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Martin boundary and exit space on the Sierpinski gasket

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Abstract We define a new Markov chain on the symbolic space representing the Sierpinski gasket (SG), and show that the corresponding Martin boundary is homeomorphic to the SG while the minimal Martin boundary is the three vertices of the SG. In addition, the harmonic structure induced by the Markov chain coincides with the canonical one on the SG. This suggests another approach to consider the existence of Laplacians on those self-similar sets for which the problem is still not settled.

Keywords Martin boundary, Markov chain, Green function, harmonic function, fractal, Sierpinski gasket

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1 Introduction

The boundary theory of Markov chains plays a central role in the theory of random walks, and the associated discrete potential theory provides a common ground for probability and analysis. Classically, the state space of a random walk under consideration is usually equipped with a group structure and the walk is reversible. More recent developments involve broader types of graphs and models, such as those arising from the study of fractals that have a self-similar structure (see [4, 5, 10, 11, 15, 18, 20]).

In [4, 5], Denker and Sato introduced a (non-irreducible) Markov chain on the tree of symbolic space of the Sierpinski gasket (SG), which moves from a state to its descendants and its neighbors' descendants. They showed that the Martin boundary and the minimal Martin boundary (also called the *exit space*) of the chain are homeomorphic to the SG. Based on this setup, they made an attempt [6] to relate this to the canonical harmonic structure and the Dirichlet form (see Kigami [12–14]). The idea of identifying the Martin boundary with the self-similar set has been carried out on the pentagasket [8] and extended to *simple post-critically finite* self-similar sets [10], and more general cases [18].

In another direction, Kaimanovich [11] observed that there is a natural hyperbolic graph structure on the symbolic space of the SG, and the SG can be identified with the hyperbolic boundary. This has also been extended by Wang and the first author to the class of self-similar sets that satisfy the open set condition [17], and results of Denker-Sato type Markov chains can also be proved on these hyperbolic graphs [18].

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In this paper we consider a new Markov chain on the space of finite words representing the SG. Our aim is twofold. First we want to have an example different from the previous ones in that the Martin boundary is homeomorphic to the SG and contains the minimal Martin boundary as a proper subset. Secondly we want the induced harmonic structure to coincide with the canonical one on the SG [14]. As is well known, there are two approaches to establishing the existence of a harmonic structure on a self-similar set: the analytic approach (see [13, 14, 23]), and the probabilistic approach (see [2, 16]). However, both approaches can only deal with limited classes of fractals. Ultimately, our goal is to use this Martin boundary consideration to study the harmonic structure on a wider class of fractals.

Our specific Markov chain $\{X_k\}_{k=0}^{\infty}$ is defined on the space of finite words Σ_* of the SG with a transition probability P (see Section 2) and initial state ϑ. Basically, on each level Σ_n, X_k moves on the non-vertex words according to the nearest neighborhood random walk. When it hits one of the three vertices, it moves to the three descendants on the next level, and continues the walk in the same way (see Figure 1). Our main conclusions are as follows.

Theorem 1.1. *$\lim_{k\to\infty} X_k = X_\infty$ P_ϑ-almost surely, where X_∞ is a $\{\dot{1}, \dot{2}, \dot{3}\}$-valued random variable. Moreover, the Martin boundary of $\{X_k\}_{k=0}^{\infty}$ is homeomorphic to the SG, and the the minimal Martin boundary is $\{\dot{1}, \dot{2}, \dot{3}\}$.*

Theorem 1.2. *The class of P-harmonic functions is 3-dimensional, and there is a natural identification with the canonical harmonic functions on the SG.*

In Theorem 1.1, the limit of $\{X_k\}_k$ and the statement concerning the minimal boundary follow easily from the general convergence theorem of Markov chains. The main effort is to establish the homeomorphism between the Martin boundary and the SG. We first find the hitting probabilities $\rho_i(x)$ from a state $x \in \Sigma_n$ to the three vertices i^n in terms of the product of some transition matrices (Theorem 2.5). We then prove the convergence of $\rho_i(\mathbf{x}|_n)$ for $\mathbf{x} \in \Sigma_\infty$ (Theorem 3.3, Corollary 3.6); this makes use of the concepts of *scrambling matrices* and the *maximum range* of a matrix, both introduced by Hajnal [9]. The convergence is used to study the limit of the Green function (Proposition 4.1) and the Martin kernel $K(x, y)$ (Proposition 5.2), which in turn enable us to establish the homeomorphism between the Martin boundary and the SG (Section 6).

It follows that the P-harmonic functions are generated by $\psi_i(x) = K(x, i^\infty), i = 1, 2, 3$. Each ψ_i has a continuous extension to the Martin boundary, i.e., to the SG, denoted by K. It is shown that these $\psi_i =: h$ satisfy the graph harmonic property on K, i.e.,

$$h(\mathbf{x}) = \frac{1}{4} \sum_{\mathbf{x} \sim_\kappa \mathbf{y}} h(\mathbf{y}), \tag{1.1}$$

for $\mathbf{x}$ being a non-boundary vertex of the level-n cells of K. (Here $\sim_\kappa$ stands for the nearest neighborhood relation.) This yields Theorem 1.2.

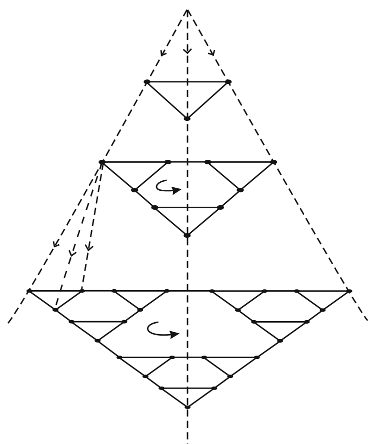

Figure 1 The random walk of the Markov chain

For a continuous function h on K, we define the average value of h on the cell K_w, $w \in \Sigma_*$, of the SG by

$$\widetilde{h}(w) = 3^{-|w|} \int_{K_w} h(\mathbf{x})\ d\mu(\mathbf{x}),$$

where μ is the standard self-similar measure on K (see Section 2 for other unexplained notation). In considering the problem of existence of the Laplacian, Strichartz [22] showed that if h is a harmonic function on K as in (1.1), then $\widetilde{h}$ is P-harmonic on Σ_*. Moreover, he used the average values to consider the corresponding Dirichlet form and showed a relation between the pointwise definition on K and the average-value definition on Σ_*. This agrees with our Martin boundary consideration. From this point of view, it will be interesting to consider the above identification problem of the Martin boundaries for more general self-similar sets, as it would potentially offer another tool to study the harmonic structure and the Laplacian on such self-similar sets.

2 Basic hitting probabilities

We first introduce some notation for the symbolic space of the SG. Let $\Sigma_n := \{i_1 \cdots i_n :\ i_j = 1, 2, 3\}$, $n \geqslant 1$, denote the set of all *words* on level n ($\Sigma_0 := \{\vartheta\}$ by convention), and let $\Sigma_* := \bigcup_{n=0}^{\infty} \Sigma_n$ be the set of finite words. For a word $x = i_1 \cdots i_n$, we let $|x| = n$ denote the *length* of x. We also use $\Sigma_n^{(i)} \subset \Sigma_n$, $i = 1, 2, 3$, to denote the three copies of Σ_{n-1} in the obvious way. On Σ_n, let $V_n := \{1^n, 2^n, 3^n\}$ denote the three vertices, and $\widetilde{\Sigma}_n := \Sigma_n \setminus V_n$. Similarly, we let Σ_∞ denote the set of infinite words $i_1 i_2 \cdots$, and let $\widetilde{\Sigma}_\infty = \Sigma_\infty \setminus \{\dot{1}, \dot{2}, \dot{3}\}$. We use x, y to denote elements in Σ_*, and $\mathbf{x}, \mathbf{y}$ for elements in Σ_∞.

Let K denote the SG and let S_1, S_2, S_3 be the three similitudes generating it. For $u = i_1 \cdots i_n \in \Sigma_n$, we let $S_u := S_{i_1} \circ \cdots \circ S_{i_n}$ denote the composition and let $K_w = S_w(K)$. We say that $u, v \in \Sigma_n$ are *neighbors*, denoted $u \sim v$, if $K_u \cap K_v \neq \emptyset$. We define a Markov chain $\{X_k\}_{k=0}^{\infty}$ on the state space Σ_* with transition probabilities

$$P(u, v) = \begin{cases} 1/3, & \text{if } u, v \in \widetilde{\Sigma}_n,\ u \sim v; \\ 1/3, & \text{if } u \in V_n,\ v = ui,\ i = 1, 2, 3; \\ 0, & \text{otherwise.} \end{cases}$$

See Figure 1 for an illustration of this Markov chain. The words of Σ_n are denoted by solid dots, and two words are neighbors if and only if they are connected by an edge. On level n, the chain starting at a state in $\widetilde{\Sigma}_n$ walks to one of its three neighbors, and when it hits a vertex, it must go down to one of its three neighbors in Σ_{n+1}. The chain moves down from Σ_n to Σ_{n+1} only through one of the three vertices in V_n.

We also use the following notation throughout the paper. A *1-cell* Δ_w in Σ_n consists of the three words of the form wi, where w is a word of length $n-1$ and $i = 1, 2, 3$. If $n \geqslant m$ and $w \in \Sigma_{n-m}$, we define the (n, m)*-cell* in Σ_n determined by the word w, denoted Δ_w^n, to be the set of words of the form

$$\{wi_1 \cdots i_m : i_1, \ldots, i_m \in \{1, 2, 3\}\}.$$

Note that the three vertices of Δ_w^n are $w1^m, w2^m, w3^m$. In particular, a 1-cell is an $(n, 1)$-cell; Σ_n is the only (n, n)-cell. Also, two words $x = i_1 \cdots i_n$, $y = j_1 \cdots j_n$ belong to the same (n, m)-cell if and only if $i_1 \cdots i_{n-m} = j_1 \cdots j_{n-m}$.

For $C \subset \Sigma_n$, we let $\rho_{x,C}$ be the probability for the chain $\{X_n\}_{n=0}^{\infty}$, starting at x, to ever reach C at some positive time. If C is a single point y, we will denote it by $\rho_{x,y}$. In the sequel we will repeatedly use the following elementary identity to set up systems of linear equations to calculate the probabilities $\rho_{x,y}$:

$$\rho_{x,y} = P(x, y) + \sum_{z \neq y} P(x, z) \rho_{z,y}. \tag{2.1}$$

For the random walk on each Σ_n, we regard $1^n, 2^n, 3^n \in V_n$ as absorbing states (via which the chain moves to Σ_{n+1} and will not return). For any $x \in \widetilde{\Sigma}_n$, let $\boldsymbol{\rho}(x) = [\rho_1(x), \rho_2(x), \rho_3(x)]$, where $\rho_i(x)$ is the probability for the chain, starting at x, to be absorbed by the vertex i^n in Σ_n, i.e., $\rho_i(x) = \rho_{x,i^n}$. If $x = j^n \in V_n$, we simply define $\rho_i(x) = \delta_{ij}$. In order to calculate these probabilities, we need some additional notation. Let a_n, b_n, c_n be the probabilities for the chain starting at 1^n2 to reach the vertices $1^n, 2^n, 3^n$, respectively; analogously let $\alpha_n, \beta_n, \gamma_n$ be the probabilities if the chain starts at 12^{n-1} instead (see Figure 2). For $n \geqslant 2$, let $\epsilon_n := 1 - a_n$. Lastly, we let $\boldsymbol{e}_i$, $i = 1, 2, 3$, be the standard basis of $\mathbb{R}^3$.

Proposition 2.1. *Let $n \geqslant 2$.*

(a) $(a_2, b_2, c_2) = (\alpha_2, \beta_2, \gamma_2) = (5/8, 1/4, 1/8)$.

