SCIENCE CHINA
Mathematics

• ARTICLES •

June 2011 Vol. 54 No. 6: 1215–1224
doi: 10.1007/s11425-011-4208-7

# Weighted integral inequalities in Orlicz martingale classes

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Received October 14, 2009; accepted December 3, 2010

**Abstract** Under appropriate conditions on Young's functions $\Phi_1$ and $\Phi_2$, we give necessary and sufficient conditions in order that weighted integral inequalities hold for Doob's maximal operator $M$ on martingale Orlicz setting. When $\Phi_1 = t^p$ and $\Phi_2 = t^q$, the inequalities revert to the ones of strong or weak $(p,q)$-type on martingale space.

**Keywords** martingale Orlicz setting, Doob's maximal operator, weighted integral inequality

**MSC(2000):** 60G46, 60G42

**Citation:** Chen W, Liu P D. Weighted integral inequalities in Orlicz martingale classes. Sci China Math, 2011, 54(6): 1215–1224, doi: 10.1007/s11425-011-4208-7

## 1 Introduction

Let $R^n$ be the $n$-dimensional real Euclidean space and $f$ a real valued measurable function, the classical Hardy-Littlewood maximal operator $M$ is defined by

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| dy,$$

where $Q$ is a non-degenerate cube with its sides paralleled to the coordinate axes and $|Q|$ is the Lebesgue measure of $Q$.

Let $u,\ v$ be two weights, i.e., positive measurable functions. As is well known, if $u = v$ and $p > 1$, [9] showed that the inequality

$$\int_{R^n} (Mf(x))^p v(x) dx \leqslant C \int_{R^n} |f(x)|^p v(x) dx, \quad \forall f \in L^p(v)$$

holds if and only if $\omega \in A_p$, i.e., for any cube $Q$ in $R^n$ with sides parallel to the coordinates

$$\left( \frac{1}{|Q|} \int_Q v(x) dx \right) \left( \frac{1}{|Q|} \int_Q v(x)^{-\frac{1}{p-1}} dx \right)^{p-1} < C. \tag{1.1}$$

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Recall that a couple $(u,v)$ of nonnegative measurable functions is said to be in $A_p$, if

$$\sup_Q \left(\frac{1}{|Q|}\int_Q u(x)dx\right)\left(\frac{1}{|Q|}\int_Q v(x)^{-\frac{1}{p-1}}dx\right)^{p-1} < \infty.$$

But $(u,v) \in A_p$ is not in general sufficient for the boundedness of $M$ from $L^p(v)$ to $L^p(u)$ (see [13] or [2, p. 395]). However, the correct necessary and sufficient condition has been established (see [13] or [3]). In fact, for $1 < p \leqslant q < \infty$, [13] established a necessary and sufficient condition in order that the weighted inequality

$$\left(\int_{R^n} (Mf(x))^q u(x)dx\right)^{\frac{1}{q}} \leqslant C\left(\int_{R^n} |f(x)|^p v(x)dx\right)^{\frac{1}{p}}, \quad f \in L^p(v) \tag{1.2}$$

holds. Without the restriction of $1 < p \leqslant q < \infty$, [12] obtained a characterization for the weak-type inequality

$$\lambda |\{Mf > \lambda\}|_u^{\frac{1}{q}} \leqslant C\left(\int_{R^n} |f(x)|^p v(x)dx\right)^{\frac{1}{p}}, \quad f \in L^p(v).$$

Let $\phi$ be a positive nondecreasing right continuous function on $R^+$ with $\phi(0+) = 0$ and $\lim_{s\to\infty}\phi(s) = \infty$, we define Young's function $\Phi$ by $\Phi(t) = \int_0^t \phi(s)ds$ and denote $q_\Phi = \inf_{t>0} \frac{t\phi(t)}{\Phi(t)}$. A Young's function $\Phi$ is said to satisfy $\triangle_2$ and $\triangle'$ condition, if there is a constant $C$ such that $\Phi(2t) \leqslant C\Phi(t)$, $\forall t > 0$ and $\Phi(st) \leqslant C\Phi(s)\Phi(t)$, $\forall s,t > 0$ respectively. For two Young's functions $\Phi_1$ and $\Phi_2$, we also denote $\Phi_1 \ll \Phi_2$, if there is a constant $C$ such that $\sum \Phi_2 \circ \Phi_1^{-1}(a_i) \leqslant C\Phi_2 \circ \Phi_1^{-1}(\sum a_i)$ holds for every nonnegative sequence $(a_i)_i$. (See [10] for the details of Young's function.)

