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A three level linearized compact difference scheme for the Cahn-Hilliard equation

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Abstract This article is devoted to the study of high order accuracy difference methods for the Cahn-Hilliard equation. A three level linearized compact difference scheme is derived. The unique solvability and unconditional convergence of the difference solution are proved. The convergence order is $O(\tau^2 + h^4)$ in the maximum norm. The mass conservation and the non-increase of the total energy are also verified. Some numerical examples are given to demonstrate the theoretical results.

Keywords Cahn-Hilliard equation, compact difference scheme, convergence, solvability, conservation, energy non-increase

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1 Introduction

Consider the Cahn-Hilliard equation

$$\frac{\partial u}{\partial t} = \triangle(\phi(u) - \alpha\triangle u), \quad \boldsymbol{x} \in \Omega, \quad t \in (0, T], \tag{1.1}$$

$$\left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} = 0, \quad \left.\frac{\partial\,(\phi(u) - \alpha\triangle u)}{\partial \nu}\right|_{\partial\Omega} = 0, \quad t \in (0, T], \tag{1.2}$$

$$u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), \quad \boldsymbol{x} \in \bar{\Omega}, \tag{1.3}$$

where $\phi(\cdot) = \psi'(\cdot)$, $\psi(u) = \frac{1}{4}\gamma(u^2 - \beta^2)^2$, Ω is a bounded domain in $\mathbb{R}^n$, u is the relative concentration difference of the mixture components, $\triangle$ is the Laplace operator, α, β and γ are positive constants, and ν is the unit normal vector to the boundary. The Cahn-Hilliard equation is a fourth order reaction-diffusion equation and was originally introduced to describe the phase separation of binary mixtures [10, 21, 37], for example, the cooling processes of alloys, glasses or polymer mixtures. More recently, it has been used to study phase transitions, interface dynamics, species competition and exclusion.

The wellposedness of the Cahn-Hilliard equation was discussed by many authors, for example, Elliott and Zheng [22], Elliott [17], Yin [46], Blowey and Elliott [8], Elliott and Luckhaus [20], Elliott and Garcke [19] and Barrett and Blowey [3]. The dynamical properties and its limit to certain free-boundary

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problems were investigated in [1, 12, 36, 38]. The linear instability has been studied recently by Burger et al. [9].

There are many of studies on numerical solutions to the normal boundary conditions and periodic boundary conditions of the Cahn-Hilliard equation.

Finite element approximations of the Cahn-Hilliard equation have been analyzed by many researchers such as Elliott and French [18], Copetti and Elliott [15], Barrett and Blowey [2, 3], Feng and Prohl [25], Du [16] and Zhang [47]. The discontinuous Galerkin methods were considered in [24, 33, 41]. The subject of a posteriori error estimates and adaptive methods for finite element approximations of the Cahn-Hilliard equation has recently been taken up by Feng and Wu [27], and Bartels and Müller [5]; see also the applications to the Cahn-Hilliard equation with a double obstacle free energy in [6, 7], and to the Allen-Cahn equation in [4, 26, 34].

The Fourier collocation method and the Fourier spectral method have been applied to the Cahn-Hilliard equation in [11, 23, 29–32, 42–45].

Sun [39] presented a three level linearized difference scheme of second order convergence in discrete L_2-norm. Choo [13, 14] developed a conservative nonlinear difference scheme, and Khiari et al. presented a nonlinear difference scheme in [35]. They proved the convergence of the difference scheme by supposing the boundedness of the difference solution. Furihata [28] constructed a stable and conservative finite difference scheme. Zhao [48] considered a two-level nonlinear and a three-level linear high accurate difference schemes for the one-dimensional Cahn-Hilliard equation and proved the conditional L_2 convergence without noticing the boundary discretization errors.

Up to now, there are many works on the second order finite difference scheme but few works on the high accuracy difference scheme for the Cahn-Hilliard equation.

In this article, we establish a compact difference scheme for the Cahn-Hilliard equation with the normal boundary conditions and prove that the compact scheme is unconditionally convergent with the convergence order of $O(\tau^2 + h^4)$ in the discrete L_∞-norm. The method is completely applicable to the periodic boundary value conditions.

The remainder of the article is arranged as follows. In Section 2, a compact difference scheme is derived for the one-dimensional Cahn-Hilliard equation. The mass conservation and the non-increasing of the total energy with the unique solvability of the difference scheme are discussed in Section 3. The L_∞ convergence of the difference solution is shown by the discrete energy method in Section 4. The outline for the two- and three-dimensional Cahn-Hilliard equation are presented in Section 5. Section 6 provides two numerical examples to verify the theoretical results. The article ends with a brief conclusion.

2 The derivation of the difference scheme

Consider the one-dimensional problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2}\left(\phi(u) - \alpha\frac{\partial^2 u}{\partial x^2}\right), \quad x \in (0, L),\ t \in (0, T], \tag{2.1}$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \quad \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0, \quad t \in (0, T], \tag{2.2}$$

$$\left.\frac{\partial}{\partial x}\left(\phi(u) - \alpha\frac{\partial^2 u}{\partial x^2}\right)\right|_{x=0} = 0, \quad \left.\frac{\partial}{\partial x}\left(\phi(u) - \alpha\frac{\partial^2 u}{\partial x^2}\right)\right|_{x=L} = 0, \quad t \in (0, T], \tag{2.3}$$

$$u(x, 0) = u_0(x), \quad x \in [0, L]. \tag{2.4}$$

We assume that the problem has a smooth solution.

Take two positive integers M and N and denote $h = L/M$, $\tau = T/N$, $\Omega_h = \{x_i \mid x_i = ih,\ 0 \leqslant i \leqslant M\}$ and $\Omega_\tau = \{t_k \mid t_k = k\tau,\ 0 \leqslant k \leqslant N\}$.

Let $\mathcal{W}_\tau = \{w \mid w = (w^0, w^1, \ldots, w^N)\}$ be a grid function space on Ω_τ. For any $w \in \mathcal{W}_\tau$, denote

$$w^{k+\frac{1}{2}} = \frac{1}{2}(w^{k+1} + w^k), \quad \delta_t w^{k+\frac{1}{2}} = \frac{1}{\tau}(w^{k+1} - w^k),$$

$$w^{\bar{k}} = \frac{1}{2}(w^{k+1} + w^{k-1}), \quad \triangle_t w^k = \frac{1}{2\tau}(w^{k+1} - w^{k-1}).$$

Let $\mathcal{V}_h = \{v \mid v = (v_0, v_1, \dots, v_M)\}$ be a grid function space on Ω_h. For any $u, v \in \mathcal{V}_h$, denote

$$u_{i+\frac{1}{2}} = \frac{1}{2}(u_{i+1} + u_i), \quad \delta_x u_{i+\frac{1}{2}} = \frac{1}{h}(u_{i+1} - u_i), \quad 0 \leqslant i \leqslant M-1;$$

$$(Au)_i = \begin{cases} \frac{5}{6}u_0 + \frac{1}{6}u_1, & i = 0, \\ \frac{1}{12}(u_{i+1} + 10u_i + u_{i-1}), & 1 \leqslant i \leqslant M-1, \\ \frac{1}{6}u_{M-1} + \frac{5}{6}u_M, & i = M; \end{cases}$$

$$(\delta_x^2 u)_i = \begin{cases} \frac{2}{h}\delta_x u_{\frac{1}{2}}, & i = 0, \\ \frac{1}{h}(\delta_x u_{i+\frac{1}{2}} - \delta_x u_{i-\frac{1}{2}}), & 1 \leqslant i \leqslant M-1, \\ -\frac{2}{h}\delta_x u_{M-\frac{1}{2}}, & i = M, \end{cases}$$

and

$$(u, v) = h\left(\frac{1}{2}u_0 v_0 + \sum_{i=1}^{M-1} u_i v_i + \frac{1}{2}u_M v_M\right), \quad \langle \delta_x u, \delta_x v\rangle = h\sum_{i=0}^{M-1}(\delta_x u_{i+\frac{1}{2}})(\delta_x v_{i+\frac{1}{2}}),$$

$$(\delta_x^2 u, \delta_x^2 v) = h\left(\frac{1}{2}(\delta_x^2 u_0)(\delta_x^2 v_0) + \sum_{i=1}^{M-1}(\delta_x^2 u_i)(\delta_x^2 v_i) + \frac{1}{2}(\delta_x^2 u_M)(\delta_x^2 v_M)\right),$$

$$\|u\|_\infty = \max_{0 \leqslant i \leqslant M}|u_i|, \quad \|u\| = \sqrt{(u, u)}, \quad |u|_1 = \sqrt{\langle \delta_x u, \delta_x u\rangle}, \quad |u|_2 = \sqrt{(\delta_x^2 u, \delta_x^2 u)}.$$

Then have

Lemma 2.1. *The following hold,*

$$(Au, v) = (u, Av), \tag{2.5}$$

$$(\delta_x^2 u, v) = (u, \delta_x^2 v) = -\langle \delta_x u, \delta_x v\rangle, \tag{2.6}$$

$$\frac{5}{12}\|u\|^2 \leqslant \|Au\|^2 \leqslant \|u\|^2. \tag{2.7}$$

If $v = \{v_i^k \mid 0 \leqslant i \leqslant M, \ 0 \leqslant k \leqslant N\}$ is a grid function on $\Omega_h^\tau = \Omega_h \times \Omega_\tau$, then we have $v^k = (v_0^k, v_1^k, \dots, v_M^k) \in \mathcal{V}_h$ and $v_i = (v_i^0, v_i^1, \dots, v_i^N) \in \mathcal{W}_\tau$. For simplicity, denote $(Au)_i$ and $(\delta_x^2 u)_i$ by Au_i and $\delta_x^2 u_i$, respectively.

We need the following lemma for the derivation of difference scheme.

Lemma 2.2. *Denote* $\alpha(s) = (1-s)^3[5 - 3(1-s)^2]$.

(I) *If* $g(x) \in C^6[x_0, x_1]$, *then it holds that*

$$\begin{aligned} &\left[\frac{5}{6}g''(x_0) + \frac{1}{6}g''(x_1)\right] - \frac{2}{h}\left[\frac{g(x_1) - g(x_0)}{h} - g'(x_0)\right] \\ &= -\frac{h}{6}g'''(x_0) + \frac{h^3}{90}g^{(5)}(x_0) + \frac{h^4}{180}\int_0^1 g^{(6)}(x_0 + sh)\alpha(s)ds \\ &= -\frac{h}{6}g'''(x_0) + \frac{h^3}{90}g^{(5)}(x_0) + \frac{h^4}{240}g^{(6)}(x_0 + \theta_0 h), \quad \theta_0 \in (0, 1). \end{aligned} \tag{2.8}$$

(II) *If* $g(x) \in C^6[x_{M-1}, x_M]$, *then it holds that*

$$\left[\frac{1}{6}g''(x_{M-1}) + \frac{5}{6}g''(x_M)\right] - \frac{2}{h}\left[g'(x_M) - \frac{g(x_M) - g(x_{M-1})}{h}\right]$$

$$= \frac{h}{6}g'''(x_M) - \frac{h^3}{90}g^{(5)}(x_M) + \frac{h^4}{180}\int_0^1 g^{(6)}(x_M - sh)\alpha(s)ds$$
$$= \frac{h}{6}g'''(x_M) - \frac{h^3}{90}g^{(5)}(x_M) + \frac{h^4}{240}g^{(6)}(x_M - \theta_M h), \quad \theta_M \in (0,1). \tag{2.9}$$

(III) *If* $g(x) \in C^6[x_{i-1}, x_{i+1}]$, *we have*

$$\frac{1}{12}[g''(x_{i+1}) + 10g''(x_i) + g''(x_{i-1})] - \frac{1}{h^2}[g(x_{i+1}) - 2g(x_i) + g(x_{i-1})]$$
$$= \frac{h^4}{360}\int_0^1 [g^{(6)}(x_i + sh) + g^{(6)}(x_i - sh)]\alpha(s)ds$$
$$= \frac{h^4}{240}g^{(6)}(x_i + \theta_i h), \quad \theta_i \in (-1,1). \tag{2.10}$$

Proof. We prove (2.8). From the Taylor expansion

$$g(x_1) = g(x_0) + hg'(x_0) + \frac{h^2}{2}g''(x_0) + \frac{h^3}{6}g'''(x_0) + \frac{h^4}{24}g^{(4)}(x_0) + \frac{h^5}{120}g^{(5)}(x_0) + \frac{h^6}{120}\int_0^1 g^{(6)}(x_0 + sh)(1-s)^5 ds,$$

we have

$$\frac{2}{h}\left[\frac{g(x_1) - g(x_0)}{h} - g'(x_0)\right] = g''(x_0) + \frac{h}{3}g'''(x_0) + \frac{h^2}{12}g^{(4)}(x_0) + \frac{h^3}{60}g^{(5)}(x_0) + \frac{h^4}{60}\int_0^1 g^{(6)}(x_0 + sh)(1-s)^5 ds. \tag{2.11}$$