(b) $\alpha_{n+1} = (5 - 3a_n)/(8 - 3a_n)$, $\beta_{n+1} = 2/(8 - 3a_n)$, *and* $\gamma_{n+1} = \beta_{n+1}/2$.

(c) $a_{n+1} = (5/2)\beta_{n+1}$,

$$\begin{bmatrix} b_{n+1} \\ c_{n+1} \end{bmatrix} = \frac{1}{8}\beta_{n+1}\cdots\beta_3 \begin{bmatrix} \left(\frac{3}{2}\right)^n + \left(\frac{1}{2}\right)^n \\ \left(\frac{3}{2}\right)^n - \left(\frac{1}{2}\right)^n \end{bmatrix}.$$

(d) $\{a_n\}, \{\beta_n\}, \{\gamma_n\}$ *are monotone increasing, while* $\{b_n\}, \{c_n\}, \{\alpha_n\}$ *are monotone decreasing.*

Proof. (a) That $(a_2, b_2, c_2) = (\alpha_2, \beta_2, \gamma_2)$ follows by definition, and their values are obtained by solving the following equations (by (2.1) and symmetry):

$$\alpha_2 = \frac{1}{3} + \frac{1}{3}\alpha_2 + \frac{1}{3}\beta_2, \qquad \beta_2 = \frac{1}{3}\alpha_2 + \frac{1}{3}\gamma_2, \qquad \gamma_2 = \frac{1}{3}\beta_2 + \frac{1}{3}\gamma_2.$$

(b) We only consider the case $n = 3$; the proof for any $n \geqslant 3$ is the same because all equations involved have exactly the same pattern. Write $u = \rho_1(121)$, $v = \rho_1(123)$. By using (2.1) and symmetry, and applying the values a_2, b_2, c_2 for Σ_2 to $\Sigma_3^{(1)}$, we have

$$\alpha_3 = \frac{1}{3}\beta_3 + \frac{1}{3}u + \frac{1}{3}v, \qquad u = b_2 + a_2\alpha_3 + c_2\alpha_3, \qquad v = c_2 + a_2\alpha_3 + b_2\alpha_3.$$

Eliminating u and v, and using $a_2 + b_2 + c_2 = 1$, we obtain

$$(2 - a_2)\alpha_3 = \beta_3 + (b_2 + c_2).$$

Applying the same argument to β_3 and γ_3, we get

$$(3 - 2a_2)\beta_3 = \alpha_3 + (b_2 + c_2)\gamma_3, \qquad (2 - 2a_2)\gamma_3 = (b_2 + c_2)\beta_3.$$

These three equations together with $a_2 + b_2 + c_2 = 1$ imply (b).

(c) Similarly we can express (a_3, b_3, c_3) in terms of a_2, b_2, c_2 and β_3 as follows:

$$\begin{aligned} a_3 &= a_2 + b_2\alpha_3 + c_2\alpha_3 = \frac{5}{2}\beta_3, \\ b_3 &= b_2\beta_3 + c_2\gamma_3 = \left(b_2 + \frac{c_2}{2}\right)\beta_3, \\ c_3 &= b_2\gamma_3 + c_2\beta_3 = \left(\frac{b_2}{2} + c_2\right)\beta_3. \end{aligned}$$

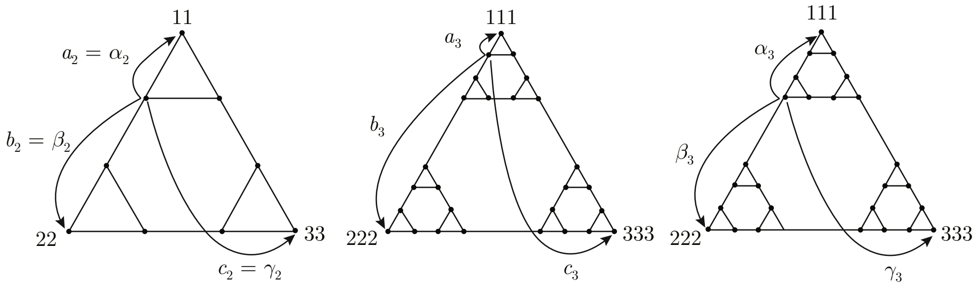

Figure 2 The probabilities a_n, b_n, c_n and $\alpha_n, \beta_n, \gamma_n$ for $n = 2, 3$

Likewise for $n \geqslant 2$,

$$a_{n+1} = \frac{5}{2}\beta_{n+1}, \qquad b_{n+1} = \left(b_n + \frac{c_n}{2}\right)\beta_{n+1}, \qquad c_{n+1} = \left(\frac{b_n}{2} + c_n\right)\beta_{n+1}.$$

Putting b_{n+1}, c_{n+1} in a matrix form, we have

$$\begin{bmatrix} b_{n+1} \\ c_{n+1} \end{bmatrix} = \beta_{n+1} \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} b_n \\ c_n \end{bmatrix}. \tag{2.2}$$

Observe that the eigenvalues of the matrix are $3/2$ and $1/2$ with corresponding eigenvectors $[1, 1]^t$ and $[-1, 1]^t$, respectively. By diagonalizing the matrix and iterating (2.2), we arrive at the second equality in (c).

(d) From (a) and (b), it follows that the statement holds for $n = 2, 3$. Let $n \geqslant 3$. Since $\beta_{n+1} \leqslant 2/5$, it can be checked directly that

$$\begin{bmatrix} b_{n+1} \\ c_{n+1} \end{bmatrix} \leqslant \frac{1}{8}\beta_n \cdots \beta_3 \begin{bmatrix} (3/2)^{n-1} + (1/2)^{n-1} \\ (3/2)^{n-1} - (1/2)^{n-1} \end{bmatrix} = \begin{bmatrix} b_n \\ c_n \end{bmatrix}.$$

This shows that $\{b_n\}, \{c_n\}$ are decreasing to 0, and hence $\{a_n\}$ is increasing to 1. Consequently, $\beta_{n+1} = 2/(8 - 3a_n)$ and $\gamma_{n+1} = \beta_{n+1}/2$ are increasing, and thus α_{n+1} is decreasing. □

As a direct consequence, we have

Corollary 2.2. *Let $a, b, c, \alpha, \beta, \gamma$ be the respective limits of $a_n, b_n, c_n, \alpha_n, \beta_n, \gamma_n$. Then $(a, b, c) = (1, 0, 0)$ and $(\alpha, \beta, \gamma) = (2/5, 2/5, 1/5)$.*

Recall that $\epsilon_n := 1 - a_n$.

Corollary 2.3. *There exists a constant $c > 0$ such that for all $n \geqslant 3$, $c(3/5)^n \leqslant \epsilon_n \leqslant (3/5)^n$.*

Proof. By Proposition 2.1(b) and (c), $\beta_3 = 16/49$ and $\beta_n \leqslant 2/5$ for $n \geqslant 2$, and thus for all $n \geqslant 3$,

$$\epsilon_n = b_n + c_n = \frac{4}{49}\beta_n \cdots \beta_4 \left(\frac{3}{2}\right)^{n-1} < \left(\frac{3}{5}\right)^n.$$

On the other hand, by Proposition 2.1(b), $\beta_k^{-1} = (8 - 3a_{k-1})/2 = (5/2)(1 + 3\epsilon_{k-1}/5)$. In view of $\sum_{k=2}^{\infty} \epsilon_k \leqslant \sum_{k=2}^{\infty} (3/5)^{k-1} < \infty$, there exists $C > 0$ such that

$$(\beta_n \cdots \beta_3)^{-1} = \left(\frac{5}{2}\right)^{n-2} \left(1 + \frac{3\epsilon_2}{5}\right) \cdots \left(1 + \frac{3\epsilon_{n-1}}{5}\right) \leqslant C \left(\frac{5}{2}\right)^{n-2}.$$

This implies that $\epsilon_n \geqslant c(3/5)^n$ for some $c > 0$, proving the corollary. □

For $n \geqslant 2$, we define

$$A_n^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ \alpha_n & \beta_n & \gamma_n \\ \alpha_n & \gamma_n & \beta_n \end{bmatrix}, \quad A_n^{(2)} = \begin{bmatrix} \beta_n & \alpha_n & \gamma_n \\ 0 & 1 & 0 \\ \gamma_n & \alpha_n & \beta_n \end{bmatrix}, \quad A_n^{(3)} = \begin{bmatrix} \beta_n & \gamma_n & \alpha_n \\ \gamma_n & \beta_n & \alpha_n \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.3}$$

Note that $A_n^{(i)}$ denotes the probabilities for the chain starting at the vertices of $\Sigma_n^{(i)}$ to reach the vertices of Σ_n, i.e.,

$$A_n^{(i)} = \begin{bmatrix} \boldsymbol{\rho}(i1^{n-1}) \\ \boldsymbol{\rho}(i2^{n-1}) \\ \boldsymbol{\rho}(i3^{n-1}) \end{bmatrix}, \quad i = 1, 2, 3. \tag{2.4}$$

Note also that each $A_n^{(i)}$ is *stochastic*, i.e., nonnegative with each row sum equal to 1. We can use this to express the probability to reach a vertex from an arbitrary point in Σ_n in terms of a matrix product.

Remark 2.4. For $i \in \{1,2,3\}$, and for any $n_1, \ldots, n_k$, the i-th row of the product of $A^{(i)}_{n_1} \cdots A^{(i)}_{n_k}$ is $\mathbf{e}_i$. This can be proved easily by induction.

Theorem 2.5. *Let $n \geqslant 2$ and $x = i_1 \cdots i_n \in \Sigma_n$. Then*

$$\boldsymbol{\rho}(x) = \boldsymbol{e}_{i_n} A_2^{(i_{n-1})} \cdots A_n^{(i_1)}.$$

Proof. For $x \in V_n$, the expression for $\boldsymbol{\rho}(x)$ follows from Remark 2.4. So we assume that $x \in \widetilde{\Sigma}_n$. First consider the case $i_{n-1} \neq i_n$. The $(n,2)$-cell that contains x is $\Delta_{i_1 \cdots i_{n-2}}$, which has vertices $i_1 \cdots i_{n-2}kk$, $k = 1,2,3$ (see Figure 3).

By first expressing $\boldsymbol{\rho}(x)$ in terms of these $\boldsymbol{\rho}(i_1 \cdots i_{n-2}kk)$ using the definition of $A_2^{(i_{n-1})}$, and then using induction, we get

$$\begin{aligned}\boldsymbol{\rho}(x) &= \boldsymbol{e}_{i_n} A_2^{(i_{n-1})} \begin{bmatrix} \boldsymbol{\rho}(i_1 \cdots i_{n-2}11) \\ \boldsymbol{\rho}(i_1 \cdots i_{n-2}22) \\ \boldsymbol{\rho}(i_1 \cdots i_{n-2}33) \end{bmatrix} \\ &= \cdots \text{ (inductively)} \\ &= \boldsymbol{e}_{i_n} A_2^{(i_{n-1})} \cdots A_{n-1}^{(i_2)} A_n^{(i_1)} \qquad \text{(by (2.4))}.\end{aligned}$$

Next we consider the case $x = i_1 \cdots i_{n-m} i^m$ for some $m \geqslant 2$, where $i_{n-m} \neq i$. By Remark 2.4 again,

$$\boldsymbol{e}_{i_n} A_2^{(i_{n-1})} \cdots A_m^{(i_{n-m+1})} = \boldsymbol{e}_i A_2^{(i)} \cdots A_m^{(i)} = \boldsymbol{e}_i.$$

Thus, a similar proof as above yields the same expression for $\boldsymbol{\rho}(x)$. □

We will give a detailed study of the convergence of the above product in the next section. In the following we prove some simple consequences of the hitting probabilities using Proposition 2.1. The next proposition is intuitively clear (see Figure 1). It is an important step in proving the limit properties of the Green function (Proposition 4.1).

