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Quantum Weyl symmetric polynomials and the center of quantum group $U_q(\mathfrak{sl}_4)$

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Abstract Suppose that q is not a root of unity, it is proved in this paper that the center of the quantum group $U_q(\mathfrak{sl}_4)$ is isomorphic to a quotient algebra of polynomial algebra with four variables and one relation.

Keywords quantized enveloping algebra, quantum group, quantum symmetric polynomials, center

MSC(2000): 16G10, 17B10, 20C30

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1 Introduction

The Drinfel'd-Jimbo quantum group $U_q(g)$ (see [6, 7, 12]), associated with a complex simple finite-dimensional Lie algebra g, plays a crucial role in the study of the quantum Yang-Baxter equations, two-dimensional solvable lattice models and the invariant of 3-manifolds to the fusion rules of conformal field theory (see, for instance, [14, 15]). In the Lie algebra case, the center of the universal enveloping algebra $U(\mathfrak{g})$ is isomorphic to the algebra $\mathcal{S}(\mathfrak{b})^W$, and the invariants of the Weyl group W are in the symmetric algebra of a Cartan subalgebra $\mathfrak{b}$ of $\mathfrak{g}$. The isomorphism is given by the Harish-Chandra homomorphism (see [4, 5, 10]). Gauger [8] gave the explicit algebraically independent generating set for the center of $U(\mathfrak{g})$, in the case, $\mathfrak{g}$ is one of the simple Lie algebras A_n, B_n, C_n, D_n (see also [1]).

When q is not the root of unity, the Harish-Chandra homomorphism can obviously be generalized to our Drinfel'd-Jimbo quantized enveloping algebra $U = U_q(\mathfrak{g})$, where we now get a homomorphism from $Z(U_q(\mathfrak{g}))$ to the Laurent polynomial algebra U^0. It is easy to show that this homomorphism is injective using the fact that the intersection of the annihilators of all finite-dimensional modules is 0. Using nontrivial homomorphism between Verma modules, one sees that the image of the Harish-Chandra homomorphism is contained in $(U^0)^W$. By analogy with the classical case we might expect equality here, but that is not quite true, and the image turns out to be $(U^0_{\rm ev})^W$, where we denote by $U^0_{\rm ev}$ the span of all k_μ with μ an "even" weight, that is, in 2Λ. Yet, to the best of our knowledge, the minimal generating set for $(U^0_{\rm ev})^W$ was not determined in general. However, if $\mathfrak{g}$ is of Type A_1, then $Z(U) \cong (U^0_{\rm ev})^W$ is generated by the quantum Casimir element and $(U^0_{\rm ev})^W$ is isomorphic to the polynomial algebra $k[x]$. This intriguing result motivates the following question: Is it true that $(U^0_{\rm ev})^W$ is a polynomial algebra?

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The answer is negative in general. In [9], Farkas proved a version of the Shephard-Todd Chevally theorem, which characterized the finite subgroup G of $GL(n,\mathbb{Z})$ such that $(U^0)^G$ is a polynomial ring.

Now our main problem is to describe explicitly the algebra $(U^0_{\rm ev})^W$. There are several results in the literature. Using representation theory, for any $\lambda\in\Lambda$, one may construct an element $z_\lambda\in Z(U_q(\mathfrak{g}))$, and then obtain a basis for $Z(U_q(\mathfrak{g}))$, see [2, 11–14]. In [3], Caldero showed that $U_q(\mathfrak{g})$ is free over its center. By Noether's theorem, $(U^0_{\rm ev})^W$ is a finitely generated algebra, see [9, 16] for details. Now the question is how to describe explicitly the minimal generating set and the relations of generators. In [17], we proved that the center of $U_q(\mathfrak{sl}_3)$ is isomorphic to a quotient algebra of polynomial algebra with three variables and one relation.

In this paper, we contribute mainly to these questions in the case that $\mathfrak{g}$ is the complex simple Lie algebra $\mathfrak{sl}_4$. We shall prove that $(U^0_{\rm ev})^W$ is generated explicitly by four generators with one relation.

2 Quantum group $U_q(\mathfrak{sl}_4)$ and Harish-Chandra homomorphism

Throughout this paper, $k=\mathbb{C}$ denotes the field of complex numbers and q is a non-zero complex number which is not a root of unity. We will adopt the following notations that agree with those of [11]. We denote by U the quantized enveloping algebra over $\mathbb{C}$ of the complex simple Lie algebra of Type A_3. Let

$$A=(a_{ij})_{3\times 3}=\begin{pmatrix}2&-1&0\\-1&2&-1\\0&-1&2\end{pmatrix}$$

be the corresponding Cartan matrix of Type A_3. Recall that the quantized enveloping algebra $U=U_q(\mathfrak{sl}_4)$ is defined as the $\mathbb{C}$ algebra generated by twelve generators $E_1,E_2,E_3,F_1,F_2,F_3,K_1,K_1^{-1},K_2,K_2^{-1},K_3,K_3^{-1}$ subject to the following relations:

$$\begin{aligned}
&K_iK_j=K_jK_i,\quad K_iK_i^{-1}=K_i^{-1}K_i=1,\quad K_iE_j=q^{a_{ij}}E_jK_i,\quad K_iF_j=q^{-a_{ij}}F_jK_i,\\
&E_iF_j-F_jE_i=\frac{\delta_{ij}(K_i-K_i^{-1})}{(q-q^{-1})},\quad E_1E_3=E_3E_1,\quad F_1F_3=F_3F_1,\\
&E_1^2E_2-(q+q^{-1})E_1E_2E_1+E_2E_1^2=0,\quad E_2^2E_1-(q+q^{-1})E_2E_1E_2+E_1E_2^2=0,\\
&E_2^2E_3-(q+q^{-1})E_2E_3E_2+E_3E_2^2=0,\quad E_3^2E_2-(q+q^{-1})E_3E_2E_3+E_2E_3^2=0,\\
&F_1^2F_2-(q+q^{-1})F_1F_2F_1+F_2F_1^2=0,\quad F_2^2F_1-(q+q^{-1})F_2F_1F_2+F_1F_2^2=0,\\
&F_2^2F_3-(q+q^{-1})F_2F_3F_2+F_3F_2^2=0,\quad F_3^2F_2-(q+q^{-1})F_3F_2F_3+F_2F_3^2=0.
\end{aligned}$$