For $\sigma = v^{-\frac{1}{p-1}}$, substituting $\Phi_1(t) = t^p$, $\Phi_2(t) = t^q$ and $f = g\sigma$ into (1.2), we recast it in the form

$$\left(\int_{R^n} (M(g\sigma)(x))^q u(x)dx\right)^{\frac{1}{q}} \leqslant C\left(\int_{R^n} |g(x)|^p \sigma(x)dx\right)^{\frac{1}{p}}, \quad g \in L^p(\sigma). \tag{1.3}$$

Therefore, for a pair of weights $(u,v)$ and two Young's functions $\Phi_1$ and $\Phi_2$, we have at least two types of weighted inequalities on Olicze spaces, i.e.,

$$\Phi_2^{-1}\left(\int_{R^n} \Phi_2(Mf(x))u(x)dx\right) \leqslant C\Phi_1^{-1}\left(C\int_{R^n} \Phi_1(|f(x)|)v(x)dx\right), \quad f \in L_{\Phi_1}(v), \tag{1.4}$$

and

$$\Phi_2^{-1}\left(\int_{R^n} \Phi_2(M(fv)(x))u(x)dx\right) \leqslant C\Phi_1^{-1}\left(C\int_{R^n} \Phi_1(|f(x)|)v(x)dx\right), \quad f \in L_{\Phi_1}(v). \tag{1.5}$$

Similarly, we also have the following weak types of weighted inequalities:

$$\Phi_2^{-1}\left(\int_{\{Mf>\lambda\}} \Phi_2(\lambda)u(x)dx\right) \leqslant C\Phi_1^{-1}\left(C\int_{R^n} \Phi_1(|f(x)|)v(x)dx\right), \quad f \in L_{\Phi_1}(v) \text{ and } \lambda > 0, \tag{1.6}$$

and

$$\Phi_2^{-1}\left(\int_{\{M(fv)>\lambda\}} \Phi_2(\lambda)u(x)dx\right) \leqslant C\Phi_1^{-1}\left(C\int_{R^n} \Phi_1(|f(x)|)v(x)dx\right), \quad f \in L_{\Phi_1}(v) \text{ and } \lambda > 0. \tag{1.7}$$

When $\Phi_1 = \Phi_2$, [14], [1] and [6] gave characterizations of the pair of weights $(u,v)$ for which (1.4)–(1.6) hold, respectively. Without the restriction of $\Phi_1 = \Phi_2$, [5] characterized the inequalities (1.5) and (1.7) in detail.

Comparing with the above results, [8] and [7] examined the boundedness of Doob's maximal operator $M$ from $L^p(u)$ to $L^q(v)$ on martingale spaces. As for martingale Orlicz setting, if $\Phi_1 = \Phi_2$, [4] gave some necessary and sufficient conditions for the inequality (1.6) to hold. Recently, [11] considered the inequality

$$\Phi_2(\lambda)|\{Mf > \lambda\}|_u \leqslant C\int_\Omega \Phi_1(|f|)vd\mu, \quad f \in L_{\Phi_1}(v) \text{ and } \lambda > 0,$$

which improved the results of [4]. In this paper, we characterize weighted integral inequalities (1.5) and (1.7) for Doob's maximal operator on martingale Orlicz setting. The rest of Section 1 consists of the preliminaries for the following sections.

Let $(\Omega,\mathcal{F},\mu)$ be a complete probability space and $(\mathcal{F}_n)_{n\geqslant 0}$ an increasing sequence of sub-$\sigma$-fields of $\mathcal{F}$ with $\mathcal{F}=\bigvee_{n\geqslant 0}\mathcal{F}_n$. A weight $\omega$ is a random variable with $\omega>0$ and $E(\omega)<\infty$. In this paper, for a Young's function $\Phi$ and a weight $\omega$, a martingale $f=(f_n)_{n\geqslant 0}\in L_\Phi(\omega)$ is meant as $f_n=E(f|\mathcal{F}_n)$ and $\int_\Omega \Phi(|f|)\omega d\mu<\infty$. The Doob's maximal operator $Mf$ is defined by $Mf=\sup_{n\geqslant 0}|f_n|$. Let $\omega$ be a weight and $B\in\mathcal{F}$, we denote $\int_\Omega \chi_B d\mu$ and $\int_\Omega \chi_B\omega d\mu$ by $|B|$ and $|B|_\omega$, respectively. Fix $(\Omega,\mathcal{F},\mu)$ and $(\mathcal{F}_n)_{n\geqslant 0}$, we always denote the family of stopping times by $\mathcal{T}$. Throughout this paper, $C$ will denote a constant not necessarily the same at each occurrence.

## 2 The inequalities involving two Young's functions

**Theorem 2.1.** *Let $(u,v)$ be a couple of weights. Suppose that $\Phi_1\ll\Phi_2$, $\Phi_2\in\triangle_2$ and $q_{\Phi_1}>1$, then the following statements are equivalent*:

(1) *There exists a positive constant $C$ such that*

$$\Phi_2^{-1}\left(\int_{\{\tau<\infty\}}\Phi_2(|E(fv|\mathcal{F}_\tau)|)ud\mu\right)\leqslant C\Phi_1^{-1}\left(C\int_\Omega \Phi_1(|f|)vd\mu\right),\quad \forall f=(f_n)\in L_{\Phi_1}(v) \text{ and } \tau\in\mathcal{T}; \tag{2.1}$$

(2) *There exists a positive constant $C$ such that*

$$\Phi_2^{-1}\left(\int_{\{\tau<\infty\}}\Phi_2(tv_\tau)ud\mu\right)\leqslant C\Phi_1^{-1}\left(C\int_{\{\tau<\infty\}}\Phi_1(t)vd\mu\right),\quad \forall \tau\in\mathcal{T} \text{ and } t>0; \tag{2.2}$$

(3) *There exists a positive constant $C$ such that*

$$\Phi_2^{-1}(\Phi_2(\lambda)|\{M(fv)>\lambda\}|_u)\leqslant C\Phi_1^{-1}\left(C\int_\Omega \Phi_1(|f|)vd\mu\right),\quad \forall f=(f_n)\in L_{\Phi_1}(v) \text{ and } \lambda>0. \tag{2.3}$$

*Proof.* We shall follow the scheme: (2) $\Leftrightarrow$ (1) $\Leftrightarrow$ (3).