Proposition 2.6. *Assume that $n \geqslant m+1$. Then $\lim_{m\to\infty} \rho_{i^m, i^n} = 1$ and $\eta_{ij}^m := \lim_{n\to\infty} \rho_{i^m, j^n}$ exists and is positive. Moreover, $\lim_{m\to\infty} \eta_{ij}^m = \delta_{ij}$.*

Proof. It follows from the definitions of the transition probability P on Σ_* that

$$\rho_{i^k, i^{k+1}} = \frac{1}{3} + \frac{2}{3} a_{k+1} = 1 - \frac{2}{3}\epsilon_{k+1}. \tag{2.5}$$

Since $\rho_{i^k, i^n} \geqslant \rho_{i^k, i^{k+1}} \rho_{i^{k+1}, i^n}$ for $k \leqslant n-1$, by induction and Corollary 2.3, we have

$$\rho_{i^m, i^n} \geqslant \prod_{k=m}^{n-1} \rho_{i^k, i^{k+1}} = \prod_{k=m}^{n-1} \left(1 - \frac{2}{3}\epsilon_{k+1}\right) \geqslant \prod_{k=m}^{n-1} \left(1 - \frac{2}{3}\left(\frac{3}{5}\right)^{k+1}\right). \tag{2.6}$$

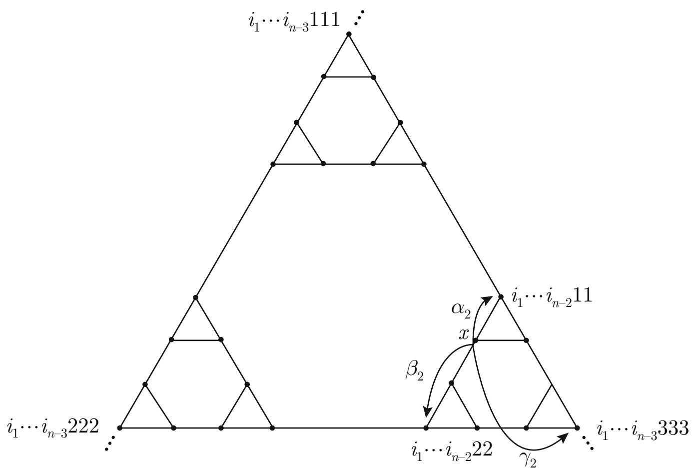

Figure 3 The (n,i)-cells $(i = 1,2,3)$ in the portion of Σ_n containing the word $x = i_1 \cdots i_n$

This implies that $\lim_{m\to\infty}\rho_{i^m,i^n}=1$.

Now to consider $\lim_{n\to\infty}\rho_{i^m,j^n}$, we assume without loss of generality that $i=1$. First we have

$$\rho_{1^m,1^{m+1}}=\frac{1}{3}+\frac{2}{3}a_{m+1}\qquad\text{and}\qquad\rho_{1^m,2^{m+1}}=\rho_{1^m,3^{m+1}}=\frac{1}{3}(1-a_{m+1}).$$

Inductively we have

$$\rho_{1^m,1^n}\ =\ \rho_{1^m,1^{n-1}}\frac{1+2a_n}{3}\ +\ \rho_{1^m,2^{n-1}}\frac{1-a_n}{3}+\rho_{1^m,3^{n-1}}\frac{1-a_n}{3}. \tag{2.7}$$

Similar expressions hold for $\rho_{1^m,2^n}$ and $\rho_{1^m,3^n}$. To put these in a matrix form, we let

$$M_k:=\frac{1}{3}\begin{bmatrix}1+2a_k & 1-a_k & 1-a_k\\ 1-a_k & 1+2a_k & 1-a_k\\ 1-a_k & 1-a_k & 1+2a_k\end{bmatrix},\quad m+2\leqslant k\leqslant n.$$

Hence from (2.7), we have

$$\begin{bmatrix}\rho_{1^m,1^n}\\ \rho_{1^m,2^n}\\ \rho_{1^m,3^n}\end{bmatrix}\ =\ M_n\cdots M_{m+2}\begin{bmatrix}(1+2a_{m+1})/3\\ (1-a_{m+1})/3\\ (1-a_{m+1})/3\end{bmatrix}. \tag{2.8}$$

For $i,j\in\{1,2,3\}$ with $i\neq j$,

$$|M_k(i,i)-1|=\frac{2}{3}\epsilon_k\qquad\text{and}\qquad|M_k(i,j)-0|=\frac{1}{3}\epsilon_k.$$

If we let $\|M\|:=\max_i\sum_j|M(i,j)|$, a norm satisfying $\|MM'\|\leqslant\|M\|\|M'\|$ for all matrices M,M', then by Corollary 2.3,

$$\lim_{\ell\to\infty}\sum_{k=\ell}^{\infty}\|M_k-I\|=\lim_{\ell\to\infty}(4/3)\sum_{k=\ell}^{\infty}\epsilon_k=0.$$

As the M_k are stochastic matrices, for any $\ell\leqslant n$, $\{\|M_n\cdots M_\ell\|\}_{\ell,n}$ is bounded by 1, and it follows easily from the above limit that $\lim_{n\to\infty}M_n\cdots M_{m+2}$ exists. Hence for $j=1,2,3$, $\lim_{n\to\infty}\rho_{1^m,j^n}$ exists.

Since $\lim_{n,k\to\infty}M_n\cdots M_k=I$ and each entry of M_i is positive, we conclude that $\lim_{n\to\infty}\rho_{i^m,j^n}>0$. Lastly, it follows from (2.8) that $\lim_{m\to\infty}\eta^m_{1j}=\delta_{1j}$. □

To conclude this section, we will estimate the probability $\rho_{\vartheta,\Delta}$ for a 1-cell $\Delta\subseteq\Sigma_m$. It is necessary in Section 5 (Proposition 5.1) to obtain an upper bound for the Green function. We use $\Delta^\ell=\Delta_{2^{m-1}}$, $\Delta^r=\Delta_{3^{m-1}}$ to denote the two 1-cells at the left and right corners of Σ_m.

Lemma 2.7. *Let $m\geqslant 2$ and Δ be any 1-cell in Σ_m. Then*

$$\rho_{1^{m-1},\Delta}\geqslant\rho_{1^{m-1},\Delta^\ell}=\rho_{1^{m-1},\Delta^r}\geqslant C_1(3/5)^m$$

for some constant $C_1>0$ (independent of m).

Proof. It is clear that $\rho_{1^{m-1},\Delta^\ell}=\rho_{1^{m-1},\Delta^r}$, and from the transition probabilities and Corollary 2.3 that

$$\rho_{1^{m-1},\Delta^\ell}\geqslant\rho_{1^{m-1},2^m}=\frac{1}{3}(b_m+c_m)\geqslant C_1(3/5)^m.$$

To prove the first inequality, we use induction. It can be verified directly for $m=2$ or 3. Assume that it holds for some $m\geqslant 2$, and consider Σ_{m+1}. For the clarity of induction, we denote $\Delta,\Delta^\ell,\Delta^r\in\Sigma_m$ by $\Delta_m,\Delta^\ell_m,\Delta^r_m$, respectively. Recall that for $i=1,2,3$,

$$\Sigma^{(i)}_{m+1}=\{(i,I):I\in\Sigma_m\},$$

i.e., $\Sigma^{(i)}_{m+1}$ are the three sub-triangles making up Σ_{m+1}. Consider the following two cases.

Case 1. $\Delta_{m+1} \subseteq \Sigma_{m+1}^{(1)}$. Note that Σ_m can be identified with $\Sigma_{m+1}^{(1)}$. We let Δ_m be the 1-cell in Σ_m corresponding to Δ_{m+1}. Then

$$\rho_{1^m,\Delta_{m+1}} \geqslant \rho_{1^{m-1},\Delta_m} \geqslant \rho_{1^{m-1},\Delta_m^l} \geqslant \rho_{1^m,\Delta_{m+1}^l} = \rho_{1^m,\Delta_{m+1}^r}.$$

(The first and third inequalities are clear by considering the paths that reach the points; the second inequality is by induction hypothesis.)

Case 2. $\Delta_{m+1} \subseteq \Sigma_{m+1}^{(2)}$. Note that any path that reaches Δ_{m+1} or Δ_{m+1}^l must first hit one of the two vertices 21^{m-1} or 23^{m-1} of $\Sigma_{m+1}^{(2)}$. Hence we can apply induction hypothesis to conclude that $\rho_{1^m,\Delta_{m+1}} \geqslant \rho_{1^m,\Delta_{m+1}^l}$. □

Proposition 2.8. *There exists a constant $C_2 > 0$ such that for any $m \geqslant 2$ and $x \in \Sigma_m$,*

$$\rho_{\vartheta,x} \geqslant C_2(3/5)^m.$$

Proof. Assume without loss of generality that $x \in \Sigma_m^{(1)}$. Let Δ be the (unique) 1-cell containing x. Then by Lemma 2.7,

$$\rho_{\vartheta,x} \geqslant \frac{1}{3}\rho_{1^{m-1},x} \geqslant \frac{1}{9}\rho_{1^{m-1},\Delta} \geqslant C\left(\frac{3}{5}\right)^m.$$

(The second inequality holds since any path reaching Δ has a probability of at least 1/3 to reach x.) □

3 Random product of the $A_n^{(i)}$

One of our main purposes in this section is to show that any random product $A_2^{(i_{n-1})}\cdots A_n^{(i_1)}$ of the matrices in (2.3) converges to a stochastic matrix whose rows are identical. To this end, we need two parameters λ and δ, both introduced by Hajnal [9], which measure the difference of the rows of a matrix.

For any stochastic matrix $M = (M(i,j))$, define

$$\lambda(M) := 1 - \min_{i_1,i_2}\sum_j \min\{M(i_1,j), M(i_2,j)\} \in [0,1].$$

M is called *scrambling* if $\lambda(M) < 1$ (see [9]). In other words, M is scrambling if and only if for every pair of rows i_1 and i_2, there exists a column j (which may depend on i_1 and i_2) such that $M(i_1,j) > 0$ and $M(i_2,j) > 0$. Also, $\lambda(M) = 0$ if and only if the rows of M are identical.

There is another parameter that measures how different the rows of M are. Define

$$\delta(M) := \max_j \max_{i_1,i_2} |M(i_1,j) - M(i_2,j)| \in [0,1].$$

In other words, $\delta(M)$ is the maximum difference between any pair of elements in the same column. Hajnal called $\delta(M)$ the *maximum range* of M. Note that $\delta(M) = 0$ if and only if $\lambda(M) = 0$.

It is not difficult to see that for any stochastic matrix M, we have $\delta(M) \leqslant \lambda(M)$. In fact Hajnal [9, Theorem 2] (see also [25, Lemma 2]) proved the more general result that for any stochastic matrices $M_1,\ldots,M_k$,

$$\delta(M_1\cdots M_k) \leqslant \prod_{i=1}^k \lambda(M_i). \tag{3.1}$$

It can also be proved that for any 2×2 or 3×3 stochastic matrix M, $\delta(M) = \lambda(M)$. We do not need such generality; instead we only need to use the following special case.