Denote by U^0 the subalgebra of U generated by $K_1,K_1^{-1},K_2,K_2^{-1},K_3,K_3^{-1}$. It follows that $U^0=\mathbb{C}[K_1,K_1^{-1},K_2,K_2^{-1},K_3,K_3^{-1}]$ is the laurent polynomial algebra over $\mathbb{C}$. Suppose that $\Pi=\{\alpha_1,\alpha_2,\alpha_3\}$ is a basis of the root system of the Lie algebra sl$_4$. Let $\Phi=\{\pm\alpha_1,\pm\alpha_2,\pm\alpha_3,\pm(\alpha_1+\alpha_2),\pm(\alpha_2+\alpha_3),\pm(\alpha_1+\alpha_2+\alpha_3)\}$ be the root system of sl$_4$. For each λ in the root lattice $\mathbb{Z}\Phi=\mathbb{Z}\Pi=\mathbb{Z}\alpha_1\oplus\mathbb{Z}\alpha_2\oplus\mathbb{Z}\alpha_3$, we can define an element K_λ in U^0 by

$$K_\lambda=K_1^{\lambda_1}K_2^{\lambda_2}K_3^{\lambda_3},\quad \text{if } \lambda=\lambda_1\alpha_1+\lambda_2\alpha_2+\lambda_3\alpha_3\in\mathbb{Z}\Phi.$$

Denote the Weyl group of Φ by W. It is well known that W is generated by the reflections $S_{\alpha_1},S_{\alpha_2},S_{\alpha_2}$ and $W\cong S_4$, the 4-symmetric group. The Weyl group W acts naturally on U^0 such that

$$w\cdot K_\lambda=K_{\omega(\lambda)},\quad \text{for all } w\in W \text{ and } \lambda\in\mathbb{Z}\Phi.$$

Set $U^0_{\rm ev}=\bigoplus_{\lambda\in\mathbb{Z}\Phi\cap 2\Lambda}kK_\lambda$, where Λ is the weight lattice. Notice that $\mathbb{Z}\Phi\cap 2\Lambda$ is a subgroup of $\mathbb{Z}\Phi\cap\Lambda$ and $\mathbb{Z}\Phi\cap 2\Lambda$ is W-stable. It follows that the action of W maps $U^0_{\rm ev}$ to itself. The following theorem is well known (see [11]).

Theorem 1.1. *There exists an algebra isomorphism between $Z(U)$ and $(U^0_{\rm ev})^W$, where $(U^0_{\rm ev})^W=\{h\in U^0_{\rm ev}\mid w\cdot h=h,\ \forall\, w\in W\}$.*

3 The center subalgebra of $U_q(\mathfrak{sl}_4)$

Let $\lambda_1, \lambda_2, \lambda_3$ be the fundamental weights corresponding to the root system Φ. Then the weight lattice $\Lambda = \mathbb{Z}\lambda_1 + \mathbb{Z}\lambda_2 + \mathbb{Z}\lambda_3$. By direct computations, we get

$$\alpha_1 = 2\lambda_1 - \lambda_2, \quad \alpha_2 = -\lambda_1 + 2\lambda_2 - \lambda_3, \quad \alpha_3 = -\lambda_2 + 2\lambda_3,$$
$$\lambda_1 = \frac{3}{4}\alpha_1 + \frac{1}{2}\alpha_2 + \frac{1}{4}\alpha_3, \quad \lambda_2 = \frac{1}{2}\alpha_1 + \alpha_2 + \frac{1}{2}\alpha_3, \quad \lambda_3 = \frac{1}{4}\alpha_1 + \frac{1}{2}\alpha_2 + \frac{3}{4}\alpha_3.$$

Also we have the Weyl group

$$W = \left\{ \begin{array}{l} e, S_{\alpha_1}S_{\alpha_2}, S_{\alpha_3}, S_{\alpha_1}S_{\alpha_2}, S_{\alpha_1}S_{\alpha_3}, S_{\alpha_2}S_{\alpha_3}, S_{\alpha_2}S_{\alpha_1}, S_{\alpha_3}S_{\alpha_2}, \\ S_{\alpha_1}S_{\alpha_2}S_{\alpha_1}, S_{\alpha_2}S_{\alpha_3}S_{\alpha_2}, S_{\alpha_1}S_{\alpha_2}S_{\alpha_3}, S_{\alpha_1}S_{\alpha_3}S_{\alpha_2}, S_{\alpha_2}S_{\alpha_1}S_{\alpha_3}, \\ S_{\alpha_3}S_{\alpha_2}S_{\alpha_1}, S_{\alpha_3}S_{\alpha_1}S_{\alpha_2}S_{\alpha_1}, S_{\alpha_1}S_{\alpha_2}S_{\alpha_3}S_{\alpha_2}, S_{\alpha_3}S_{\alpha_2}S_{\alpha_1}S_{\alpha_3}, \\ S_{\alpha_2}S_{\alpha_1}S_{\alpha_3}S_{\alpha_2}, S_{\alpha_1}S_{\alpha_2}S_{\alpha_3}S_{\alpha_1}, S_{\alpha_2}S_{\alpha_1}S_{\alpha_2}S_{\alpha_3}S_{\alpha_2}, S_{\alpha_1}S_{\alpha_3} \\ S_{\alpha_2}S_{\alpha_1}S_{\alpha_3}, S_{\alpha_3}S_{\alpha_2}S_{\alpha_1}S_{\alpha_3}S_{\alpha_2}, S_{\alpha_1}S_{\alpha_3}S_{\alpha_2}S_{\alpha_3}S_{\alpha_1}S_{\alpha_2} \end{array} \right\}.$$