(1) $\Rightarrow$ (2) Substituting $f=t\chi_{\{\tau<\infty\}}$ into (2.1), we have (2.2).

(2) $\Rightarrow$ (1) If $\tau\equiv n$ for some $n\in N$, we shall show that (2.1) is valid. Fix $f=(f_n)\in L_{\Phi_1}(v)$. For each $k\in Z$ and $j\in Z$, let

$$B_{k,j}=\{2^k<|E(fv|\mathcal{F}_n)|\leqslant 2^{k+1}\}\cap\{2^j<E(v|\mathcal{F}_n)\leqslant 2^{j+1}\}.$$

Then $B_{k,j}\in\mathcal{F}_n$. Moreover, $\{B_{k,j}\}_{k,j}$ is a family of disjoint sets and

$$\{2^k<|E(fv|\mathcal{F}_n)|\leqslant 2^{k+1}\}=\bigcup_{j\in Z}B_{k,j}.$$

Trivially $E(fv|\mathcal{F}_n)=\hat{E}_v(f|\mathcal{F}_n)E(v|\mathcal{F}_n)$. It follows that

$$2^k\leqslant \operatorname*{essinf}_{B_{k,j}}|E(fv|\mathcal{F}_n)|\leqslant \operatorname*{essinf}_{B_{k,j}}\hat{E}_v(|f||\mathcal{F}_n)\operatorname*{esssup}_{B_{k,j}}E(v|\mathcal{F}_n).$$

Thus

$$\begin{aligned}\int_{\{\tau<\infty\}}\Phi_2(|E(fv|\mathcal{F}_\tau)|)ud\mu&=\int_\Omega \Phi_2(|E(fv|\mathcal{F}_n)|)ud\mu\\&\leqslant C\sum_{k\in Z,j\in Z}\int_{B_{k,j}}\Phi_2(2^k)ud\mu\\&\leqslant C\sum_{k\in Z,j\in Z}\int_{B_{k,j}}\Phi_2\Big(\operatorname*{essinf}_{B_{k,j}}\hat{E}_v(|f||\mathcal{F}_n)\operatorname*{esssup}_{B_{k,j}}E(v|\mathcal{F}_n)\Big)ud\mu.\end{aligned}$$

For $l \in Z$, let

$$E_l = \Big\{(k,j):\ 2^l < \operatorname*{essinf}_{B_{k,j}} \hat{E}_v(|f||\mathcal{F}_n) \leqslant 2^{l+1}\Big\}$$

and $\tau^{(l)} = \inf\{m : \hat{E}_v(|f||\mathcal{F}_m) > 2^l\}$. For $k,j \in Z$, we also define $\tau_{k,j} = n\chi_{B_{k,j}} + \infty\chi_{B^C_{k,j}}$. It is evident that

$$\bigcup_{(k,j)\in E_l} B_{k,j} \subseteq \{\hat{M}_v(|f|) > 2^l\} = \{\tau^{(l)} < \infty\},$$

and $v_n\chi_{B_{k,j}} = v_{\tau_{k,j}}\chi_{B_{k,j}}$. Thus, it follows from (2.2) and $\Phi_1 \ll \Phi_2$ that

$$\begin{aligned}
&\int_{\{\tau<\infty\}} \Phi_2(|E(fv|\mathcal{F}_n)|)ud\mu \\
&\quad\leqslant C\sum_{l\in Z}\sum_{(k,j)\in E_l}\int_{B_{k,j}} \Phi_2\Big(\operatorname*{essinf}_{B_{k,j}} \hat{E}_v(|f||\mathcal{F}_n) \operatorname*{esssup}_{B_{k,j}} E(v|\mathcal{F}_n)\Big)ud\mu \\
&\quad\leqslant C\sum_{l\in Z}\sum_{(k,j)\in E_l}\int_{B_{k,j}} \Phi_2(2^{l-2}E(v|\mathcal{F}_n))ud\mu \\
&\quad\leqslant C\sum_{l\in Z}\sum_{(k,j)\in E_l} \Phi_2\circ\Phi_1^{-1}\Big(C\int_{B_{k,j}} \Phi_1(2^{l-2})vd\mu\Big) \\
&\quad\leqslant C\sum_{l\in Z} \Phi_2\circ\Phi_1^{-1}\Big(C\int_{\{\tau^{(l)}<\infty\}} \Phi_1(2^{l-2})vd\mu\Big) \\
&\quad\leqslant C\sum_{l\in Z} \Phi_2\circ\Phi_1^{-1}\big(C\Phi_1(2^{l-2})|\{\tau^{(l)}<\infty\}|_v\big) \\
&\quad= C\sum_{l\in Z} \Phi_2\circ\Phi_1^{-1}\Big(C\Phi_1(2^{l-2})\Big|\Big\{\hat{M}_v\Big(\frac{|f|}{2}\Big) > 2^{l-1}\Big\}\Big|_v\Big) \\
&\quad\leqslant C\Phi_2\circ\Phi_1^{-1}\Big(C\sum_{l\in Z}\Phi_1(2^{l-2})\Big|\Big\{\hat{M}_v\Big(\frac{|f|}{2}\Big) > 2^{l-1}\Big\}\Big|_v\Big) \\
&\quad\leqslant C\Phi_2\circ\Phi_1^{-1}\Big(C\sum_{l\in Z}(\Phi_1(2^{l-1})-\Phi_1(2^{l-2}))\Big|\Big\{\hat{M}_v\Big(\frac{|f|}{2}\Big) > 2^{l-1}\Big\}\Big|_v\Big) \\
&\quad\leqslant C\Phi_2\circ\Phi_1^{-1}\Big(C\int_\Omega \Phi_1\Big(\hat{M}_v\Big(\frac{|f|}{2}\Big)\Big)vd\mu\Big).
\end{aligned}$$