Lemma 3.1. *For $k \geqslant 2$ and $i = 1,2,3$, each $A_k^{(i)}$ defined in* (2.3) *is a scrambling matrix and*

$$\lambda(A_k^{(i)}) = \delta(A_k^{(i)}) = 1 - \alpha_k < 1.$$

Moreover, for any $m \geqslant 2$ and $k_1,\ldots,k_m \geqslant 2$, we have

$$\delta(A_{k_m}^{(i_m)}A_{k_{m-1}}^{(i_{m-1})}\cdots A_{k_1}^{(i_1)}) \leqslant \left(\frac{3}{5}\right)^m.$$

Proof. It is straightforward to check that $\lambda(A_k^{(i)}) = 1 - \alpha_k$. Since $1 - \alpha_k = \beta_k + \gamma_k$, we also have

$$\delta(A_k^{(i)}) = \max\{1 - \alpha_k, \beta_k, \gamma_k\} = 1 - \alpha_k.$$

For the second part, we use (3.1), and observe that $1 - \alpha_k = \beta_k + \gamma_k < 3/5$. □

For $i = \{1, 2, 3\}$, let

$$A^{(i)} := \lim_{n\to\infty} A_n^{(i)}. \tag{3.2}$$

The limit exists because $\alpha_n \to 2/5$, $\beta_n \to 2/5$ and $\gamma_n \to 1/5$ by Corollary 2.2. In fact,

$$A^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 2/5 & 2/5 & 1/5 \\ 2/5 & 1/5 & 2/5 \end{bmatrix}, \quad A^{(2)} = \begin{bmatrix} 2/5 & 2/5 & 1/5 \\ 0 & 1 & 0 \\ 1/5 & 2/5 & 2/5 \end{bmatrix}, \quad A^{(3)} = \begin{bmatrix} 2/5 & 1/5 & 2/5 \\ 1/5 & 2/5 & 2/5 \\ 0 & 0 & 1 \end{bmatrix}, \tag{3.3}$$

with all $A^{(i)}$ being invertible. We fix a sequence $i_1 i_2 \cdots \in \Sigma_\infty$, and let

$$T_n = A_2^{(i_n)} \cdots A_{n-k+1}^{(i_{k+1})} A_{n-k+2}^{(i_k)} \cdots A_{n+1}^{(i_1)} =: Q_{n,k} R_{n,k}, \tag{3.4}$$

where $R_{n,k}$ is the product of the last k matrices. Define $R_{\infty,k} := \lim_{n\to\infty} R_{n,k}$.

Lemma 3.2. *With the above notation, then*

(a) *the diagonal entries of $R_{n,k}$ are positive, and for each $s \in \{1, 2, 3\}$, $R_{n,k}(s,t) = \delta_{st}$ if and only if $i_j = s$ for all $j = 1, \dots, k$;*

(b) *$R_{\infty,k} = A^{(i_k)} \cdots A^{(i_1)}$, and* (a) *holds the same for $R_{\infty,k}$.*

Proof. (a) Observe that $R_{n,k+1} = A_{n-k+1}^{(i_{k+1})} R_{n,k}$, and

$$R_{n,k+1}(i,i) \geqslant A_{n-k+1}^{(i_{k+1})}(i,i) R_{n,k}(i,i) \geqslant \beta_{n-k+1} R_{n,k}(i,i) > 0.$$

The first statement follows by induction on k.

For the second statement, the sufficiency follows directly from (2.3). To prove the necessity, we use induction. Assume that $R_{n,k+1}(1,t) = \delta_{1t}$. Suppose that $i_{k+1} \neq 1$. Then

$$A_{n-k+1}^{(i_{k+1})} = \begin{bmatrix} \beta_{n-k+1} & \alpha_{n-k+1} & \gamma_{n-k+1} \\ 0 & 1 & 0 \\ \gamma_{n-k+1} & \alpha_{n-k+1} & \beta_{n-k+1} \end{bmatrix} \quad \text{or} \quad A_{n-k+1}^{(i_{k+1})} = \begin{bmatrix} \beta_{n-k+1} & \gamma_{n-k+1} & \alpha_{n-k+1} \\ \gamma_{n-k+1} & \beta_{n-k+1} & \alpha_{n-k+1} \\ 0 & 0 & 1 \end{bmatrix}.$$

In either case,

$$\begin{bmatrix} 1 & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix} = R_{n,k+1} = A_{n-k+1}^{(i_{k+1})} R_{n,k}$$

would imply

$$R_{n,k} = \begin{bmatrix} * & 0 & 0 \\ * & 0 & 0 \\ * & 0 & 0 \end{bmatrix},$$

contradicting the first part that the diagonal is positive. Thus $i_{k+1} = 1$. Now by equating the first rows of both sides of $R_{n,k+1} = A_{n-k+1}^{(1)} R_{n,k}$, we get $R_{n,k}(1,t) = \delta_{1t}$. By induction hypothesis, $i_j = 1$ for all $j = 1, \dots, k$. The proofs for $s = 2$ or 3 are similar.

(b) The existence of the limit $R_{\infty,k}$ follows from (3.2). The proof for the other properties can be established similarly. □

Theorem 3.3. *For any $i_1i_2\cdots\in\Sigma_\infty$, let $T_n = A_2^{(i_n)}\cdots A_{n+1}^{(i_1)}$ be as in (3.4). Then the limit*

$$T_\infty := \lim_{n\to\infty} T_n = \lim_{k\to\infty} A^{(i_k)}\cdots A^{(i_1)} \tag{3.5}$$

exists. Moreover, T_∞ is stochastic and the rows of T_∞ are identical.

Proof. For $\epsilon > 0$, using Lemma 3.1, we can let k be sufficiently large so that

$$\delta(R_{n,k}) \leqslant \left(\frac{3}{5}\right)^k < \frac{\epsilon}{2}.$$

This implies that for $i = 2, 3$,

$$R_{n,k}(1,1) - \frac{\epsilon}{2} < R_{n,k}(i,1) < R_{n,k}(1,1) + \frac{\epsilon}{2}. \tag{3.6}$$

By Lemma 3.2, there exists n_0 such that for $n \geqslant n_0$, $\|R_{n,k} - R_{\infty,k}\| < \epsilon/2$, where $\|\cdot\|$ is the norm defined in the proof of Proposition 2.6. We write $T_n = Q_{n,k}R_{n,k}$ as in (3.4). By using (3.6) and the fact that the row sums of $Q_{n,k}$ are 1, we get

$$\begin{aligned} T_n(1,1) &= \sum_{k=1}^{3} Q_{n,k}(1,k)R_{n,k}(k,1) \\ &\leqslant Q_{n,k}(1,1)R_{n,k}(1,1) + Q_{n,k}(1,2)\left(R_{n,k}(1,1)+\frac{\epsilon}{2}\right) + Q_{n,k}(1,3)\left(R_{n,k}(1,1)+\frac{\epsilon}{2}\right) \\ &\leqslant R_{n,k}(1,1) + \frac{\epsilon}{2}. \end{aligned}$$

Similarly, $R_{n,k}(1,1) - \epsilon/2 \leqslant T_n(1,1)$ and thus $|T_n(1,1) - R_{n,k}(1,1)| \leqslant \epsilon/2$. Hence

$$|T_n(1,1) - R_{\infty,k}(1,1)| < \epsilon, \quad \forall\ n \geqslant n_0,$$

proving that $\{T_n(1,1)\}$ is a Cauchy sequence. The same proof holds for the other $\{T_n(i,j)\}$. This proves the existence of the limit (3.5).

Since for each $n \geqslant 2$, the product $A_2^{(i_n)}A_3^{(i_{n-1})}\cdots A_{n+1}^{(i_1)}$ is stochastic, it follows that T_∞ must also be stochastic.

Finally, by Lemma 3.1, $\delta(A_2^{(i_n)}\cdots A_{n+1}^{(i_1)}) \leqslant (3/5)^n \to 0$. Thus the rows of T_∞ must be identical. □

Define

$$L_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad L_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Corollary 3.4. *For $i_1i_2\cdots\in\Sigma_\infty$, we have $\lim_{n\to\infty} T_n = L_s$ for some $s \in \{1,2,3\}$ if and only if $i_j = s$ for all $j \in \mathbb{N}$.*

Proof. We prove the lemma for the case $s = 1$; the cases $s = 2, 3$ are the same. Assume $\lim_{n\to\infty} T_n = L_1$. Fix any $k \in \mathbb{N}$ and let $n \geqslant k+2$. By Theorem 3.3, we have

$$\lim_{n\to\infty} Q_{n,k} := Q_{\infty,k} = \begin{bmatrix} a & b & 1-a-b \\ a & b & 1-a-b \\ a & b & 1-a-b \end{bmatrix}. \tag{3.7}$$

By assumption, we have $L_1 = Q_{\infty,k}R_{\infty,k}$. By comparing the entries and making use of the fact that the diagonal of $R_{\infty,k}$ is positive (Lemma 3.2(b)), we have $a = 1, b = 0$ and $R_{\infty,k} = I$, the identity. The last statement of Lemma 3.2(b) implies that $i_j = 1$ for all $j = 1, \ldots, k$. Since k is arbitrary, the assertion follows.

To prove the converse, we assume that $i_j = 1$ for all $j \in \mathbb{N}$. Note that

$$T_\infty = \lim_{n\to\infty} T_n = \Big(\lim_{n\to\infty} A_2^{(1)}\cdots A_n^{(1)}\Big)\Big(\lim_{n\to\infty} A_{n+1}^{(1)}\Big).$$

It follows that $T_\infty = T_\infty A^{(1)}$, where $A^{(1)}$ is given by (3.3). Also by Theorem 3.3, T_∞ has the same form as the matrix in (3.7). It is straightforward to solve the system of equations $T_\infty = T_\infty A^{(1)}$ to obtain that $a = 1$, and hence $b = 0$ and $1 - a - b = 0$. Therefore $\lim_{n\to\infty} T_n = L_1$. $\square$

As a consequence we have

Proposition 3.5. *Let T_∞ be the matrix product for $\mathbf{x} = i_1 i_2 \cdots \in \Sigma_\infty$ as above, and let T'_∞ be the corresponding matrix product for $\mathbf{y} = j_1 j_2 \cdots$ with $i_1 \neq j_1$. Then $T_\infty = T'_\infty$ if and only if $\mathbf{x} = ijj\cdots$ and $\mathbf{y} = jii\cdots$ with $i \neq j$.*

Proof. Let

$$T_n = A_2^{(i_n)} A_3^{(i_{n-1})} \cdots A_{n+1}^{(i_1)} =: \widetilde{T}_n A_{n+1}^{(i_1)},$$

where $\widetilde{T}_n$ is defined in an obvious way. Let $T'_n =: \widetilde{T}'_n A_{n+1}^{(j_1)}$ be defined similarly. Using Theorem 3.3, we have

$$\widetilde{T}_\infty = \begin{bmatrix} a & b & 1-a-b \\ a & b & 1-a-b \\ a & b & 1-a-b \end{bmatrix} \qquad \text{and} \qquad \widetilde{T}'_\infty = \begin{bmatrix} c & d & 1-c-d \\ c & d & 1-c-d \\ c & d & 1-c-d \end{bmatrix}.$$

Hence

$$T_\infty = \lim_{n\to\infty} T_n = \Big(\lim_{n\to\infty} \widetilde{T}_n\Big)\Big(\lim_{n\to\infty} A_{n+1}^{(i_1)}\Big) = \widetilde{T}_\infty A^{(i_1)},$$

where $A^{(i_1)}$ is defined in (3.2). Similarly, $T'_\infty = \widetilde{T}'_\infty A^{(j_1)}$.