It is known that the Weyl group is isomorphic to the symmetric group S_4, and the actions of the reflections $S_{\alpha_1}, S_{\alpha_2}, S_{\alpha_3}$ on $\alpha_1, \alpha_2, \alpha_3$ are given by

$$S_{\alpha_1}: \alpha_1 \mapsto -\alpha_1, \quad \alpha_2 \mapsto \alpha_1 + \alpha_2, \quad \alpha_3 \mapsto \alpha_3;$$
$$S_{\alpha_2}: \alpha_1 \mapsto \alpha_1 + \alpha_2, \quad \alpha_2 \mapsto -\alpha_2, \quad \alpha_3 \mapsto \alpha_2 + \alpha_3;$$
$$S_{\alpha_3}: \alpha_1 \mapsto \alpha_1, \quad \alpha_2 \mapsto \alpha_2 + \alpha_3, \quad \alpha_3 \mapsto -\alpha_3.$$

In order to describe the generators and generate relations of $(U_{\rm ev}^0)^W$, we need the following proposition.

Proposition 3.1. $\mathbb{Z}\Phi \cap 2\Lambda = \mathbb{Z}(2\alpha_1) + \mathbb{Z}(2\alpha_2) + \mathbb{Z}(\alpha_1 + \alpha_3)$.

Proof. For any $2n_1\alpha_1 + 2n_2\alpha_2 + n_3(\alpha_1 + \alpha_3) \in \mathbb{Z}(2\alpha_1) + \mathbb{Z}(2\alpha_2) + \mathbb{Z}(\alpha_1 + \alpha_3)$, $n_1, n_2, n_3 \in \mathbb{Z}$, it is clear that $2n_1\alpha_1 + 2n_2\alpha_2 + n_3(\alpha_1 + \alpha_3) \in \mathbb{Z}\Phi$. Since $2\alpha_1 = 4\lambda_1 - 2\lambda_2 \in 2\Lambda$, $2\alpha_2 = -2\lambda_1 + 4\lambda_2 - 2\lambda_3 \in 2\Lambda, \alpha_1 + \alpha_3 = 2\lambda_1 - 2\lambda_2 + 2\lambda_3 \in 2\Lambda$. It follows that $\mathbb{Z}(2\alpha_1) + \mathbb{Z}(2\alpha_2) + \mathbb{Z}(\alpha_1 + \alpha_3) \subseteq \mathbb{Z}\Phi \cap 2\Lambda$. Conversely, suppose that $x \in \mathbb{Z}\Phi \cap 2\Lambda$, and write $x = m_1 2\lambda_1 + m_2 2\lambda_2 + m_3 2\lambda_3$, $m_1, m_2, m_3 \in \mathbb{Z}$. Then

$$\begin{aligned} x &= \frac{3m_1 + 2m_2 + m_3}{2}\alpha_1 + (m_1 + 2m_2 + m_3)\alpha_2 + \frac{m_1 + 2m_2 + 3m_3}{2}\alpha_3 \\ &= \left(\frac{3m_1 + 2m_2 + m_3}{2} - \frac{m_1 + 2m_2 + 3m_3}{2}\right)\alpha_1 + (m_1 + 2m_2 + m_3)\alpha_2 \\ &\quad + \frac{m_1 + 2m_2 + 3m_3}{2}(\alpha_1 + \alpha_3) \\ &= (m_1 - m_3)\alpha_1 + (m_1 + 2m_2 + m_3)\alpha_2 + \frac{m_1 + 2m_2 + 3m_3}{2}(\alpha_1 + \alpha_3). \end{aligned}$$

Since $x \in \mathbb{Z}\Phi$, we have $\frac{3m_1+2m_2+m_3}{2}, \frac{m_1+2m_2+3m_3}{2}, m_1 + 2m_2 + m_3$ are integers. It follows that $\frac{(3m_1+m_3)}{2} \in \mathbb{Z}$, i.e., $3m_1 + m_3 \in \mathbb{Z}$ and hence $3m_1 + m_3 + m_1 - m_3 = 4m_1$. Then $m_1 - m_3 \in 2\mathbb{Z}$ and so $m_1 + m_3 \in 2\mathbb{Z}$, i.e., $m_1 + 2m_2 + m_3 \in 2\mathbb{Z}$. Thus we have $\mathbb{Z}\Phi \cap 2\Lambda \subseteq \mathbb{Z}(2\alpha_1) + \mathbb{Z}(2\alpha_2) + \mathbb{Z}(\alpha_1 + \alpha_3)$. All these show that $\mathbb{Z}\Phi \cap 2\Lambda = \mathbb{Z}(2\alpha_1) + \mathbb{Z}(2\alpha_2) + \mathbb{Z}(\alpha_1 + \alpha_3)$.