Note that $q_{\Phi_1} > 1$, we have

$$\int_{\{\tau<\infty\}} \Phi_2(|E(fv|\mathcal{F}_n)|)ud\mu \leqslant C\Phi_2\circ\Phi_1^{-1}\Big(C\int_\Omega \Phi_1(|f|)vd\mu\Big) \leqslant \Phi_2\Big(C\Phi_1^{-1}\Big(C\int_\Omega \Phi_1(|f|)vd\mu\Big)\Big).$$

If $\tau \in \mathcal{T}$ is arbitrary, we shall show that (2.1) is still valid. Fix $\tau \in \mathcal{T}$, let $B_k = \{\tau = k\}$ and $\tau_k \equiv k,\ k \in N$. Obviously

$$\begin{aligned}
\int_{\{\tau<\infty\}} \Phi_2(|E(fv|\mathcal{F}_\tau)|)ud\mu &= \sum_{k\in N}\int_{B_k} \Phi_2(|E(fv|\mathcal{F}_\tau)|)ud\mu \\
&= \sum_{k\in N}\int_\Omega \Phi_2(|E(fv\chi_{B_k}|\mathcal{F}_k)|)ud\mu.
\end{aligned}$$

Using $\Phi_1 \ll \Phi_2$, we obtain that

$$\begin{aligned}
\int_{\{\tau<\infty\}} \Phi_2(|E(fv|\mathcal{F}_\tau)|)ud\mu &\leqslant C\sum_{k\in N} \Phi_2\circ\Phi_1^{-1}\Big(C\int_\Omega \Phi_1(|f\chi_{B_k}|)vd\mu\Big) \\
&\leqslant C\Phi_2\circ\Phi_1^{-1}\Big(C\sum_{k\in N}\int_\Omega \Phi_1(|f\chi_{B_k}|)vd\mu\Big)
\end{aligned}$$

$$\leqslant \Phi_2\bigg(C\Phi_1^{-1}\bigg(C\int_\Omega \Phi_1(|f|)vd\mu\bigg)\bigg).$$

(1) $\Rightarrow$ (3) Fix $f=(f_n)\in L_{\Phi_1}(vd\mu)$ and $\lambda>0$. Let $\tau=\inf\{n:\ |E(fv|\mathcal{F}_n)|>\lambda\}$. It follows from (2.1) that

$$\begin{aligned}\Phi_2^{-1}(\Phi_2(\lambda)|\{M(fv)>\lambda\}|_u) &= \Phi_2^{-1}\bigg(\int_{\{M(fv)>\lambda\}}\Phi_2(\lambda)ud\mu\bigg)\\ &\leqslant \Phi_2^{-1}\bigg(\int_{\{\tau<\infty\}}\Phi_2(|E(fv|\mathcal{F}_\tau)|)ud\mu\bigg)\\ &\leqslant C\Phi_1^{-1}\bigg(C\int_\Omega \Phi_1(|f|)vd\mu\bigg).\end{aligned}$$

(3) $\Rightarrow$ (1) It suffices to prove that (2.1) holds for $\tau\equiv n,$ for every $n\in N$. Fix $n\in N$, for $B\in\mathcal{F}_n$, let $g=f\chi_B$. Trivially, $E(gv|\mathcal{F}_n)=E(fv|\mathcal{F}_n)\chi_B$. Thus

$$\{|E(fv|\mathcal{F}_n)|>\lambda\}\cap B\subset\{M(gv)>\lambda\}.$$

In virtue of (2.3), we have

$$\begin{aligned}\Phi_2(\lambda)\int_{\{|E(fv|\mathcal{F}_n)|>\lambda\}\cap B}ud\mu &\leqslant \Phi_2(\lambda)\int_{\{M(gv)>\lambda\}}ud\mu\\ &\leqslant C\Phi_2\circ\Phi_1^{-1}\bigg(C\int_\Omega \Phi_1(|g|)vd\mu\bigg)=C\Phi_2\circ\Phi_1^{-1}\bigg(C\int_B \Phi_1(|f|)vd\mu\bigg).\end{aligned}$$