Consider the case $i_1 = 1$ and $j_1 = 2$. Then $T_\infty = T'_\infty$ if and only if $\widetilde{T}_\infty A^{(1)} = \widetilde{T}'_\infty A^{(2)}$. Solving this linear system yields

$$a = \frac{1}{3}(c - d - 1), \qquad b = \frac{1}{3}(c + 8d + 2).$$

Since $a \geqslant 0$, we obtain $c \geqslant d + 1$, which forces $a = d = 0, b = c = 1$. That is, $\tilde{T}_\infty = L_2$ and $\tilde{T}'_\infty = L_1$. It follows from Corollary 3.4 that $i_k = 2$ and $j_k = 1$ for all $k \geqslant 2$. The other cases can be proved similarly. $\square$

For $\mathbf{x} \in \Sigma_\infty$, we define

$$\boldsymbol{\rho}(\mathbf{x}) = [\rho_1(\mathbf{x}), \rho_2(\mathbf{x}), \rho_3(\mathbf{x})] := \lim_{n\to\infty} \boldsymbol{\rho}(\mathbf{x}|_n). \tag{3.8}$$

It follows from Theorems 2.5 and 3.3 that the limit exists. In terms of the SG, $\boldsymbol{\rho}(\mathbf{x})$ can be realized as the probabilities for some random walk starting at $\mathbf{x}$ in the SG to reach the three vertices.

Let S_i, $i = 1, 2, 3$, be the three similitudes defining the Sierpinski gasket K. Let $\pi : \Sigma_\infty \to K$ be the standard projection defined by

$$\pi(\mathbf{x}) = \lim_{n\to\infty} S_{i_1} \circ \cdots \circ S_{i_n}(x_0) \quad \text{for} \quad \mathbf{x} = i_1 i_2 \cdots \in \Sigma_\infty,$$

where $x_0 \in \mathbb{R}^d$ is arbitrary and the definition is independent of x_0. We say that $\mathbf{x}, \mathbf{y} \in \Sigma_\infty$ are π-*equivalent*, denoted by $\mathbf{x} \sim_\pi \mathbf{y}$, if $\pi(\mathbf{x}) = \pi(\mathbf{y})$. Note that if we write $\mathbf{y} = j_1 j_2 \cdots$, then $\mathbf{x} \sim_\pi \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$ if and only if there exists some $m > 0$ such that $i_p = j_p$ for all $1 \leqslant p \leqslant m$ and $i_{m+1} i_{m+2} \cdots = \ell\dot{k}, j_{m+1} j_{m+2} \cdots = k\dot{\ell}$ for some $k \neq \ell$.

It follows easily from Proposition 3.5 that

Corollary 3.6. *For $\mathbf{x}, \mathbf{y} \in \Sigma_\infty$, $\boldsymbol{\rho}(\mathbf{x}) = \boldsymbol{\rho}(\mathbf{y})$ if and only if $\mathbf{x} \sim_\pi \mathbf{y}$.*

The proof of Theorem 3.3 also implies the following continuity property of $\boldsymbol{\rho}(\mathbf{x})$.

Corollary 3.7. *Let $\mathbf{x} \in \Sigma_\infty$. Then for any $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that for any $\mathbf{y} \in \Sigma_\infty$ with $\mathbf{x}|_n = \mathbf{y}|_n$, we have $|\boldsymbol{\rho}(\mathbf{x}) - \boldsymbol{\rho}(\mathbf{y})| \leqslant \epsilon$.*

4 Green function

Recall that the *Green function* for a Markov chain is

$$G(x,y) := \sum_{n=0}^{\infty} P^n(x,y),$$

where $P^n(x,y)$ is the n-step transition probability from x to y, with $P^0(x,x) := 1$. $G(x,y)$ is the expected number of visits from x to y. It is related to the hitting probabilities by the simple formula

$$G(x,y) = \begin{cases} \rho_{x,y}/(1-\rho_{y,y}), & x \neq y, \\ 1/(1-\rho_{x,x}), & x = y. \end{cases} \tag{4.1}$$

Note that our random walk satisfies

$$G(i^{n-1},x) = \frac{1}{3}(G(i^{n-1}j,x) + G(i^{n-1}k,x)), \quad x \in \widetilde{\Sigma}_n, \quad i \neq j,k. \tag{4.2}$$

We let

$$c := \lim_{n\to\infty} \rho_{i^{n-1}j,i^{n-1}j}, \quad i \neq j, \tag{4.3}$$

and

$$c' := \lim_{n\to\infty} \rho_{i^{n-1}j,i^{n-1}k}, \quad \text{where } i,j,k \text{ are distinct.} \tag{4.4}$$

These limits exist as it is easy to show that the sequences are increasing. To obtain a crude estimate for c we notice that if the chain starts at $i^{n-1}j$, it has a probability of $1/9$ of jumping to one of the neighboring non-vertex points and returning directly. Also it can be absorbed by i^n in one step. Thus, $2/9 < c < 2/3$.

Let $\sigma(i_1i_2\cdots) = (i_2i_3\cdots)$ be the *shift operator* on the symbolic space Σ_∞. Our main purpose in this section is to prove

Proposition 4.1. *Let* $\mathbf{x} = i^q i_{q+1}\cdots \in \Sigma_\infty$, *with* $i_{q+1} \neq i$. *Then for* $j,k \in \{1,2,3\}\setminus\{i\}, j \neq k$, *we have*

(a) $\lim_{n\to\infty} G(j^{n-1},\mathbf{x}|_n) = c_1(2\rho_j(\sigma(\mathbf{x})) + \rho_k(\sigma(\mathbf{x})))$,

(b) $\lim_{n\to\infty} G(i^{n-1},\mathbf{x}|_n) = c_1(c_2\rho_i(\sigma(\mathbf{x})) + 2\rho_j(\sigma(\mathbf{x})) + 2\rho_k(\sigma(\mathbf{x})))$,

where $c_1 = 2/(15(1-c))$, $c_2 = (5/2)(c+c')$, *and* c, c' *are defined as in* (4.3) *and* (4.4), *respectively.*

We need some additional notation to prove the proposition. For $x = i_1\cdots i_n$, let $\Delta^n_{i_1\cdots i_{n-m}}$ denote the (n,m)-cell $(m<n)$ containing x (as defined in Section 2); it has vertices $i_1\cdots i_{n-m}j^m$, $j=1,2,3$. For $y \in \Sigma_n \setminus \Delta^n_{i_1\cdots i_{n-m}}$, we use the following notation:

$$\mathbf{G}(\Delta^n_{i_1\cdots i_{n-m}},y) = \begin{bmatrix} G(i_1\cdots i_{n-m}1^m,y) \\ G(i_1\cdots i_{n-m}2^m,y) \\ G(i_1\cdots i_{n-m}3^m,y) \end{bmatrix}. \tag{4.5}$$

We use $Q_{n,k}$ to denote $A_2^{(i_n)}\cdots A_{n-k+1}^{(i_{k+1})}$ as in the previous section. Then

$$\mathbf{G}(\Delta^n_{i_1\cdots i_{n-1}},y) = Q_{n-1,n-m}\mathbf{G}(\Delta^n_{i_1\cdots i_{n-m}},y). \tag{4.6}$$

This amounts to saying that starting at a vertex of $\Delta^n_{i_1\cdots i_{n-1}}$, the chain has to go through one of the vertices of $\Delta^n_{i_1\cdots i_{n-m}}$ before it reaches y (as $y \notin \Delta^n_{i_1\cdots i_{n-m}}$); the probabilities of reaching these vertices are given by $Q_{n-1,n-m}$ (by Theorem 2.5).

Proof of (a). Note that if $\mathbf{x} = i^q i_{q+1}i_{q+2}\cdots$ and $y = j_1\cdots j_n$ with $j_1 \neq i$, then $y \notin \Delta^n_i$, and (4.6) implies

$$G(\mathbf{x}|_n,y) = \boldsymbol{e}_{i_n}\mathbf{G}(\Delta^n_{i^q i_{q+1}\cdots i_{n-1}},y) = \boldsymbol{e}_{i_n}Q_{n-1,1}\mathbf{G}(\Delta^n_i,y).$$

Note that by Theorems 2.5 and 3.3,

$$\lim_{n\to\infty} \boldsymbol{e}_{i_n} Q_{n-1,1} = \lim_{n\to\infty} \boldsymbol{e}_{i_n} A_2^{(i_{n-1})} \cdots A_{n-1}^{(i_2)} = \boldsymbol{\rho}(\sigma(\mathbf{x})).$$

In view of $G(x,y)=G(y,x)$ for $x,y\in\widetilde{\Sigma}_n$ and the above, it is easy to see that

$$\lim_{n\to\infty} G(j^{n-1},\mathbf{x}|_n) = \rho_j(\sigma(\mathbf{x}))\lim_{n\to\infty} G(j^{n-1},ij^{n-1}) + \rho_k(\sigma(\mathbf{x}))\lim_{n\to\infty} G(j^{n-1},ik^{n-1}).$$

To evaluate the limits on the right-hand side, we note that

$$\begin{aligned}\lim_{n\to\infty} G(j^{n-1},ij^{n-1}) &= \frac{1}{3}\lim_{n\to\infty}(G(j^{n-1}i,ij^{n-1})+G(j^{n-1}k,ij^{n-1}))\\ &= \frac{1}{3}\lim_{n\to\infty}\left(\frac{\beta_n}{1-\rho_{j^{n-1}i,j^{n-1}i}}+\frac{\beta_n}{1-\rho_{j^{n-1}k,j^{n-1}k}}\right)\\ &= \frac{2(2/5)}{3(1-c)} = 2c_1. \end{aligned} \tag{4.7}$$

By the same argument, we have

$$\lim_{n\to\infty} G(j^{n-1},ik^{n-1}) = \frac{2}{3(1-c)}\left(\frac{1}{5}\right) = c_1.$$

It follows that $\lim_{n\to\infty} G(j^{n-1},\mathbf{x}|_n) = c_1(2\rho_j(\sigma(\mathbf{x}))+\rho_k(\sigma(\mathbf{x})))$. □

To prove (b) we need some technical adjustments for the hitting probabilities $\boldsymbol{\rho}(x)$ and the corresponding $A_n^{(i)}$ in (2.3). Consider Σ_n with the three vertices i^n, j^n, k^n. We identify $i^{n-1}j, i^{n-1}k$, the two neighboring states of i^n, as i_*^n and use it to replace i^n as an absorbing state. Let $\widehat{\Sigma}_n$ be the modified level-n states. We define the corresponding $\widetilde{\alpha}_n, \widetilde{\beta}_n, \widetilde{\gamma}_n$, $\widetilde{A}_n^{(i)}$ and $\widetilde{\boldsymbol{\rho}}$, etc., as in Section 2. It is clear that for $\mathbf{x}=i_1i_2\cdots\in\Sigma_\infty$, we have $\widetilde{\boldsymbol{\rho}}(\sigma(\mathbf{x}|_n)) = \boldsymbol{e}_{i_n}\widetilde{A}_2^{(i_{n-1})}\cdots\widetilde{A}_{n-1}^{(i_2)} =: \boldsymbol{e}_{i_n}\widetilde{Q}_{n-1,1}$. Moreover,

$$\lim_{n\to\infty}\widetilde{A}_n^{(i)} = \lim_{n\to\infty} A_n^{(i)} = A^{(i)} \quad\text{and}\quad \lim_{n\to\infty}\widetilde{Q}_{n-1,1} = \lim_{n\to\infty} Q_{n-1,1}. \tag{4.8}$$

Proof of (b). We denote $i_{q+1}=\ell$. Let us first consider $G(\mathbf{x}|_n, i^{n-1}j)$. From (4.6), we have

$$G(\mathbf{x}|_n, i^{n-1}j) = \boldsymbol{e}_{i_n} Q_{n-1,q+1}\mathbf{G}(\Delta^n_{i^q\ell}, i^{n-1}j). \tag{4.9}$$

We cannot use (4.6) directly to expand $\mathbf{G}(\Delta^n_{i^q\ell}, i^{n-1}j)$ further in the form of a matrix product as in the proof of (a) because $i^{n-1}j$ is in the $(n,n-q)$-cell $\Delta^n_{i^q}$. We need some slight modification to continue the reduction.