By Proposition 3.1 we know that $U_{\rm ev}^0 = k[K_1^{\pm 2}, K_2^{\pm 2}, K_1^{\pm 1}K_3^{\pm 1}]$. Now for any

$$f(K_1, K_2, K_3) = \sum_{t,s,m \in \mathbb{Z}} a_{t,s,m} K_1^{2t} K_2^{2s} (K_1 K_3)^m \in U_{\rm ev}^0,$$

we define

$$\deg_{K_1} f = \max\{2t + m | a_{t,s,m} \neq 0, t, s, m \in \mathbb{Z}\},$$
$$\deg_{K_2} f = \max\{2s | a_{t,s,m} \neq 0, t, s, m \in \mathbb{Z}\},$$
$$\deg_{K_3} f = \max\{m | a_{t,s,m} \neq 0, t, s, m \in \mathbb{Z}\}.$$

For any $t, s \in \mathbb{Z}$, set

$$\begin{aligned} f_{t,s,m} = {} & K_1^{2t+m}K_2^{2s}K_3^{m} + K_1^{2t+m}K_2^{2s}K_3^{2s-m} + K_1^{2t+m}K_2^{2t+2m-2s}K_3^{m} \\ & + K_1^{2t+m}K_2^{2t+2m-2s}K_3^{2t+m-2s} + K_1^{2t+m}K_2^{2t}K_3^{2s-m} \\ & + K_1^{2t+m}K_2^{2t}K_3^{2t+m-2s} + K_1^{2s-2t-m}K_2^{2s}K_3^{m} + K_1^{2s-2t-m}K_2^{2s}K_3^{2s-m} \\ & + K_1^{2s-2t-m}K_2^{-2t}K_3^{m} + K_1^{2s-2t-m}K_2^{-2t}K_3^{-2t-m} \\ & + K_1^{2s-2t-m}K_2^{2s-2t-2m}K_3^{2s-m} + K_1^{2s-2t-m}K_2^{2s-2t-2m}K_3^{-2t-m} \\ & + K_1^{m-2s}K_2^{2t+2m-2s}K_3^{m} + K_1^{m-2s}K_2^{2t+2m-2s}K_3^{2t+m-2s} \\ & + K_1^{m-2s}K_2^{-2t}K_3^{-2t-m} + K_1^{m-2s}K_2^{-2t}K_3^{m} + K_1^{m-2s}K_2^{-2s}K_3^{2t+m-2s} \\ & + K_1^{m-2s}K_2^{-2s}K_3^{-2t-m} + K_1^{-m}K_2^{2t}K_3^{2s-m} + K_1^{-m}K_2^{2t}K_3^{2t+m-2s} \\ & + K_1^{-m}K_2^{2s-2t-2m}K_3^{2s-m} + K_1^{-m}K_2^{2s-2t-2m}K_3^{-2t-m} \\ & + K_1^{-m}K_2^{-2s}K_3^{2t+m-2s} + K_1^{-m}K_2^{-2s}K_3^{-2t-m}. \end{aligned}$$

We shall call $f_{t,s,m}, t, s, m \in \mathbb{Z}$ the quantum Weyl symmetric polynomials.

Propsition 3.2. *The invariant subalgebra* $(U^0_{\mathrm{ev}})^W$ *is spanned by* $f_{t,s,m}$*, where* $t, s, m \in \mathbb{Z}$.

Proof. On the one hand, it is easy to see that the actions of the generators $S_{\alpha_1}, S_{\alpha_2}$ and S_{α_3} of W on the element $f_{t,s,m}$ are invariant, and so the subspace V spanned all $f_{t,s,m}$ of U^0_{ev} is in $(U^0_{\mathrm{ev}})^W$, i.e., $\sum_{t,s,m\in\mathbb{Z}} a_{t,s,m} f_{t,s,m} \in (U^0_{\mathrm{ev}})^W$.

On the other hand, take

$$\sum_{t,s,m\in\mathbb{Z}} a_{t,s,m}K_1^{2t}K_1^{2s}(K_1K_3)^m = \sum_{t,s,m\in\mathbb{Z}} a_{t,s,m}K_1^{2t+m}K_1^{2s}K_3^{m} \in (U^0_{\mathrm{ev}})^W, \quad a_{t,s,m} \in k.$$

Since the action of S_{α_1} on this element is invariant, we get

$$\sum_{t,s,m\in\mathbb{Z}} a_{t,s,m}K_1^{2t+m}K_2^{2s}K_3^{m} = \sum_{t,s,m\in\mathbb{Z}} a_{t,s,m}K_1^{2s-2t-m}K_2^{2s}K_3^{m}.$$

It follows that $a_{t,s,m} = a_{(s-t-m),s,m}$. Similarly, using the actions of the element of W, we get $a_{t,(t+m-s),m} = a_{t,s,m}, a_{t-s+m,s,2s-m)} = a_{t,s,m}, \ldots$

Thus we have

$$\sum_{t,s,m\in\mathbb{Z}} a_{t,s,m}K_1^{2t+m}K_2^{2s}K_3^{m} = \sum_{t,s,m\in\mathbb{Z}} a_{t,s,m}f_{t,s,m} \in V.$$