Thus

$$\begin{aligned}\int_{\{\tau<\infty\}}\Phi_2(|E(fv|\mathcal{F}_\tau)|)ud\mu &= \int_\Omega \Phi_2(|E(fv|\mathcal{F}_n)|)ud\mu\\ &\leqslant \sum_{k\in Z}\int_{\{2^k<|E(fv|\mathcal{F}_n)|\leqslant 2^{k+1}\}}\Phi_2(|E(fv|\mathcal{F}_n)|)ud\mu\\ &\leqslant C\sum_{k\in Z}\int_{\{2^k<|E(fv|\mathcal{F}_n)|\leqslant 2^{k+1}\}}\Phi_2(2^k)ud\mu\\ &= C\sum_{k\in Z}\Phi_2(2^k)\int_{(\{|E(fv|\mathcal{F}_n)|>2^k\}\cap\{2^k<|E(fv|\mathcal{F}_n)|\leqslant 2^{k+1}\})}ud\mu\\ &\leqslant C\sum_{k\in Z}\Phi_2\circ\Phi_1^{-1}\bigg(C\int_{\{2^k<|E(fv|\mathcal{F}_n)|\leqslant 2^{k+1}\}}\Phi_1(|f|)vd\mu\bigg)\\ &\leqslant C\Phi_2\circ\Phi_1^{-1}\bigg(C\sum_{k\in Z}\int_{\{2^k<|E(fv|\mathcal{F}_n)|\leqslant 2^{k+1}\}}\Phi_1(|f|)vd\mu\bigg)\\ &\leqslant C\Phi_2\circ\Phi_1^{-1}\bigg(C\int_\Omega \Phi_1(|f|)vd\mu\bigg)\\ &\leqslant \Phi_2\bigg(C\Phi_1^{-1}\bigg(C\int_\Omega \Phi_1(|f|)vd\mu\bigg)\bigg),\end{aligned}$$

which implies (2.1).

**Theorem 2.2.** *Let $(u,v)$ be a couple of weights. Suppose that $\Phi_1\ll\Phi_2$, $\Phi_2\in\triangle_2$ and $q_{\Phi_1}>1$, then the following statements are equivalent*:

(1) *There exists a positive constant $C$ such that*

$$\Phi_2^{-1}\bigg(\int_{\{\tau<\infty\}}\Phi_2(M(tv\chi_{\{\tau<\infty\}}))ud\mu\bigg)\leqslant C\Phi_1^{-1}(C\Phi_1(t)|\{\tau<\infty\}|_v),\quad \forall\,\tau\in\mathcal{T}\ \text{and}\ t>0;\tag{2.4}$$

(2) *There exists a positive constant $C$ such that*

$$\Phi_2^{-1}\bigg(\int_\Omega \Phi_2(M(fv))ud\mu\bigg)\leqslant C\Phi_1^{-1}\bigg(C\int_\Omega \Phi_1(|f|)vd\mu\bigg),\quad \forall\,f=(f_n)\in L_{\Phi_1}(vd\mu).\tag{2.5}$$

*Proof.* (1) $\Rightarrow$ (2) Let $f \in L_{\Phi_1}(vd\mu)$. For all $k \in Z$, define stopping times $\tau_k = \inf\{n : |f_n| > 2^k\}$. Set

$$A_{k,j} = \{\tau_k < \infty\} \cap \{2^j < E(v|\mathcal{F}_{\tau_k}) \leqslant 2^{j+1}\};$$
$$B_{k,j} = \{\tau_k < \infty, \tau_{k+1} = \infty\} \cap \{2^j < E(v|\mathcal{F}_{\tau_k}) \leqslant 2^{j+1}\}, \quad j \in Z.$$

Then $A_{k,j} \in \mathcal{F}_{\tau_k}, B_{k,j} \subseteq A_{k,j}$. Moreover, $\{B_{k,j}\}_{k,j}$ is a family of disjoint sets and

$$\{2^k < Mf \leqslant 2^{k+1}\} = \{\tau_k < \infty, \tau_{k+1} = \infty\} = \bigcup_{j \in Z} B_{k,j}, \quad k \in Z.$$

Note that $E(fv|\mathcal{F}_{\tau_k}) = \hat{E}_v(f|\mathcal{F}_{\tau_k})E(v|\mathcal{F}_{\tau_k})$, we have

$$2^k \leqslant \operatorname*{essinf}_{A_{k,j}} |E(fv|\mathcal{F}_{\tau_k})| \leqslant \operatorname*{essinf}_{A_{k,j}} \hat{E}_v(|f||\mathcal{F}_{\tau_k}) \operatorname*{esssup}_{A_{k,j}} E(v|\mathcal{F}_{\tau_k}).$$

It follows that

$$\begin{aligned}\int_\Omega \Phi_2(M(fv))ud\mu &\leqslant C \sum_{k \in Z, j \in Z} \int_{B_{k,j}} \Phi_2(2^k)ud\mu \\ &\leqslant C \sum_{k \in Z, j \in Z} \int_{B_{k,j}} \Phi_2\Big(\operatorname*{ess\,inf}_{A_{k,j}} \hat{E}_v(|f||\mathcal{F}_{\tau_k}) \operatorname*{ess\,sup}_{A_{k,j}} E(v|\mathcal{F}_{\tau_k})\Big)ud\mu.\end{aligned}$$

For $l \in Z$, let

$$E_l = \Big\{(k,j) : \ 2^l < \operatorname*{ess\,inf}_{A_{k,j}} \hat{E}_v(|f||\mathcal{F}_{\tau_k}) \leqslant 2^{l+1}\Big\}$$

and

$$\tau^{(l)} = \inf\{n : \hat{E}_v(|f||\mathcal{F}_n) > 2^l\}.$$

It is clear that

$$\bigcup_{(k,j) \in E_l} B_{k,j} \subseteq \bigcup_{(k,j) \in E_l} A_{k,j} \subseteq \{\hat{M}_v(|f|) > 2^l\} = \{\tau^{(l)} < \infty\}.$$

