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The Schwarz-Pick lemma for planar harmonic mappings

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Abstract The classical Schwarz-Pick lemma for holomorphic mappings is generalized to planar harmonic mappings of the unit disk D completely. (I) For any $0 < r < 1$ and $0 \leqslant \rho < 1$, the author constructs a closed convex domain $E_{r,\rho}$ such that

$$F(\overline{\Delta}(z,r)) \subset e^{i\alpha}E_{r,\rho} = \{e^{i\alpha}z : z \in E_{r,\rho}\}$$

holds for every $z \in D$, $w = \rho e^{i\alpha}$ and harmonic mapping F with $F(D) \subset D$ and $F(z) = w$, where $\Delta(z,r)$ is the pseudo-disk of center z and pseudo-radius r; conversely, for every $z \in D$, $w = \rho e^{i\alpha}$ and $w' \in e^{i\alpha}E_{r,\rho}$, there exists a harmonic mapping F such that $F(D) \subset D$, $F(z) = w$ and $F(z') = w'$ for some $z' \in \partial\Delta(z,r)$. (II) The author establishes a Finsler metric $\mathcal{H}_z(u)$ on the unit disk D such that

$$\mathcal{H}_{F(z)}(e^{i\theta}F_z(z) + e^{-i\theta}F_{\overline{z}}(z)) \leqslant \frac{1}{1-|z|^2}$$

holds for any $z \in D$, $0 \leqslant \theta \leqslant 2\pi$ and harmonic mapping F with $F(D) \subset D$; furthermore, this result is precise and the equality may be attained for any values of z, θ, $F(z)$ and $\arg(e^{i\theta}F_z(z) + e^{-i\theta}F_{\overline{z}}(z))$.

Keywords harmonic mappings, Schwarz-Pick lemma, Finsler metric

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1 Introduction

A harmonic mapping is a complex-valued harmonic function defined on a domain in the complex plane. Harmonic mappings have interesting links with geometric function theory, minimal surfaces and locally quasiconformal mappings. For a survey of harmonic mappings in the plane, see [2].

For a continuously differentiable function $f(z)$, where $z = x + iy$, we use the common notations for its formal derivatives

$$f_z = \frac{1}{2}(f_x - if_y), \quad f_{\overline{z}} = \frac{1}{2}(f_x + if_y).$$

Then f is a harmonic mapping if and only if f is twice continuously differentiable and

$$\Delta f = f_{xx} + f_{yy} = 4f_{z\overline{z}} = 0.$$

Denote

$$\Lambda_f = \max_{0\leqslant\theta\leqslant2\pi} |e^{i\theta}f_z + e^{-i\theta}f_{\overline{z}}| = |f_z| + |f_{\overline{z}}|,$$

$$\lambda_f = \min_{0 \leqslant \theta \leqslant 2\pi} |e^{i\theta} f_z + e^{-i\theta} f_{\overline{z}}| = ||f_z| - |f_{\overline{z}}||.$$

Denote the unit disc $\{z : |z| < 1\}$ by D and a disk with the center at the origin and the radius r by D_r. The classical Schwarz-Pick lemma [1, 7, 8] is formulated as follows.

Schwarz-Pick lemma. *Let f be a holomorphic mapping such that $f(D) \subset D$. Then,*

$$\frac{|f(z_1) - f(z_2)|}{|1 - \overline{f(z_2)} f(z_1)|} \leqslant \frac{|z_1 - z_2|}{|1 - \overline{z_2} z_1|} \tag{1.1}$$

holds for $z_1, z_2 \in D$, and

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leqslant \frac{1}{1 - |z|^2} \tag{1.2}$$

holds for $z \in D$.

Using the notations

$$d_p(z_1, z_2) = \frac{|z_1 - z_2|}{|1 - \overline{z_2} z_1|}$$

for the pseudo-distance between $z_1, z_2 \in D$, and $\Delta(z, r) = \{\zeta \in D : d_p(\zeta, z) < r\}$, $z \in D$ and $0 < r < 1$, for the pseudo-disk with center z and pseudo-radius r, (1.1) may be written in the following form:

$$f(\Delta(z, r)) \subset \Delta(f(z), r). \tag{1.1$'$}$$

For a harmonic mapping F on the unit disk such that $F(D) = D$ and $F(0) = 0$, it is known [3] that

$$|F(z)| \leqslant \frac{4}{\pi} \arctan |z| \tag{1.3}$$

holds for $z \in D$, and

$$\Lambda_f(0) \leqslant \frac{4}{\pi}. \tag{1.4}$$

Since the composition $F \circ f$ of a harmonic mapping F and a holomorphic mapping f is harmonic, if the condition $F(0) = 0$ is replaced by $F(z) = 0$ for some z, as consequences of (1.3) and (1.4),

$$|F(\zeta)| \leqslant \frac{4}{\pi} \arctan d_p(\zeta, z) \tag{1.5}$$

holds for $\zeta \in D$, and

$$\Lambda_F(z) \leqslant \frac{4}{\pi} \frac{1}{1 - |z|^2}. \tag{1.6}$$

Unfortunately, the composition $f \circ F$ of a harmonic mapping F and a holomorphic mapping f do not need to be harmonic, so it is a serious problem to seek the estimates corresponding to (1.1)$'$ and (1.2) for a harmonic mapping F without the assumption $F(z) = 0$.

This paper gives a complete solution to the problem. (I) For any $0 < r < 1$ and $0 \leqslant \rho < 1$, the author constructs a closed convex domain $E_{r,\rho}$, which contains ρ and is symmetric to the real axis, with the following properties: Let $z \in D$ and $w = \rho e^{i\alpha}$ be given. For every harmonic mapping F with $F(D) \subset D$ and $F(z) = w$, we have $F(\overline{\Delta}(z, r)) \subset e^{i\alpha} E_{r,\rho} = \{e^{i\alpha}\zeta : \zeta \in E_{r,\rho}\}$; conversely, for every $w' \in e^{i\alpha} E_{r,\rho}$, there exists a harmonic mapping F such that $F(D) \subset D$, $F(z) = w$ and $F(z') = w'$ for some $z' \in \partial\Delta(z, r)$. This is the Schwarz-Pick lemma for harmonic mappings corresponding to (1.1) or (1.1)$'$. (II) The author establishes a Finsler metric $\mathcal{H}_z(u)$ on the unit disk D such that for any harmonic mapping F with $F(D) \subset D$,

$$\mathcal{H}_{F(z)}(e^{i\theta} F_z(z) + e^{-i\theta} F_{\overline{z}}(z)) \leqslant \frac{1}{1 - |z|^2}$$

holds for $z \in D$ and $0 \leqslant \theta \leqslant 2\pi$. Furthermore, the author gives examples to show that the equality can be attained for any values of z, $F(z)$, θ and $\arg\{e^{i\theta} F_z(z) + e^{-i\theta} F_{\overline{z}}(z)\}$. This is the Schwarz-Pick lemma for harmonic mappings corresponding to (1.2). As a consequence,

$$\frac{\Lambda_F(z)}{h_{|F(z)|}(\pi/2)} \leqslant \frac{1}{1 - |z|^2}$$

holds for $z \in D$, where $h_\rho(\pi/2)$ is decreasing from $4/\pi$ to 0 as ρ increasing from 0 to 1, and $h_\rho(\pi/2) \approx \sqrt{2}\sqrt{1-\rho^2}$ as $\rho \to 1$.

Finally, by using (1.9), the author generalizes the classical Landau theorem for bounded holomorphic functions to the harmonic case. This improves the known results of Chen et al. [5] and Liu [6].

2 An extremal problem of a functional on $L^\infty[0, 2\pi]$

For $0 < r < 1$, $\mu > 0$ and a real number λ, define

$$A_{r,\lambda,\mu}(\theta) = \frac{1}{\mu}\left(\frac{1}{1+r^2-2r\sin\theta} - \lambda\right), \quad 0 \leqslant \theta \leqslant 2\pi,$$

and

$$R(r,\lambda,\mu) = \frac{1}{2\pi}\int_0^{2\pi} \frac{A_{r,\lambda,\mu}(\theta)}{\sqrt{1+A^2_{r,\lambda,\mu}(\theta)}}\,d\theta, \quad I(r,\lambda,\mu) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{\sqrt{1+A^2_{r,\lambda,\mu}(\theta)}}\,d\theta.$$

Lemma 1. *Let $0 < r < 1$ be fixed. Then, there exists a unique pair of real functions $\lambda = \lambda(r,a,b)$ and $\mu = \mu(r,a,b) > 0$, defined on the upper half disk $\{(a,b) : a^2+b^2 < 1,\ b > 0\}$ and analytic in the real sense, such that $R(r,\lambda(r,a,b),\mu(r,a,b)) = a$ and $I(r,\lambda(r,a,b),\mu(r,a,b)) = b$ for any point (a,b) in the half disk.*

Proof. Denote $g(\theta) = (1+A^2_{r,\lambda,\mu}(\theta))^{-3/2}$ for $0 \leqslant \theta \leqslant 2\pi$. Then

$$\frac{\partial R}{\partial \lambda} = -\frac{1}{2\pi\mu}\int_0^{2\pi} g(\theta)d\theta, \quad \frac{\partial R}{\partial \mu} = -\frac{1}{2\pi\mu}\int_0^{2\pi} A_{r,\lambda,\mu}(\theta)g(\theta)d\theta,$$
$$\frac{\partial I}{\partial \lambda} = \frac{1}{2\pi\mu}\int_0^{2\pi} A_{r,\lambda,\mu}(\theta)g(\theta)d\theta, \quad \frac{\partial I}{\partial \mu} = \frac{1}{2\pi\mu}\int_0^{2\pi} A^2_{r,\lambda,\mu}(\theta)g(\theta)d\theta.$$

It is easy to see that

(i) for the fixed μ, $R(r,\lambda,\mu) \to -1$ or 1 according to $\lambda \to +\infty$ or $\lambda \to -\infty$;

(ii) $\partial R/\partial\lambda < 0$ for any λ and $\mu > 0$, and $R(r,\lambda,\mu)$ is strictly decreasing as a function of λ for a fixed μ;

(iii) by the convexity of the square function, $\frac{\partial R}{\partial \lambda}\frac{\partial I}{\partial \mu} - \frac{\partial R}{\partial \mu}\frac{\partial I}{\partial \lambda} < 0$ for any λ and $\mu > 0$.

Let (a,b) in the upper half disk be given. By (i) and (ii), there exists a unique function $\lambda_a(\mu)$ such that $R(r,\lambda_a(\mu),\mu) = a$ for $\mu > 0$. Furthermore, by (ii), one may use the implicit function theorem, and sees that $\lambda_a(\mu)$ is a continuous function of μ and

$$\lambda_a'(\mu) = -\left(\frac{\partial R}{\partial \mu}\Big/\frac{\partial R}{\partial \lambda}\right)_{(\lambda_a(\mu),\mu)}.$$

Thus, by (ii) and (iii),

$$\frac{dI(r,\lambda_a(\mu),\mu)}{d\mu} = \left(\left(\frac{\partial R}{\partial \lambda}\frac{\partial I}{\partial \mu} - \frac{\partial R}{\partial \mu}\frac{\partial I}{\partial \lambda}\right)\Big/\frac{\partial R}{\partial \lambda}\right)_{(\lambda_a(\mu),\mu)} > 0 \quad \text{for} \quad \mu > 0.$$

This shows that $I(r,\lambda_a(\mu),\mu)$ is strictly increasing as a function of μ on $(0,\infty)$.