From the Taylor expansion

$$g''(x_1) = g''(x_0) + hg'''(x_0) + \frac{h^2}{2}g^{(4)}(x_0) + \frac{h^3}{6}g^{(5)}(x_0) + \frac{h^4}{6}\int_0^1 g^{(6)}(x_0 + sh)(1-s)^3 ds,$$

we have

$$\frac{5}{6}g''(x_0) + \frac{1}{6}g''(x_1) = g''(x_0) + \frac{h}{6}g'''(x_0) + \frac{h^2}{12}g^{(4)}(x_0) + \frac{h^3}{36}g^{(5)}(x_0) + \frac{h^4}{36}\int_0^1 g^{(6)}(x_0 + sh)(1-s)^3 ds. \tag{2.12}$$

Subtracting (2.11) from (2.12), we get (2.8). Similarly, we can prove (2.9). The proof of (2.10) can refer to [40]. □

Let $v = \phi(u) - \alpha\frac{\partial^2 u}{\partial x^2}$. Then Problem (2.1)–(2.4) is equivalent to the following problem of second order equations,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 v}{\partial x^2}, \quad x \in (0,L),\ t \in (0,T], \tag{2.13}$$
$$v = \phi(u) - \alpha\frac{\partial^2 u}{\partial x^2}, \quad x \in (0,L), t \in (0,T], \tag{2.14}$$
$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \quad \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0, \quad \left.\frac{\partial v}{\partial x}\right|_{x=0} = 0, \quad \left.\frac{\partial v}{\partial x}\right|_{x=L} = 0, \quad t \in (0,T], \tag{2.15}$$
$$u(x,0) = u_0(x), \quad x \in [0,L]. \tag{2.16}$$

From (2.13)–(2.15), we can obtain

$$\left.\frac{\partial^3 u}{\partial x^3}\right|_{x=0} = 0, \quad \left.\frac{\partial^3 u}{\partial x^3}\right|_{x=L} = 0, \quad \left.\frac{\partial^3 v}{\partial x^3}\right|_{x=0} = 0, \quad \left.\frac{\partial^3 v}{\partial x^3}\right|_{x=L} = 0, \quad t \in (0,T], \tag{2.17}$$

$$\left.\frac{\partial^5 u}{\partial x^5}\right|_{x=0} = 0, \quad \left.\frac{\partial^5 u}{\partial x^5}\right|_{x=L} = 0, \quad \left.\frac{\partial^5 v}{\partial x^5}\right|_{x=0} = 0, \quad \left.\frac{\partial^5 v}{\partial x^5}\right|_{x=L} = 0, \quad t \in (0, T]. \tag{2.18}$$

Define, on $\Omega_h \times \Omega_\tau$, the following grid functions,

$$U_i^k = u(x_i, t_k), \quad V_i^k = v(x_i, t_k), \quad 0 \leqslant i \leqslant M,\ 0 \leqslant k \leqslant N.$$

Considering (2.13) and (2.14) at the point $(x_i, t_{\frac{1}{2}})$, using the Taylor expansion and Lemma 2.2, we can obtain

$$A\delta_t U_i^{\frac{1}{2}} = \delta_x^2 V_i^{\frac{1}{2}} + p_i^0, \quad 0 \leqslant i \leqslant M, \tag{2.19}$$

$$AV_i^{\frac{1}{2}} = A\phi\left(u(x_i, 0) + \frac{\tau}{2} u_t(x_i, 0)\right) - \alpha\delta_x^2 U_i^{\frac{1}{2}} + q_i^0, \quad 0 \leqslant i \leqslant M, \tag{2.20}$$

and there exists a constant c_1 independent of h and τ such that

$$|p_i^0| \leqslant c_1(\tau^2 + h^4), \quad |q_i^0| \leqslant c_1(\tau^2 + h^4), \quad 0 \leqslant i \leqslant M. \tag{2.21}$$

Considering (2.13) and (2.14) at the point (x_i, t_k), using the Taylor expansion with the integration remainder term and Lemma 2.2, we can obtain

$$A\triangle_t U_i^k = \delta_x^2 V_i^k + p_i^k, \quad 0 \leqslant i \leqslant M, \quad 1 \leqslant k \leqslant N-1, \tag{2.22}$$

$$AV_i^k = A\phi(U_i^k) - \alpha\delta_x^2 U_i^{\bar{k}} + q_i^k, \quad 0 \leqslant i \leqslant M, \quad 1 \leqslant k \leqslant N-1, \tag{2.23}$$

and there exists a constant c_2 independent of h and τ such that

$$|p_i^k| \leqslant c_2(\tau^2 + h^4), \quad 0 \leqslant i \leqslant M, \quad 1 \leqslant k \leqslant N-1, \tag{2.24}$$

$$|q_i^k| \leqslant c_2(\tau^2 + h^4), \quad 0 \leqslant i \leqslant M, \quad 1 \leqslant k \leqslant N-1, \tag{2.25}$$

$$|\triangle_t q_i^k| \leqslant c_2(\tau^2 + h^4), \quad 0 \leqslant i \leqslant M, \quad 2 \leqslant k \leqslant N-2. \tag{2.26}$$

From (2.16), we have

$$U_i^0 = u_0(x_i), \quad 0 \leqslant i \leqslant M. \tag{2.27}$$

Omitting the small terms p_i^k, q_i^k in the equations (2.19)–(2.20) and (2.22)–(2.23), noticing (2.27), and replacing the grid functions U_i^k and V_i^k with their numerical approximations u_i^k and v_i^k, respectively, we obtain the compact difference scheme for (2.13)–(2.16) as follows,

$$A\delta_t u_i^{\frac{1}{2}} = \delta_x^2 v_i^{\frac{1}{2}}, \quad 0 \leqslant i \leqslant M, \tag{2.28}$$

$$Av_i^{\frac{1}{2}} = A\phi\left(u(x_i, 0) + \frac{\tau}{2} u_t(x_i, 0)\right) - \alpha\delta_x^2 u_i^{\frac{1}{2}}, \quad 0 \leqslant i \leqslant M, \tag{2.29}$$

$$A\triangle_t u_i^k = \delta_x^2 v_i^k, \quad 0 \leqslant i \leqslant M, \quad 1 \leqslant k \leqslant N-1, \tag{2.30}$$

$$Av_i^k = A\phi(u_i^k) - \alpha\delta_x^2 u_i^{\bar{k}}, \quad 0 \leqslant i \leqslant M, \quad 1 \leqslant k \leqslant N-1, \tag{2.31}$$

$$u_i^0 = u_0(x_i), \quad 0 \leqslant i \leqslant M. \tag{2.32}$$

We have the following theorem.

Theorem 2.1. *Finite difference scheme* (2.28)–(2.32) *is equivalent to*

$$A^2\delta_t u_i^{\frac{1}{2}} = \delta_x^2\left(A\phi\left(u(x_i, 0) + \frac{\tau}{2} u_t(x_i, 0)\right) - \alpha\delta_x^2 u_i^{\frac{1}{2}}\right), \qquad 0 \leqslant i \leqslant M, \tag{2.33}$$

$$A^2\triangle_t u_i^k = \delta_x^2(A\phi(u_i^k) - \alpha\delta_x^2 u_i^{\bar{k}}), \qquad 0 \leqslant i \leqslant M, \quad 1 \leqslant k \leqslant N-1, \tag{2.34}$$

$$u_i^0 = u_0(x_i), \quad 0 \leqslant i \leqslant M, \tag{2.35}$$

and

$$v_i^{\frac{1}{2}} = A\phi\bigg(u(x_i,0) + \frac{\tau}{2}u_t(x_i,0)\bigg) - \alpha\delta_x^2 u_i^{\frac{1}{2}} - \frac{h^2}{12}A\delta_t u_i^{\frac{1}{2}}, \quad 0 \leqslant i \leqslant M, \tag{2.36}$$

$$v_i^k = A\phi(u_i^k) - \alpha\delta_x^2 u_i^{\bar{k}} - \frac{h^2}{12}A\triangle_t u_i^k, \quad 0 \leqslant i \leqslant M,\ 1 \leqslant k \leqslant N-1. \tag{2.37}$$

Proof. Acting A on both sides of (2.28) and substituting (2.29) into the result, we obtain (2.33). Since

$$Av_i^{\frac{1}{2}} = v_i^{\frac{1}{2}} + \frac{h^2}{12}\delta_x^2 v_i^{\frac{1}{2}},$$

we have

$$v_i^{\frac{1}{2}} = Av_i^{\frac{1}{2}} - \frac{h^2}{12}\delta_x^2 v_i^{\frac{1}{2}} = A\phi\bigg(u(x_i,0) + \frac{\tau}{2}u_t(x_i,0)\bigg) - \alpha\delta_x^2 u_i^{\frac{1}{2}} - \frac{h^2}{12}A\delta_t u_i^{\frac{1}{2}}.$$

This is (2.36). Similarly, we can get (2.34) and (2.37). This completes the proof. □

We construct the difference scheme (2.33)–(2.35) for (2.1)–(2.4). At each time level, it is a system of penta-diagonal linear algebraic equations.

Theorem 2.1 implies that the analysis of numerical solution to the difference system (2.33)–(2.35) can be converted to that to the difference scheme (2.28)–(2.32).

Remark 2.1. If we use $\delta_x^2 u_i^k$ instead of $\delta_x^2 u_i^{\bar{k}}$ in (2.34), we will obtain an unconditionally unstable difference scheme. This can be checked similarly to applying von Neumann method to the difference scheme $\triangle_t u_i^k = -\alpha\delta_x^4 u_i^k$.

3 The conservation and unique solvability of the difference scheme

The solution to (2.1)–(2.4) satisfies the mass conservation law [16]

$$\frac{d}{dt}\int_0^L u(x,t)dx = 0, \quad 0 \leqslant t \leqslant T \tag{3.1}$$

and the non-increase of the total energy

$$\frac{d}{dt}\bigg[\int_0^L \psi(u(x,t))dx + \frac{\alpha}{2}\int_0^L \bigg(\frac{\partial u(x,t)}{\partial x}\bigg)^2 dx\bigg] \leqslant 0, \quad 0 \leqslant t \leqslant T, \tag{3.2}$$

or

$$\frac{d}{dt}\bigg[(\psi(u(\cdot,t)),1) - \frac{\alpha}{2}\bigg(\frac{\partial^2 u(\cdot,t)}{\partial x^2}, u(\cdot,t)\bigg)\bigg] \leqslant 0, \quad 0 \leqslant t \leqslant T.$$

We point out that the solution to our difference scheme has the same properties.

Since $\psi'(u) = \phi(u)$, we have

$$\begin{aligned}\psi(U_i^1) &= \psi(U_i^0) + \int_{U_i^0}^{U_i^1}\psi'(s)ds = \psi(U_i^0) + \int_{U_i^0}^{U_i^1}\phi(s)ds\\ &= \psi(U_i^0) + \phi\bigg(u(x_i,0) + \frac{\tau}{2}u_t(x_i,0)\bigg)(U_i^1 - U_i^0) + O(\tau^3), \quad 0 \leqslant i \leqslant M,\\ \psi(U_i^{k+1}) &= \psi(U_i^{k-1}) + \int_{U_i^{k-1}}^{U_i^{k+1}}\psi'(s)ds = \psi(U_i^{k-1}) + \int_{U_i^{k-1}}^{U_i^{k+1}}\phi(s)ds\\ &= \psi(U_i^{k-1}) + \phi(U_i^k)(U_i^{k+1} - U_i^{k-1}) + O(\tau^3), \quad 0 \leqslant i \leqslant M, \quad 1 \leqslant k \leqslant N-1.\end{aligned}$$

Define

$$\psi_i^0 = \psi(u_i^0), \quad 0 \leqslant i \leqslant M,$$

$$\psi_i^1 = \psi_i^0 + A\phi\bigg(u(x_i,0) + \frac{\tau}{2}u_t(x_i,0)\bigg) \cdot A^2(u_i^1 - u_i^0), \quad 0 \leqslant i \leqslant M,$$
$$\psi_i^{k+1} = \psi_i^{k-1} + A\phi(u_i^k) \cdot A^2(u_i^{k+1} - u_i^{k-1}), \quad 0 \leqslant i \leqslant M, \quad 1 \leqslant k \leqslant N-1.$$

Let

$$F^k = h\bigg(\frac{1}{2}A^2u_0^k + \sum_{i=1}^{M-1} A^2u_i^k + \frac{1}{2}A^2u_M^k\bigg), \quad 0 \leqslant k \leqslant N,$$
$$\bar{G}^k = (\psi^k, 1) - \frac{\alpha}{2}\left(\delta_x^2 u^k, A^2 u^k\right), \quad 0 \leqslant k \leqslant N, \tag{3.2}$$
$$G^0 = \bar{G}^0, \tag{3.3}$$
$$G^k = \frac{1}{2}(\bar{G}^k + \bar{G}^{k-1}), \quad 1 \leqslant k \leqslant N. \tag{3.4}$$

Then we have

Theorem 3.1. *The solution to difference scheme* (2.33)–(2.35) *satisfies*

$$F^k \equiv F^0, \quad 1 \leqslant k \leqslant N, \tag{3.5}$$
$$G^k \leqslant G^{k-1}, \quad 1 \leqslant k \leqslant N. \tag{3.6}$$

Proof. Taking the product of (2.33) with 1, we can obtain

$$F^1 = F^0. \tag{3.7}$$

Taking the product of (2.34) with 1, we can obtain

$$F^{k+1} = F^{k-1}, \quad 1 \leqslant k \leqslant N-1. \tag{3.8}$$

Combining (3.7) with (3.8), we have (3.5).