By adopting the notation set up above, we use $\widetilde{\Delta}^n_{i^q}$ to denote the modified $(n,n-q)$-cell. It follows that

$$\begin{aligned}\mathbf{G}(\Delta^n_{i^q\ell}, i^{n-1}j) &= \widetilde{A}^{(\ell)}_{n-q}\mathbf{G}(\widetilde{\Delta}^n_{i^q}, i^{n-1}j)\\ &= \widetilde{A}^{(\ell)}_{n-q}\tilde{A}^{(i)}_{n-q+1}\mathbf{G}(\widetilde{\Delta}^n_{i^{q-1}}, i^{n-1}j)\\ &= \cdots\\ &= \widetilde{A}^{(\ell)}_{n-q}\widetilde{A}^{(i)}_{n-q+1}\cdots\tilde{A}^{(i)}_{n-1}\mathbf{G}(\widetilde{\Delta}^n_{i}, i^{n-1}j). \end{aligned} \tag{4.10}$$

Now, by using $G(x,y)=G(y,x)$ for $x,y\in\widetilde{\Sigma}_n$ again, we get

$$\begin{aligned}\mathbf{G}(i^{n-1}j,\mathbf{x}|_n) &= \boldsymbol{e}_{i_n}Q_{n-1,q+1}\mathbf{G}(\Delta^n_{i^q\ell}, i^{n-1}j)\\ &= \boldsymbol{e}_{i_n}A_2^{(i_{n-1})}\cdots A_{n-q-1}^{(i_{q+2})}\tilde{A}^{(\ell)}_{n-q}\tilde{A}^{(i)}_{n-q+1}\cdots\tilde{A}^{(i)}_{n-1}\mathbf{G}(\widetilde{\Delta}^n_i, i^{n-1}j).\end{aligned}$$

By (4.8) and Theorems 2.5 and 3.3,

$$\lim_{n\to\infty}\boldsymbol{e}_{i_n}A_2^{(i_{n-1})}\cdots A_{n-q-1}^{(i_{q+2})}\tilde{A}^{(\ell)}_{n-q}\tilde{A}^{(i)}_{n-q+1}\cdots\tilde{A}^{(i)}_{n-1} = \boldsymbol{\rho}(\sigma(\mathbf{x})).$$

To evaluate $\lim_{n\to\infty}\mathbf{G}(\widetilde{\Delta}_i^n, i^{n-1}j)$, we first observe that the three vertices of $\widetilde{\Delta}_i$ are $v_1 = ij^{n-1}, v_2 = ik^{n-1}$ and $i_*^n = \{i^{n-1}j, i^{n-1}k\} =: \{v_3, v_4\}$. We have

$$\lim_{n\to\infty} G(v_1, i^{n-1}j) = \lim_{n\to\infty} \frac{\rho_{ij^{n-1},i^{n-1}j}}{1-\rho_{i^{n-1}j,i^{n-1}j}} = \frac{2/5}{1-c},$$

and the same for $\lim_{n\to\infty} G(v_2, i^{n-1}j)$. Also

$$\lim_{n\to\infty} G(v_3, i^{n-1}j) - 1 = \lim_{n\to\infty} \frac{1}{1-\rho_{i^{n-1}j,i^{n-1}j}} - 1 = \frac{c}{1-c},$$

and

$$\lim_{n\to\infty} G(v_4, i^{n-1}j) = \lim_{n\to\infty} \frac{\rho_{i^{n-1}k,i^{n-1}j}}{1-\rho_{i^{n-1}j,i^{n-1}j}} = \frac{c'}{1-c}.$$

Note that if the chain starting at $\mathbf{x}|_n$ hits i_*^n before hitting v_1 and v_2, the conditional probability that it first hits either v_3 or v_4 tends to $1/2$ as $n\to\infty$. Thus,

$$\lim_{n\to\infty}\mathbf{G}(\widetilde{\Delta}_i^n, i^{n-1}j) = (1-c)^{-1}[(c+c')/2, 2/5, 2/5]^t.$$

Combining the above and (4.2), we have

$$\lim_{n\to\infty} G(i^{n-1}, \mathbf{x}|_n) = c_1((5/2)(c+c')\rho_i(\sigma(\mathbf{x})) + 2\rho_j(\sigma(\mathbf{x})) + 2\rho_k(\sigma(\mathbf{x}))).$$

This completes the proof of (b). □

We remark that if $\mathbf{x} = ijj\cdots \in \Sigma_\infty$, then $\rho_i(\sigma(\mathbf{x})) = 0 = \rho_k(\sigma(\mathbf{x}))$ and $\rho_j(\sigma(\mathbf{x})) = 1$. Thus the formulas in (a) and (b) coincide. Moreover, we have

Corollary 4.2. *Assume* $\mathbf{x} = i_1 i_2\cdots \in \Sigma_\infty \setminus \{\dot{1}, \dot{2}, \dot{3}\}$ *with* $i_1 = i$. *Define* $u_j := \lim_{n\to\infty} G(j^{n-1}, \mathbf{x}|_n)$. *Then for* $j \neq i$, *we have* $u_i \geqslant u_j$, *and equality holds if and only if* $\mathbf{x} = ijj\cdots$.

Proof. Using Proposition 4.1, we have

$$\begin{aligned} u_i - u_j &= c_1((5/2)(c+c')\rho_i(\sigma(\mathbf{x})) + 2\rho_j(\sigma(\mathbf{x})) + 2\rho_k(\sigma(\mathbf{x})\big) - c_1\big(2\rho_j(\sigma(\mathbf{x})) + \rho_k(\sigma(\mathbf{x}))) \\ &= c_1((5/2)(c+c')\rho_i(\sigma(\mathbf{x})) + \rho_k(\sigma(\mathbf{x}))) \geqslant 0. \end{aligned}$$

Equality holds if and only if $\rho_i(\sigma(\mathbf{x})) = 0 = \rho_k(\sigma(\mathbf{x}))$ and $\rho_j(\sigma(\mathbf{x})) = 1$. By Proposition 3.5 this happens if and only if $\mathbf{x} = ijj\cdots$. □

5 Martin kernel

We define the Martin kernel as

$$K(x,y) := \frac{G(x,y)}{G(\vartheta,y)}, \quad x,y\in\Sigma_*.$$

Proposition 5.1. *There exists* $C > 0$ *such that for any* $x, y \in \Sigma_*$ *with* $|x| \leqslant |y|$,

$$K(x,y) \leqslant C(5/3)^{|x|}.$$

Proof. Since $G(\vartheta, y) \geqslant G(\vartheta,x)G(x,y)$, by using Lemma 2.8 we have

$$K(x,y) \leqslant \frac{G(x,y)}{G(\vartheta,x)G(x,y)} = \frac{1}{G(\vartheta,x)} = \frac{1-\rho_{x,x}}{\rho_{\vartheta,x}} \leqslant \frac{1}{\rho_{\vartheta,x}} \leqslant C\left(\frac{5}{3}\right)^{|x|}.$$

□

Let $R = 5/3$ and define a function $\varrho : \Sigma_* \times \Sigma_* \to \mathbb{R}^+ \cup \{0\}$ as

$$\varrho(x, y) = \sum_{n=0}^{\infty} \left(\frac{r}{R}\right)^n \sup_{z \in \Sigma_n} |K(z, x) - K(z, y)|,$$

where $0 < r < 1$. By Proposition 5.1, the series converges. Since the chain is transient, ϱ is a metric on Σ_*, called a *Martin metric*. It is straightforward to verify that with this Martin metric, a sequence $\{x_n\}$ in Σ_* is ϱ-Cauchy if and only if x_n is eventually equal to some $x \in \Sigma_*$, or

$$|x_n| \to \infty \quad \text{and} \quad \lim_{n\to\infty} K(z, x_n) \quad \text{exists for every } z \in \Sigma_*. \tag{5.1}$$

We say that two ϱ-Cauchy sequences $\{x_n\}, \{y_n\}$ are *ϱ-equivalent*, denoted $\{x_n\} \sim_\varrho \{y_n\}$, if $\lim_{n\to\infty} \varrho(x_n, y_n) = 0$. We denote the ϱ-equivalence class of $\{x_n\}$ by $[\![\{x_n\}]\!]$. Let $\overline{\Sigma}_*$ be the collection of all ϱ-equivalence classes in Σ_* and call it the *Martin space*. We call $\mathcal{M} := \partial\overline{\Sigma}_* = \overline{\Sigma}_* \setminus \Sigma_*$ the *Martin boundary*.

We remark that this metric is topologically equivalent to the one defined in [7], both metrics define the same collection of Cauchy sequences and their completions are also topologically equivalent.

First let us reformulate $K(x, y)$ according to our specific transition probabilities. Let $x \in \Sigma_m$ and $y \in \Sigma_n$ with $m < n$. In order for the chain starting at x to reach y, it must first reach one of the three vertices in $V_m = \{1^m, 2^m, 3^m\}$ and then jump to level $m+1$. Recall that for $x \in \widetilde{\Sigma}_m$ and $i = 1, 2, 3$, ρ_{x,i^m} is the probability for the chain starting at x to be absorbed by i^m (see Proposition 2.5).

For $x \in \Sigma_m$, $n \geqslant m+2$, and $j = 1, 2, 3$, we let

$$b_j^{m,n-1}(x) := \begin{cases} \displaystyle\sum_{i=1}^{3} \rho_{x,i^m} \rho_{i^m, j^{n-1}}, & \text{if } x \in \widetilde{\Sigma}_m, \\ \rho_{x,j^{n-1}}, & \text{if } x \in V_m. \end{cases} \tag{5.2}$$

These are the probabilities for the chain starting at $x \in \Sigma_m$ to reach the three vertices in V_{n-1}.

Proposition 5.2. *Let $m \in \mathbb{N}$ and $x \in \Sigma_m$. For $n \geqslant m+2$, let $y \in \widetilde{\Sigma}_n$. Then*

$$K(x, y) = \frac{\sum_{i=1}^3 b_i^{m,n-1}(x) G(i^{n-1}, y)}{(1/3)\sum_{i=1}^3 G(i^{n-1}, y)}. \tag{5.3}$$

Moreover for $\mathbf{y} \in \Sigma_\infty$,

(a) *$K(x, \mathbf{y}) := \lim_{n\to\infty} K(x, \mathbf{y}|_n)$ exists;*

(b) *for any $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that for any $\mathbf{y}' \in \Sigma_\infty$ with $\mathbf{y}|_n = \mathbf{y}'|_n$, we have*

$$|K(x, \mathbf{y}) - K(x, \mathbf{y}')| < \epsilon.$$

Proof. The symmetry yields $G(\vartheta, y) = (1/3)\sum_{i=1}^3 G(i^{n-1}, y)$, which gives the denominator. The numerator follows after regrouping the right-hand side of the following expression:

$$G(x, y) = \sum_{i=1,2,3} \rho_{x,i^m} G(i^m, y) = \sum_{i,j=1,2,3} \rho_{x,i^m} \rho_{i^m,j^{n-1}} G(j^{n-1}, y).$$

For (a) we need to observe that $\lim_{n\to\infty} b_i^{m,n-1}(x)$ and $\lim_{n\to\infty} G(i^{n-1}, \mathbf{y}|_n)$ exist (by Proposition 2.6 and Proposition 4.1 respectively). For (b), we make use of, in addition, Corollary 3.7. □

6 Martin boundary

For $\mathbf{x} = i_1 i_2 \cdots$, $\mathbf{y} = j_1 j_2 \cdots \in \Sigma_\infty$, since $\{K(z, \mathbf{x}|_n)\}_n$ and $\{K(z, \mathbf{y}|_n)\}_n$ are Cauchy sequences for each $z \in \Sigma_*$ (Proposition 5.2), $\{\mathbf{x}|_n\}$ and $\{\mathbf{y}|_n\}$ are ϱ-Cauchy sequences. Hence $\varrho(\mathbf{x}|_n, \mathbf{y}|_n)$ is a Cauchy sequence of real numbers. We can extend the Martin metric ϱ to Σ_∞ by defining

$$\varrho(\mathbf{x}, \mathbf{y}) := \lim_{n\to\infty} \varrho(\mathbf{x}|_n, \mathbf{y}|_n).$$