It follows from (2.4), $\Phi_1 \ll \Phi_2$ and $q_{\Phi_1} > 1$ that

$$\begin{aligned}\int_\Omega \Phi_2(M(fv))ud\mu &\leqslant C \sum_{l \in Z} \sum_{(k,j) \in E_l} \int_{B_{k,j}} \Phi_2\Big(\operatorname*{ess\,inf}_{A_{k,j}} \hat{E}_v(|f||\mathcal{F}_{\tau_k}) \operatorname*{ess\,sup}_{A_{k,j}} E(v|\mathcal{F}_{\tau_k})\Big)ud\mu \\ &\leqslant C \sum_{l \in Z} \sum_{(k,j) \in E_l} \int_{B_{k,j}} \Phi_2(2^{l-2}E(v|\mathcal{F}_{\tau_k}))ud\mu \\ &\leqslant C \sum_{l \in Z} \sum_{(k,j) \in E_l} \int_{B_{k,j}} \Phi_2(M(2^{l-2}v\chi_{\{\tau^{(l)}<\infty\}}))ud\mu \\ &\leqslant C \sum_{l \in Z} \int_{\{\tau^{(l)}<\infty\}} \Phi_2(M(2^{l-2}v\chi_{\{\tau^{(l)}<\infty\}}))ud\mu \\ &\leqslant C \sum_{l \in Z} \Phi_2 \circ \Phi_1^{-1}(C\Phi_1(2^{l-2})|\{\tau^{(l)} < \infty\}|_v) \\ &= C \sum_{l \in Z} \Phi_2 \circ \Phi_1^{-1}\left(C\Phi_1(2^{l-2})\left|\left\{\hat{M}_v\left(\frac{|f|}{2}\right) > 2^{l-1}\right\}\right|_v\right) \\ &\leqslant \Phi_2\left(C\Phi_1^{-1}\left(C\int_\Omega \Phi_1(|f|)vd\mu\right)\right).\end{aligned}$$

(2) $\Rightarrow$ (1) It is trivial and we omit it.

**Corollary 2.3.** *Let $(u,v)$ be a couple of weights and $1 < p \leqslant q < \infty$. Suppose that $\sigma = v^{-\frac{1}{p-1}} \in L^1(\Omega)$, then the following statements are equivalent:*

(1) *There exists a positive constant $C$ such that*

$$\bigg(\int_{\{\tau<\infty\}}(|E(f|\mathcal{F}_\tau)|)^q u d\mu\bigg)^{\frac{1}{q}} \leqslant C\bigg(\int_\Omega |f|^p v d\mu\bigg)^{\frac{1}{p}}, \quad \forall\, f=(f_n)\in L^p(v) \text{ and } \tau\in\mathcal{T};$$

(2) *There exists a positive constant $C$ such that*

$$\bigg(\int_{\{\tau<\infty\}}(\sigma_\tau)^q u d\mu\bigg)^{\frac{1}{q}} \leqslant C|\{\tau<\infty\}|_\sigma^{\frac{1}{p}}, \quad \forall\,\tau\in\mathcal{T};$$

(3) *There exists a positive constant $C$ such that*

$$\lambda|\{Mf>\lambda\}|_u^{\frac{1}{q}} \leqslant C\bigg(\int_\Omega |f|^p v d\mu\bigg)^{\frac{1}{p}}, \quad \forall\, f=(f_n)\in L^p(v) \text{ and } \lambda>0.$$

**Corollary 2.4.** *Let $(u,v)$ be a couple of weights and $1<p\leqslant q<\infty$. Suppose that $\sigma=v^{-\frac{1}{p-1}}\in L^1(\Omega)$, then the following statements are equivalent*:

(1) *There exists a positive constant $C$ such that*

$$\bigg(\int_{\{\tau<\infty\}}(M(\sigma\chi_{\{\tau<\infty\}}))^q u d\mu\bigg)^{\frac{1}{q}} \leqslant C|\{\tau<\infty\}|_\sigma^{\frac{1}{p}}, \quad \forall\,\tau\in\mathcal{T};$$

(2) *There exists a positive constant $C$ such that*

$$\bigg(\int_\Omega (Mf)^q u d\mu\bigg)^{\frac{1}{q}} \leqslant C\bigg(\int_\Omega |f|^p v d\mu\bigg)^{\frac{1}{p}}, \quad f\in L^p(v).$$

For the couple of weights $(u,\sigma)$, substituting $\Phi_1=t^p$ and $\Phi_2=t^q$ into Theorems 2.1 and 2.2, we have Corollaries 2.3 and 2.4, respectively.