Taking the product of (2.33) with $(A\phi(u(x_i,0) + \frac{\tau}{2}u_t(x_i,0)) - \alpha\delta_x^2 u_i^{\frac{1}{2}})$ and noticing

$$A\phi\bigg(u(x_i,0) + \frac{\tau}{2}u_t(x_i,0)\bigg) \cdot A^2\delta_t u_i^{\frac{1}{2}} = \delta_t\psi_i^{\frac{1}{2}}, \quad 0 \leqslant i \leqslant M,$$

we can obtain

$$\bar{G}^1 \leqslant \bar{G}^0. \tag{3.9}$$

Taking the product of (2.34) with $(A\phi(u_i^k) - \alpha\delta_x^2 u_i^{\bar{k}})$ and using

$$A\phi(u_i^k) \cdot A^2\triangle_t u_i^k = \triangle_t\psi_i^k, \quad 0 \leqslant i \leqslant M, \quad 1 \leqslant k \leqslant N-1,$$

we can obtain

$$\bar{G}^{k+1} \leqslant \bar{G}^{k-1}, \quad 1 \leqslant k \leqslant N-1. \tag{3.10}$$

Combining (3.9) with (3.10), we have (3.6). □

Remark 3.1. If we define

$$\hat{G}^0 = h\bigg(\frac{1}{2}\psi_0^0 + \sum_{i=1}^{M-1}\psi_i^0 + \frac{1}{2}\psi_M^0\bigg) - \frac{\alpha}{2}(\delta_x^2u^0, A^2u^0),$$
$$\hat{G}^k = h\bigg(\frac{1}{2}\psi_0^{k-\frac{1}{2}} + \sum_{i=1}^{M-1}\psi_i^{k-\frac{1}{2}} + \frac{1}{2}\psi_M^{k-\frac{1}{2}}\bigg) - \frac{\alpha}{2}(\delta_x^2u^{k-\frac{1}{2}}, A^2u^{k-\frac{1}{2}}), \quad 0 \leqslant k \leqslant N,$$

we have the following non-increase of the total energy,

$$\hat{G}^k \leqslant \hat{G}^{k-1}, \quad 1 \leqslant k \leqslant N.$$

Next, we discuss the unique solvability of the difference scheme.

Theorem 3.2. *Difference scheme* (2.33)–(2.35) *has a unique solution.*

Proof. Consider the homogenous system of (2.33),

$$\frac{1}{\tau}A^2u_i^1 = -\frac{\alpha}{2}\delta_x^2(\delta_x^2u_i^1), \qquad 0 \leqslant i \leqslant M. \tag{3.11}$$

Taking the inner product (3.11) with u^1, we have

$$\frac{1}{\tau}(A^2u^1, u^1) + \frac{\alpha}{2}(\delta_x^2(\delta_x^2u^1), u^1) = 0.$$

Using the basic identities (2.5)–(2.6), we obtain

$$\frac{1}{\tau}(Au^1, Au^1) + \frac{\alpha}{2}(\delta_x^2u^1, \delta_x^2u^1) = 0.$$

Consequently,

$$u_i^1 = 0, \quad 0 \leqslant i \leqslant M.$$

Thus, the system (2.33) determines $u^1 = (u_0^1, u_1^1, \ldots, u_M^1)$ uniquely.

Now, suppose u^{k-1} and u^k have been uniquely determined. Consider the homogenous system of (2.34),

$$\frac{1}{2\tau}A^2u_i^{k+1} = -\frac{\alpha}{2}\delta_x^2(\delta_x^2u_i^{k+1}), \qquad 0 \leqslant i \leqslant M,$$

we can obtain

$$u_i^{k+1} = 0, \quad 0 \leqslant i \leqslant M,$$

i.e., the system (2.34) has a unique solution u^{k+1}. This completes the proof. □

4 The convergence of the finite difference scheme

In this section, we will prove the convergence of the finite difference scheme (2.33)–(2.35).

Lemma 4.1. *Let* $u = (u^0, u^1, \ldots, u^N) \in \mathcal{W}_\tau$ *and* $U = (U^0, U^1, \ldots, U^N) \in \mathcal{W}_\tau$. *Then there are* $\rho \in (0,1)$ *and*

$$\xi \in (\min\{\rho u^{k+1} + (1-\rho)u^{k-1}, \rho U^{k+1} + (1-\rho)U^{k-1}\}, \max\{\rho u^{k+1} + (1-\rho)u^{k-1}, \rho U^{k+1} + (1-\rho)U^{k-1}\})$$

dependent on k *such that*

$$\begin{aligned}\Delta_t[\phi(U^k) - \phi(u^k)] = {}& \phi'(\rho u^{k+1} + (1-\rho)u^{k-1})\Delta_t(U^k - u^k) \\ & + \phi''(\xi)[\rho(U^{k+1} - u^{k+1}) + (1-\rho)(U^{k-1} - u^{k-1})]\Delta_tU^k.\end{aligned}$$

Proof. We have

$$\begin{aligned}\Delta_t[\phi(U^k) - \phi(u^k)] &= \frac{1}{2\tau}\{[\phi(U^{k+1}) - \phi(u^{k+1})] - [\phi(U^{k-1}) - \phi(u^{k-1})]\} \\ &= \frac{1}{2\tau}\{[\phi(U^{k-1} + 2\tau\Delta_tU^k) - \phi(u^{k-1} + 2\tau\Delta_tu^k)] - [\phi(U^{k-1}) - \phi(u^{k-1})]\} \\ &= \phi'(U^{k-1} + 2\rho\tau\Delta_tU^k)\Delta_tU^k - \phi'(u^{k-1} + 2\rho\tau\Delta_tu^k)\Delta_tu^k \qquad (4.1) \\ &= \phi'(u^{k-1} + 2\rho\tau\Delta_tu^k)\Delta_t(U^k - u^k) \\ &\quad + [\phi'(U^{k-1} + 2\rho\tau\Delta_tU^k) - \phi'(u^{k-1} + 2\rho\tau\Delta_tu^k)]\Delta_tU^k \\ &= \phi'(\rho u^{k+1} + (1-\rho)u^{k-1})\Delta_t(U^k - u^k) \\ &\quad + [\phi'(\rho U^{k+1} + (1-\rho)U^{k-1}) - \phi'(\rho u^{k+1} + (1-\rho)u^{k-1})]\Delta_tU^k \\ &= \phi'(\rho u^{k+1} + (1-\rho)u^{k-1})\Delta_t(U^k - u^k) \\ &\quad + \phi''(\xi)[\rho(U^{k+1} - u^{k+1}) + (1-\rho)(U^{k-1} - u^{k-1})]\Delta_tU^k. \qquad (4.2)\end{aligned}$$

In obtaining (4.1), we consider $\phi(U^{k-1} + 2\tau\rho\Delta_tU^k) - \phi(u^{k-1} + 2\tau\rho\Delta_tu^k)$ as a function of $\rho \in [0,1]$ and then use the differential mid-value theorem. In obtaining (4.2), we apply the differential mid-value theorem again. This completes the proof. □

Lemma 4.2. *The following hold,*

$$\sum_{l=1}^{k} u^l \Delta_t v^l = \begin{cases} \dfrac{1}{2\tau}(u^1v^2 - u^1v^0), \quad k=1, \\ \dfrac{1}{2\tau}(u^kv^{k+1} + u^{k-1}v^k - u^0v^1 - u^1v^0) - \displaystyle\sum_{l=1}^{k-1}(\Delta_t u^l)v^l, \quad 2 \leqslant k \leqslant N-1, \end{cases}$$

and

$$\sum_{l=1}^{k} u^l \Delta_t v^l = \begin{cases} \dfrac{1}{2\tau}(u^1v^2 - u^1v^0), \quad k=1, \\ \dfrac{1}{2\tau}(u^kv^{k+1} + u^{k-1}v^k - u^2v^1 - u^1v^0) - \displaystyle\sum_{l=2}^{k-1}(\Delta_t u^l)v^l, \quad 2 \leqslant k \leqslant N-1. \end{cases}$$

Proof. Since

$$\begin{aligned}\sum_{l=1}^{k} u^l \Delta_t v^l &= \frac{1}{2\tau}\sum_{l=1}^{k} u^l(v^{l+1} - v^{l-1}) \\ &= \frac{1}{2\tau}\left(\sum_{l=1}^{k} u^l v^{l+1} - \sum_{l=1}^{k} u^l v^{l-1}\right) = \frac{1}{2\tau}\left(\sum_{l=2}^{k+1} u^{l-1} v^{l} - \sum_{l=0}^{k-1} u^{l+1} v^{l}\right),\end{aligned}$$

we have

$$\begin{aligned}\sum_{l=1}^{k} u^l \Delta_t v^l &= \frac{1}{2\tau}(u^kv^{k+1} + u^{k-1}v^k - u^0v^1 - u^1v^0) - \sum_{l=1}^{k-1}(\Delta_t u^l)v^l \\ &= \frac{1}{2\tau}(u^kv^{k+1} + u^{k-1}v^k - u^2v^1 - u^1v^0) - \sum_{l=2}^{k-1}(\Delta_t u^l)v^l.\end{aligned}$$

This completes the proof. □

Lemma 4.3 (See [49]). *For any grid function $v \in \mathcal{V}_h$, there is a constant κ_1 such that the following holds,*

$$\|v\|_\infty \leqslant \kappa_1 \|v\|^{3/4} \left(|\, v|_2 + \|v\|\right)^{1/4}.$$

Let

$$\begin{aligned}
&c_3 = \max_{0\leqslant x\leqslant L, 0\leqslant t\leqslant T} |u(x,t)|, \quad c_4 = \max_{|u|\leqslant c_3+1} |\phi'(u)|, \quad c_5 = \max_{|u|\leqslant c_3+1} |\phi''(u)|, \\
&c_6 = \max\left\{\left(1+\frac{8c_4^2}{\alpha^2}\right)\left(4+\frac{4c_4^2}{\alpha}\right), \quad \frac{8}{\alpha}\left(\frac{6}{5}c_4^2 + c_5^2 + \frac{1}{2}\right)\right\}, \\
&c_7 = 2\left[\left(T^2 + \frac{T}{\alpha}\right)c_1^2 + 2\left(1+\frac{1}{\alpha}\right)Tc_2^2\right]L, \qquad c_8 = \left[4\left(\frac{T}{\alpha} + \frac{4}{\alpha^2}\right) + \left(16 + \frac{8T}{\alpha} + \frac{2}{\alpha^2}\right)c_2^2\right]L, \\
&c_9 = \left(1+\frac{8c_4^2}{\alpha^2}\right)c_7 + c_8, \qquad c = 2\kappa_1\sqrt{c_9 L}\exp\left(\frac{c_6 T}{2}\right).
\end{aligned} \tag{4.3}$$

Theorem 4.1. *Assume that the solution $u(x,t)$ to* (2.1)–(2.4) *is sufficiently smooth. Then the solution to difference scheme* (2.33)–(2.35) *unconditionally converges to the solution to* (2.1)–(2.4) *in the discrete L_∞-norm and the rate of convergence is the order of $O(\tau^2 + h^4)$ when h and τ are small. More precisely, denote*

$$e_i^k = U_i^k - u_i^k, \quad 0 \leqslant i \leqslant M, \quad 0 \leqslant k \leqslant N.$$

Then, there exists a positive constant c such that, if $\tau^2 + h^4 \leqslant 1/c$, then the following estimate holds,

$$\|e^k\|_\infty \leqslant c(\tau^2 + h^4), \quad 0 \leqslant k \leqslant N. \tag{4.4}$$

Proof. Let c be defined in (4.3). Denote

$$f_i^k = V_i^k - v_i^k, \quad 0 \leqslant i \leqslant M, \quad 0 \leqslant k \leqslant N.$$

Subtracting (2.28)–(2.32) from (2.19)–(2.20), (2.22)–(2.23) and (2.27), respectively, we obtain the following error system,

$$A\delta_t e_i^{\frac{1}{2}} = \delta_x^2 f_i^{\frac{1}{2}} + p_i^0, \quad 0 \leqslant i \leqslant M, \tag{4.5}$$

$$Af_i^{\frac{1}{2}} = -\alpha\delta_x^2 e_i^{\frac{1}{2}} + q_i^0, \quad 0 \leqslant i \leqslant M, \tag{4.6}$$

$$A\triangle_t e_i^k = \delta_x^2 f_i^k + p_i^k, \quad 0 \leqslant i \leqslant M, \quad 1 \leqslant k \leqslant N-1, \tag{4.7}$$

$$Af_i^k = A(\phi(U_i^k) - \phi(u_i^k)) - \alpha\delta_x^2 e_i^{\bar{k}} + q_i^k, \quad 0 \leqslant i \leqslant M, \quad 1 \leqslant k \leqslant N-1, \tag{4.8}$$

$$e_i^0 = 0, \quad 0 \leqslant i \leqslant M. \tag{4.9}$$

(I) We estimate $\|Ae^1\|$ and $|\,e^1|_2$.