The author claims that $I(r,\lambda_a(\mu),\mu) \to 0$ as $\mu \to 0$, and $I(r,\lambda_a(\mu),\mu) \to \sqrt{1-a^2}$ as $\mu \to +\infty$.

There exists a subsequence of μ_n such that $\lambda_a(\mu_n)$ tends to ∞ or has a finite limit l. In both cases, one has $\sqrt{1+A^2_{r,\lambda_a(\mu_n),\mu_n}(\theta)}$ tends to ∞ except for two values of θ at most and, by the Lebesgue's dominated convergence theorem, $I(r,\lambda_a(\mu_n),\mu_n) \to 0$. The first claim is proved since $I(r,\lambda_a(\mu),\mu)$ is strictly increasing.

If there exists a sequence $\mu_n \to \infty$ such that $\lambda_a(\mu_n)/\mu_n \to \infty$, then $A_{r,\lambda_a(\mu_n),\mu_n}(\theta) \to \infty$ uniformly for $0 \leqslant \theta \leqslant 2\pi$, and $|a| = |R(r,\lambda_a(\mu_n),\mu_n)| \to 1$, a contradiction. This shows that $\lambda_a(\mu)/\mu$ is bounded as $\mu_n \to \infty$. If μ_n is a sequence such that $\lambda_a(\mu_n)/\mu_n$ tends to a finite limit l, then $A_{r,\lambda_a(\mu_n),\mu_n}(\theta) \to -l$

uniformly for $0 \leqslant \theta \leqslant 2\pi$, and $a = R(r, \lambda_0(\mu_n), \mu_n) \to -l/\sqrt{1+l^2}$. Consequently, $l = -a/\sqrt{1+a^2}$. This shows that $\lambda_a(\mu)/\mu \to -a/\sqrt{1-a^2}$ and $I(r, \lambda_a(\mu), \mu) \to \sqrt{1-a^2}$ as $\mu \to \infty$. The second claim is proved.

It is proved that $I(r, \lambda_a(\mu), \mu)$ is continuous and strictly increasing from 0 to $\sqrt{1-a^2}$ as μ is increasing from 0 to $+\infty$. Thus, there exists a unique μ such that $I(r, \lambda_a(\mu), \mu) = b$ since $0 < b < \sqrt{1-a^2}$. We have proved that there exist a unique pair of functions $\lambda = \lambda(r, a, b)$ and $\mu = \mu(r, a, b)$ such that $R(r, \lambda(a, b), \mu(a, b)) = a$ and $I(r, \lambda(a, b), \mu(a, b)) = b$ on the upper half disk. The real analyticity of $\lambda = \lambda(r, a, b)$ and $\mu = \mu(r, a, b)$ is asserted by the implicit function theorem. The lemma is proved. □

Let a and b be two numbers such that $0 \leqslant b < 1$, $-1 < a < 1$ and $a^2 + b^2 < 1$. Let $\mathcal{U}_{a,b}$ denote the class of real-valued functions $u \in L^\infty[0, 2\pi]$ satisfying the following conditions:

$$\|u\|_\infty \leqslant 1, \quad \frac{1}{2\pi}\int_0^{2\pi} u(\theta)d\theta = a, \quad \frac{1}{2\pi}\int_0^{2\pi} \sqrt{1-u^2(\theta)}d\theta \geqslant b.$$

Every function $u \in L^\infty[0, 2\pi]$ defines a harmonic function

$$U(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2} u(\theta)d\theta \quad \text{for} \quad z \in D.$$

Let $0 < r < 1$ and define a functional L_r on $L^\infty[0, 2\pi]$ by

$$L_r(u) = U(ri) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1+r^2-2r\sin\theta} u(\theta)d\theta.$$

Then we have the following result.

Theorem 1. *For any a, b and r satisfying the above conditions, there exists a unique extremal function $u_{a,b,r} \in \mathcal{U}_{a,b}$ such that L_r attains its maximum on $\mathcal{U}_{a,b}$ at $u_{a,b,r}$.*

Proof. Let a, b and r be fixed. First assume that $b > 0$. From Lemma 1, one has unique $\lambda = \lambda(r, a, b)$ and $\mu = \mu(r, a, b) > 0$ such that $R(r, \lambda, \mu) = a$ and $I(r, \lambda, \mu) = b$. Let

$$u_0(\theta) = \frac{A_{r,\lambda,\mu}(\theta)}{\sqrt{1+A^2_{r,\lambda,\mu}(\theta)}}.$$

Then $\|u_0\|_\infty < 1$ and

$$\frac{1}{2\pi}\int_0^{2\pi} u_0(\theta)d\theta = R(r, \lambda, \mu) = a, \quad \frac{1}{2\pi}\int_0^{2\pi} \sqrt{1-u_0^2(\theta)}d\theta = I(r, \lambda, \mu) = b.$$

This means that $u_0 \in \mathcal{U}_{a,b}$.

Let $u \in \mathcal{U}_{a,b}$ and let U and U_0 be the harmonic functions corresponding to u and u_0 respectively. Then

$$\begin{aligned}\frac{U_0(ri)-U(ri)}{1-r^2} &= \frac{1}{2\pi}\int_0^{2\pi} \frac{u_0(\theta)-u(\theta)}{1+r^2-2r\sin\theta}d\theta \\ &\geqslant \frac{1}{2\pi}\int_0^{2\pi} \frac{u_0(\theta)-u(\theta)}{1+r^2-2r\sin\theta}d\theta - \frac{\lambda}{2\pi}\int_0^{2\pi}(u_0(\theta)-u(\theta))d\theta \\ &\quad + \frac{\mu}{2\pi}\int_0^{2\pi}\left(\sqrt{1-u_0^2(\theta)} - \sqrt{1-u^2(\theta)}\right)d\theta.\end{aligned}$$

By the convexity of the function $\sqrt{1-x^2}$,

$$\sqrt{1-u^2(\theta)} - \sqrt{1-u_0^2(\theta)} \leqslant \frac{u_0(\theta)}{\sqrt{1-u_0^2(\theta)}}(u_0(\theta)-u(\theta)).$$

Thus,

$$\frac{U_0(ri)-U(ri)}{1-r^2} \geqslant \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{1+r^2-2r\sin\theta} - \lambda - \frac{\mu u_0(\theta)}{\sqrt{1-u_0^2(\theta)}}\right)(u_0(\theta)-u(\theta))d\theta = 0.$$

Thus $U_0(ri) \geqslant U(ri)$ with equality if and only if $u_0(\theta) = u(\theta)$ almost everywhere. This shows that $u_0(\theta)$ is the unique extremal function, which will be denoted by $u_{a,b,r}(\theta)$.

The case that $b = 0$ is much simpler. Let

$$u_0(\theta) = \begin{cases} 1, & -a\pi/2 < \theta < \pi + a\pi/2; \\ -1, & \pi + a\pi/2 < \theta < 2\pi - a\pi/2. \end{cases}$$

The author wants to show that u_0 is just the unique extremal function, which will be denoted by $u_{a,0,r}(\theta)$.

It is obvious that $u_0 \in \mathcal{U}_{a,0}$. For $u \in \mathcal{U}_{a,0}$ and $0 < r < 1$, one has

$$\begin{aligned} U_0(ri) - U(ri) &= \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 + r^2 - 2r\sin\theta} (u_0(\theta) - u(\theta))\, d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{1 - r^2}{1 + r^2 - 2r\sin\theta} - \frac{1 - r^2}{1 + r^2 + 2r\sin(a\pi/2)} \right) (u_0(\theta) - u(\theta))\, d\theta. \end{aligned}$$

If $-a\pi/2 < \theta < \pi + a\pi/2$, then

$$\frac{1 - r^2}{1 + r^2 - 2r\sin\theta} - \frac{1 - r^2}{1 + r^2 + 2r\sin(b\pi/2)} > 0, \quad u_0(\theta) - u(\theta) \geqslant 0.$$

The opposite inequalities occur in the case that $\pi + a\pi/2 < \theta < 2\pi - a\pi/2$. Thus, for $0 \leqslant \theta \leqslant 2\pi$, one always has

$$\left(\frac{1 - r^2}{1 + r^2 - 2r\sin\theta} - \frac{1 - r^2}{1 + r^2 + 2r\sin(a\pi/2)} \right) (u_{a,0}(\theta) - u(\theta)) \geqslant 0.$$

This shows that $U_{a,0}(ri) \geqslant U(ri)$ with the equality if and only if $u(\theta) = u_{a,0}(\theta)$ almost everywhere. The theorem is proved. □

3 The Schwarz-Pick lemma for harmonic mappings (I)

Let a and b be two real numbers with $a^2 + b^2 < 1$, and $0 < r < 1$. If $b \geqslant 0$, $u_{a,b,r}$ has been defined in Theorem 1. Now, define

$$v_{a,b,r}(\theta) = \sqrt{1 - u_{a,b,r}^2(\theta)} \quad \text{for} \ \ 0 \leqslant \theta \leqslant 2\pi,$$

and

$$\begin{aligned} V_{a,b,r}(z) &= \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} v_{a,b,r}(\theta) d\theta, \\ U_{a,b,r}(z) &= \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} u_{a,b,r}(\theta) d\theta, \\ F_{a,b,r}(z) &= U_{a,b,r}(z) + iV_{a,b,r}(z), \quad z \in D. \end{aligned}$$

For $b < 0$, let

$$U_{a,b,r}(z) = U_{a,-b,r}(z), \quad V_{a,b,r}(z) = -V_{a,-b,r}(z).$$

Then, the harmonic mapping $F_{a,b,r}(z) = U_{a,b,r}(z) + iV_{a,b,r}(z)$ satisfies $F_{a,b,r}(0) = a+bi$ and $F_{a,b,r}(D) \in D$. The functions $F_{a,b,r}$ are the extremal functions in the following theorem. It is not difficult to obtain

$$U_{a,0,r}(z) = \frac{2}{\pi} \left[\arg \frac{e^{i(\pi + a\pi/2)} - z}{e^{-ia\pi/2} - z} - \pi \left(1 - \frac{a}{2} \right) \right], \quad V_{a,0,r}(z) \equiv 0,$$

and

$$F_{a,0,r}(ri) = U_{a,0,r}(ri) = \frac{4}{\pi} \arctan \frac{r + \sin(a\pi/2)}{\cos(a\pi/2)} - a.$$

Lemma 2. *Let $F = U + iV$ be a harmonic mapping such that $F(D) \subset D$ and $F(0) = a + bi$. Then, for $0 < r < 1$ and $0 \leqslant \theta \leqslant 2\pi$,*

$$U(re^{i\theta}) \leqslant U_{a,b,r}(ri)$$

with equality at some point $re^{i\theta}$ if and only if $F(z) = F_{a,b,r}(e^{i(\pi/2-\theta)}z)$. Furthermore, $U(z) < U_{a,b,r}(ri)$ for $|z| < r$.

