{b, c, d, b^{-1}, c^{-1}, d^{-1}\}$. Similarly if we define $\lambda := (2\ 4)$, $\mu := (1\ 3)$, $\nu := (1\ 4)$, $\rho := (1\ 4\ 3)$ and $\Gamma_i := \mathrm{Cay}(A_4, S_i), i = 7, \ldots, 10$, we can easily check that $\sigma_\eta, \sigma_\alpha$ and σ_λ force Γ_7 to be equivalent to Γ_8, Γ_9 and Γ_{10}. We also can see that $\sigma_\mu, \sigma_\theta, \sigma_\nu, \sigma_\rho$ and $\sigma_{\rho^{-1}}$ are all in $\mathrm{Aut}(A_4, S_7)$ which make $\mathrm{Aut}(A_4, S_7)$ to act transitively on S_7, i.e., Γ_i is normal arc-transitive for $7 \leqslant i \leqslant 10$.

And finally if $|S| = 8$, the only possibility for S is $S_{11} = \{a, b, c, d, a^{-1}, b^{-1}, c^{-1}, d^{-1}\}$. But every automorphism of A_4 permutes the 3-cycles, thus $\mathrm{Aut}(A_4) = \mathrm{Aut}(A_4, S_{11})$, which is transitive on S_{11}, proving that $\Gamma_{11} = \mathrm{Cay}(A_4, S_{11})$ is also normal arc-transitive.

(3) ⇒ (1) Follows from Results 2.2 and 2.4. □

According to Result 2.6, if $\Gamma = \mathrm{Cay}(D_{2n}, S)$ is a normal edge-transitive Cayley graph on the dihedral group D_{2n}, then $\mathrm{Aut}(D_{2n}, S)$ acts transitively on S. Hence no $\frac{1}{2}$ arc-transitive Cayley graph arise from the group D_{4p} and D_{2p} where p is a prime number.

Therefore regarding the results of this paper obtained so far with the observation that any group of order p is abelian and any group of order $2p$ is either abelian or dihedral, and classification of non-abelian groups of order $4p$ which is mentioned at the first part of this section, we can state the following theorem.

Theorem 3.12. *Let Γ be a connected Cayley graph of order $2^{\delta}p$, where p is a prime number and $0 \leqslant \delta \leqslant 2$. Γ is normal $\frac{1}{2}$ arc-transitive if and only if Γ is a normal edge-transitive Cayley graph of a group isomorphic to F_{4p}.*

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