(1) Taking the product of (4.5) with $Ae^{\frac{1}{2}}$, we have

$$(A\delta_t e^{\frac{1}{2}}, Ae^{\frac{1}{2}}) = (\delta_x^2 f^{\frac{1}{2}}, Ae^{\frac{1}{2}}) + (p^0, Ae^{\frac{1}{2}}).$$

Taking the product of (4.6) with $\frac{1}{\alpha}Af^{\frac{1}{2}}$, we have

$$\frac{1}{\alpha}\|Af^{\frac{1}{2}}\|^2 = -(\delta_x^2 e^{\frac{1}{2}}, Af^{\frac{1}{2}}) + \frac{1}{\alpha}(q^0, Af^{\frac{1}{2}}).$$

Adding the two equalities above, we obtain

$$(A\delta_t e^{\frac{1}{2}}, Ae^{\frac{1}{2}}) + \frac{1}{\alpha}\|Af^{\frac{1}{2}}\|^2 = (p^0, Ae^{\frac{1}{2}}) + \frac{1}{\alpha}(q^0, Af^{\frac{1}{2}}) \leqslant (p^0, Ae^{\frac{1}{2}}) + \frac{1}{\alpha}\left(\frac{1}{4}\|q^0\|^2 + \|Af^{\frac{1}{2}}\|^2\right).$$

Noticing (4.9), we have

$$\frac{1}{2\tau}\|Ae^1\|^2 \leqslant \frac{1}{2}(p^0, Ae^1) + \frac{1}{4\alpha}\|q^0\|^2 \leqslant \frac{1}{2}\left(\frac{\tau}{2}\|p^0\|^2 + \frac{1}{2\tau}\|Ae^1\|^2\right) + \frac{1}{4\alpha}\|q^0\|^2.$$

Using (2.21), we obtain

$$\|Ae^1\|^2 \leqslant \tau^2\|p^0\|^2 + \frac{\tau}{\alpha}\|q^0\|^2 \leqslant \left(\tau^2 + \frac{\tau}{\alpha}\right)c_1^2 L(\tau^2 + h^4)^2 \leqslant \left(T^2 + \frac{T}{\alpha}\right)c_1^2 L(\tau^2 + h^4)^2. \tag{4.10}$$

(2) Taking the product of (4.5) with $A\delta_t e^{\frac{1}{2}}$, we have

$$\|A\delta_t e^{\frac{1}{2}}\|^2 = (\delta_x^2 f^{\frac{1}{2}}, A\delta_t e^{\frac{1}{2}}) + (p^0, A\delta_t e^{\frac{1}{2}}).$$

Taking the product of (4.6) with $\delta_x^2\delta_t e^{\frac{1}{2}}$, we have

$$(Af^{\frac{1}{2}}, \delta_x^2\delta_t e^{\frac{1}{2}}) = -\alpha(\delta_x^2 e^{\frac{1}{2}}, \delta_x^2\delta_t e^{\frac{1}{2}}) + (q^0, \delta_x^2\delta_t e^{\frac{1}{2}}).$$

Adding the two equalities above, we obtain

$$\|A\delta_t e^{\frac{1}{2}}\|^2 + \alpha(\delta_x^2 e^{\frac{1}{2}}, \delta_x^2\delta_t e^{\frac{1}{2}}) = (p^0, A\delta_t e^{\frac{1}{2}}) + (q^0, \delta_x^2\delta_t e^{\frac{1}{2}}) \leqslant \|A\delta_t e^{\frac{1}{2}}\|^2 + \frac{1}{4}\|p^0\|^2 + (q^0, \delta_x^2\delta_t e^{\frac{1}{2}}).$$

Noticing (4.9), we have

$$\frac{\alpha}{2\tau}|\,e^1|_2^2 \leqslant \frac{1}{4}\|p^0\|^2 + \frac{1}{\tau}(q^0, \delta_x^2 e^1) \leqslant \frac{1}{4}\|p^0\|^2 + \frac{1}{\tau}\left(\frac{1}{\alpha}\|q^0\|^2 + \frac{\alpha}{4}|e^1|_2^2\right),$$

or

$$|\,e^1|_2^2 \leqslant \frac{\tau}{\alpha}\|p^0\|^2 + \frac{4}{\alpha^2}\|q^0\|^2.$$

Using (2.21), we obtain

$$|e^1|_2^2 \leqslant \left(\frac{\tau}{\alpha}+\frac{4}{\alpha^2}\right)c_1^2L(\tau^2+h^4)^2 \leqslant \left(\frac{T}{\alpha}+\frac{4}{\alpha^2}\right)c_1^2L(\tau^2+h^4)^2. \tag{4.11}$$

(II) Suppose (4.4) is true for k from 0 to m $(1 \leqslant m \leqslant N-1)$. Then, when $\tau^2+h^4 \leqslant 1/c$, we have

$$\|e^k\|_\infty \leqslant c(\tau^2+h^4) \leqslant 1, \quad 1 \leqslant k \leqslant m.$$

Then it follows that

$$\|u^k\|_\infty = \|U^k-(U^k-u^k)\|_\infty \leqslant \|U^k\|_\infty + \|e^k\|_\infty \leqslant c_3+1, \quad 1 \leqslant k \leqslant m \tag{4.12}$$

and

$$|\phi(U_i^k)-\phi(u_i^k)| \leqslant c_4|e_i^k|, \quad 0 \leqslant i \leqslant M, \quad 1 \leqslant k \leqslant m. \tag{4.13}$$

Consequently, we arrive at

$$|A(\phi(U_i^k)-\phi(u_i^k))| \leqslant A|\phi(U_i^k)-\phi(u_i^k)| \leqslant c_4A|e_i^k|, \quad 0 \leqslant i \leqslant M, \quad 1 \leqslant k \leqslant m$$

and

$$\|A(\phi(U^k)-\phi(u^k))\| \leqslant c_4\|Ae^k\|, \quad 1 \leqslant k \leqslant m. \tag{4.14}$$

Using Lemma 4.1, we have

$$\begin{aligned}|\Delta_t[\phi(U_i^k)-\phi(u_i^k)]| &\leqslant c_4|\Delta_t(U_i^k-u_i^k)| + c_5(|U_i^{k+1}-u_i^{k+1}|+|U_i^{k-1}-u_i^{k-1}|)\\ &= c_4|\Delta_t e_i^k| + c_5(|e_i^{k+1}|+|e_i^{k-1}|), \quad 0 \leqslant i \leqslant M, \quad 1 \leqslant k \leqslant m-1.\end{aligned} \tag{4.15}$$

We will prove that (4.4) is valid for $k=m+1$.

(1) Taking the product of (4.7) with $Ae^{\bar{k}}$, we have

$$(A\Delta_t e^k, Ae^{\bar{k}}) = (\delta_x^2 f^k, Ae^{\bar{k}}) + (p^k, Ae^{\bar{k}}).$$

Taking the product of (4.8) with $\frac{1}{\alpha}Af^k$, we have

$$\frac{1}{\alpha}\|Af^k\|^2 = \frac{1}{\alpha}(A(\phi(U^k)-\phi(u^k)), Af^k) - (\delta_x^2 e^{\bar{k}}, Af^k) + \frac{1}{\alpha}(q^k, Af^k).$$

Adding the two equalities above and using (4.14), we obtain

$$\begin{aligned}(A\Delta_t e^k, Ae^{\bar{k}}) + \frac{1}{\alpha}\|Af^k\|^2 &= \frac{1}{\alpha}(A(\phi(U^k)-\phi(u^k)), Af^k) + (p^k, Ae^{\bar{k}}) + \frac{1}{\alpha}(q^k, Af^k)\\ &\leqslant \frac{1}{\alpha}\|A(\phi(U^k)-\phi(u^k))\|\cdot\|Af^k\| + \|p^k\|\cdot\|Ae^{\bar{k}}\| + \frac{1}{\alpha}\|q^k\|\cdot\|Af^k\|\\ &\leqslant \frac{1}{2\alpha}(c_4^2\|Ae^k\|^2 + \|Af^k\|^2) + \frac{1}{2}(\|p^k\|^2 + \|Ae^{\bar{k}}\|^2)\\ &\quad + \frac{1}{2\alpha}(\|q^k\|^2 + \|Af^k\|^2), \quad 1 \leqslant k \leqslant m.\end{aligned}$$

Consequently,

$$\frac{1}{4\tau}(\|Ae^{k+1}\|^2 - \|Ae^{k-1}\|^2) \leqslant \frac{c_4^2}{2\alpha}\|Ae^k\|^2 + \frac{1}{2}\|Ae^{\bar{k}}\|^2 + \frac{1}{2}\|p^k\|^2 + \frac{1}{2\alpha}\|q^k\|^2, \quad 1 \leqslant k \leqslant m.$$

Replacing the superscript k by l in the inequality above and summing up for l from 1 to k, we have

$$\frac{1}{4\tau}(\|Ae^{k+1}\|^2 + \|Ae^k\|^2 - \|Ae^1\|^2 - \|Ae^0\|^2)$$

$$\leqslant \frac{c_4^2}{2\alpha}\sum_{l=1}^{k}\|Ae^l\|^2+\frac{1}{2}\sum_{l=1}^{k}\|Ae^{\bar{l}}\|^2+\frac{1}{2}\sum_{l=1}^{k}\|p^l\|^2+\frac{1}{2\alpha}\sum_{l=1}^{k}\|q^l\|^2,\quad 1\leqslant k\leqslant m,$$

or,

$$\begin{aligned}&\|Ae^{k+1}\|^2+\|Ae^k\|^2\\&\quad\leqslant \|Ae^1\|^2+\frac{2c_4^2\tau}{\alpha}\sum_{l=1}^{k}\|Ae^l\|^2+2\tau\sum_{l=1}^{k}\|Ae^{\bar{l}}\|^2+2\tau\sum_{l=1}^{k}\|p^l\|^2+\frac{2\tau}{\alpha}\sum_{l=1}^{k}\|q^l\|^2,\quad 1\leqslant k\leqslant m.\end{aligned}$$

Suppose $\tau\leqslant 1/2$. Using (2.24)–(2.25) and (4.10), we have

$$\|Ae^{k+1}\|^2+\|Ae^k\|^2\leqslant 2\left(2+\frac{2c_4^2}{\alpha}\right)\tau\sum_{l=1}^{k}\|Ae^l\|^2+c_7(\tau^2+h^4)^2,\quad 1\leqslant k\leqslant m. \tag{4.16}$$

(2) Taking the product of (4.7) with $A\Delta_t e^k$, we have

$$\|A\Delta_t e^k\|^2=(\delta_x^2 f^k,A\Delta_t e^k)+(p^k,A\Delta_t e^k),\quad 1\leqslant k\leqslant m.$$

Taking the product of (4.8) with $\delta_x^2\Delta_t e^k$, we have

$$\begin{aligned}(Af^k,\delta_x^2\Delta_t e^k)&=(A(\phi(U^k)-\phi(u^k)),\delta_x^2\Delta_t e^k)\\&\quad-\alpha(\delta_x^2 e^{\bar{k}},\delta_x^2\Delta_t e^k)+(q^k,\delta_x^2\Delta_t e^k),\quad 1\leqslant k\leqslant m.\end{aligned}$$

Adding the two equalities above, we obtain

$$\|A\Delta_t e^k\|^2+\alpha(\delta_x^2 e^{\bar{k}},\delta_x^2\Delta_t e^k)=(A(\phi(U^k)-\phi(u^k)),\delta_x^2\Delta_t e^k)+(p^k,A\Delta_t e^k)+(q^k,\delta_x^2\Delta_t e^k),$$

or,

$$\begin{aligned}\|A\Delta_t e^k\|^2+\frac{\alpha}{4\tau}(|e^{k+1}|_2^2-|e^{k-1}|_2^2)&\leqslant (A(\phi(U^k)-\phi(u^k)),\delta_x^2\Delta_t e^k)+\frac{1}{2}\|p^k\|^2\\&\quad+\frac{1}{2}\|A\Delta_t e^k\|^2+(q^k,\delta_x^2\Delta_t e^k),\quad 1\leqslant k\leqslant m.\end{aligned}$$