Proof. By the symmetry and passing through a rotation, one may assume that $b \geqslant 0$ and $\theta = \pi/2$. Let $F(e^{i\theta}) = u(\theta) + iv(\theta)$ be the non-tangential boundary value of F. Then, $u \in \mathcal{U}_{a,b}$. Applying Theorem 1, one has $U(ri) \leqslant U_{a,b,r}(ri)$ with equality if and only if $u(\theta) = u_{a,b,r}(\theta)$ a.e. If $u = u_{a,b,r}$, then $U = U_{a,b,r}$, $v(\theta) \leqslant v_{a,b,r}(\theta)$ for $0 \leqslant \theta \leqslant 2\pi$, and

$$b = \frac{1}{2\pi}\int_0^{2\pi} v(\theta)d\theta \leqslant \frac{1}{2\pi}\int_0^{2\pi} v_{a,b,r}(\theta)d\theta = b.$$

Thus, $v = v_{a,b,r}$ a.e. and $F = F_{a,b,r}$.

By the maximum principle, $U(z) \leqslant U_{a,b,r}(ri)$ for $|z| < r$. If the equality holds for some z with $|z| < r$, then U must be equal to $U_{a,b,r}(ri)$ identically. However, by what one proved above, it results $U = U_{a,b,r}$. It is not possible since $U_{a,b,r}$ is not a constant. This shows that $U(z) < U_{a,b,r}(ri)$ for $|z| < r$. The proof of the lemma is complete. □

Lemma 3. *For fixed $0 < r < 1$ and $z \in D$, $F_{a,b,r}(z)$, as a function of variables a and b, is analytic in the real sense on the open half disk $\{(a,b) : b > 0,\ a^2 + b^2 < 1\}$ and is continuous to the real radius.*

Proof. Let $0 < r < 1$ and $z \in D$ be fixed. It is obvious that $F_{a,b,r}(z)$ is analytic in the real sense on the open half disk, since it is determined there by the functions $\lambda(r,a,b)$ and $\mu(r,a,b)$ formulated in Lemma 1, which are analytic in the real sense on the open half disk $\{(a,b) : b > 0,\ a^2 + b^2 < 1\}$. One only needs to prove the continuity at the points of the real diameter.

Let $-1 < a_0 < 1$ be given. The author wants to prove that $U_{a,b,r}(z)$ and $V_{a,b,r}(z)$ is continuous at $(a_0, 0)$. Recall that

$$u_{a_0,0,r}(\theta) = \begin{cases} 1, & -a_0\pi/2 < \theta < \pi + a_0\pi/2, \\ -1, & \pi + a_0\pi/2 < \theta < 2\pi - a_0\pi/2, \end{cases}$$

and that $v(a_0, 0, r) = 0$.

Assume that there exists a sequence $(a_n, b_n) \to (a_0, 0)$ with $b_n > 0$ such that $\mu_n = \mu(a_n, b_n)$ has a positive lower bound. Since

$$\frac{1}{2\pi}\int_0^{2\pi}\left(1 + \left(\frac{1}{\mu_n(1 + r^2 - 2r\sin\theta)} - \frac{\lambda_n}{\mu_n}\right)^2\right)^{-1/2} d\theta = I(r, \lambda_n, \mu_n) = b_n \to 0,$$

then $\lambda_n/\mu_n \to \infty$ and, consequently, $\lambda_n \to \infty$. One may assume that $\lambda_n \to +\infty$. Then,

$$u_{a_n,b_n,r} = \frac{\frac{1}{\lambda_n(1+r^2-2r\sin\theta)} - 1}{(\frac{\mu_n^2}{\lambda_n^2} + (\frac{1}{\lambda_n(1+r^2-2r\sin\theta)} - 1)^2)^{1/2}} \to -1,$$

uniformly for $0 \leqslant \theta \leqslant 2\pi$, and $a_n \to -1$, a contradiction. This proves that $\mu(r,a,b) \to 0$ as $(a,b) \to (a_0, 0)$ with $b > 0$.

Now, the author wants to prove that

$$\lambda(r,a,b) \to \lambda_0 = \frac{1}{1 + r^2 + 2r\sin(a_0\pi/2)},$$

as $(a,b) \to (a_0, 0)$ with $b > 0$. On the contrary, assume that there is a sequence $(a_n, b_n) \to (a_0, 0)$ with $b_n > 0$ such that $\lambda_n = \lambda(a_n, b_n) \to \lambda' \neq \lambda_0$. If $\lambda' = \infty$, then, as the above, $|a_n| \to 1$, a contradiction. In the case that λ' is finite, one has

$$u_{a_n,b_n,r}(\theta) = \frac{\frac{1}{1+r^2-2r\sin\theta} - \lambda_n}{(\mu_n^2 + (\frac{1}{1+r^2-2r\sin\theta} - \lambda_n)^2)^{1/2}} \to \operatorname{sgn}\left\{\frac{1}{1 + r^2 - 2r\sin\theta} - \lambda'\right\},$$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} u_{a_n,b_n,r}(\theta)d\theta \to \frac{1}{2\pi}\int_0^{2\pi} \text{sgn}\Big\{\frac{1}{1+r^2-2r\sin\theta} - \lambda_0\Big\}d\theta$$

$$= \begin{cases} -1, & \lambda' \geqslant 1/(1-r^2); \\ 1, & \lambda' \leqslant 1/(1+r^2); \\ a', & \lambda' = \dfrac{1}{1+r^2+2r\sin(a'\pi/2)}, \ -1 < a' < 1, \ a' \neq a_0. \end{cases}$$

This contradicts $a_n \to a_0$. Thus, $a' = a$ and $l = \lambda_0$. This shows that

$$\lambda(r,a,b) \to \lambda_0 = \frac{1}{1+r^2+2r\sin(a_0\pi/2)}$$

as $(a,b) \to (a_0,0)$ with $b > 0$.

It is proved that $\mu(r,a,b) \to 0$ and $\lambda(r,a,b) \to \lambda_0$ as $(a,b) \to (a_0,0)$ with $b > 0$. Thus,

$$u_{a,b,r}(\theta) \to \text{sgn}\left\{\frac{1}{1+r^2-2r\sin\theta} - \lambda_0\right\} = u_{a_0,0,r}(\theta), \quad v_{a,b,r}(\theta) \to v_{a_0,0,r}(\theta) \equiv 0,$$

and, consequently, $U_{a,b,r}(z) \to U_{a_0,0,r}(z)$ and $V_{a,b,r}(z) \to V_{a_0,0,r}(z) = 0$ as $(a,b) \to (a_0,0)$ with $b > 0$. By the symmetry, $F_{a,b,r}(z) \to F_{a,0,r}(z)$ as $(a,b) \to (a_0,0)$ with $b \neq 0$. The continuity of $F_{a,b,r}(z)$ at $(a_0,0)$ is proved. The proof of the lemma is complete. □

For $-\pi \leqslant \beta \leqslant \pi$ and real number σ, denote the straight line $l(\beta,\sigma)$ and closed half plane $P(\beta,\sigma)$ by

$$l(\beta,\sigma) = \{w = u+iv : \text{Re}\{we^{-i\beta}\} = u\cos\beta + v\sin\beta = \sigma\}$$

and

$$P(\beta,\sigma) = \{w = u+iv : \text{Re}\{we^{-i\beta}\} = u\cos\beta + v\sin\beta \leqslant \sigma\}.$$

Theorem 2. *Let $0 < r < 1$ and $0 \leqslant \rho < 1$. Denote*

$$P_\beta = P(\beta, U_{\rho\cos\beta,-\rho\sin\beta,r}(ri)), \quad l_\beta = l(\beta, U_{\rho\cos\beta,-\rho\sin\beta,r}(ri)),$$

and define

$$E_{r,\rho} = \bigcap_{-\pi\leqslant\beta\leqslant\pi} P_\beta, \qquad \Gamma_{r,\rho} : w = f_{r,\rho}(\beta) = e^{i\beta}F_{\rho\cos\beta,-\rho\sin\beta,r}(ri), \quad -\pi \leqslant \beta \leqslant \pi.$$

Then:

(i) *$E_{r,\rho}$ is a closed convex domain and symmetrical with respect to the real axis, and ρ is an interior point of $E_{r,\rho}$;*

(ii) *For any harmonic mapping F such that $F(D) \subset D$ and $F(0) = \rho$, one has $F(\overline{D}_r) \subset E_{r,\rho}$, where $\overline{D}_r = \{z \in \mathbb{C} : |z| \leqslant r\}$; conversely, for any $w' \in E_{r,\rho}$, there is a harmonic mapping such that $F(D) \subset D$, $F(0) = \rho$ and $F(ri) = w'$.*

(iii) *$\Gamma_{r,\rho}$ is a convex Jordan closed curve and $\partial E_{r,\rho} = \Gamma_{r,\rho}$.*

Proof. Denote

$$P'_\beta = P(0, U_{\rho\cos\beta,-\rho\sin\beta,r}(ri)), \quad l'_\beta = l(0, U_{\rho\cos\beta,-\rho\sin\beta,r}(ri)).$$

P_β and l_β are obtained from P'_β and l'_β by an anti-clockwise rotation of angle β.

It is obvious that $E_{r,\rho}$ is a bounded closed convex set and symmetrical with respect to the real axis. For $-\pi \leqslant \beta \leqslant \pi$, $f_{r,\rho}(\beta) \in l_\beta$ since $F_{\rho\cos\beta,-\rho\sin\beta,r}(ri) \in l'_\beta$, and $E_{r,\rho} \subset P_\beta$ by the definition of $E_{r,\rho}$. This shows that $f_{r,\rho}(\beta) \in l_\beta \cap \partial E_{r,\rho}$ and $\Gamma_{r,\rho} \subset \partial E_{r,\rho}$. One has

$$f_{r,\rho}(0) = \frac{4}{\pi}\arctan\frac{r+\sin(\rho\pi/2)}{\cos(\rho\pi/2)} - \rho > \rho,$$

$$f_{r,\rho}(\pi) = -\frac{4}{\pi}\arctan\frac{r-\sin(\rho\pi/2)}{\cos(\rho\pi/2)} - \rho < \rho,$$

and, by Lemma 2,

$$\mathrm{Im}\{f_{r,\rho}(\pi/2)\} = U_{0,\rho,r}(ri) > U_{0,\rho,r}(0) = 0,$$
$$\mathrm{Im}\{f_{r,\rho}(-\pi/2)\} = -U_{0,-\rho,r}(ri) < -U_{0,-\rho,r}(0) = 0.$$

Thus, $E_{r,\rho}$ is a closed domain with ρ as its interior point. (i) is proved.