Let

$$\begin{aligned} x_1 &= \frac{1}{4}f_{0,1,1} = K_1K_2^2K_3 + K_1K_3 + K_1K_3^{-1} + K_1^{-1}K_3 + K_1^{-1}K_3^{-1} + K_1^{-1}K_2^{-2}K_3^{-1}, \\ x_2 &= \frac{1}{6}f_{-1,1,3} = K_1K_2^2K_3^3 + K_1K_2^2K_3^{-1} + K_1K_2^{-2}K_3^{-1} + K_1^{-3}K_2^{-2}K_3^{-1}, \\ x_3 &= \frac{1}{6}f_{1,1,1} = K_1^3K_2^2K_3 + K_1^{-1}K_2^2K_3 + K_1^{-1}K_2^{-2}K_3 + K_1^{-1}K_2^{-2}K_3^{-3}, \\ x_4 &= \frac{1}{2}f_{0,1,2} = K_1^2K_2^2K_3^2 + K_1^2K_2^2 + K_2^2K_3^2 + K_1^2 + K_2^2 + K_3^2 + K_1^{-2} \\ &\qquad + K_2^{-2} + K_3^{-2} + K_1^{-2}K_2^{-2} + K_2^{-2}K_3^{-2} + K_1^{-2}K_2^{-2}K_3^{-2}. \end{aligned}$$

We call x_1, x_2, x_3, x_4 the quantum elementary Weyl symmetric polynomials. Now we have the following theorem.

Theorem 3.3. *The quantum elementary Weyl symmetric polynomial* x_1, x_2, x_3, x_4 *are the minimal generators of* $(U^0_{\mathrm{ev}})^W$.

Proof. Firstly, we prove x_1, x_2, x_3, x_4 are the generators of $(U^0_{\mathrm{ev}})^W$. Notice that $x_1, x_2, x_3, x_4 \in (U^0_{\mathrm{ev}})^W$, so by Proposition 3.2 we only need to prove that $f_{t,s,m}$ is generated by x_1, x_2, x_3, x_4 for all $t, s, m \in \mathbb{Z}$.

When $t = s = m = 0$, $f_{t,s,m} = 24 \in k$, it is clear.

When $t \neq 0$ or $s \neq 0$, by the expression of $f_{t,s,m}$, it follows that there must exist a monomial such that the degrees of K_1, K_2, K_3 are all larger than zero. We may suppose that the term is the first one, i.e., $2t + m > 0, 2s > 0, 2m > 0$, where $s, t, m \in \mathbb{Z}$. Set $w = \max\{\deg_{K_i} f_{t,s,m} \mid i = 1, 2, 3\}$. Notice that if a monomial appears, then the others certainly appear in $f_{t,s,m}$. Conveniently, for any $t, s, m \in \mathbb{Z}$, we denote $f_{t,s,m}$ only by a monomial containing w. For example, denote by x_1 $K_1K_2^2K_3 + \cdots$. In the following, we shall use induction on w to prove that $f_{t,s,m}$ can be generated by x_1, x_2, x_3, x_4.

When $w = 2$, then $f_{t,s,m} = f_{0,1,1} = 4x_1$ or $f_{t,s,m} = f_{0,1,2} = 2x_4$, which are clearly generated by x_1, x_2, x_3, x_4.

When $w = 3$, there are also two terms satisfying the condition:

$$f_{t,s,m} = f_{-1,1,3} = 6x_1, \quad f_{t,s,m} = f_{1,1,1} = 6x_3.$$

They are clearly generated by x_1, x_2, x_3, x_4.

Suppose that when $w < n\,(n \geqslant 4)$, for any $t, s, m \in \mathbb{Z}$, $f_{t,s,m}$ with $w = \max\{\deg_{K_i} f_{t,s,m} \mid i = 1, 2, 3\}$ are generated by x_1, x_2, x_3, x_4. Now we prove the case $w = n$, and we break the proof into seveval cases:

If n is even.

Case 1. $n = \deg_{K_1} f_{t,s,m} > \deg_{K_i} f_{t,s,m}\,(i = 2, 3)$. By our assumption for t, s, m, it is clear that we only need to consider that $n = 2t + m, n > 2s > m > 0$.

$$\begin{aligned} K_1^n K_2^{2s} K_3^m + \cdots &= (K_1^{n-3} K_2^{2s-2} K_3^{m-1} + \cdots)x_3 - (K_1^{n-4} K_2^{2s} K_3^m + \cdots) \\ &\quad - (K_1^{n-4} K_2^{n+m-2s} K_3^m + \cdots) - (K_1^{n-4} K_2^{2s-4} K_3^{m-4} + \cdots). \end{aligned}$$

By the inductive hypothesis we know $K_1^{n-3}K_2^{2s-2}K_3^{m-1} + \cdots, K_1^{n-4}K_2^{2s}K_3^m + \cdots, K_1^{n-4}K_2^{n+m-2s}K_3^m + \cdots, K_1^{n-4}K_2^{2s-4}K_3^{m-4} + \cdots$ can be generated by x_1, x_2, x_3, x_4. So $K_1^nK_2^{2s}K_3^m + \cdots$ can be generated by x_1, x_2, x_3, x_4.