Replacing the superscript k by l in the above inequality and summing up for l from 1 to k, we get

$$\begin{aligned}&\frac{1}{2}\sum_{l=1}^{k}\|A\Delta_t e^l\|^2+\frac{\alpha}{4\tau}(|e^{k+1}|_2^2+|e^k|_2^2-|e^1|_2^2-|e^0|_2^2)\\&\quad\leqslant\sum_{l=1}^{k}(A(\phi(U^l)-\phi(u^l)),\delta_x^2\Delta_t e^l)+\frac{1}{2}\sum_{l=1}^{k}\|p^l\|^2+\sum_{l=1}^{k}(q^l,\delta_x^2\Delta_t e^l),\quad 1\leqslant k\leqslant m.\end{aligned} \tag{4.17}$$

Owing to Lemma 4.2, we have

$$\begin{aligned}\sum_{l=1}^{k}\left(A(\phi(U^l)-\phi(u^l)),\delta_x^2\Delta_t e^l\right)&=\frac{1}{2\tau}[(A(\phi(U^k)-\phi(u^k)),\delta_x^2 e^{k+1})+(A(\phi(U^{k-1})-\phi(u^{k-1})),\delta_x^2 e^k)\\&\quad-(A(\phi(U^0)-\phi(u^0)),\delta_x^2 e^1)-(A(\phi(U^1)-\phi(u^1)),\delta_x^2 e^0)]\\&\quad-\sum_{l=1}^{k-1}(\Delta_t A(\phi(U^l)-\phi(u^l)),\delta_x^2 e^l),\quad 1\leqslant k\leqslant m.\end{aligned}$$

Applying the Cauchy-Schwarz inequality and noticing (4.9), (4.14)–(4.15) and (2.7), we have

$$\left|\sum_{l=1}^{k}(A(\phi(U^l)-\phi(u^l)),\delta_x^2\Delta_t e^l)\right|$$

$$\leqslant \frac{1}{2\tau}[\|A(\phi(U^k)-\phi(u^k))\|\cdot|e^{k+1}|_2+\|A(\phi(U^{k-1})-\phi(u^{k-1}))\|\cdot|e^k|_2] \\ +\sum_{l=1}^{k-1}\|\Delta_t A(\phi(U^l)-\phi(u^l))\|\cdot|e^l|_2 \\ \leqslant \frac{1}{2\tau}[c_4\|Ae^k\|\cdot|e^{k+1}|_2+c_4\|Ae^{k-1}\|\cdot|e^k|_2] \\ +\sum_{l=1}^{k-1}(c_4\|\Delta_t e^l\|+c_5\|e^{l+1}\|+c_5\|e^{l-1}\|)\cdot|e^l|_2,\quad 1\leqslant k\leqslant m,$$

or

$$\left|\sum_{l=1}^{k}\left(A(\phi(U^l)-\phi(u^l)),\delta_x^2\Delta_t e^l\right)\right| \\ \leqslant \frac{1}{2\tau}(c_4\|Ae^k\|\cdot|e^{k+1}|_2+c_4\|Ae^{k-1}\|\cdot|e^k|_2) \\ +\sqrt{\frac{12}{5}}\sum_{l=1}^{k-1}(c_4\|A\Delta_t e^l\|+c_5\|Ae^{l+1}\|+c_5\|Ae^{l-1}\|)\cdot|e^l|_2,\quad 1\leqslant k\leqslant m. \tag{4.18}$$

For the last term of (4.17), using Lemma 4.2, we have

$$\sum_{l=1}^{k}(q^l,\delta_x^2\Delta_t e^l)=\frac{1}{2\tau}((q^k,\delta_x^2e^{k+1})+(q^{k-1},\delta_x^2e^k)-(q^2,\delta_x^2e^1)-(q^1,\delta_x^2e^0))-\sum_{l=2}^{k-1}(\Delta_t q^l,\delta_x^2e^l) \\ \leqslant \frac{1}{2\tau}(\|q^k\|\cdot|e^{k+1}|_2+\|q^{k-1}\|\cdot|e^k|_2+\|q^2\|\cdot|e^1|_2+\|q^1\|\cdot|e^0|_2) \\ +\sum_{l=2}^{k-1}\|\Delta_t q^l\|\cdot|e^l|_2,\qquad 1\leqslant k\leqslant m. \tag{4.19}$$

Substituting (4.18) and (4.19) into (4.17), noticing (4.9), we get

$$\frac{1}{2}\sum_{l=1}^{k}\|A\Delta_t e^l\|^2+\frac{\alpha}{4\tau}(|e^{k+1}|_2^2+|e^k|_2^2-|e^1|_2^2-|e^0|_2^2) \\ \leqslant \frac{1}{2\tau}(c_4\|Ae^k\|\cdot|e^{k+1}|_2+c_4\|Ae^{k-1}\|\cdot|e^k|_2) \\ +\sqrt{\frac{12}{5}}\sum_{l=1}^{k-1}(c_4\|A\Delta_t e^l\|+c_5\|Ae^{l+1}\|+c_5\|Ae^{l-1}\|)\cdot|e^l|_2+\frac{1}{2}\sum_{l=1}^{k}\|p^l\|^2 \\ +\frac{1}{2\tau}(\|q^k\|\cdot|e^{k+1}|_2+\|q^{k-1}\|\cdot|e^k|_2+\|q^2\|\cdot|e^1|_2)+\sum_{l=2}^{k-1}\|\Delta_t q^l\|\cdot|e^l|_2 \\ \leqslant \frac{1}{2\tau}\left(\frac{2c_4^2}{\alpha}\|Ae^k\|^2+\frac{\alpha}{8}|e^{k+1}|_2^2+\frac{2c_4^2}{\alpha}\|Ae^{k-1}\|^2+\frac{\alpha}{8}|e^k|_2^2\right) \\ +\sum_{l=1}^{k-1}\left(\frac{1}{2}\|A\Delta_t e^l\|^2+\frac{6}{5}c_4^2|e^l|_2^2+\frac{6}{5}\|Ae^{l+1}\|^2+\frac{1}{2}c_5^2|e^l|_2^2+\frac{6}{5}\|Ae^{l-1}\|^2+\frac{1}{2}c_5^2|e^l|_2^2\right) \\ +\frac{1}{2}\sum_{l=1}^{k}\|p^l\|^2+\frac{1}{2\tau}\left(\frac{2}{\alpha}\|q^k\|^2+\frac{\alpha}{8}|e^{k+1}|_2^2+\frac{2}{\alpha}\|q^{k-1}\|^2+\frac{\alpha}{8}|e^k|_2+\frac{1}{2\alpha}\|q^2\|^2+\frac{\alpha}{2}|e^1|_2^2\right) \\ +\frac{1}{2}\sum_{l=2}^{k-1}(\|\Delta_t q^l\|^2+|e^l|_2^2),\quad 1\leqslant k\leqslant m.$$

Consequently, we have

$$|e^{k+1}|_2^2+|e^k|_2^2\leqslant\frac{8c_4^2}{\alpha^2}(\|Ae^k\|^2+\|Ae^{k-1}\|^2)+\frac{8}{\alpha}\left(\frac{6}{5}c_4^2+c_5^2\right)\tau\sum_{l=1}^{k-1}|e^l|_2^2+4|e^1|_2^2$$

$$
\begin{aligned}
&+\frac{4\tau}{\alpha}\sum_{l=1}^{k}\|p^l\|^2+8\|q^k\|^2+8\|q^{k-1}\|^2+\frac{2}{\alpha^2}\|q^2\|^2+\frac{4\tau}{\alpha}\sum_{l=2}^{k-1}(\|\Delta_t q^l\|^2+|e^l|_2^2)\\
\leqslant{}&\frac{8c_4^2}{\alpha^2}(\|Ae^k\|^2+\|Ae^{k-1}\|^2)+\left[\frac{8}{\alpha}\left(\frac{6}{5}c_4^2+c_5^2\right)+\frac{8}{\alpha}\right]\tau\sum_{l=1}^{k-1}|e^l|_2^2\\
&+c_8(\tau^2+h^4)^2,\quad 1\leqslant k\leqslant m.
\end{aligned}\tag{4.20}
$$

It follows from (4.16) that

$$
\|Ae^k\|^2+\|Ae^{k-1}\|^2\leqslant 2\left(2+\frac{2c_4^2}{\alpha}\right)\tau\sum_{l=1}^{k-1}\|Ae^l\|^2+c_7(\tau^2+h^4)^2,\quad 1\leqslant k\leqslant m.\tag{4.21}
$$

Substituting (4.21) into (4.20) gives

$$
\begin{aligned}
|e^{k+1}|_2^2+|e^k|_2^2\leqslant{}&\frac{8c_4^2}{\alpha^2}\left[2\left(2+\frac{2c_4^2}{\alpha}\right)\tau\sum_{l=1}^{k-1}\|Ae^l\|^2+c_7(\tau^2+h^4)^2\right]\\
&+\left[\frac{8}{\alpha}\left(\frac{6}{5}c_4^2+c_5^2\right)+\frac{4}{\alpha}\right]\tau\sum_{l=1}^{k-1}|e^l|_2^2+c_8(\tau^2+h^4)^2,\quad 1\leqslant k\leqslant m.
\end{aligned}\tag{4.22}
$$

Let

$$
E^k=|e^{k+1}|_2^2+|e^k|_2^2+\|Ae^{k+1}\|^2+\|Ae^k\|^2,\quad 0\leqslant k\leqslant N-1.
$$

Adding (4.22) with (4.16), we get

$$
E^k\leqslant c_6\tau\sum_{l=1}^{k-1}E^l+c_9(\tau^2+h^4)^2,\quad 1\leqslant k\leqslant m.
$$

Using the discrete Gronwall inequality yields

$$
E^m\leqslant c_9\exp(c_6T)\cdot(\tau^2+h^4)^2.
$$

Consequently,

$$
|e^{m+1}|_2^2+\|Ae^{m+1}\|^2\leqslant c_9\exp(c_6T)\cdot(\tau^2+h^4)^2,
$$

or

$$
|e^{m+1}|_2^2+\frac{5}{12}\|e^{m+1}\|^2\leqslant c_9\exp(c_6T)\cdot(\tau^2+h^4)^2.
$$

Applying Lemma 4.3, we may can obtain

$$
\|e^{m+1}\|_\infty\leqslant\kappa_1\|e^{m+1}\|^{3/4}(|\,e^{m+1}|_2+\|e^{m+1}\|)^{1/4}\leqslant c(\tau^2+h^4).
$$

Therefore, (4.4) is valid for $k=m+1$. This completes the proof. □

5 The compact difference scheme for the two-dimensional problem

Consider the two-dimensional problem

$$
\frac{\partial u}{\partial t}=\triangle(\phi(u)-\alpha\triangle u),\quad \boldsymbol{x}\in\Omega,\quad t\in(0,T],\tag{5.1}
$$

$$
\left.\frac{\partial u}{\partial\nu}\right|_{\partial\Omega}=0,\qquad \left.\frac{\partial\,(\phi(u)-\alpha\triangle u)}{\partial\nu}\right|_{\partial\Omega}=0,\quad t\in(0,T],\tag{5.2}
$$

$$
u(\boldsymbol{x},0)=u_0(\boldsymbol{x}),\quad \boldsymbol{x}\in\bar{\Omega},\tag{5.3}
$$

where $\Omega=(0,L_1)\times(0,L_2)$ and suppose that it has a smooth solution.

Take the three integers M_1, M_2, N and denote $h_1 = L_1/M_1, h_2 = L_2/M_2, \tau = T/N, x_i = ih_1, y_j = jh_2, t_k = k\tau$, $\hat{\Omega}_h = \{(x_i, y_j) \mid 0 \leqslant i \leqslant M_1, 0 \leqslant j \leqslant M_2\}$ and $\hat{\Omega}_\tau = \{t_k \mid 0 \leqslant k \leqslant N\}$. Suppose $u = \{u_{ij}^k \mid 0 \leqslant i \leqslant M_1, 0 \leqslant j \leqslant M_2,\ 0 \leqslant k \leqslant N\}$ is a grid function on $\hat{\Omega}_h^\tau = \hat{\Omega}_h \times \hat{\Omega}_\tau$. Introduce the following notations,

$$u_{ij}^{k+\frac{1}{2}} = \frac{1}{2}(u_{ij}^{k+1} + u_{ij}^k), \quad u_{ij}^{k+\frac{1}{2}} = \frac{1}{2}(u_{ij}^{k+1} + u_{ij}^k),$$
$$u_{ij}^{\bar{k}} = \frac{1}{2}(u_{ij}^{k+1} + u_{ij}^{k-1}), \quad \triangle_t u_{ij}^k = \frac{1}{2\tau}(u_{ij}^{k+1} - u_{ij}^{k-1}),$$
$$\delta_x u_{i+\frac{1}{2},j}^k = \frac{1}{h_1}(u_{i+1,j}^k - u_{ij}^k), \quad \delta_y u_{i,j+\frac{1}{2}}^k = \frac{1}{h_2}(u_{i,j+1}^k - u_{ij}^k),$$
$$A_1 u_{ij}^k = \begin{cases} \frac{5}{6}u_{0,j}^k + \frac{1}{6}u_{1,j}^k, & i = 0,\ 0 \leqslant j \leqslant M_2, \\ \frac{1}{12}(u_{i+1,j}^k + 10u_{ij}^k + u_{i-1,j}^k), & 1 \leqslant i \leqslant M_1 - 1,\ 0 \leqslant j \leqslant M_2, \\ \frac{1}{6}u_{M_1-1,j}^k + \frac{5}{6}u_{M_1,j}^k, & i = M_1,\ 0 \leqslant j \leqslant M_2, \end{cases}$$
$$A_2 u_{ij}^k = \begin{cases} \frac{5}{6}u_{i,0}^k + \frac{1}{6}u_{i,1}^k, & 0 \leqslant i \leqslant M_1,\ j = 0, \\ \frac{1}{12}(u_{i,j+1}^k + 10u_{ij}^k + u_{i,j-1}^k), & 0 \leqslant i \leqslant M_1,\ 1 \leqslant j \leqslant M_2 - 1, \\ \frac{1}{6}u_{i,M_2-1}^k + \frac{5}{6}u_{i,M_2}^k, & 0 \leqslant i \leqslant M_1,\ j = M_2; \end{cases}$$