## 3 The inequalities involving one Young's function

**Proposition 3.1.** *Let $(u,v)$ be a couple of weights. Suppose that $\Phi\in\triangle_2$ and $q_\Phi>1$, then the following statements are equivalent*:

(1) *There exists a positive constant $C$ such that*

$$\int_{\{\tau<\infty\}}\Phi(|E(fv|\mathcal{F}_\tau)|)u d\mu \leqslant C\int_\Omega \Phi(|f|)v d\mu, \quad \forall\, f=(f_n)\in L_\Phi(vd\mu) \text{ and } \tau\in\mathcal{T}; \tag{3.1}$$

(2) *There exists a positive constant $C$ such that*

$$\int_{\{\tau<\infty\}}\Phi(tv_\tau)u d\mu \leqslant C\Phi(t)|\{\tau<\infty\}|_v, \quad \forall\,\tau\in\mathcal{T} \text{ and } t>0; \tag{3.2}$$

(3) *There exists a positive constant $C$ such that*

$$\Phi(\lambda)|\{M(fv)>\lambda\}|_u \leqslant C\int_\Omega \Phi(|f|)v d\mu, \quad \forall\, f=(f_n)\in L_\Phi(vd\mu) \text{ and } \lambda>0. \tag{3.3}$$

**Proposition 3.2.** *Let $(u,v)$ be a couple of weights. Suppose that $\Phi\in\triangle_2$ and $q_\Phi>1$, then the following statements are equivalent*:

(1) *There exists a positive constant $C$ such that*

$$\int_{\{\tau<\infty\}}\Phi(M(tv\chi_{\{\tau<\infty\}}))u d\mu \leqslant C\Phi(t)|\{\tau<\infty\}|_v, \quad \forall\,\tau\in\mathcal{T}; \tag{3.4}$$

(2) *There exists a positive constant* $C$ *such that*

$$\int_\Omega \Phi(M(fv))ud\mu \leqslant C\int_\Omega \Phi(|f|)vd\mu, \quad \forall f=(f_n)\in L_\Phi(vd\mu). \tag{3.5}$$

Propositions 3.1 and 3.2 follow from Theorems 2.1 and 2.2 respectively.

**Theorem 3.3.** *Let* $(u,v)$ *be a couple of weights. Suppose that* $\Phi\in\triangle'$ *and* $q_\Phi>1$, *then the following statements are equivalent*:

(1) *There exists a positive constant* $C$ *such that*

$$\int_{\{\tau<\infty\}} \Phi(|E(fv|\mathcal{F}_\tau)|)ud\mu \leqslant C\int_\Omega \Phi(|f|)vd\mu, \quad \forall f=(f_n)\in L_\Phi(vd\mu) \text{ and } \tau\in\mathcal{T}; \tag{3.6}$$

(2) *There exists a positive constant* $C$ *such that*

$$\int_{\{\tau<\infty\}} \Phi(v_\tau)ud\mu \leqslant C|\{\tau<\infty\}|_v, \quad \forall \tau\in\mathcal{T}; \tag{3.7}$$

(3) *There exists a positive constant* $C$ *such that*

$$\Phi(\lambda)|\{M(fv)>\lambda\}|_u \leqslant C\int_\Omega \Phi(|f|)vd\mu, \quad \forall f=(f_n)\in L_\Phi(vd\mu) \text{ and } \lambda>0. \tag{3.8}$$

*Proof.* It suffices to prove (2) $\Rightarrow$ (1), when $\tau\equiv n$ for some $n\in N$. Fix $f=(f_n)\in L_\Phi(vd\mu)$. Following the process (2) $\Rightarrow$ (1) in Theorem 2.1, we have

$$\begin{aligned}\int_{\{\tau<\infty\}} \Phi(|E(fv|\mathcal{F}_\tau)|)ud\mu &= \int_\Omega \Phi(|E(fv|\mathcal{F}_n)|)ud\mu \\ &\leqslant C\sum_{k\in Z,j\in Z}\int_{B_{k,j}}\Phi(2^k)ud\mu \\ &\leqslant C\sum_{k\in Z,j\in Z}\Phi\Big(\operatorname*{essinf}_{B_{k,j}}\hat{E}_v(|f||\mathcal{F}_n)\Big)\int_{B_{k,j}}\Phi(E(v|\mathcal{F}_n))ud\mu.\end{aligned}$$

It is clear that $\vartheta$ is a measure on $X=Z^2$ with

$$\vartheta(k,j)=\int_{B_{k,j}}\Phi(E(v|\mathcal{F}_n))ud\mu.$$

For the above $f\in L_\Phi(vd\mu)$, define

$$Tf(k,j)=\Phi\Big(\operatorname*{essinf}_{B_{k,j}}\hat{E}_v(|f||\mathcal{F}_n)\Big),$$

and denote

$$E_\lambda=\Big\{(k,j):\Phi\Big(\operatorname*{essinf}_{B_{k,j}}\hat{E}_v(|f||\mathcal{F}_n)\Big)>\lambda\Big\} \quad\text{and}\quad G_\lambda=\bigcup_{(k,j)\in E_\lambda}B_{k,j}$$

for each $\lambda>0$. Recall that $G_\lambda\in\mathcal{F}_n$, we have $\tau_\lambda\in\mathcal{T}$ and $\{\tau_\lambda<\infty\}=G_\lambda$, where $\tau_\lambda=n\chi_{G_\lambda}+\infty\chi_{G_\lambda}^c$. Thus

$$\begin{aligned}|\{Tf>\lambda\}|_\vartheta &= \sum_{(k,j)\in E_\lambda}\int_{B_{k,j}}\Phi(E(v|\mathcal{F}_n))ud\mu \\ &\leqslant \int_{G_\lambda}\Phi(E(v|\mathcal{F}_n))ud\mu \\ &= \int_{\{\tau_\lambda<\infty\}}\Phi(E(v|\mathcal{F}_{\tau_\lambda}))ud\mu\end{aligned}$$

$$\leqslant C|\{\tau_\lambda < \infty\}|_v = C|G_\lambda|_v$$
$$\leqslant C|\{\Phi(\hat{M}_v(|f|)) > \lambda\}|_v.$$