Case 2. $n = \deg_{K_2} f_{t,s,m} > \deg_{K_i} f_{t,s,m}\,(i = 1, 3)$. As in Case 1, we only need to consider that $n = 2s > 2t + m, m > 0$.

$$\begin{aligned} K_1^{2n+m} K_2^n K_3^m + \cdots &= (K_1^{2t+m-1} K_2^{n-2} K_3^{m-1} + \cdots)x_1 - (K_1^{2t+m} K_2^{n-2} K_3^m + \cdots) \\ &\quad - (K_1^{2t+m} K_2^{n-2} K_3^{m-2} + \cdots) - (K_1^{2t+m-2} K_2^{n-2} K_3^m + \cdots) \\ &\quad - (K_1^{2t+m-2} K_2^{n-2} K_3^{m-2} + \cdots) - (K_1^{2t+m-2} K_2^{n-4} K_3^{m-2} + \cdots). \end{aligned} \tag{3.1}$$

Remark. When $2t + 2m = 2n - 2$, (3.1) is

$$\begin{aligned} K_1^{2n+m} K_2^n K_3^m + \cdots &= \frac{1}{2}[(K_1^{2t+m-1} K_2^{n-2} K_3^{m-1} + \cdots)x_1 - (K_1^{2t+m} K_2^{n-2} K_3^{m-2} + \cdots) \\ &\quad - (K_1^{2t+m-2} K_2^{n-2} K_3^m + \cdots) - 2(K_1^{2t+m-2} K_2^{n-2} K_3^{m-2} + \cdots)]. \end{aligned}$$

By the inductive hypothesis we know that

$$\begin{aligned} &K_1^{2t+m-1} K_2^{n-2} K_3^{m-1} + \cdots, \quad K_1^{2t+m} K_2^{n-2} K_3^m + \cdots, \\ &K_1^{2t+m} K_2^{n-2} K_3^{m-2} + \cdots, \quad K_1^{2t+m-2} K_2^{n-2} K_3^m + \cdots, \\ &K_1^{2t+m-2} K_2^{n-2} K_3^{m-2} + \cdots, \quad K_1^{2t+m-2} K_2^{n-4} K_3^{m-2} + \cdots \end{aligned}$$

can be generated by x_1, x_2, x_3, x_4. So $K_1^{2n+m}K_2^nK_3^m + \cdots$ can be generated by x_1, x_2, x_3, x_4.

Case 3. $n = \deg_{K_3} f_{t,s,m} > \deg_{K_i} f_{t,s,m}\,(i = 1, 2)$. In this case it is clear that we only need to consider that $n = m > 2s > 2t + m > 0$. Then the proof is similar to Case 1.

Case 4. $n = \deg_{K_1} f_{t,s,m} = \deg_{K_2} f_{t,s,m} > \deg_{K_3} f_{t,s,m}$. In this case we only need to consider that $n = 2t + m = 2s > m > 0$ and $m \in 2\mathbb{Z}$.

$$K_1^n K_2^n K_3^m + \cdots = (K_1^{n-3} K_2^{n-2} K_3^{m-1} + \cdots)x_3 - (K_1^{n-4} K_2^n K_3^m + \cdots)$$

$$- (K_1^{n-4}K_2^{n-4}K_3^m + \cdots) - (K_1^{n-4}K_2^{n-4}K_3^{m-4} + \cdots). \tag{3.2}$$

Remark. When $n = 4$, (3.2) is $K_1^nK_2^nK_3^m + \cdots = \frac{1}{2}[(K_1^{n-3}K_2^{n-2}K_3^{m-1} + \cdots)x_3 - 2(K_1^mK_2^mK_3^m + \cdots)]$.

By the inductive hypothesis we know that $K_1^{n-3}K_2^{n-2}K_3^{m-1} + \cdots, K_1^{n-4}K_2^{n-4}K_3^m + \cdots, K_1^{n-4}K_2^{n-4}K_3^{m-4} + \cdots$ can both be generated by x_1, x_2, x_3, x_4. Notice that by Case 2 $K_1^{n-4}K_2^nK_3^m + \cdots$ is also generated by x_1, x_2, x_3, x_4. Thus, $K_1^nK_2^nK_3^m + \cdots$ can be generated by x_1, x_2, x_3, x_4.

Case 5. $n = \deg_{K_2} f_{t,s,m} = \deg_{K_3} f_{t,s,m} > \deg_{K_1} f_{t,s,m}$. In this case it is clear that we only need to consider that $n = 2s = m > 2t + m > 0$. Then the proof is similar to that of Case 4.

Case 6. $n = \deg_{K_i} f_{t,s,m}$ $(i = 1, 2, 3)$. In this case we only need to consider that $n = 2s = 2t + m = m$.

$$\begin{aligned} K_1^nK_2^nK_3^n + \cdots = (K_1^{n-1}K_2^{n-2}K_3^{n-3} + \cdots)x_2 - 2(K_1^nK_2^nK_3^4 + \cdots) \\ - (K_1^{n-4}K_2^{n-4}K_3^{n-4} + \cdots). \end{aligned} \tag{3.3}$$

By the inductive hypothesis we know that $K_1^{n-1}K_2^{n-2}K_3^{n-3} + \cdots, K_1^{n-4}K_2^{n-4}K_3^{n-4} + \cdots$ can both be generated by x_1, x_2, x_3, x_4. Notice that by Case 4 $K_1^nK_2^nK_3^4 + \cdots$ is also generated by x_1, x_2, x_3, x_4. Thus, $K_1^nK_2^nK_3^n + \cdots$ can be generated by x_1, x_2, x_3, x_4.

If n is odd.

Case 1. $n = \deg_{K_1} f_{t,s,m} > \deg_{K_i} f_{t,s,m}$ $(i = 2, 3)$. It is easy to see that we only need to consider that $n = 2t + m > 2s > m > 0$.

$$\begin{aligned} K_1^nK_2^{2s}K_3^m + \cdots = (K_1^{n-3}K_2^{2s-2}K_3^{m-1} + \cdots)x_3 - (K_1^{n-4}K_2^{2s}K_3^m + \cdots) \\ - (K_1^{n-4}K_2^{n+m-2s}K_3^m + \cdots) - (K_1^{n-4}K_2^{2s-4}K_3^{m-4} + \cdots). \end{aligned}$$

By the inductive hypothesis we know $K_1^{n-3}K_2^{2s-2}K_3^{m-1} + \cdots, K_1^{n-4}K_2^{2s}K_3^m + \cdots, K_1^{n-4}K_2^{n+m-2s}K_3^m + \cdots, K_1^{n-4}K_2^{2s-4}K_3^{m-4} + \cdots$ can be generated by x_1, x_2, x_3, x_4. So $K_1^nK_2^{2s}K_3^m + \cdots$ can be generated by x_1, x_2, x_3, x_4.