$$\delta_x^2 u_{ij}^k = \begin{cases} \frac{2}{h_1}\delta_x u_{\frac{1}{2},j}^k, & i = 0,\ 0 \leqslant j \leqslant M_2, \\ \frac{1}{h_1}(\delta_x u_{i+\frac{1}{2},j}^k - \delta_x u_{i-\frac{1}{2},j}^k), & 1 \leqslant i \leqslant M_1 - 1,\ 0 \leqslant j \leqslant M_2, \\ -\frac{2}{h_1}\delta_x u_{M_1-\frac{1}{2},j}^k, & i = M_1,\ 0 \leqslant j \leqslant M_2, \end{cases}$$
$$\delta_y^2 u_{ij}^k = \begin{cases} \frac{2}{h_2}\delta_y u_{i,\frac{1}{2}}^k, & 0 \leqslant i \leqslant M_1,\ j = 0, \\ \frac{1}{h_2}(\delta_y u_{i,j+\frac{1}{2}}^k - \delta_y u_{i,j-\frac{1}{2}}^k), & 0 \leqslant i \leqslant M_1, 1 \leqslant j \leqslant M_2 - 1, \\ -\frac{2}{h_2}\delta_y u_{i,M_2-\frac{1}{2}}^k, & 0 \leqslant i \leqslant M_1,\ j = M_2. \end{cases}$$

Let $v = \phi(u) - \triangle u$. Then the problem (5.1)–(5.3) is equivalent to the following problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}, \quad (x, y) \in \Omega, \quad 0 < t \leqslant T, \tag{5.4}$$
$$v = \phi(u) - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}, \quad (x, y) \in \Omega, \quad 0 < t \leqslant T, \tag{5.5}$$
$$\frac{\partial u}{\partial x}(0, y, t) = \frac{\partial u}{\partial x}(L_1, y, t) = \frac{\partial v}{\partial x}(0, y, t) = \frac{\partial v}{\partial x}(L_1, y, t) = 0, \quad 0 \leqslant y \leqslant L_2,\ 0 \leqslant t \leqslant T, \tag{5.6}$$
$$\frac{\partial u}{\partial y}(x, 0, t) = \frac{\partial u}{\partial y}(x, L_2, t) = \frac{\partial v}{\partial y}(x, 0, t) = \frac{\partial v}{\partial y}(x, L_2, t) = 0, \quad 0 \leqslant x \leqslant L_1,\ 0 \leqslant t \leqslant T, \tag{5.7}$$
$$u(x, y, 0) = u_0(x, y), \quad (x, y) \in \bar{\Omega}. \tag{5.8}$$

It follows from (5.4)–(5.7) that

$$\frac{\partial^3 u}{\partial x^3}(0, y, t) = \frac{\partial^3 u}{\partial x^3}(L_1, y, t) = \frac{\partial^3 v}{\partial x^3}(0, y, t) = \frac{\partial^3 v}{\partial x^3}(L_1, y, t) = 0, \quad 0 \leqslant y \leqslant L_2,\ 0 \leqslant t \leqslant T, \tag{5.9}$$

$$\frac{\partial^5 u}{\partial x^5}(0,y,t) = \frac{\partial^5 u}{\partial x^5}(L_1,y,t) = \frac{\partial^5 v}{\partial x^5}(0,y,t) = \frac{\partial^5 v}{\partial x^5}(L_1,y,t) = 0, \quad 0 \leqslant y \leqslant L_2,\ 0 \leqslant t \leqslant T, \tag{5.10}$$

$$\frac{\partial^3 u}{\partial y^3}(x,0,t) = \frac{\partial^3 u}{\partial y^3}(x,L_2,t) = \frac{\partial^3 v}{\partial y^3}(x,0,t) = \frac{\partial^3 v}{\partial y^3}(x,L_2,t) = 0, \quad 0 \leqslant x \leqslant L_1,\ 0 \leqslant t \leqslant T, \tag{5.11}$$

$$\frac{\partial^5 u}{\partial y^5}(x,0,t) = \frac{\partial^5 u}{\partial y^5}(x,L_2,t) = \frac{\partial^5 v}{\partial y^5}(x,0,t) = \frac{\partial^5 v}{\partial y^5}(x,L_2,t) = 0, \quad 0 \leqslant x \leqslant L_1,\ 0 \leqslant t \leqslant T. \tag{5.12}$$

Define the grid functions

$$U_{ij}^k = u(x_i,y_j,t_k), \quad V_{ij}^k = v(x_i,y_j,t_k), \quad 0 \leqslant i \leqslant M_1,\ 0 \leqslant j \leqslant M_2,\ 0 \leqslant k \leqslant N.$$

Considering (5.4)–(5.5) at the points $(x_i,y_j,t_{\frac{1}{2}})$ and (x_i,y_j,t_k), then using Lemma 2.2 with (5.6)–(5.7), (5.9)–(5.12), and the Taylor's expansion with the integration remainder term, we can obtain

$$A_1A_2\delta_t U_{ij}^{\frac{1}{2}} = A_2\delta_x^2 V_{ij}^{\frac{1}{2}} + A_1\delta_y^2 V_{ij}^{\frac{1}{2}} + \hat{p}_{ij}^0, \quad 0 \leqslant i \leqslant M_1,\ 0 \leqslant j \leqslant M_2, \tag{5.13}$$

$$A_1A_2V_{ij}^{\frac{1}{2}} = A_1A_2\phi\left(u(x_i,y_j,0) + \frac{\tau}{2}u_t(x_i,y_j,0)\right) - \alpha(A_2\delta_x^2 U_{ij}^{\frac{1}{2}} + A_1\delta_y^2 U_{ij}^{\frac{1}{2}}) + \hat{q}_{ij}^0,$$
$$0 \leqslant i \leqslant M_1,\ 0 \leqslant j \leqslant M_2, \tag{5.14}$$

$$A_1A_2\triangle_t U_{ij}^k = A_2\delta_x^2 V_{ij}^k + A_1\delta_y^2 V_{ij}^k + \hat{p}_{ij}^k, \quad 0 \leqslant i \leqslant M_1,\ 0 \leqslant j \leqslant M_2,\ 1 \leqslant k \leqslant N-1, \tag{5.15}$$

$$A_1A_2V_{ij}^k = A_1A_2\phi(U_{ij}^k) - \alpha(A_2\delta_x^2 U_{ij}^{\bar{k}} + A_1\delta_y^2 U_{ij}^{\bar{k}}) + \hat{q}_{ij}^k,$$
$$0 \leqslant i \leqslant M_1,\ 0 \leqslant j \leqslant M_2, \quad 1 \leqslant k \leqslant N-1, \tag{5.16}$$

where there exists a constant $\hat{c}_1$ independent of h_1, h_2 and τ such that

$$|\hat{p}_{ij}^k| \leqslant \hat{c}_1(\tau^2 + h_1^4 + h_2^4), \quad 0 \leqslant i \leqslant M_1,\ 0 \leqslant j \leqslant M_2,\ 0 \leqslant k \leqslant N-1,$$
$$|\hat{q}_{ij}^k| \leqslant \hat{c}_1(\tau^2 + h_1^4 + h_2^4), \quad 0 \leqslant i \leqslant M_1,\ 0 \leqslant j \leqslant M_2,\ 0 \leqslant k \leqslant N-1,$$
$$|\triangle_t \hat{q}_{ij}^k| \leqslant \hat{c}_1(\tau^2 + h_1^4 + h_2^4), \quad 0 \leqslant i \leqslant M_1,\ 0 \leqslant j \leqslant M_2,\ 2 \leqslant k \leqslant N-2.$$

Omitting the small terms in (5.13)–(5.16) and noting the initial condition

$$U_{ij}^0 = u_0(x_i,y_j,0), \quad 0 \leqslant i \leqslant M_1,\ 0 \leqslant j \leqslant M_2,$$

we construct for (5.4)–(5.8) the following compact difference scheme,

$$A_1A_2\delta_t u_{ij}^{\frac{1}{2}} = A_2\delta_x^2 v_{ij}^{\frac{1}{2}} + A_1\delta_y^2 v_{ij}^{\frac{1}{2}}, \quad 0 \leqslant i \leqslant M_1,\ 0 \leqslant j \leqslant M_2, \tag{5.17}$$

$$A_1A_2v_{ij}^{\frac{1}{2}} = A_1A_2\phi(u(x_i,y_j,0) + \frac{\tau}{2}u_t(x_i,y_j,0)) - \alpha(A_2\delta_x^2 u_{ij}^{\frac{1}{2}} + A_1\delta_y^2 u_{ij}^{\frac{1}{2}}),$$
$$0 \leqslant i \leqslant M_1,\ 0 \leqslant j \leqslant M_2, \tag{5.18}$$

$$A_1A_2\triangle_t u_{ij}^k = A_2\delta_x^2 v_{ij}^k + A_1\delta_y^2 v_{ij}^k,\ 0 \leqslant i \leqslant M_1,\ 0 \leqslant j \leqslant M_2,\ 1 \leqslant k \leqslant N-1, \tag{5.19}$$

$$A_1A_2v_{ij}^k = A_1A_2\phi(u_{ij}^k) - \alpha(A_2\delta_x^2 u_{ij}^{\bar{k}} + A_1\delta_y^2 u_{ij}^{\bar{k}}),$$
$$0 \leqslant i \leqslant M_1,\ 0 \leqslant j \leqslant M_2,\ 1 \leqslant k \leqslant N-1, \tag{5.20}$$

$$u_{ij}^0 = u_0(x_i,y_j,0), \quad 0 \leqslant i \leqslant M_1,\ 0 \leqslant j \leqslant M_2. \tag{5.21}$$

Acting A_1A_2 on both sides of (5.17) and inserting (5.18) into the result, we obtain

$$(A_1A_2)^2\delta_t u_{ij}^{\frac{1}{2}} = (A_2\delta_x^2 + A_1\delta_y^2)\Bigg[A_1A_2\phi\left(u(x_i,y_j,0) + \frac{\tau}{2}u_t(x_i,y_j,0)\right)$$
$$-\alpha(A_2\delta_x^2 u_{ij}^{\frac{1}{2}} + A_1\delta_y^2 u_{ij}^{\frac{1}{2}})\Bigg], \quad 0 \leqslant i \leqslant M_1,\ 0 \leqslant j \leqslant M_2. \tag{5.22}$$

Acting A_1A_2 on both sides of (5.19) and inserting (5.20) into the result, we get

$$(A_1A_2)^2\triangle_t u_{ij}^k = (A_2\delta_x^2 + A_1\delta_y^2)[A_1A_2\phi(u_{ij}^k) - \alpha(A_2\delta_x^2 u_{ij}^{\bar{k}} + A_1\delta_y^2 u_{ij}^{\bar{k}})],$$

$$0 \leqslant i \leqslant M_1,\ 0 \leqslant j \leqslant M_2,\ 1 \leqslant k \leqslant N-1. \tag{5.23}$$

We construct the difference scheme (5.21)–(5.23) for the problem (5.1)–(5.3).