Let F be a harmonic mapping such that $F(D) \subset D$ and $F(0) = \rho$. For $-\pi \leqslant \beta \leqslant \pi$, let $F_\beta = e^{-i\beta}F$. Then, $F_\beta(D) \subset D$ and $F_\beta(0) = \rho(\cos\beta - i\sin\beta)$. Using Lemma 3 to the harmonic mapping F_β, one has $F_\beta(\overline{D}_r) \subset P(0, U_{\rho\cos\beta,-\rho\sin\beta,r}(ri))$ and, consequently, $F(\overline{D}_r) \subset P(\beta, U_{\rho\cos\beta,-\rho\sin\beta,r}(ri))$. This shows the first half of (ii).

The boundary of $E_{r,\rho}$ is a convex Jordan closed curve and is denoted by $\gamma_{r,\rho}$. The part of γ on the upper half-plane or lower half-plane is denoted by $\gamma^+_{r,\rho}$ or $\gamma^-_{r,\rho}$ respectively.

The curve $\Gamma_{r,\rho}$ is continuous by Lemma 3. Assume that there exist $0 \leqslant \beta_1 < \beta_2 \leqslant \pi$ such that $w_0 = f_{r,\rho}(\beta_1) = f_{r,\rho}(\beta_2)$. Then, $\beta_2 - \beta_1 < \pi$ and w_0 is the vertex of the angular domain $P_{\beta_1} \cap P_{\beta_2}$. Further, it is easy to see that $w_0 \in l_\beta$ and $f_{r,\rho}(\beta) = w_0$ for $\beta_1 < \beta < \beta_2$, since $w_0 \in \partial E_{r,\rho} \subset P_\beta$ and $f_{r,\rho}(\beta) \in l_\beta \cap \partial E_{r,\rho} \subset l_\beta \cap P_{\beta_1} \cap P_{\beta_2}$. $f_{r,\rho}(\beta)$ is analytic on $(0,\pi)$ in the real sense by Lemma 3. Then, one concludes that $f_{r,\rho}(\beta) = w_0$ for $0 < \beta < \pi$ and, by the continuity, $f_{r,\rho}(0) = f_{r,\rho}(\pi) = w_0$. A contraction, since $f_{r,\rho}(0) > f_{r,\rho}(\pi)$. This shows that $\Gamma^+_{r,\rho} : w = f_{r,\rho}(\beta)$, $0 \leqslant \beta \leqslant \pi$, is a Jordan curve. Furthermore, $\Gamma^+_{r,\rho} = \gamma^+_{r,\rho}$ since $\mathrm{Im}\{f_{r,\rho}(\pi/2)\} > 0$. By the same reason, $\Gamma^-_{r,\rho} : w = f_{r,\rho}(\beta)$, $-\pi \leqslant \beta \leqslant 0$, is a Jordan curve, and $\Gamma^-_{r,\rho} = \gamma^-_{r,\rho}$. It is proved that $\Gamma_{r,\rho}$ is a Jordan closed curve and $\Gamma_{r,\rho} = \gamma_{r,\rho}$. This shows (iii).

For $w' \in E_{r,\rho}$, draw a straight line l passing through w' and intersect $\partial E_{r,\rho}$ at w_1 and w_2. Let $w' = k_1 w_1 + k_2 w_2$ with $k_1, k_2 \geqslant 0$ and $k_1 + k_2 = 1$, and let $w_1 = f_{r,\rho}(\beta_1)$ and $w_2 = f_{r,\rho}(\beta_2)$. Then, the harmonic mapping $F = k_1 F_{\rho\cos\beta_1,-\rho\sin\beta_1,r} + k_2 F_{\rho\cos\beta_2,-\rho\sin\beta_2,r}$ satisfies $F(D) \subset D$, $F(0) = \rho$ and $F(ri) = w'$. This shows the second half of (ii). The theorem is proved. □

Now, the author formulates the general version of the above theorem.

Theorem 3. *Let $z, w = \rho e^{i\alpha} \in D$ and $0 < r < 1$ be given. Then, for every harmonic mapping F with $F(D) \subset D$ and $F(z) = w$, one has $F(\Delta(z,r)) \subset e^{i\alpha}E_{r,\rho} = \{e^{i\alpha}\zeta : \zeta \in E_{r,\rho}\}$; conversely, for every $w' \in e^{i\alpha}E_{r,\rho}$, there exists a harmonic mapping F and a point $z' \in \partial\Delta(z,r)$ such that $F(D) \subset D$, $F(z) = w$ and $F(z') = w'$.*

4 The limit of $r^{-1}(F_{a,b,r}(ri) - (a+bi))$ as $r \to 0$

Lemma 4. *Let*

$$\mathcal{R}(\sigma,\tau) = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\frac{(\sigma + 2\tau\sin\theta)d\theta}{\sqrt{1+(\sigma+2\tau\sin\theta)^2}}, \quad \mathcal{I}(\sigma,\tau) = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\frac{d\theta}{\sqrt{1+(\sigma+2\tau\sin\theta)^2}}.$$

Then, there exist a unique pair of functions $\sigma = \sigma(a,b)$ and $\tau = \tau(a,b) > 0$, defined on the upper half disk $\{(a,b) : a^2 + b^2 < 1,\ b > 0\}$, such that

$$\mathcal{R}(\sigma(a,b),\tau(a,b)) = a \quad \text{and} \quad \mathcal{I}(\sigma(a,b),\tau(a,b)) = b$$

hold for any point in this half disk. Furthermore, $\sigma(a,b)$ and $\tau(a,b)$ are analytical in the real sense on the half disk.

Proof. A direct calculation gives

$$\frac{\partial\mathcal{R}}{\partial\sigma} = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\frac{d\theta}{\sqrt{(1+(\sigma+2\tau\sin\theta)^2)^3}}, \quad \frac{\partial\mathcal{R}}{\partial\tau} = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\frac{2\sin\theta d\theta}{\sqrt{(1+(\sigma+2\tau\sin\theta)^2)^3}},$$
$$\frac{\partial\mathcal{I}}{\partial\sigma} = -\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\frac{(\sigma+2\tau\sin\theta)d\theta}{\sqrt{(1+(\sigma+2\tau\sin\theta)^2)^3}}, \quad \frac{\partial\mathcal{I}}{\partial\tau} = -\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\frac{2\sin\theta(\sigma+2\tau\sin\theta)d\theta}{\sqrt{(1+(\sigma+2\tau\sin\theta)^2)^3}}.$$

One asserts:

(i) $\partial\mathcal{R}/\partial\sigma > 0$ for any σ and $\tau > 0$, and $R(\sigma,\tau)$ is strictly increasing as a function of σ for a fixed $\tau > 0$;

(ii) for the fixed $\tau > 0$, $\mathcal{R}(\sigma,\tau) \to 1$ or -1 according to $\sigma \to +\infty$ or $\sigma \to -\infty$, respectively;

(iii) $D = \frac{\partial\mathcal{R}}{\partial\sigma}\frac{\partial\mathcal{I}}{\partial\tau} - \frac{\partial\mathcal{R}}{\partial\tau}\frac{\partial I}{\partial\sigma} < 0$ for any σ and $\tau > 0$;

(iv) $\mathcal{I}(\sigma,\tau) \to 0$ as $\tau \to +\infty$ uniformly for $-\infty < \sigma < +\infty$.

(i) and (ii) are obvious. (iii) is proved by using the convexity of the square function. To prove (iv), assume to the contrary that there exist a sequence $\tau_n \to +\infty$ and σ_n such that $|\mathcal{I}(\sigma_n,\tau_n)| \geqslant \delta > 0$ for $n = 1, 2, \dots$ Without loss of generality, one may assume that σ_n/τ_n has a limit l (l may be ∞). If $|l| > 2$, it is obvious that

$$\frac{1}{\sqrt{1 + (\sigma_n + 2\tau_n\sin\theta)^2}} = \frac{1}{\sqrt{1 + \tau_n^2(\sigma_n/\tau_n + 2\sin\theta)^2}} \to 0$$

uniformly for $-\pi/2 \leqslant \theta \leqslant \pi/2$. If $|l| \leqslant 2$, one has $\sigma_n/\tau_n + 2\sin\theta \to l + 2\sin\theta$ uniformly for $- \leqslant \theta \leqslant 2\pi$, and, consequently,

$$\frac{1}{\sqrt{1 + (\sigma_n + 2\tau_n\sin\theta)^2}} \to 0$$

for every $-\pi/2 \leqslant \theta \leqslant \pi/2$ except $\theta = -\arcsin(l/2)$. In both cases, one has $\mathcal{I}(\sigma_n\tau_n) \to 0$ (in the second case, Lebesgue's dominated convergence theorem is used), and obtain a contradiction. This shows (iv).

Let (a,b), in the upper half disk, be given. By (i) and (ii), there exists a unique function $\sigma = \sigma_a(\tau)$ such that $\mathcal{R}(\sigma_a(\tau),\tau) = a$ for $\tau > 0$. Furthermore, by the implicit function theorem, $\sigma_a(\tau)$ is a continuous function of τ and

$$\sigma_a'(\tau) = -\left(\frac{\partial\mathcal{R}}{\partial\tau}\Big/\frac{\partial\mathcal{R}}{\partial\sigma}\right)_{(\sigma_a(\tau),\tau)}.$$

Thus, by (i) and (iii),

$$\frac{d\mathcal{I}(\sigma_a(\tau),\tau)}{d\tau} = \left(\frac{\partial\mathcal{R}}{\partial\sigma}\frac{\partial\mathcal{I}}{\partial\tau} - \frac{\partial\mathcal{R}}{\partial\tau}\frac{\partial\mathcal{I}}{\partial\sigma}\right)_{(\sigma_a(\tau),\tau)}\Big/\frac{\partial\mathcal{R}}{\partial\sigma}(\sigma_a(\tau),\tau) < 0, \quad \tau > 0.$$

This shows that $\mathcal{I}(\sigma_a(\tau),\tau)$ is strictly decreasing for $\tau > 0$. It is easy to see that $\lim_{\tau\to 0}\sigma_a(\tau) = l$ exists and is finite, and $l/\sqrt{1 + l^2} = a$. Thus,

$$\lim_{\tau\to 0}\mathcal{I}(\sigma_a(\tau),\tau) = \frac{1}{\sqrt{1 + l^2}} = \sqrt{1 - a^2}.$$

On the other hand, $\mathcal{I}(\sigma_a(\tau)),\tau) \to 0$ as $\tau \to +\infty$ by (iv). Thus, there exists a unique $\tau > 0$ such that $I(\sigma_a(\tau)),\tau) = b$. This shows the existence and uniqueness of the functions $\sigma(a,b)$ and $\tau(a,b)$. By using the implicit function theorem, one concludes that the two functions are real analytical functions of (a,b), since $\mathcal{R}(\sigma,\tau)$ and $\mathcal{I}(\sigma,\tau)$ are real analytical functions of (σ,τ). The lemma is proved. □

Lemma 5. *Let $\lambda = \lambda(r,a,b)$ and $\mu = \mu(r,a,b)$ be the functions defined in Lemma* 1. *Then, for any fixed a, b with $a^2 + b^2 < 1$ and $b > 0$,*

$$-\frac{\lambda(r,a,b) - 1}{\mu(r,a,b)} \to \sigma(a,b), \quad \frac{r}{\mu(r,a,b)} \to \tau(a,b), \quad \text{as } r \to 0,$$

where $\sigma(a,b)$ and $\tau(a,b)$ are functions defined in Lemma 4.