Case 2. $n = \deg_{K_3} f_{t,s,m} > \deg_{K_i} f_{t,s,m}$ $(i = 1, 2)$. In this case we only need to consider $n = m > 2s > 2t + m > 0$. Then the proof is similar to that of Case 1.

To sum up, $m = n$, for any $t, s, m \in \mathbb{Z}$, $f_{t,s,m}$ can be generated by x_1, x_2, x_3, x_4. By comparing the degrees, it is obvious that x_1, x_2, x_3, x_4 cannot be generated by the others, hence x_1, x_2, x_3, x_4 are the minimal generators of $(U_{\text{ev}}^0)^W$.

We need the following lemmas before proving the main theorem of this paper.

Lemma 3.4. *The quantum elementary Weyl symmetric polynomials x_1, x_2, x_3, x_4 satisfy the relation*:

$$x_4^2 - 4x_1^2 - 2x_1x_2 - x_2x_3 - 2x_1x_3 + 8x_4 + 16 = 0.$$

Proof. By a direct computation it is easy to get the above conclusion. We omit its proof.

Suppose that $f \in (U_{\text{ev}}^0)^W$, then $f \in k[K_1, K_2, K_3]$. Using a lexicographic order, the first term is called the highest term of f. For example, the highest term of x_1 is $K_1K_2^2K_3$, the highest term of x_2 is $K_1K_2^2K_3^3$, the highest term of x_3 is $K_1^3K_2^2K_3$. Generally, the highest term of $x_1^{m_1}x_2^{m_2}x_3^{m_3}x_4^{m_4}$ is $K_1^{m_1+m_2+3m_3+2m_4}K_2^{2m_1+2m_2+2m_3+2m_4}K_3^{m_1+3m_2+m_3+2m_4}$, where $m_1, m_2, m_3, m_4 \in \mathbb{N}$.

Lemma 3.5. *For any $m_i, m_i' \in \mathbb{N}, i = 1, 2, 3, 4$, we have that the highest term of $x_1^{m_1}x_2^{m_2}x_3^{m_3}x_4^{m_4}$ is equal to the highest term of $x_1^{m_1'}x_2^{m_2'}x_3^{m_3'}x_4^{m_4'}$ if and only if there exists $d \in \mathbb{Z}$ such that $m_1' = m_1, m_2' = m_2 + d, m_3' = m_3 - d, m_4' = m_4 + 2d$.*

Proof. " $\Leftarrow$ " Notice that the highest terms of $x_1^{m_1}x_2^{m_2}x_3^{m_3}x_4^{m_4}$ and $x_1^{m_1'}x_2^{m_2'}x_3^{m_3'}x_4^{m_4'}$ are

$$K_1^{m_1+m_2+3m_3+2m_4}K_2^{2m_1+2m_2+2m_3+2m_4}K_3^{m_1+3m_2+m_3+2m_4}$$

and $K_1^{m_1'+m_2'+3m_3'+2m_4'}K_2^{2m_1'+2m_2'+2m_3'+2m_4'}K_3^{m_1'+3m_2'+m_3'+2m_4'}$ respectively. It follows that

$$m_1 + m_2 + 3m_3 + 2m_4 = m_1' + m_2' + d + 3m_3' + 3d + 2m_4' - 4d$$

$$
\begin{aligned}
&= m_1' + m_2' + 3m_3' + 2m_4', \\
2m_1 + 2m_2 + 2m_3 + 2m_4 &= 2m_1' + 2m_2' + 2d + 2m_3' + 2d + 2m_4' - 4d \\
&= 2m_1' + 2m_2' + 2m_3' + 2m_4', \\
m_1 + 3m_2 + m_3 + 2m_4 &= m_1' + 3m_2' + 3d + m_3' + d + 2m_4' - 4d \\
&= m_1' + 3m_2' + m_3' + 2m_4'.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
&K_1^{m_1+m_2+3m_3+2m_4} K_2^{2m_1+2m_2+2m_3+2m_4} K_3^{m_1+3m_2+m_3+2m_4} \\
&\quad = K_1^{m_1'+m_2'+3m_3'+2m_4'} K_2^{2m_1'+2m_2'+2m_3'+2m_4'} K_3^{m_1'+3m_2'+m_3'+2m_4'},
\end{aligned} \tag{3.4}
$$

i.e., the highest term of $x_1^{m_1}x_2^{m_2}x_3^{m_3}x_4^{m_4}$ is equal to the highest term of $x_1^{m_1'}x_2^{m_2'}x_3^{m_3'}x_4^{m_4'}$.