Let

$$\omega_{ij} = \begin{cases} 1, & 1 \leqslant i \leqslant M_1-1, 1 \leqslant j \leqslant M_2-1, \\ \frac{1}{4}, & (i,j)=(0,0),(M_1,0),(0,M_2),(M_1,M_2), \\ \frac{1}{2}, & \text{otherwise}, \end{cases}$$

$$(u,v) = h_1 h_2 \sum_{i=0}^{M_1}\sum_{j=0}^{M_2} \omega_{ij} u_{ij} v_{ij}, \quad \|u\| = \sqrt{(u,u)},$$

$$|u|_2 = \sqrt{\|\delta_x^2 u + \delta_y^2 u\|^2}, \qquad \|u\|_\infty = \max_{0\leqslant i\leqslant M_1, 0\leqslant j\leqslant M_2} |u_{ij}|.$$

The solution to (5.1)–(5.3) satisfies the energy decreasing

$$\frac{d}{dt}\left\{ \int_0^{L_1}\int_0^{L_2} \psi(u(x,y,t))dxdy + \frac{1}{2}\alpha \int_0^{L_1}\int_0^{L_2} \left[\left(\frac{\partial u(x,y,t)}{\partial x}\right)^2 + \left(\frac{\partial u(x,y,t)}{\partial y}\right)^2\right] dxdy \right\} \leqslant 0, \quad 0 \leqslant t \leqslant T,$$

or

$$\frac{d}{dt}\left\{ (\psi(u(\cdot,\cdot,t)),1) - \frac{\alpha}{2}\left[\left(\frac{\partial^2 u(\cdot,\cdot,t)}{\partial x^2}, u(\cdot,\cdot,t)\right) + \left(\frac{\partial^2 u(\cdot,\cdot,t)}{\partial y^2}, u(\cdot,\cdot,t)\right)\right]\right\} \leqslant 0, \quad 0 \leqslant t \leqslant T. \tag{5.24}$$

Since $\psi'(u) = \phi(u)$, we have

$$\begin{aligned} \psi(U_{ij}^1) &= \psi(U_{ij}^0) + \int_{U_{ij}^0}^{U_{ij}^1} \psi'(s)ds = \psi(U_{ij}^0) + \int_{U_{ij}^0}^{U_{ij}^1} \phi(s)ds \\ &= \psi(U_{ij}^0) + \phi\left(u(x_i,y_j,0) + \frac{\tau}{2}u_t(x_i,y_j,0)\right)(U_{ij}^1 - U_{ij}^0) + O(\tau^3), \quad 0 \leqslant i \leqslant M_1, \quad 0 \leqslant j \leqslant M_2, \\ \psi(U_{ij}^{k+1}) &= \psi(U_{ij}^{k-1}) + \int_{U_{ij}^{k-1}}^{U_{ij}^{k+1}} \psi'(s)ds = \psi(U_{ij}^{k-1}) + \int_{U_{ij}^{k-1}}^{U_{ij}^{k+1}} \phi(s)ds \\ &= \psi(U_{ij}^{k-1}) + \phi(U_{ij}^k)(U_{ij}^{k+1} - U_{ij}^{k-1}) + O(\tau^3), \quad 0 \leqslant i \leqslant M_1, \quad 0 \leqslant j \leqslant M_2, \quad 1 \leqslant k \leqslant N-1. \end{aligned}$$

Define

$$\begin{aligned} &\psi_{ij}^0 = \psi(u_{ij}^0), \quad 0 \leqslant i \leqslant M_1, \quad 0 \leqslant j \leqslant M_2, \\ &\psi_{ij}^1 = \psi_{ij}^0 + A_1A_2\phi\left(u(x_i,y_j,0) + \frac{\tau}{2}u_t(x_i,y_j,0)\right) \cdot (A_1A_2)^2(u_{ij}^1 - u_{ij}^0), \quad 0 \leqslant i \leqslant M_1, \quad 0 \leqslant j \leqslant M_2, \\ &\psi_{ij}^{k+1} = \psi_{ij}^{k-1} + A_1A_2\phi(u_{ij}^k) \cdot (A_1A_2)^2(u_{ij}^{k+1} - u_{ij}^{k-1}), \quad 0 \leqslant i \leqslant M_1, \quad 0 \leqslant j \leqslant M_2, \quad 1 \leqslant k \leqslant N-1 \end{aligned}$$

and

$$\tilde{G}^k = (\psi^k, 1) - \frac{\alpha}{2}(A_2\delta_x^2 u^k + A_1\delta_y^2 u^k, (A_1A_2)^2 u^k), \quad 0 \leqslant k \leqslant N. \tag{5.25}$$

Let

$$W^0 = \tilde{G}^0, \quad W^k = \frac{1}{2}(\tilde{G}^k + \tilde{G}^{k-1}), \quad 1 \leqslant k \leqslant N.$$

We can prove that the solution to difference scheme (5.21)–(5.23) satisfies the energy decreasing

$$W^k \leqslant W^{k-1}, \quad 1 \leqslant k \leqslant N. \tag{5.26}$$

In order to prove the convergence of the difference scheme (5.21)–(5.23), we need the following lemma.

Lemma 5.1 (See [49]). *For any grid function v on $\hat{\Omega}_h = \{(x_i, y_j) \mid 0 \leqslant i \leqslant M_1, 0 \leqslant j \leqslant M_2\}$, there is a positive constant κ_2 such that*

$$\|v\|_\infty \leqslant \kappa_2 \|v\|^{\frac{1}{2}} (|v|_2 + \|v\|)^{\frac{1}{2}}.$$

Using Lemma 5.1 and similarly to the analysis in Section 4, we can prove

Theorem 5.1. *Assume that the solution $u(x, y, t)$ to* (5.1)–(5.3) *is sufficiently smooth. Then the solution to difference scheme* (5.21)–(5.23) *unconditionally converges to the solution to* (5.1)–(5.3) *in the maximum norm and the rate of convergence is the order of $O(\tau^2 + h_1^4 + h_2^4)$ when h_1, h_2 and τ are small.*

Remark 5.1. The method in this section can apply to the three dimensional Cahn-Hilliard equation by using the following three dimension space norm relation [49]

$$\|v\|_\infty \leqslant \kappa_3 \|v\|^{\frac{1}{4}} (|v|_2 + \|v\|)^{\frac{3}{4}},$$

where v is any grid function v on $\check{\Omega}_h \equiv \{(x_i, y_j, z_l) \mid 0 \leqslant i \leqslant M_1, 0 \leqslant j \leqslant M_2, 0 \leqslant l \leqslant M_3\}$.

Remark 5.2. If we let A_1 and A_2 be the identity operator I in the difference scheme (5.21)–(5.23), we will get a new difference scheme for (5.1)–(5.3) as follows:

$$\delta_t u_{ij}^{\frac{1}{2}} = (\delta_x^2 + \delta_y^2)\left[\phi\left(u(x_i, y_j, 0) + \frac{\tau}{2} u_t(x_i, y_j, 0)\right) - \alpha(\delta_x^2 + \delta_y^2) u_{ij}^{\frac{1}{2}}\right],$$
$$0 \leqslant i \leqslant M_1,\ 0 \leqslant j \leqslant M_2, \tag{5.27}$$

$$\triangle_t u_{ij}^k = (\delta_x^2 + \delta_y^2)[\phi(u_{ij}^k) - \alpha(\delta_x^2 + \delta_y^2) u_{ij}^{\bar{k}}],$$
$$0 \leqslant i \leqslant M_1,\ 0 \leqslant j \leqslant M_2,\ 1 \leqslant k \leqslant N - 1, \tag{5.28}$$

$$u_{ij}^0 = u_0(x_i, y_j, 0), \quad 0 \leqslant i \leqslant M_1,\ 0 \leqslant j \leqslant M_2. \tag{5.29}$$

Similarly to the proof of Theorem 5.1, we can show that the solution to the difference scheme (5.27)–(5.29) is unconditionally convergent to the solution to (5.1)–(5.3) with the convergence order of $O(\tau^2 + h_1^2 + h_2^2)$ in the maximum norm and the solution meets the mass conservation and the energy non-increasing. The differece scheme (5.28) is just as that presented in [39], where only L_2 convergence was proved.

6 Numerical experiments

Example 1 (1D problem). Let $L = 1, T = 0.5, \alpha = 0.5, \beta = \sqrt{2}, \gamma = 1$ and $u_0(x) = 100x^2(1-x)^2(x - 0.5)$ in (1.1)–(1.3) as in [28].

We compute the numerical solutions to this problem by the difference scheme (2.33)–(2.35). Take $h = L/M, \tau = T/N$. Denote the difference solution by $\{u_i^k(h, \tau) \mid 0 \leqslant i \leqslant M, 0 \leqslant k \leqslant N\}$. Suppose

$$\max_{0 \leqslant k \leqslant N} \max_{0 \leqslant i \leqslant M} |u(x_i, t_k) - u_i^k(h, \tau)| = O(\tau^p) + O(h^q).$$

If $O(\tau^p)$ is sufficiently small, we have

$$\max_{0 \leqslant i \leqslant M} \left| u_i^N(h, \tau) - u_{2i}^N\left(\frac{h}{2}, \tau\right) \right| = O(h^q).$$

Denote

$$E(h) = \max_{0 \leqslant i \leqslant M} \left| u_i^N(h, \tau) - u_{2i}^N\left(\frac{h}{2}, \tau\right) \right|.$$

Then

$$\log_2 \frac{E(h)}{E(\frac{h}{2})} \approx q.$$

Some numerical results are presented in Table 1. If $O(h^q)$ is sufficiently small, we have

$$\max_{0 \leqslant i \leqslant M} \left| u_i^N(h, \tau) - u_i^{2N}\left(h, \frac{\tau}{2}\right) \right| = O(\tau^p).$$

Denote

$$F(\tau) = \max_{0 \leqslant i \leqslant M} \left| u_i^N(h, \tau) - u_i^{2N}\left(h, \frac{\tau}{2}\right) \right|.$$

Then

$$\log_2 \frac{F(\tau)}{F(\frac{\tau}{2})} \approx p.$$

Some numerical results are presented in Table 1. From Tables 1 and 2, we see that $p = 2, q = 4$ and the difference scheme (2.33)–(2.35) is convergent with the convergence order of $O(\tau^2 + h^4)$. This is accordance with Theorem 4.1. Figure 1 shows the decrease of the total energy G^k defined by (3.3)–(3.4). This is accordance with Theorem 3.1.

Table 1 Some numerical results of (2.33)–(2.35) when $\tau = T/36000$

h	$E(h)$	$\log_2 \frac{E(h)}{E(h/2)}$
$\frac{1}{5}$	4.945e−3	4.4532
$\frac{1}{10}$	2.258e−4	3.9634
$\frac{1}{20}$	1.447e−5	4.0119
$\frac{1}{40}$	8.971e−7	*
$\frac{1}{80}$	*	*

Table 2 Some numerical results of (2.33)–(2.35) when $h = L/2000$

τ	$F(\tau)$	$\log_2 \frac{F(\tau)}{F(\tau/2)}$
$\frac{T}{200}$	1.012e−2	1.9628
$\frac{T}{400}$	2.596e−3	1.5413
$\frac{T}{800}$	8.920e−4	2.0093
$\frac{T}{1600}$	2.216e−4	*
$\frac{T}{3200}$	*	*

Table 3 Some numerical results of (6.1)–(6.2) when $\tau = T/1000$

h	$E(h)$	$\log_2 \frac{E(h)}{E(h/2)}$
$\frac{1}{100}$	3.961e−4	2.1235
$\frac{1}{200}$	9.089e−5	1.8151
$\frac{1}{400}$	2.583e−5	1.8884
$\frac{1}{800}$	6.976e−6	*
$\frac{1}{1600}$	*	*

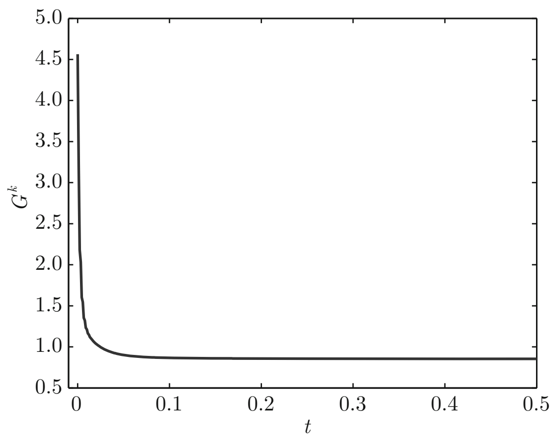

Figure 1 The discrete total energy of the numerical solution obtained by the scheme (2.33)–(2.35) with $\alpha = 0.5, T = 0.5, h = 1/30$ and $\tau = 1/900$. The initial state is $u_0(x) = 100x^2(1-x)^2(x-0.5)$.

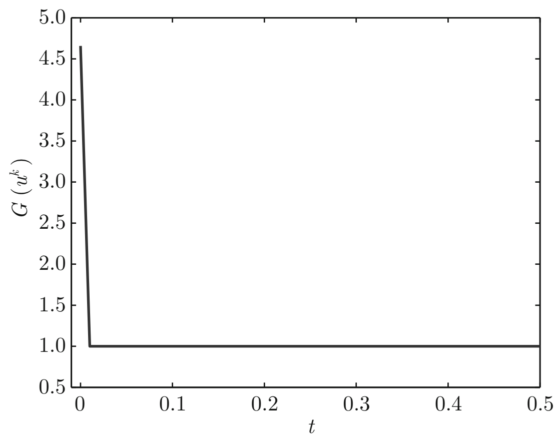

Figure 2 The discrete total energy (6.3) of the numerical solution obtained by the scheme (6.1)–(6.2) with $T = 0.5, \alpha = 0.5, h = 1/100$ and $\tau = 1/100$. The initial state is $u_0(x) = 100x^2(1-x)^2(x-0.5)$.