Proof. Let a and b be fixed. Then

$$A_{r_n,\lambda_n,\mu_n}(\theta) = \frac{1}{\sqrt{1 + \frac{(\lambda_n-1)^2}{\mu_n^2}}}(1 + o(1)), \quad \frac{A_{r_n,\lambda_n,\mu_n}(\theta)}{\sqrt{1 + A^2_{r_n,\lambda_n,\mu_n}(\theta)}} = -\frac{\frac{\lambda_n-1}{\mu_n}}{\sqrt{1 + \frac{(\lambda_n-1)^2}{\mu_n^2}}} + o(1),$$

where $o(1)$ denotes a quantity which tends to 0, as $n \to \infty$, uniformly for $0 \leqslant \theta \leqslant 2\pi$. Thus,

$$a = R(r_n,\lambda_n,\mu_n) = -\frac{\frac{\lambda_n-1}{\mu_n}}{\sqrt{1 + \frac{(\lambda_n-1)^2}{\mu_n^2}}} + o(1), \quad b = I(r_n,\lambda_n,\mu_n) = \frac{1}{\sqrt{1 + \frac{(\lambda_n-1)^2}{\mu_n^2}}} + o(1).$$

Consequently, $a^2 + b^2 = 1 + o(1)$, a contradiction.

In the case that $r_n/\mu_n \to \infty$, write

$$A_{r_n,\lambda_n,\mu_n}(\theta) = \frac{r_n}{\mu_n}\left(-\frac{\lambda_n - 1}{r_n} + \frac{2\sin\theta - r_n}{1 + r_n^2 - 2r_n\sin\theta}\right),$$

and, without loss of generality, one may assume that $(\lambda_n - 1)/r_n$ has a limit l (l may be ∞). Then, $A_{r_n,\lambda_n,\mu_n}(\theta) \to \infty$ for $0 \leqslant \theta \leqslant 2\pi$ with two exceptions at most. Using Lebesgue's dominated convergence theorem, one obtains $b = I(r_n, \lambda_n, \mu_n) \to 0$, a contradiction. (i) is proved.

To prove (ii), assume to the contrary that there exists a sequence $r_n \to 0$ such that $(\lambda_n - 1)/\mu_n \to \infty$. Then,

$$A_{r_n,\lambda_n,\mu_n}(\theta) = -\frac{\lambda_n - 1}{\mu_n} + \frac{r_n}{\mu_n}\frac{2\sin\theta - r_n}{1 + r_n^2 - 2r_n\sin\theta} \to \infty$$

uniformly for $0 \leqslant \theta \leqslant 2\pi$, since r_n/μ_n is bounded by (i). Thus, $b = I(r_n, \lambda_n, \mu_n) \to 0$, a contradiction. This shows (ii).

On the the basis of (i) and (ii), one can prove the conclusion of the lemma. Assume to the contrary that there exist $r_n \to 0$ and a $\delta > 0$ such that

$$\left|\frac{\lambda_n - 1}{\mu_n} + \sigma(a,b)\right| \geqslant \delta \quad \text{or} \quad \left|\frac{r_n}{\mu_n} - \tau(a,b)\right| \geqslant \delta.$$

By (i) and (ii) proved above, without of loss generality, one may assume that $-(\lambda_n - 1)/\mu_n$ and r_n/μ_n have finite limits σ' and $\tau' > 0$ respectively, which satisfy

$$|\sigma' - \sigma(a,b)| \geqslant \delta \quad \text{or} \quad |\tau' - \tau(a,b)| \geqslant \delta.$$

Then,

$$A_{r_n,\lambda_n,\mu_n}(\theta) = -\frac{\lambda_n - 1}{\mu_n} + \frac{r_n}{\mu_n}\frac{2\sin\theta - r_n}{1 + r_n^2 - 2r_n\sin\theta} \to \sigma' + 2\tau'\sin\theta$$

uniformly for $0 \leqslant \theta \leqslant 2\pi$ and, consequently,

$$a = R(r_n, \lambda_n, \mu_n) \to \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \frac{(\sigma' + 2\tau'\sin\theta)d\theta}{\sqrt{1 + (\sigma' + 2\tau'\sin\theta)^2}} = \mathcal{R}(\sigma', \tau'),$$

$$b = I(r_n, \lambda_n, \mu_n) \to \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \frac{d\theta}{\sqrt{1 + (\sigma' + 2\tau'\sin\theta)^2}} = \mathcal{I}(\sigma', \tau').$$

It follows that $a = \mathcal{R}(\sigma', \tau')$ and $b = \mathcal{I}(\sigma', \tau')$. By Lemma 4, σ' and τ' must be equal to $\sigma(a,b)$ and $\tau(a,b)$ respectively. A contradiction, and the lemma is proved. □

Theorem 4. *Let a, b be fixed real numbers with $a^2 + b^2 < 1$ and $b \geqslant 0$ Then,*

$$\lim_{r\to 0}\frac{F_{a,b,r}(ri) - (a + bi)}{r} = \begin{cases} \dfrac{2}{\pi}\displaystyle\int_{-\pi/2}^{\pi/2} \frac{\sin\theta(\sigma + 2\tau\sin\theta)d\theta}{\sqrt{1 + (\sigma + 2\tau\sin\theta)^2}} + \frac{2i}{\pi}\int_{-\pi/2}^{\pi/2} \frac{\sin\theta d\theta}{\sqrt{1 + (\sigma + 2\tau\sin\theta)^2}}, & b > 0,\\ \dfrac{4}{\pi}\cos\dfrac{a\pi}{2}, & b = 0, \end{cases}$$

where $\sigma = \sigma(a,b)$ and $\tau = \tau(a,b)$ are defined in Lemma 4.

Proof. First assume that $b > 0$. Then, let $\lambda = \lambda(r,a,b)$ and $\mu = \mu(r,a,b)$ be defined in Lemma 1. Using Lemma 5, one has, as $r \to 0$,

$$\begin{aligned} A_{r,\lambda,\mu}(\theta) &= -\frac{\lambda - 1}{\mu} + \frac{2r\sin\theta - r^2}{\mu(1 + r^2 - 2r\sin\theta)} = -\frac{\lambda - 1}{\mu} + \frac{2r\sin\theta}{\mu} + O(r) \\ &= \sigma(a,b) + 2\tau(a,b)\sin\theta + o(1), \end{aligned}$$

and consequently,

$$\frac{1}{\sqrt{1+A_{r,\lambda,\mu}^2(\theta)}} = \frac{1}{\sqrt{1+(\sigma+2\tau\sin\theta)^2}} + o(1),$$

$$\frac{A_{r,\lambda,\mu}(\theta)}{\sqrt{1+A_{r,\lambda,\mu}^2(\theta)}} = \frac{\sigma+2\tau\sin\theta}{\sqrt{1+(\sigma+2\tau\sin\theta)^2}} + o(1),$$

where $o(1)$ is a quantity which tends to 0, uniformly for $0 \leqslant \theta \leqslant 2\pi$, as $r \to 0$. Thus,

$$\begin{aligned} F_{a,b,r}(ri) - (a+bi) &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\left(\frac{1-r^2}{1+r^2-2r\sin\theta} - 1\right)\frac{A_{r,\lambda,\mu}(\theta)d\theta}{\sqrt{1+A_{r,\lambda,\mu}^2(\theta)}} \\ &\quad + \frac{i}{\pi}\int_{-\pi/2}^{\pi/2}\left(\frac{1-r^2}{1+r^2-2r\sin\theta} - 1\right)\frac{d\theta}{\sqrt{1+A_{r,\lambda,\mu}^2(\theta)}} \\ &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\frac{2r(\sin\theta - r)}{1+r^2-2r\sin\theta}\left(\frac{(\sigma+2\tau\sin\theta)}{\sqrt{1+(\sigma+2\tau\sin\theta)^2}} + o(1)\right)d\theta \\ &\quad + \frac{i}{\pi}\int_{-\pi/2}^{\pi/2}\frac{2r(\sin\theta - r)}{1+r^2-2r\sin\theta}\left(\frac{1}{\sqrt{1+(\sigma+2\tau\sin\theta)^2}} + o(1)\right)d\theta. \end{aligned}$$

This shows the lemma for $b > 0$.

For $b = 0$, recall that $u_{a,0,r}$ is defined in the proof of Theorem 1. Recall that

$$U_{a,0,r}(z) = \frac{2}{\pi}\left[\arg\frac{e^{i(\pi+a\pi/2)} - z}{e^{-ia\pi/2} - z} - \pi\left(1 - \frac{a}{2}\right)\right], \quad V_{a,0,r}(z) \equiv 0,$$

and

$$F_{a,0,r}(ri) = U_{a,0,r}(ri) = \frac{4}{\pi}\arctan\frac{r+\sin(a\pi/2)}{\cos(a\pi/2)} - a.$$

Thus,

$$\lim_{r\to 0}\frac{F_{a,0,r}(ri) - a}{r} = \frac{d}{dr}\left(\frac{4}{\pi}\arctan\frac{r+\sin(a\pi/2)}{\cos(a\pi/2)}\right)_{r=0} = \frac{4}{\pi}\cos\frac{a\pi}{2}.$$

The theorem is proved. □

5 The Schwarz-Pick lemma for harmonic mappings (II)

For $0 < \rho < 1$ and $0 \leqslant \alpha \leqslant \pi/2$, define

$$h_\rho(\alpha) = \begin{cases} \dfrac{4}{\pi}\cos\dfrac{\rho\pi}{2}, & \alpha = 0, \\ \dfrac{2}{\pi}\displaystyle\int_{-\pi/2}^{\pi/2}\frac{\sin\theta(\sigma+2\tau\sin\theta)d\theta}{\sqrt{1+(\sigma+2\tau\sin\theta)^2}}, & 0 < \alpha \leqslant \pi/2, \end{cases}$$

and

$$k_\rho(\alpha) = \begin{cases} 0, & \alpha = 0, \\ -\dfrac{2}{\pi}\displaystyle\int_{-\pi/2}^{\pi/2}\frac{\sin\theta d\theta}{\sqrt{1+(\sigma+2\tau\sin\theta)^2}}, & 0 < \alpha \leqslant \pi/2, \end{cases}$$

where $\sigma = \sigma(a,b)$ and $\tau = \tau(a,b)$, with $a = \rho\cos\alpha$ and $b = \rho\sin\alpha$, are defined in Lemma 4.

Lemma 6. *Let h_ρ and k_ρ be defined as above. Then*:

(i) *As $\alpha \to 0$, one has*

$$h_\rho(\alpha) \to \frac{4}{\pi}\cos\frac{\rho\pi}{2}, \quad k_\rho(\alpha) \to 0;$$

(ii) $h_\rho(\alpha)$ *is strictly increasing for* $0 \leqslant \alpha \leqslant \pi/2$;

(iii) $k(0) = k(\pi/2) = 0$, $k_\rho(\alpha) > 0$ *for* $0 < \alpha < \pi/2$.