Lemma 6.1. *If $\mathbf{x} \sim_\pi \mathbf{y}$, then $\lim_{n\to\infty} K(z,\mathbf{x}|_n) = \lim_{n\to\infty} K(z,\mathbf{y}|_n)$ for all $z \in \Sigma_*$. It follows that $\Sigma_\infty/_{\sim_\pi}$ is in the Martin boundary and ϱ is well defined on $\Sigma_\infty/_{\sim_\pi}$.*

Proof. We write $\mathbf{x} = i_1 \cdots i_m \ell \dot{k}$ and $\mathbf{y} = i_1 \cdots i_m k \dot{\ell}$. Then for n sufficiently large and $i = 1, 2, 3$,

$$G(i^{n-1}, \mathbf{x}|_n) = \boldsymbol{e}_k A_2^{(k)} \cdots A_{n-m-2}^{(k)} A_{n-m-1}^{(\ell)} \mathbf{G}(i^{n-1}, \Delta_{i_1\cdots i_m}),$$
$$G(i^{n-1}, \mathbf{y}|_n) = \boldsymbol{e}_\ell A_2^{(\ell)} \cdots A_{n-m-2}^{(\ell)} A_{n-m-1}^{(k)} \mathbf{G}(i^{n-1}, \Delta_{i_1\cdots i_m}).$$

According to Proposition 3.5,

$$\lim_{n\to\infty} G(i^{n-1}, \mathbf{x}|_n) = \lim_{n\to\infty} G(i^{n-1}, \mathbf{y}|_n).$$

Therefore, $\lim_{n\to\infty} K(z,\mathbf{x}|_n) = \lim_{n\to\infty} K(z,\mathbf{y}|_n)$. It follows that $\varrho(\mathbf{x},\mathbf{y}) = 0$ and hence ϱ is well defined on $\Sigma_\infty/\sim_\pi$. □

We will show that ϱ is a metric on $\Sigma_\infty/_{\sim_\pi}$. The main difficulty lies in showing that if $\mathbf{x} \not\sim_\pi \mathbf{y}$, then $\varrho(\mathbf{x},\mathbf{y}) > 0$ (Proposition 6.3). We need a lemma.

Lemma 6.2. *Suppose $\mathbf{x} = i_1 i_2 \cdots$, $\mathbf{y} = j_1 j_2 \cdots \in \Sigma_\infty \backslash \{\dot{1}, \dot{2}, \dot{3}\}$, and $\mathbf{x} \not\sim_\pi \mathbf{y}$. Assume $k \neq i_1, j_1$, and let*

$$r_i := \lim_{n\to\infty} \frac{G(i^{n-1}, \mathbf{x}|_n)}{G(k^{n-1}, \mathbf{x}|_n)} \quad \textit{and} \quad s_i := \lim_{n\to\infty} \frac{G(i^{n-1}, \mathbf{y}|_n)}{G(k^{n-1}, \mathbf{y}|_n)}.$$

Then $(r_i, r_j) \neq (s_i, s_j)$ for distinct $i, j \neq k$.

Proof. We use the same notation of Proposition 4.1. First we consider the case $i_1 = j_1$. For convenience we let 1 be the common index, $k = 3$, and write

$$\boldsymbol{\rho}(\sigma(\mathbf{x})) = [\rho_1, \rho_2, \rho_3] \qquad \text{and} \qquad \boldsymbol{\rho}(\sigma(\mathbf{y})) = [\eta_1, \eta_2, \eta_3].$$

Since $\mathbf{x} \not\sim_\pi \mathbf{y}$, Corollary 3.6 implies that $[\rho_1, \rho_2, \rho_3] \neq [\eta_1, \eta_2, \eta_3]$. We observe that $2\rho_3 + \rho_2 > 0$ (otherwise by Corollary 3.4, $\mathbf{x} = \dot{1}$) and $2\eta_3 + \eta_2 > 0$. Hence by Proposition 4.1, the limits defining r_i and s_i exist.

Suppose on the contrary that $(r_1, r_2) = (s_1, s_2)$. Then by using Proposition 4.1 and a direct calculation we get

$$\frac{\rho_1}{\eta_1} = \frac{\rho_2}{\eta_2} = \frac{\rho_3}{\eta_3}.$$

It follows that $[\rho_1, \rho_2, \rho_3] = c[\eta_1, \eta_2, \eta_3]$ for some number c. As these are probability weights, we have $c = 1$. This leads to a contradiction and completes the proof for the case $i_1 = j_1 = 1$.

Next we consider the case $i_1 \neq j_1$. Without loss of generality, we let $i_1 = 1$, $j_1 = 2$ and $k = 3$. Since $\mathbf{x} \not\sim_\pi \mathbf{y}$, we have $\mathbf{x} \neq 1\dot{2}$ or $\mathbf{y} \neq 2\dot{1}$. We also abbreviate the notation in the lemma as: $r_i = u_i/u_k$, $s_i = v_i/v_k$. Suppose on the contrary that $(r_1, r_2) = (s_1, s_2)$. Then this, together with Corollary 4.2, would imply

$$\frac{u_2}{u_3} \leqslant \frac{u_1}{u_3} = \frac{v_1}{v_3} \leqslant \frac{v_2}{v_3} = \frac{u_2}{u_3}.$$

Hence $u_1 = u_2$ and $v_1 = v_2$. By Corollary 4.2 again, we would have $\mathbf{x} = 1\dot{2}$ and $\mathbf{y} = 2\dot{1}$, a contradiction. The proof is complete. □

Proposition 6.3. *ϱ is a metric on $\Sigma_\infty/_{\sim_\pi}$.*

Proof. In view of Lemma 6.1, the only part we need to show is that if $\mathbf{x} \not\sim_\pi \mathbf{y}$, then $\varrho(\mathbf{x},\mathbf{y}) > 0$. By (5.1), it suffices to show that there exists $z_0 \in \Sigma_{m_0}$ such that

$$\lim_{n\to\infty} |K(z_0, \mathbf{x}|_n) - K(z_0, \mathbf{y}|_n)| > 0. \tag{6.1}$$

We first consider the case $\mathbf{x}, \mathbf{y} \notin \{\dot{1}, \dot{2}, \dot{3}\}$. Let $\mathbf{x} = i_1 i_2 \cdots$, $\mathbf{y} = j_1 j_2 \cdots$ and assume $i_1, j_1 \in \{1, 2\}$. Suppose on the contrary that (6.1) does not hold. Hence for $\epsilon > 0$ and for any $m > 0$, we have

$$\sup_{z\in\Sigma_m} |K(z, \mathbf{x}|_n) - K(z, \mathbf{y}|_n)| \leqslant \frac{\epsilon}{2} \tag{6.2}$$

for sufficiently large n. Then Lemma 6.2 with $k = 3$ together with (5.3) yields

$$\left|\frac{b_1^{m,n}(z)r_1 + b_2^{m,n}(z)r_2 + b_3^{m,n}(z)}{(1/3)(r_1 + r_2 + 1)} - \frac{b_1^{m,n}(z)s_1 + b_2^{m,n}(z)s_2 + b_3^{m,n}(z)}{(1/3)(s_1 + s_2 + 1)}\right| < \epsilon \tag{6.3}$$

for all $z \in \Sigma_m$ and for all m. Letting $z = 3^m$, we have $\lim_{m\to\infty} b_j^{m,n}(z) = \delta_{3j}$ (by Lemma 2.6). This implies that $r_1 + r_2 = s_1 + s_2$, and hence the denominators are equal. Next, choose $z = 1^m$. Then $\lim_{m\to\infty} b_j^{m,n}(z) = \delta_{1j}$, and we get $r_1 = s_1$. A similar argument shows that $r_2 = s_2$. This contradicts that $(r_1, r_2) \neq (s_1, s_2)$.

Next we consider the case that $\mathbf{x}, \mathbf{y} \in \{\dot{1}, \dot{2}, \dot{3}\}$. Assume $\mathbf{x} = \dot{1}$ and $\mathbf{y} = \dot{2}$. Then take $z = 1^m$. By using (4.1), it is straightforward to show that $|K(z, 1^n) - K(z, 2^n)| = 3|\rho_{1^m,1^n} - \rho_{1^m,2^n}|$. Since $\lim_{m\to\infty} \rho_{1^m,1^n} = 1$ and $\lim_{m\to\infty} \rho_{1^m,2^n} = 0$, (6.1) follows.

Finally we consider $\mathbf{x} \in \{\dot{1}, \dot{2}, \dot{3}\}$, $\mathbf{y} \notin \{\dot{1}, \dot{2}, \dot{3}\}$. We assume that $\mathbf{x} = \dot{1}$. Then by using $\lim_{m\to\infty} b_j^{m,n}(1^m) = \delta_{1j}$ and the approximation of $K(1^m, \mathbf{y}|_n)$ as in (6.3), we see that $\lim_{n\to\infty} K(1^m, \mathbf{y}|_n) < 3$ (as $s_1 + s_2 > 0$ by Corollary 3.4). Hence (6.1) follows. □

Recall that the standard metric d on Σ_∞ is defined as $d(\mathbf{x}, \mathbf{y}) = r^{-\max\{n:\, \mathbf{x}|_n = \mathbf{y}|_n\}}$, where $0 < r < 1$. Let $\mathcal{Q}$ be the induced quotient topology on $\Sigma_\infty/_{\sim_\pi}$. It follows that the Sierpinski gasket with the Euclidean norm $|\cdot|$ is homeomorphic to $(\Sigma_\infty/_{\sim_\pi}, \mathcal{Q})$. To prove our main theorem, we need to establish the second and third homeomorphisms below:

$$(K, |\cdot|) \cong (\Sigma_\infty/_{\sim\pi}, \mathcal{Q}) \cong (\Sigma_\infty/_{\sim\pi}, \rho) \cong (\mathcal{M}, \rho).$$

Theorem 6.4. *The Martin boundary of $\{X_n\}_{n=0}^\infty$ is homeomorphic to the Sierpinski gasket K.*

Proof. We first identify $(\Sigma_\infty/_{\sim\pi}, \rho)$ with $(\mathcal{M}, \rho)$. Define $\varphi : \Sigma_\infty/_{\sim_\pi} \to \mathcal{M}$ by $\varphi(\mathbf{x}) = [\![\{\mathbf{x}|_n\}]\!]$. It follows from Lemma 6.1 and Proposition 6.3 that the map φ is well defined and injective. We show that φ is surjective. This follows from a diagonal argument as follows: Let $\{w_n\}$ be a ϱ-Cauchy sequence in Σ_* with limit $\mathbf{w} \in \mathcal{M}$. Since Σ_1 is the finite set $\{1, 2, 3\}$, there exists a subsequence $\{w_n^{(1)}\}$ of $\{w_n\}$ such that $w_n^{(1)}|_1 = i_1$ for all n. Denote the first element of this subsequence by $\{w_1^{(1)}\}$. For the same reason, there exists a subsequence $\{w_n^{(2)}\}$ of $\{w_n^{(1)} : n \geqslant 2\}$ such that $w_n^{(2)}|_2 = i_1 i_2$ for all $n \geqslant 2$. Denote the first element of this subsequence by $w_2^{(2)}$. Inductively, for each $k \geqslant 1$, there exists a subsequence $\{w_n^{(k)}\}_n$ of $\{w_n^{(k-1)} : n \geqslant k\}$ such that $w_n^{(k)}|_k = i_1 i_2 \cdots i_k$ for all $n \geqslant k$, and we denote the first element of this subsequence by $w_k^{(k)}$. Using a diagonal argument, we extract the ϱ-Cauchy subsequence $\{w_n^{(n)}\}_n$. Clearly its ϱ-limit is also $\mathbf{w}$.