Therefore

$$\begin{aligned}\int_{\{\tau<\infty\}} \Phi(|E(fv|\mathcal{F}_n)|)ud\mu &\leqslant C\int_0^\infty |\{Tf>\lambda\}|_\vartheta d\lambda \\ &\leqslant C\int_0^\infty |\{\Phi(\hat{M}_v(|f|))>\lambda\}|_v d\lambda \\ &= C\int_\Omega \Phi(\hat{M}_v(|f|))vd\mu \leqslant C\int_\Omega \Phi(|f|)vd\mu. \end{aligned} \tag{3.9}$$

**Theorem 3.4.** *Let $(u,v)$ be a couple of weights. Suppose that $\Phi\in\triangle'$ and $q_\Phi>1$, then the following statements are equivalent*:

(1) *There exists a positive constant $C$ such that*

$$\int_{\{\tau<\infty\}} \Phi(M(v\chi_{\{\tau<\infty\}}))ud\mu \leqslant C|\{\tau<\infty\}|_v, \quad \forall\,\tau\in\mathcal{T}; \tag{3.10}$$

(2) *There exists a positive constant $C$ such that*

$$\int_\Omega \Phi(M(fv))ud\mu \leqslant C\int_\Omega \Phi(|f|)vd\mu, \quad \forall\, f=(f_n)\in L_\Phi(vd\mu). \tag{3.11}$$

*Proof.* (1) $\Rightarrow$ (2) Let $f\in L_\Phi(vd\mu)$. As the process (1) $\Rightarrow$ (2) in Theorem 2.2, we have

$$\begin{aligned}\int_\Omega \Phi(M(fv))ud\mu &= \sum_{k\in Z}\int_{\{2^k<M(fv)\leqslant 2^{k+1}\}} \Phi(M(fv))ud\mu \\ &\leqslant C\sum_{k\in Z}\int_{\{2^k<M(fv)\leqslant 2^{k+1}\}} \Phi(2^k)ud\mu \\ &= C\sum_{k\in Z, j\in Z} \Phi(2^k)|B_{k,j}|_u \\ &\leqslant C\sum_{k\in Z, j\in Z} \Phi\Big(\operatorname*{essinf}_{A_{k,j}} \hat{E}_v(|f||\mathcal{F}_{\tau_k})\Big)\int_{B_{k,j}} \Phi(E(v|\mathcal{F}_{\tau_k}))ud\mu. \end{aligned}$$

It is clear that $\vartheta$ is a measure on $X=Z^2$ with $\vartheta(k,j)=\int_{B_{k,j}}\Phi(E(v|\mathcal{F}_{\tau_k}))ud\mu$. For the above $f\in L_\Phi(vd\mu)$, define

$$Tf(k,j)=\Phi\Big(\operatorname*{essinf}_{A_{k,j}} \hat{E}_v(|f||\mathcal{F}_{\tau_k})\Big),$$

and denote

$$E_\lambda=\Big\{(k,j):\Phi\Big(\operatorname*{essinf}_{A_{k,j}} \hat{E}_v(|f||\mathcal{F}_{\tau_k})\Big)>\lambda\Big\} \quad\text{and}\quad G_\lambda=\bigcup_{(k,j)\in E_\lambda} A_{k,j}$$

for each $\lambda>0$. Then we have

$$\begin{aligned}|\{Tf>\lambda\}|_\vartheta &= \sum_{(k,j)\in E_\lambda}\int_{B_{k,j}} \Phi(E(v|\mathcal{F}_{\tau_k}))ud\mu \\ &\leqslant \sum_{(k,j)\in E_\lambda}\int_{B_{k,j}} \Phi(E(v\chi_{G_\lambda}|\mathcal{F}_{\tau_k}))ud\mu \\ &\leqslant \int_{G_\lambda} \Phi(M(v\chi_{G_\lambda}))ud\mu. \end{aligned}$$

Let $\tau = \inf\{n : \Phi(\hat{E}_v(|f||\mathcal{F}_n)) > \lambda\}$, we have $G_\lambda \subseteq \{\Phi(\hat{M}_v(|f|)) > \lambda\} = \{\tau < \infty\}$. It follows from (3.10) that

$$\begin{aligned} |\{Tf > \lambda\}|_\vartheta &\leqslant \int_{G_\lambda} \Phi(M(v\chi_{G_\lambda}))ud\mu \\ &\leqslant \int_{\{\tau<\infty\}} \Phi(M(v\chi_{\{\tau<\infty\}}))ud\mu \\ &\leqslant C|\{\tau < \infty\}|_v = C|\{\Phi(\hat{M}_v(|f|)) > \lambda\}|_v. \end{aligned}$$

In the way of (3.9), we have

$$\int_\Omega \Phi(M(fv))ud\mu \leqslant C\int_\Omega \Phi(|f|)vd\mu.$$

(2) $\Rightarrow$ (1) It is trivial and we omit it.

**Acknowledgements** This work was supported by National Natural Science Foundation of China (Grant Nos. 10671147, 11071190).

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