Proof. If there exists a positive sequence $\alpha_n \to 0$ such that $\tau_n = \tau(\rho\cos\alpha_n, \rho\sin\alpha_n)$ is bounded, then, $\sigma_n = \sigma(\rho\cos\alpha_n, \rho\sin\alpha_n)$ must be unbounded, since otherwise $\rho\sin\alpha_n = \mathcal{I}(\sigma_n, \tau_n)$ will be bounded from below. By choosing a subsequence, one may assume that $\sigma_n \to \infty$. However, this results in $\rho\cos\alpha_n = \mathcal{R}(\sigma_n, \tau_n) \to 1$, a contradiction. This shows that $\tau = \tau(\rho\cos\alpha, \rho\sin\alpha) \to \infty$ as $\alpha \to 0$.

It is easy to see that if there exists a positive sequence $\alpha_n \to 0$ such that $\sigma_n/\tau_n \to 0$, or $+\infty$, or $-\infty$, then $\rho\cos\alpha_n = \mathcal{R}(\sigma_n, \tau_n) \to 0$, or 1, or -1, respectively, which contradicts $0 < \rho < 1$. So, $\sigma(\rho\cos\alpha, \rho\sin\alpha)/\tau(\rho\cos\alpha, \rho\sin\alpha)$ is bounded from above and below. Let $\alpha_n \to 0$ be a sequence such that

$$\sigma_n/\tau_n = \sigma(\rho\cos\alpha_n, \rho\sin\alpha_n)/\tau(\rho\cos\alpha_n, \rho\sin\alpha_n) \to l \neq 0.$$

Then,

$$\rho\cos\alpha_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \frac{(\sigma_n + 2\tau_n\sin\theta)d\theta}{\sqrt{1 + (\sigma_n + 2\tau_n\sin\theta)^2}} \to \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \frac{(l + 2\sin\theta)d\theta}{|l + 2\sin\theta|},$$

and consequently,

$$\frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \frac{(l + 2\sin\theta)d\theta}{|l + 2\sin\theta|} = \rho.$$

This implies $l = 2\sin(\rho\pi/2)$ and it is proved that

$$\frac{\sigma(\rho\cos\alpha, \rho\sin\alpha)}{\tau(\rho\cos\alpha, \rho\sin\alpha)} \to 2\sin(\rho\pi/2) \quad \text{as} \quad \alpha \to 0.$$

Thus, as $\alpha \to 0$, one has

$$k_\rho(\alpha) \to 0,$$

and

$$h_\rho(\alpha) \to \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} \frac{\sin\theta(\sin(\rho\pi/2) + \sin\theta)d\theta}{|\sin(\rho\pi/2) + \sin\theta|} = \frac{2}{\pi}\int_{-\rho\pi/2}^{\pi/2} \sin\theta d\theta - \frac{2}{\pi}\int_{-\pi/2}^{-\rho\pi/2} \sin\theta d\theta = \frac{4}{\pi}\cos\frac{\rho\pi}{2}.$$

This shows (i).

Now, one proceeds to prove (ii). Let

$$T = \int_{-\pi/2}^{\pi/2} \frac{\sin\theta d\theta}{\sqrt{1 + (\sigma + 2\tau\sin\theta)^2}},$$

and

$$T_j = \int_{-\pi/2}^{\pi/2} \frac{\sin^j\theta d\theta}{\sqrt{(1 + (\sigma + 2\tau\sin\theta)^2)^3}}, \quad j = 0, 1, 2.$$

It follows from $\mathcal{R}(\sigma(a,b), \tau(a,b)) = a$ and $\mathcal{I}(\sigma(a,b), \tau(a,b)) = b$ that

$$\frac{\partial\sigma}{\partial a} = \frac{1}{D}\frac{\partial\mathcal{I}}{\partial\tau}, \quad \frac{\partial\sigma}{\partial b} = -\frac{1}{D}\frac{\partial\mathcal{R}}{\partial\tau}, \quad \frac{\partial\tau}{\partial a} = -\frac{1}{D}\frac{\partial\mathcal{I}}{\partial\sigma}, \quad \frac{\partial\tau}{\partial b} = \frac{1}{D}\frac{\partial\mathcal{R}}{\partial\sigma},$$

and consequently,

$$\frac{d\sigma}{d\alpha} = -\frac{\rho\sin\alpha}{D}\frac{\partial\mathcal{I}}{\partial\tau} - \frac{\rho\cos\alpha}{D}\frac{\partial\mathcal{R}}{\partial\tau}, \quad \frac{d\tau}{d\alpha} = \frac{\rho\sin\alpha}{D}\frac{\partial\mathcal{I}}{\partial\sigma} + \frac{\rho\cos\alpha}{D}\frac{\partial\mathcal{R}}{\partial\sigma},$$

where

$$D = \frac{\partial\mathcal{R}}{\partial\sigma}\frac{\partial\mathcal{I}}{\partial\tau} - \frac{\partial\mathcal{R}}{\partial\tau}\frac{\partial\mathcal{I}}{\partial\sigma}.$$

Thus,

$$h_\rho'(\alpha) = -\left(\frac{\rho\sin\alpha}{D}\frac{\partial\mathcal{I}}{\partial\tau} + \frac{\rho\cos\alpha}{D}\frac{\partial\mathcal{R}}{\partial\tau}\right)\int_{-\pi/2}^{\pi/2} \frac{\sin\theta d\theta}{\sqrt{(1 + (\sigma + 2\tau\sin\theta)^2)^3}}$$

$$+\left(\frac{\rho\sin\alpha}{D}\frac{\partial\mathcal{I}}{\partial\sigma}+\frac{\rho\cos\alpha}{D}\frac{\partial\mathcal{R}}{\partial\sigma}\right)\int_{-\pi/2}^{\pi/2}\frac{2\sin^2\theta d\theta}{\sqrt{(1+(\sigma+2\tau\sin\theta)^2)^3}}$$
$$=\frac{2\rho}{\pi D}(T_0T_2-T_1^2)(\cos\alpha-\sigma\sin\alpha)=\frac{4\tau T}{\pi^2 D}(T_0T_2-T_1^2),$$

since $\rho(\cos\alpha-\sigma\sin\alpha)=a-\sigma b=\mathcal{R}(\sigma,\tau)-\sigma\mathcal{I}(\sigma,\tau)=2\tau T/\pi$.

Since $\mathcal{R}(0,\tau)=0$ for $\tau>0$ and $\partial\mathcal{R}(\sigma,\tau)/\partial\sigma>0$ for $\tau>0$ and any real σ, one has $\sigma(0,b)=0$, $\sigma(a,b)>0$ for $a>0$, and $\sigma(a,b)<0$ for $a<0$. It is easy to see that $T=0$ if $\sigma=0$, $T<0$ if $\sigma>0$ and $T>0$ if $\sigma<0$. Therefore, $T=0$ if $\alpha=\pi/2$, $T<0$ if $0<\alpha<\pi/2$. It is indicated in the proof of Lemma 4 that $D<0$ for any σ and $\tau>0$. Using the Schwarz inequality gives $T_0T_2-T_1^2>0$ for any σ and $\tau>0$. Thus, one concludes that $h_\rho'(\alpha)>0$ for $0<\alpha<\pi/2$. This shows (ii).

Since $k_\rho(\alpha)=-(2/\pi)T$, (iii) follows from what we indicated above for T. The lemma is proved. □

Lemma 7. *Let $0<\rho<1$. For $0\leqslant\alpha\leqslant\pi/2$, define*

$$\omega_\rho(\alpha)=\alpha+\arctan\frac{k_\rho(\alpha)}{h_\rho(\alpha)}.$$

Then:

(i) $\omega_\rho(\alpha)=\alpha$ *for* $\alpha=0,\pi/2$, $\omega_\rho(\alpha)>\alpha$ *for* $0<\alpha<\pi/2$;

(ii) $\omega_\rho(\alpha)$ *is strictly increasing for* $0\leqslant\alpha\leqslant\pi/2$.

Proof. (i) follows from (iii) in Lemma 6.

For given $0\leqslant\alpha<\beta\leqslant\pi/2$, let

$$z_\alpha=e^{i\alpha}(h_\rho(\alpha)+ik_\rho(\alpha)),\quad z_\beta=e^{i\beta}(h_\rho(\beta)+ik_\rho(\beta)).$$

Note that $\arg z_\alpha=\omega_\rho(\alpha)$ and $\arg z_\beta=\omega_\rho(\beta)$. Using Lemma 2 to the function $e^{i(\beta-\alpha)}F_{\rho\cos\alpha,\rho\sin\alpha,r}$, one has

$$\mathrm{Re}\{e^{i(\beta-\alpha)}F_{\rho\cos\alpha,\rho\sin\alpha,r}(ri)\}\leqslant\mathrm{Re}\{F_{\rho\cos\beta,\rho\sin\beta,r}(ri)\},$$

and consequently,

$$\mathrm{Re}\left\{\frac{e^{i(\beta-\alpha)}(F_{\rho\cos\alpha,\rho\sin\alpha,r}(ri)-\rho e^{i\alpha})}{r}\right\}\leqslant\mathrm{Re}\left\{\frac{F_{\rho\cos\beta,\rho\sin\beta,r}(ri)-\rho e^{i\beta}}{r}\right\}.$$

Letting $r\to 0$ and using Theorem 4 give

$$\mathrm{Re}\{e^{i(\beta-\alpha)}(h_\rho(\alpha)-ik_\rho(\alpha))\}\leqslant h_\rho(\beta).$$

Thus,

$$p_\alpha=\mathrm{Re}\{e^{-i\beta}z_\alpha\}\leqslant h_\rho(\beta).$$

Symmetrically, $p_\beta=\mathrm{Re}\{e^{-i\alpha}z_\beta\}\leqslant h_\rho(\alpha)$. Note $k_\rho(\alpha),k_\rho(\beta)\geqslant 0$ by (iii) in Lemma 6.