Let $\mathbf{x} := i_1 i_2 \cdots \in \Sigma_\infty$. We claim that $\varphi(\mathbf{x}) = \mathbf{w}$; that is, $[\![\{\mathbf{x}|_n\}]\!] = [\![\{w_n^{(n)}\}]\!]$, or equivalently, $\lim_{n\to\infty} \varrho(\mathbf{x}|_n, w_n^{(n)}) = 0$. This follows from

$$\lim_{n\to\infty} |K(z, \mathbf{x}|_n) - K(z, w_n^{(n)})| = \lim_{n\to\infty} |K(z, \mathbf{x}|_n) - K(z, \mathbf{x}|_n j_{n+1} \cdots j_{n+\ell})| = 0,$$

for all $z \in \Sigma_m$ (by Proposition 5.2(a) and (b)). Hence we can identify $(\Sigma_\infty/_{\sim_\pi}, \varrho)$ with $(\mathcal{M}, \varrho)$.

Next we let $\iota : (\Sigma_\infty, d) \to (\Sigma_\infty/_{\sim_\pi}, \varrho)$ be the natural map. In terms of the metric d on Σ_∞, Proposition 5.2(b) implies that for each $x \in \Sigma_*$, $K(x, \cdot)$ is continuous on (Σ_∞, d) and hence ι is continuous. Being a continuous surjection on the compact space $(\Sigma_\infty/_{\sim_\pi}, \varrho)$, it induces a homeomorphism $\tilde{\iota} : (\Sigma_\infty/_{\sim\pi}, \mathcal{Q}) \to (\Sigma_\infty/_{\sim\pi}, \rho)$. □

Our next objective is to identify the minimal Martin boundary $\mathcal{M}_{\min}$ (see [24]) of our Markov chain. For this, we use a result in [21, p. 235]. Let $\{E_k\}$ be an increasing sequence of events such that the state space $E = \bigcup_{k=1}^\infty E_k$ and assume that for each k, the Green function $G(\cdot, E_k)$ is bounded. Let L_k be the *last hitting time* defined as

$$L_k(\omega) := \sup\{n \geqslant 0 : X_n(\omega) \in E_k\}.$$

Then $L_k \leqslant L_{k+1}$ on $\{L_k < \infty\}$ and $\lim_{k\to\infty} L_k = \infty$ a.s. Define

$$Z_k := X_{L_k}.$$

Theorem 6.5. $\lim_{n\to\infty} X_n(\omega) = X_\infty(\omega)$ *P_ϑ-almost surely, where X_∞ is a $\{\dot{1},\dot{2},\dot{3}\}$-valued random variable. Moreover,* $\mathcal{M}_{\min} = \{\dot{1},\dot{2},\dot{3}\}$.

Proof. Since $\{X_n\}$ is a transient Markov chain, it converges P_ϑ-almost surely to an $\mathcal{M}_{\min}$-valued random variable X_∞ [7, Theorems 4 and 5]. It remains to show that $\mathcal{M}_{\min} = \{\dot{1},\dot{2},\dot{3}\}$. Define

$$E_k := \bigcup_{n\leqslant k} \Sigma_n.$$

Then for each k, $G(\cdot, E_k)$ is bounded, and $Z_k = X_{L_k}$ takes values in $\{1^k, 2^k, 3^k\}$. Since

$$L_k(\omega) = \inf\{n \geqslant 0 : X_n(\omega) \in \Sigma_{k+1}\} - 1,$$

it is a stopping time. Therefore, $\{Z_k\}$ is also a Markov chain. It follows that $Z_k(\omega) \to Z_\infty(\omega)$ as $k \to \infty$ P_ϑ-almost surely, where Z_∞ takes values $\{\dot{1},\dot{2},\dot{3}\}$. Now

$$|X_n(\omega) - Z_n(\omega)| = |X_n(\omega) - X_{L_n}(\omega)| \to 0, \quad P_\vartheta\text{-a.s.}$$

Hence $\lim_{n\to\infty} X_n(\omega) \in \{\dot{1},\dot{2},\dot{3}\}$ P_ϑ-a.s. This completes the proof. □

7 Harmonic functions

We call $h : \Sigma_* \to \mathbb{R}$ a *harmonic* (or *P-harmonic*) function on Σ_* if

$$\mathbf{P}h(x) := \sum_{y\in\Sigma_*} P(x,y)h(y) = h(x), \quad \forall\ x \in \Sigma_*.$$

It is well known that for any $\mathbf{y} \in \mathcal{M}$, $K(\cdot, \mathbf{y})$ is a harmonic function on Σ_* and any bounded harmonic function h has a unique integral representation

$$h(x) = \int_{\mathcal{M}_{\min}} K(x,\mathbf{y})\phi(\mathbf{y})d\nu(\mathbf{y}),$$

where ν is the measure on $\mathcal{M}_{\min}$ representing the constant harmonic function 1, and ϕ is some bounded ν-integrable function. In our case, $\mathcal{M}_{\min} = \{\dot{1},\dot{2},\dot{3}\}$ and $\nu(\dot{k}) = 1/3$ for $k = 1, 2, 3$. We define

$$\psi_i(x) := K(x, i^\infty), \qquad x \in \Sigma_*.$$

Then all the bounded harmonic functions are linear combinations of the $\psi_i, i = 1, 2, 3$.

Proposition 7.1. *For the above ψ_i, the extension*

$$\psi_i(\mathbf{x}) := \lim_{n\to\infty} \psi_i(\mathbf{x}|_n), \quad \mathbf{x} \in \mathcal{M},$$

is well defined and is continuous on $\mathcal{M}$.

Proof. We show that $\psi_i(\mathbf{x}|_n) = K(\mathbf{x}|_n, i^\infty)$ is a Cauchy sequence, and the limit follows. For $m < n$,

$$\begin{aligned}
&|K(\mathbf{x}|_n, 1^\infty) - K(\mathbf{x}|_m, 1^\infty)| \\
&\quad = \lim_{k\to\infty} |K(\mathbf{x}|_n, 1^k) - K(\mathbf{x}|_m, 1^k)| \\
&\quad = 3 \lim_{k\to\infty} |G(\mathbf{x}|_n, 1^k) - G(\mathbf{x}|_m, 1^k)| \\
&\quad = 3 \lim_{k\to\infty} \left| \sum_{j=1}^{3} \rho_{\mathbf{x}|_n, j^n} \rho_{j^n, 1^k} - \sum_{j=1}^{3} \rho_{\mathbf{x}|_m, j^m} \rho_{j^m, 1^k} \right| \\
&\quad = 3 \left| \sum_{j=1}^{3} \rho_{\mathbf{x}|_n, j^n} \eta_{j,1}^n - \sum_{j=1}^{3} \rho_{\mathbf{x}|_m, j^m} \eta_{j,1}^m \right|,
\end{aligned}$$

where the last equality follows from Proposition 2.6. Thus by Proposition 2.6 and Theorem 3.3, $\{K(\mathbf{x}|_n, 1^\infty)\}_n$ is a Cauchy sequence and hence ψ_i is well defined on $\mathcal{M}$. The above estimation also shows that ψ_i is uniformly continuous on Σ_*, and hence it has a continuous extension to its closure $\mathcal{M}$. □

Let K denote the SG, $K_w = S_w(K)$ for $w \in \Sigma_*$, $\mathbf{v}_j = wj^\infty$ correspond to the three vertices of K_w, and $\mathbf{v}_{jk}$ correspond to the intersection of K_{wj} and K_{wk}. We have the "$1/5 - 2/5$ rule" as in [23].

Proposition 7.2. *Assume the above notation and let $j, k, \ell \in \{1, 2, 3\}$ be distinct. Then*

$$\psi_i(\mathbf{v}_{jk}) = \frac{2}{5}\psi_i(\mathbf{v}_j) + \frac{2}{5}\psi_i(\mathbf{v}_k) + \frac{1}{5}\psi_i(\mathbf{v}_\ell), \qquad i = 1, 2, 3.$$

Proof. Fix m and for any $n > m$, let $\Delta^n_w \subseteq \Sigma_n$ be an $(n, n-m)$-cell with vertices $v^n_j = wj^{n-m}$, where $j = 1, 2, 3$ and $w \in \Sigma_m$. Consider the following six words around the center of Δ^n_w: $v^n_{jk} := wjk^{n-m-1}$. We have

$$\begin{aligned} K(v^n_{jk}, i^\infty) &= \lim_{q\to\infty} K(v^n_{jk}, i^q) = \lim_{q\to\infty} \frac{G(v^n_{jk}, i^q)}{G(\vartheta, i^q)} = \frac{1}{3}\lim_{q\to\infty} G(v^n_{jk}, i^q) \\ &= \frac{1}{3}\lim_{q\to\infty}(\alpha_{n-m}G(v^n_j, i^q) + \beta_{n-m}G(v^n_k, i^q) + \gamma_{n-m}G(v^n_\ell, i^q)) \\ &= \alpha_{n-m}K(v^n_j, i^\infty) + \beta_{n-m}K(v^n_k, i^\infty) + \gamma_{n-m}K(v^n_\ell, i^\infty). \end{aligned}$$

Since $\lim_{n\to\infty}\alpha_{n-m} = \lim_{n\to\infty}\beta_{n-m} = 2/5$ and $\lim_{n\to\infty}\gamma_{n-m} = 1/5$, the result follows. □

The above $\mathbf{v}_{jk}$ has four neighboring vertices in $K_{wj} \cup K_{wk}$. We denote this neighborhood relation by $\sim_\kappa$. By solving the system of equations above, we obtain the following graph harmonic property of ψ_i on the SG.

Corollary 7.3. *Let $\mathbf{x} = \mathbf{v}_{jk}$ be given as above. Then*

$$\psi_i(\mathbf{x}) = \frac{1}{4}\sum\nolimits_{\mathbf{y}\sim_\kappa\mathbf{x}}\psi_i(\mathbf{y}), \qquad i = 1, 2, 3.$$

We remark that the above graph harmonic property is the defining property for the canonical harmonic functions considered by Kigami [14]. We call them the K-harmonic functions; they are of three dimension. It follows from above that the K-harmonic functions are generated by ψ_i, $i = 1, 2, 3$. On the other hand, in [22], Strichartz showed that for a continuous function φ on K, if we define the average

$$h(w) = 3^{-|w|}\int_{K_w}\varphi(\mathbf{y})\, d\mu(\mathbf{y}), \quad w \in \Sigma_*,$$

then φ is K-harmonic if and only if h satisfies the identities

$$h(u) = \frac{1}{3}\sum_{u\sim v} h(v), \qquad u, v \in \Sigma_n \setminus V_n, \quad n \geqslant 2,$$

and thus h is a P-harmonic function under the present random walk.

The existence of the *Laplacian* is still a major open question in the analysis of fractals. The harmonic structures and random product of matrices can be used to induce Laplacians [14, 16]. It is seen from the above that on the Sierpinski gasket, the harmonic functions obtained through this Martin boundary approach coincide with the classical ones obtained by the minimal energy approach [12]. For the latter approach, Kigami introduced the class of *post-critically finite* (*p.c.f.*) self-similar sets [13], and extended the theory to a certain subclass of *strongly symmetric* p.c.f. sets. It is still not clear whether the symmetry condition can be removed, as it hinges on the existence of a self-similar energy identity. In the present consideration, it changes the problem to the identification of the Martin boundaries and the self-similar sets. It is likely that this method can offer another approach to showing the existence of the Laplacian on more general self-similar sets.

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