Table 4 Some numerical results of (5.21)–(5.23) when $\tau = T/700000$

h	$\widetilde{E}(h)$	$\log_2 \frac{\widetilde{E}(h)}{\widetilde{E}(h/2)}$
$\frac{1}{5}$	1.184e−4	4.0976
$\frac{1}{10}$	6.915e−6	4.0340
$\frac{1}{20}$	4.221e−7	*
$\frac{1}{40}$	*	*

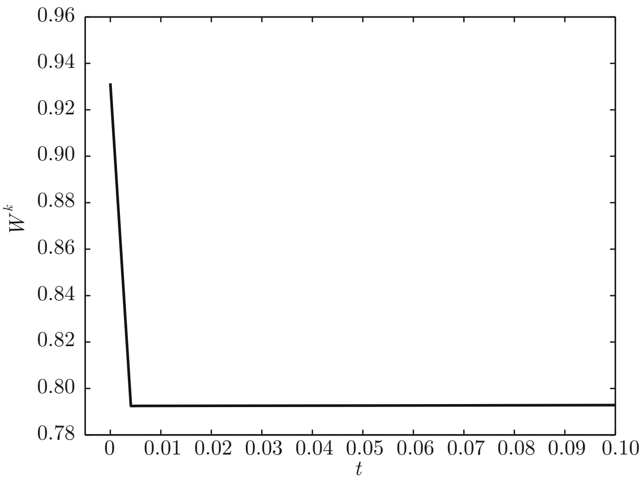

Figure 3 The discrete total energy (5.26) of the numerical solution obtained by the scheme (5.27)–(5.29) with $T = 0.1, \alpha = 0.1, \beta = \sqrt{2}, \gamma = 1, h = 1/20$ and $\tau = T/50000$. The initial state is $u_0(x, y) = \cos(\pi x)\cos(\pi y)$.

We compare our difference scheme with Furihata's scheme. Let

$$\Phi(u,v) = \gamma\left[\frac{1}{4}(u^3 + u^2 v + uv^2 + v^3) - \frac{\beta^2}{2}(u+v)\right].$$

Then we have

$$\Phi(u,v)(u-v) = \psi(u) - \psi(v), \quad \Phi(u,u) = \phi(u).$$

Furihata's scheme for (2.1)–(2.4) in [28] is as follows,

$$\delta_t u_i^{k+\frac{1}{2}} = \delta_x^2[\Phi(u_i^{k+1}, u_i^k) - \alpha\delta_x^2 u_i^{k+\frac{1}{2}}], \quad 0 \leqslant i \leqslant M, 0 \leqslant k \leqslant N-1, \tag{6.1}$$

$$u_i^0 = u_0(x_i), \quad 0 \leqslant i \leqslant M. \tag{6.2}$$

Some numerical results are presented in Table 3. We see that the difference scheme (6.1)–(6.2) is convergent at most with second order in space. For the proof of the convergence, Furihata needed a condition $\tau = O(h^2)$. Figure 2 shows the decrease of the total energy $G(u^k)$ defined by

$$G(u^k) = h\left[\frac{1}{2}\psi(u_0^k) + \sum_{i=1}^{M-1}\psi(u_i^k) + \frac{1}{2}\psi(u_M^k)\right] + \frac{\alpha}{2}h\sum_{i=0}^{M-1}(\delta_x u_{i+\frac{1}{2}}^k)^2, \quad 0 \leqslant k \leqslant N. \tag{6.3}$$

Since (6.1)–(6.2) is a system of nonlinear equations at each time level, it needs much more CPU time compared with our difference scheme (2.33)–(2.35).

Example 2 (2D problem). Let $L_1 = 1, L_2 = 1, T = 0.1, \alpha = 0.1, \beta = \sqrt{2}, \gamma = 1, u_0(x,y) = \cos(\pi x)\cos(\pi y)$ in (5.1)–(5.3). Take $h_1 = h_2 = h$. We compute the numerical solution to the problem by the difference scheme (5.21)–(5.23). Denote the difference solution by $\{u_{ij}^k(h,\tau) \,|\, 0 \leqslant i \leqslant M_1, 0 \leqslant j \leqslant M_2, 0 \leqslant k \leqslant N\}$. For sufficiently small fixed τ, denote

$$\widetilde{E}(h) = \max_{0\leqslant i\leqslant M_1, 0\leqslant j\leqslant M_2}\left|u_{ij}^N(h,\tau) - u_{2i,2j}^N\left(\frac{h}{2},\tau\right)\right|.$$

Some numerical results are presented in Table 4. We can see that the difference scheme (5.21)–(5.23) is convergent with the order 4 in spatial step sizes h. The energy W^k defined in (5.26) is plotted in Figure 3. We see that W^k is not increasing.

7 Conclusion

In this article, we constructed in detail a three level linearized compact difference scheme for the one-dimensional Cahn-Hilliard equation. The mass conservation and the non-decrease of the total energy of the difference solution were presented. The unique solvability and convergence of the difference scheme in H^2-norm and maximum-norm were showed by the discrete energy method. The outline for the high-dimensional problem was also given. The numerical results verified the theoretical results. The comparison was made with the existing difference scheme. The method in this article can apply to the periodic boundary value problem.

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References

1 Alikakos N D, Bates P W, Chen X. Convergence of the Cahn-Hilliard equation to the Hele-shaw model. Arch Ration Mech Anal, 1994, 128: 165–205
2 Barrett J, Blowey J. An error bound for the finite element approximation of the Cahn-Hilliard equation with logarithmic free energy. Numer Math, 1995, 72: 1–20
3 Barrett J, Blowey J. Finite element approximation of the Cahn-Hilliard equation with concentration dependent mobility. Math Comput, 1999, 68: 487–517
4 Bartels S. A posteriori error analysis for time-dependent Ginzburg-Landau type equations. Numer Math, 2005, 99: 557–583
5 Bartels S, Müller R. A posteriori error controlled local resolution of evolving interfaces for generalized Cahn-Hilliard equations. Interfaces Free Bound, 2010, 12: 45–73
6 Bañas L, Nürnberg R. Adaptive finite element methods for Cahn-Hilliard equations. J Comput Appl Math, 2008, 218: 2–11
7 Baňnas L, Nürnberg R. A posteriori estimates for the Cahn-Hilliard equation with obstacle free energy. Math Model Numer Anal, 2009, 43: 1003–1026
8 Blowey J, Elliott C. The Cahn-Hilliard gradient theory for phase separation with non-smooth free energy. Part I: Mathematical analysis. European J Appl Math, 1991, 2: 233–279
9 Burger M, Chu S Y, Markowich P A, et al. The Willmore functional and instabilities in the Cahn-Hilliard equation. Commun Math Sci, 2008, 6: 309–329
10 Cahn J W, Hilliard J E. Free energy of nonuniform system, I. Interfacial free energy. J Chem Phys, 1958, 28: 258–267
11 Chai S M, Zou Y K, Gong C C. Spectral method for a class of Cahn-Hilliard equation with nonconstant mobility. Commun Math Res, 2009, 25: 9–18
12 Chen X. Global asymptotic limit of solutions of the Cahn-Hilliard equation. J Differential Geom, 1996, 44: 262–311
13 Choo S M, Chung S K. Conservative nonlinear difference scheme for the Cahn-Hilliard equation. Comput Math Appl, 1998, 36: 31–39
14 Choo S M, Chung S K, Kim K I. Conservative nonlinear difference scheme for the Cahn-Hilliard equation, II. Comput Math Appl, 2000, 39: 229–243
15 Copetti M, Elliott C. Numerical analysis of the Cahn-Hilliard equation with a logarithmic free energy. Numer Math, 1992, 63: 39–65
16 Du Q, Nicolaides R A. Numerical analysis of a continuum model of phase transtion. SIAM J Numer Anal, 1991, 28: 1310–1322
17 Elliott C. The Cahn-Hilliard model for the kinetics of phase separation. In: Rodrigues J, ed. Mathematical Models for Phase Change Problems. International Series of Numerical Mathematics, vol. 88. Basel: Birkhäser, 1989, 35–73
18 Elliott C, French D. Numerical studies of the Cahn-Hilliard equation for phase separation. IMA J Appl Math, 1987, 38: 97–128
19 Elliott C, Garcke H. On the Cahn-Hilliard equation with degenerate mobility. SIAM J Math Anal, 1996, 27: 404–423

20 Elliott C M, Luckhaus S. Ageneralized equation for phase separation of a multi-component mixture with interfacial free energy. IMA Preprint Series, 887. Institute for Mathematics and its Applications, University of Minnesota, Minneapolis, Minnesota, 1991

21 Elliott C M, French D. Numerical studies of the Cahn-Hilliard equation for phase separation. IMA J Appl Math, 1987, 38: 97–128

22 Elliott C, Zheng S. On the Cahn-Hilliard equation. Arch Ration Mech Anal, 1986, 96: 339–357

23 Feng W M, Yu P, Hu S Y, et al. A Fourier spectral moving mesh method for the Cahn-Hilliard equation with elasticity. Commun Comput Phys, 2009, 5: 582–599

24 Feng X, Karakashian O A. Fully discrete dynamic mesh discontinuous Galerkin methods for the Cahn-Hilliard equation of phase transition. Math Comput, 2007, 76: 1093–1117

25 Feng X, Prohl A. Error analysis of a mixed finite element method for the Cahn-Hilliard equation. Numer Math, 2004, 99: 47–84

26 Feng X, Wu H J. A posteriori error estimates and an adaptive finite element method for the Allen-Cahn equation and the mean curvature flow. J Sci Comput, 2005, 24: 121–146

27 Feng X, Wu H J. A posteriori error estimates for finite element approximations of the Cahn-Hilliard equation and the Hele-Shaw flow. J Comput Math, 2008, 26: 767–796

28 Furihata D. A stable and conservation finite difference scheme for the Cahn-Hilliard equation. Numer Math, 2001, 87: 675–699

29 He C Y, Zhang F Y. The long-time behavior of Four pseudo-spectral method of the nonlinear Cahn-Hilliard equation. J Natur Sci Heilongjiang Univ, 2006, 23: 779–786

30 He L P. Error estimation of a class of stable spectral approximation to the Cahn-Hilliard equation. J Sci Comput, 2009, 41: 461–482

31 He L P, Liu Y X. A class of stable spectral methods for the Cahn-Hilliard equation. J Comput Phys, 2009, 228: 5101–5110

32 He Y N, Liu Y X. Stability and convergence of the spectral Galerkin method for the Cahn-Hilliard equation. Numer Methods Partial Differ Equ, 2008, 24: 1485–1500

33 Kay D, Styles V, Süli E. Discontinuous Galerkin finite element approximation of the Cahn-Hilliard equation with convection. SIAM J Numer Anal, 2009, 47: 2660–2685

34 Kessler D, Nochetto R H, Schmidt A. A posteriori error control for the Allen-Cahn problem: Circumventing Gronwalls inequality. Math Model Numer Anal, 2004, 38: 129–142

35 Khiari N, Achouri T, Ben Mohamed M L, et al. Finite difference approximate solutions for the Cahn-Hilliard equation. Numer Methods Partial Differ Equ, 2007, 23: 437–455

36 Nicolaenko B, Scheurer B, Temam R. Some global dynamical properties of a class of pattern formation equations. Comm Partial Differ Equ, 1989, 14: 245–297

37 Novick-Cohen A. Energy methods for the Cahn-Hilliard equation. IMA Preprint, 1985, 157

38 Pego R L. Front migration in the nonlinear Cahn-Hilliard equation. Proc R Soc London Ser A Math Phys Eng Sci, 1989, 422: 261–278

39 Sun Z Z. A second-order accurate linearized difference scheme for the two-dimensional Cahn-Hilliard equation. Math Comput, 1995, 64: 1463–1471

40 Sun Z Z. On the compact difference scheme for heat equation with Neumann boundary conditions. Numer Methods Partial Differ Equ, 2009, 25: 1320–1341

41 Wells G N, Kuhl E, Garikipati K. A discontinuous Galerkin method for the Cahn-Hilliard equation. J Comput Phys, 2006, 218: 860–877

42 Yang X F. Numerical approximations of Allen-Cahn and Cahn-Hilliard equations. Discrete Contin Dyn Syst, 2010, 28: 1669–1691

43 Ye X. The Fourier collocation method for the Cahn-Hilliard equation. Comput Math Appl, 2002, 44: 213–229

44 Ye X, Cheng X. The Fourier spectral method for the Cahn-Hilliard equation. Appl Math Comput, 2005, 171: 345–357

45 Ye X D, Cheng X L. The Fourier spectral method for the Cahn-Hilliard equation. Appl Math Comput, 2005, 171: 345–357

46 Yin J. On the existence of nonnegative continuous solution of the Cahn-Hilliard equation. J Differ Equ, 1992, 97: 310–327

47 Zhang T. Finite element analysis for the Cahn-Hilliard equation (in Chinese). Math Numer Sin, 2006, 28: 281–292

48 Zhao R. High order accurate numerical methods for Cahn-Hilliard equations. Master's Degree Thesis. Qingdao: Chinese Ocean University, 2009

49 Zhou Y L. Applications of Discrete Functional Analysis of Finite Difference Method. New York: International Academic Publishers, 1990