Since $\omega_\rho(\alpha)<\alpha+\pi/2$, to prove that $\omega_\rho(\alpha)<\omega_\rho(\beta)$, one may assume that $\omega_\rho(\beta)=\arg z_\beta<\alpha+\pi/2$. Then, $p_\beta>0$, $z_\beta=p_\beta e^{i\alpha}+i|z_\beta-p_\beta e^{i\alpha}|e^{i\alpha}$ and, consequently,

$$h_\rho(\beta)=\mathrm{Re}\{e^{-i\beta}z_\beta\}=p_\beta\cos(\beta-\alpha)+|z_\beta-p_\beta e^{i\alpha}|\sin(\beta-\alpha).$$

On the other hand,

$$p_\alpha=h_\rho(\alpha)\cos(\beta-\alpha)+k_\rho(\alpha)\sin(\beta-\alpha).$$

Thus, it follows from $p_\alpha\leqslant h_\rho(\beta)$ and $p_\beta\leqslant h_\rho(\alpha)$ that

$$\begin{aligned}p_\beta\cos(\alpha-\beta)+|z_\beta-p_\beta e^{i\alpha}|\sin(\beta-\alpha)&\geqslant h_\rho(\alpha)\cos(\beta-\alpha)+k_\rho(\alpha)\sin(\beta-\alpha)\\&\geqslant p_\beta\cos(\beta-\alpha)+k_\rho(\alpha)\sin(\beta-\alpha),\end{aligned}$$

from which one obtains $k_\rho(\alpha) \leqslant |z_\beta - p_\beta e^{i\alpha}|$. Now, one has

$$\omega_\rho(\beta) = \arg z_\beta = \alpha + \arctan \frac{|z_\beta - p_\beta e^{i\alpha}|}{p_\beta} \geqslant \alpha + \arctan \frac{k_\rho(\alpha)}{h_\rho(\alpha)} = \omega_\rho(\alpha).$$

This shows that $\omega_\rho(\alpha)$ is increasing for $0 \leqslant \alpha \leqslant \pi/2$.

Assume that $\omega_\rho(\alpha_2) = \omega_\rho(\alpha_1)$ for some α_1, α_2 with $0 < \alpha_1 < \alpha_2 < \pi/2$. Then, $\omega(\alpha)$ is a constant for $\alpha_1 \leqslant \alpha \leqslant \alpha_2$. Since $\mathcal{R}(\sigma,\tau)$ and $\mathcal{I}(\sigma,\tau)$ are analytical functions of (σ,τ) in the real sense and, by Lemma 4, $\sigma(a,b)$ and $\tau(a,b)$ are analytical functions of variables (a,b) in the real sense, one sees that $h_\rho(\alpha)$, $k_\rho(\alpha)$ and $\omega_\rho(\alpha)$ are analytical functions for $0 < \alpha < \pi/2$ in the real sense. Thus, $\omega(\alpha)$ is a constant for $\alpha_1 \leqslant \alpha \leqslant \alpha_2$, which implies that $\omega(\alpha)$ is a constant on $(-\pi/2, 0)$. However, $\omega_\rho(0) = 0$ and $\omega_\rho(\pi/2) = \pi/2$. Then one receives a contradiction since ω is continuous. This shows (ii), and the lemma is proved. □

Lemma 8. *Let $0 < \rho < 1$. For $0 \leqslant \alpha \leqslant \pi/2$, define*

$$g_\rho(\alpha) = (h_\rho^2(\alpha) + k_\rho^2(\alpha))^{1/2}.$$

Then:

(i) $g_\rho(\alpha)$ *is strictly increasing for* $0 \leqslant \alpha \leqslant \pi/2$;

(ii) $g(\omega_\rho^{-1}(\alpha)) \leqslant h_\rho(\alpha)$ *for* $0 \leqslant \alpha \leqslant \pi/2$.

Proof. Let $0 < \beta < \pi/2$ be given. Since $\omega(\beta) > \beta$, by the continuity of ω, there exists a $\delta > 0$ such that $\omega(\alpha) > \beta$ for $\beta - \delta < \alpha < \beta$. Assume that $\alpha \in (\beta - \delta, \beta)$. As in the proof of the above lemma, denoting $z_\alpha = e^{i\alpha}(h_\rho(\alpha) + ik_\rho(\alpha))$, one has

$$p_\alpha = \mathrm{Re}\{e^{-i\beta} z_\alpha\} \leqslant h_\rho(\beta).$$

Since $\beta + \pi/2 > \omega(\beta) > \omega(\alpha) > \beta$ by the above lemma,

$$\begin{aligned} g_\rho^2(\alpha) &= |z_\alpha|^2 = |e^{-i\beta} z_\alpha|^2 = p_\alpha^2(1 + \tan^2(\omega(\alpha) - \beta)) \\ &< h_\rho^2(\beta)(1 + \tan^2(\omega(\beta) - \beta)) = h_\rho^2(\beta) + k_\rho^2(\beta) = g_\rho^2(\beta). \end{aligned}$$

It is proved that $g_\rho(\alpha) < g_\rho(\beta)$ if α is very close to β from left. Also, one can prove that $g_\rho(\alpha) > g_\rho(\beta)$ if α is very close to β from right. Thus, by a normal argument one can conclude that $g_\rho(\alpha)$ is strictly decreasing for $0 \leqslant \alpha \leqslant \pi/2$. (i) is proved.

To prove (ii), let $0 \leqslant \alpha \leqslant \pi/2$, $\beta = \omega_\rho^{-1}(\alpha)$ and $z_\beta = z^{i\beta}(h_\rho(\beta) + ik_\rho(\beta))$. Then, $\arg z_\beta = \omega_\rho(\beta) = \alpha$ and, as in the proof of the above lemma, $g_\rho(\omega_\rho^{-1}(\alpha)) = |z_\beta| = \mathrm{Re}\{e^{-i\alpha} z_\beta\} \leqslant h_\rho(\alpha)$. This shows (ii). The lemma is proved. □

Lemma 9. *Let $w = F(z) = U(z) + iV(z)$, $z = x + iy$, be a harmonic mapping such that $F(D) \subset D$. If $\rho = F(0) > 0$ and $F_x(0) = te^{-i\alpha}$ with $t > 0$, then*

$$|F_x(0)| \leqslant g_\rho(\omega^{-1}(\kappa(\alpha))), \tag{5.1}$$

where

$$\kappa(\alpha) = \begin{cases} |\alpha|, & |\alpha| \leqslant \pi/2; \\ \pi - |\alpha|, & \pi/2 \leqslant |\alpha| \leqslant \pi. \end{cases}$$

Proof. First assume that $0 \leqslant \alpha \leqslant \pi/2$. Let $0 \leqslant \beta = \omega^{-1}(\alpha) \leqslant \alpha$. Applying Lemma 2 to $e^{i\beta}F$, one has, for $x > 0$,

$$\mathrm{Re}\{e^{i\beta} F(x)\} \leqslant U_{\rho\cos\beta, \rho\sin\beta, x}(xi),$$

and consequently,

$$\mathrm{Re}\left\{e^{i\beta} \frac{F(x) - \rho}{x}\right\} \leqslant \frac{U_{\rho\cos\beta, \rho\sin\beta, x}(xi) - \rho\cos\beta}{x}.$$

Letting $x \to 0$, by Theorem 4, one obtains

$$t\cos(\alpha-\beta) \leqslant h_\rho(\beta).$$

On the other hand,

$$\alpha-\beta = \omega(\beta)-\beta = \arctan\frac{k_\rho(\beta)}{h_\rho(\beta)},$$

$$\frac{1}{\cos^2(\alpha-\beta)} = 1+\tan^2(\alpha-\beta) = 1+\frac{k_\rho^2(\beta)}{h_\rho^2(\beta)}.$$

Thus,

$$t \leqslant (h_\rho^2(\beta)+k_\rho^2(\beta))^{1/2} = g_\rho(\omega^{-1}(\alpha)).$$

This shows (5.1) for $0 \leqslant \alpha \leqslant \pi/2$.

If $-\pi/2 \leqslant \alpha \leqslant 0$ or $\pi/2 \leqslant |\alpha| \leqslant \pi$, one can prove (5.1) by considering the mapping G defined by $G(z) = \overline{F(z)}$, $G(z) = F(-z)$ or $G(z) = \overline{F(-z)}$ for $z \in D$. The lemma is proved. □

Now, define a Finsler metric $\mathcal{H}$ on the unit disk D. For $z \in D$ and $u \in \mathbb{C} \setminus \{0\}$, define

$$\mathcal{H}_0(u) = \frac{\pi}{4}|u|, \quad \mathcal{H}_z(u) = \frac{|u|}{g_{|z|}(\omega_{|z|}^{-1}(\kappa(\arg\frac{u}{z})))}.$$

Theorem 5. *Let F be a harmonic mapping such that $F(D) \subset D$ and $z \in D$. Then,*

$$\mathcal{H}_{|F(z)|}(e^{i\theta}F_z(z)+e^{-i\theta}F_{\overline{z}}(z)) \leqslant \frac{1}{1-|z|^2} \tag{5.2}$$

holds for $z \in D$ and $0 \leqslant \theta \leqslant 2\pi$.

Proof. Let $z \in D$ and $0 \leqslant \theta \leqslant 2\pi$ be fixed. If $F(z) = 0$, then

$$\mathcal{H}_{|F(z)|}(e^{i\theta}F_z(z)+e^{-i\theta}F_{\overline{z}}(z)) = \frac{\pi}{4}|e^{i\theta}F_z(z)+e^{-i\theta}F_{\overline{z}}(z)| \leqslant \frac{\pi}{4}\Lambda_F(z),$$

and (5.2) follows from (1.6). Now, assume that $F(z) = \rho e^{i\theta_0}$ with $\rho > 0$, and $e^{i\theta}F_z(z)+e^{-i\theta}F_{\overline{z}}(z) = te^{i(\theta_0+\alpha)}$ with $t > 0$. Let

$$\phi(\zeta) = \frac{z+e^{i\theta}\zeta}{1+e^{i\theta}\overline{z}\zeta} \quad \text{for} \quad \zeta = \xi+i\eta \in D,$$

and $G = e^{-i\theta_0}F\circ\phi$. Then, G is a harmonic mapping such that $G(D) \in D$, $G(0) = \rho$, and

$$e^{-i\theta_0}(1-|z|^2)(e^{i\theta}F_z(z)+e^{-i\theta}F_{\overline{z}}(z)) = G_\zeta(0)+G_{\overline{\zeta}}(0) = G_\xi(0).$$

Note that $G_\xi(0) \neq 0$ and $\arg G_\xi(0) = \alpha$. Thus, using Lemma 9 to the mapping G, one obtains

$$(1-|z|^2)|e^{i\theta}F_z(z)+e^{-i\theta}F_{\overline{z}}(z)| = |G_\xi(0)| \leqslant g_\rho(\omega_\rho^{-1}(\kappa(\alpha))).$$

(5.2) follows and the theorem is proved. □

6 Λ_F and λ_F

Theorem 6. *Let $F(z)$ be a harmonic mapping such that $F(D) \subset D$. Then,*

$$\frac{\Lambda_F(z)}{h_{|F(z)|}(\pi/2)} \leqslant \frac{1}{1-|z|^2} \tag{6.1}$$

and

$$\frac{\lambda_F(z)}{\cos(|F(z)|\pi/2)} \leqslant \frac{4}{\pi}\frac{1}{(1-|z|^2)} \tag{6.2}$$

holds for $z \in D$.

Proof. Since $g_\rho(\omega_\rho^{-1}(\kappa(\alpha))) \leqslant h_\rho(\kappa(\alpha)) \leqslant h_\rho(\pi/2)$ for $\rho > 0$ and $-\pi \leqslant \alpha \leqslant \pi$ by (ii) of Lemma 8 and 6, (6.1) is a consequence of (5.2). Note that $h_0(\pi/2)$ is regarded as $4/\pi$. To prove (6.2), let $z \in D$ be fixed. If $F(z) = 0$, (6.2) follows from (6.1). Now, assume that $F(z) \neq 0$ and $\lambda_F(z) = ||F_z(z)| - |F_{\overline{z}}(z)|| > 0$. Then, there exists a θ such that

$$\arg\{e^{i\theta} F_z(z) + e^{i\theta} F_{\overline{z}}(z)\} = \arg F(z).$$

By (5.2), one has

$$\lambda_F(z) \leqslant |e^{i\theta} F_z(z) + e^{i\theta} F_{\overline{z}}(z)| \leqslant \frac{g_{|F(z)|}(0)}{1-|z|^2} = \frac{h_{|F(z)|}(0)}{1-|z|^2} = \frac{4}{\pi} \frac{\cos(|F(z)|\pi/2)}{(1-|z|^2)}.$$

This shows (6.2) and the theorem is proved. □

According the definition of h_ρ,

$$h_\rho\left(\frac{\pi}{2}\right) = \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} \frac{\sin\theta(\sigma + 2\tau\sin\theta)d\theta}{\sqrt{1+(\sigma+2\tau\sin\theta)^2}},$$

where $\sigma = \sigma(0,\rho)$ and $\tau = \tau(0,\rho)$ are defined in Lemma 4. It is easy to see that $\sigma(0,\rho) = 0$. Thus,

$$h_\rho\left(\frac{\pi}{2}\right) = \frac{4\tau}{\pi} \int_{-\pi/2}^{\pi/2} \frac{\sin^2\theta d\theta}{\sqrt{1+4\tau^2\sin^2\theta}},$$

where $\tau = \tau_\rho = \tau(0,\rho)$ is the unique number such that

$$\frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \frac{d\theta}{\sqrt{1+4\tau^2\sin^2\theta}} = \rho.$$

It is obvious that

(i) τ_ρ and $h_\rho(\pi/2)$ are continuous as functions of ρ, and are strictly decreasing as ρ is increasing;

(ii) $\tau_\rho \to \infty$ and $h_\rho(\pi/2) \to 4/\pi$ as $\rho \to 0$, $\tau_\rho \to 0$ and $h_\rho(\pi/2) \to 0$ as $\rho \to 1$.

It follow from (i) and (ii) that $h_\rho(\pi/2) \leqslant 4/\pi$ for $0 \leqslant \rho < 1$. Thus, one has the following consequence: If F is a harmonic mapping of D into itself, then

$$\Lambda_F(z) \leqslant \frac{4}{\pi} \frac{1}{1-|z|^2} \tag{6.3}$$

holds for $z \in D$.

Furthermore, a simple calculation gives $dh_\rho(\pi/2)/d\rho = -1/\tau$ and $\rho = 1 - \tau^2 + O(\tau^4)$ as $\tau \to 0$. Thus,

$$\lim_{\rho\to1} \frac{h_\rho(\pi/2)}{\sqrt{1-\rho^2}} = \lim_{\rho\to1} \frac{\sqrt{1-\rho^2}}{\rho\tau} = \sqrt{2}\lim_{\tau\to0} \frac{\sqrt{1-\rho}}{\tau} = \sqrt{2}.$$

This shows that $h_\rho(\pi/2) \sim \sqrt{2}\sqrt{1-\rho^2}$ as $\rho \to 1$. It seems that $h_\rho(\pi/2)/\sqrt{1-\rho^2}$ is increasing from $4/\pi$ to $\sqrt{2}$ as ρ increases from 0 to 1.

As a consequence of (6.2), for K-quasiregular harmonic mappings, one has the following theorem.

Theorem 7. *If F is a K-quasiregular harmonic mapping of D into itself, then*

$$\frac{\Lambda_F(z)}{\cos(|F(z)|\pi/2)} \leqslant \frac{4}{\pi} \frac{K}{(1-|z|^2)} \tag{6.4}$$

holds for $z \in D$.

7 Examples and applications

It is known that the equality in (1.6) (the special case $F(z) = 0$ of (5.2)) may be attained. The following examples show that the equality in (5.2) can be attained for any values of z, $F(z)$, θ and $\arg\left\{(e^{i\theta}F_z(z) + e^{-i\theta}F_{\overline{z}}(z))/F(z)\right\}$. Without loss of generality, consider the case $z = 0$, $\theta = \pi/2$ and $F(z) = \rho > 0$ only.

Example 1. For $0 < \rho < 1$, define

$$F(z) = \frac{2}{\pi}\left[\arg\frac{e^{i(\pi+\rho\pi/2)} - z}{e^{-i\rho\pi/2} - z} - \pi\left(1 - \frac{\rho}{2}\right)\right].$$

Then, for the real y,

$$F(yi) = \frac{4}{\pi}\arctan\frac{y + \sin(\rho\pi/2)}{\cos(\rho\pi/2)} - \rho.$$

One has

$$e^{i\pi/2}F_z(0) + e^{-i\pi/2}F_{\overline{z}}(0) = i(F_z(0) - F_{\overline{z}}(0)) = F_y(0) = \frac{4}{\pi}\cos\frac{\rho\pi}{2}.$$

Using (5.2) to the mapping F, $z = 0$ and $\theta = \pi/2$, one finds that the equality holds, where $\arg\{e^{i\theta}F_z(z) + e^{-i\theta}F_{\overline{z}}(z)\} = 0$. If one takes $\theta = -\pi/2$, then the equality holds also with $\arg\{e^{i\theta}F_z(z) + e^{-i\theta}F_{\overline{z}}(z)\} = \pi$.

Example 2. For the given $0 < \rho < 1$ and $0 < \alpha \leqslant \pi/2$, let $\beta = \omega_\rho^{-1}(\alpha)$, $\sigma = \sigma(\rho\cos\beta, \rho\sin\beta)$ and $\tau = \tau(\rho\cos\beta, \rho\sin\beta)$ be defined in Lemma 4. Define

$$F(z) = \frac{e^{i\beta}}{2\pi}\int_{-\pi}^{\pi}\frac{1 - |z|^2}{|e^{i\theta} - z|^2}\frac{(\sigma + 2\tau\sin\theta) - i}{\sqrt{1 + (\sigma + 2\tau\sin\theta)^2}}d\theta.$$

One has $F(D) \subset D$, $F(0) = \rho$, and

$$\begin{aligned}
e^{i\pi/2}F_z(0) + e^{-i\pi/2}F_{\overline{z}}(0) &= i(F_z(0) - F_{\overline{z}}(0)) = F_y(0) \\
&= \frac{e^{i\beta}}{\pi}\lim_{y\to 0}\frac{1}{y}\int_{-\pi/2}^{\pi/2}\left(\frac{1 - y^2}{1 + y^2 - 2y\sin\theta} - 1\right)\frac{(\sigma + 2\tau\sin\theta) - i}{\sqrt{1 + (\sigma + 2\tau\sin\theta)^2}}d\theta \\
&= e^{i\beta}(h_\rho(\beta) + ik_\rho(\beta)).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\arg\{e^{i\pi/2}F_z(0) + e^{-i\pi/2}F_{\overline{z}}(0)\} = \beta + \arctan\frac{k_\rho(\beta)}{h_\rho(\beta)} = \omega_\rho(\beta) = \alpha, \\
&|e^{i\pi/2}F_z(0) + e^{-i\pi/2}F_{\overline{z}}(0)| = g_\rho(\omega_\rho^{-1}(\alpha)),
\end{aligned}$$

and this mapping F makes the equality in (5.2) be true for $z = 0$, $F(z) = \rho$, $\theta = \pi/2$ and

$$\arg\{e^{i\pi/2}F_z(0) + e^{-i\pi/2}F_{\overline{z}}(0)\} = \alpha.$$

For any α other than 0 and π, let F be the mapping defined for $\kappa(\alpha)$ as the above. Then, the mapping $\overline{F(z)}$, $F(-z)$ or $\overline{F(-z)}$ makes the equality in (5.2) be true for $\arg\{e^{i\pi/2}F_z(0) + e^{-i\pi/2}F_{\overline{z}}(0)\} = \alpha$.

For bounded holomorphic mappings on the unit disk, the classical Koebe type theorem of Landau [4] says: Let f be a holomorphic function on the unit disc D with $f(0) = 0$ and $f'(0) = 1$. If $|f(z)| < M$ for $z \in D$, then f is univalent on $D_{\rho_0} = \{z : |z| < \rho_0\}$ with

$$\rho_0 = \frac{1}{M + \sqrt{M^2 - 1}},$$

and $f(D\rho_0)$ covers a disc D_{R_0} with

$$R_0 = M\left(\frac{1}{M + \sqrt{M^2 - 1}}\right)^2.$$

Moreover, this result is sharp. Recently, Chen et al. [5] generalized this classical theorem to the harmonic mapping, and later Liu [6] improved their results. Now, by using (4.5) and a theorem of Liu in [6], the author proves the following Landau theorem for harmonic mappings, which has better estimates than theirs.

Lemma 10 [6]. *Let f be a harmonic mapping on the unit disk D such that $f(0) = 0$ and $\lambda_f(0) = 1$. If $\Lambda_f(z) \leqslant \Lambda$ for $z \in D$, then f is univalent in a disk D_{ρ_0} with*

$$\rho_0 = \frac{1}{1 + \Lambda - \frac{1}{\Lambda}},$$

and $f(D_{\rho_0})$ covers a disk D_{R_0} with

$$R_0 = 1 + \left(\Lambda - \frac{1}{\Lambda}\right) \log \frac{\Lambda - \frac{1}{\Lambda}}{1 + \Lambda - \frac{1}{\Lambda}} > \frac{\rho_0}{2}.$$

Theorem 8. *Let f be a harmonic mapping on the unit disk D such that $f(0) = 0$, $f_{\overline{z}}(0) = 0$, $f_z(0) = 1$, and $|f(z)| < M$ for $z \in D$. Then, f is univalent on a disk D_{ρ_0} with*

$$\rho_0 = \frac{1}{\sqrt{2}} \frac{1}{1 + \frac{8M}{\pi} - \frac{\pi}{8M}},$$

and $f(D_{\rho_0})$ contains a disc D_{R_0} with $R_0 = \rho_0/2$.

Proof. Let

$$F(z) = \sqrt{2} f\left(\frac{z}{\sqrt{2}}\right), \quad z \in D.$$

Then, F is a harmonic mapping on D such that $F(0) = 0$, $\lambda_F(0) = \Lambda_F(0) = 1$ and $F(z) < M$ for $z \in D$. Using (6.3) to the mapping f/M, one obtains

$$\Lambda_F(z) = \Lambda_f\left(\frac{z}{\sqrt{2}}\right) \leqslant \frac{4}{\pi} \frac{M}{1 - |z|^2/2} < \frac{8M}{\pi}, \quad z \in D.$$

Thus, by Lemma 10, F is univalent in a disk $D_{\rho'}$ with

$$\rho' = \frac{1}{1 + \frac{8M}{\pi} - \frac{\pi}{8M}},$$

and $F(D_{\rho'})$ covers a disk $D_{R'}$ with $R' > \rho'/2$. The theorem is proved. □

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