" $\Rightarrow$ " If the highest term of $x_1^{m_1}x_2^{m_2}x_3^{m_3}x_4^{m_4}$ is equal to the highest term of $x_1^{m_1'}x_2^{m_2'}x_3^{m_3'}x_4^{m_4'}$, then we have

$$
\begin{aligned}
m_1 + m_2 + 3m_3 + 2m_4 &= m_1' + m_2' + 3m_3' + 2m_4', \\
2m_1 + 2m_2 + 2m_3 + 2m_4 &= 2m_1' + 2m_2' + 2m_3' + 2m_4', \\
m_1 + 3m_2 + m_3 + 2m_4 &= m_1' + 3m_2' + m_3' + 2m_4'.
\end{aligned}
$$

Thus we have $m_1 = m_1', m_2 - m_3 = m_2' - m_3, 2m_3 + m_4 = 2m_3' + m_4'$. It follows that

$$m_2' = m_2 - (m_3 - m_3'), \quad m_4' = m_4 + 2(m_3 - m_3').$$

Noticing that $m_3' = m_3 - (m_3 - m_3')$, set $m_3 - m_3' = d$. Thus we have

$$
\begin{cases}
m_1' = m_1, \\
m_2' = m_2 - d, \\
m_3' = m_3 - d, \\
m_4' = m_4 + 2d.
\end{cases}
$$

Hence we complete the proof. □

By Lemma 3.5, we can define a relation in $\mathbb{N}^4$ as follows: $(m_1, m_2, m_3, m_4) \sim (m_1', m_2', m_3', m_4')$, if there exists $d \in \mathbb{Z}$ such that

$$m_1' = m_1, m_2' = m_2 - d, m_3' = m_3 - d, m_4' = m_4 + 2d.$$

It is clear that $\sim$ is an equivalence relation. Denote by $\mathbb{N}^4/\sim$ the quotient set corresponding to $\sim$, i.e.,

$$\mathbb{N}^4/\sim = \{[m_1, m_2, m_3, m_4] \mid m_1, m_2, m_3, m_4 \in \mathbb{N}\},$$

where

$$[m_1, m_2, m_3, m_4] = \{(m_1'.m_2', m_3', m_4') \in \mathbb{N}^4 \mid (m_1', m_2', m_3', m_4') \sim (m_1, m_2, m_3, m_4)\}.$$

Now, let $k[y_1, y_2, y_3, y_4]$ be the polynomial algebra over y_1, y_2, y_3, y_4. For any $f = f(y_1, y_2, y_3, y_4) = \sum_{i,j,s,t\in\mathbb{N}} a_{i,j,s,t} y_1^i y_2^j y_3^s y_4^t \in k[y_1, y_2, y_3, y_4]$, let

$$P = \{(i, j, s, t) \mid a_{i,j,s,t} \neq 0\}.$$

Then P is a finite set. Thus there exists $r \in \mathbb{Z}^+$ such that

$$P = ([t_1, s_1, m_1, n_1] \cap P) \cup \cdots \cup ([t_r, s_r, m_r, n_r] \cap P),$$

where $(t_i, s_i, m_i, n_i) \in \mathbb{N}^4, i = 1, \ldots, r$. Note that every $[t_i, s_i, m_i, n_i] \cap P$ is a finite set, so we can properly choose $[t_i, s_i, m_i, n_i]$ such that for any $(t, s, m, n) \in [t_i, s_i, m_i, n_i] \cap P$ we have $s_i - s \geqslant 0$. This means that if $(t,s,m,n) \in [t_i,s_i,m_i,n_i] \cap P$, then there exists $d \geqslant 0$ such that $s_i - s = m_i - m = d$. Thus we can write f as

$$f = f_1 + f_2 + \cdots + f_r, \quad \text{where } f_i = \sum_{0 \leqslant d \leqslant \min\{s_i,m_i\}} a_{i,d} y_1^{t_i} y_2^{s_i-d} y_3^{m_i-d} y_4^{n_i+2d}.$$

Let

$$d_i = \max\{0 \leqslant d \leqslant \min\{s_i,m_i\} \mid a_{i,d} \neq 0\}$$

Then we have $\deg_{y_4} f_i = n_i + 2d_i$.

Now we can give the main theorem.

Theorem 3.6. *Using the notations above, we have*

$$Z(U_q(\mathfrak{sl}_4)) \cong k[y_1,y_2,y_3,y_4]/I,$$

where I is an ideal generated by one element

$$y_4^2 - 4y_1^2 - 2y_1y_2 - y_2y_3 - 2y_1y_3 + 8y_4 + 16.$$

Proof. By the Harish-Chandra isomorphism we only need to prove $(U_{\mathrm{ev}}^0)^W \cong k[y_1,y_2,y_3,y_4]/I$. Define a map

$$\sigma : k[y_1,y_2,y_3,y_4] \longrightarrow (U_{\mathrm{ev}}^0)^W, \quad y_1 \mapsto x_1, \quad y_2 \mapsto x_2, \quad y_3 \mapsto x_3, \quad y_4 \mapsto x_4.$$

By Theorem 3.3 we know σ is surjection, thus we only need to prove $\ker\sigma = I$.

In fact, on the one hand, since

$$\begin{aligned}\sigma(y_4^2 - 4y_1^2 - 2y_1y_2 - y_2y_3 - 2y_1y_3 + 8y_4 + 16)\\ = x_4^2 - 4x_1^2 - 2x_1x_2 - x_2x_3 - 2x_1x_3 + 8x_4 + 16 = 0,\end{aligned}$$

it follows that $y_4^2 - 4y_1^2 - 2y_1y_2 - y_2y_3 - 2y_1y_3 + 8y_4 + 16 \in \ker\sigma$, thus we have $I \subseteq \ker\sigma$.

On the other hand, let $f(y_1,y_2,y_3,y_4) = \sum_{i,j,s,t\in\mathbb{N}} a_{i,j,s,t} y_1^i y_2^j y_3^s y_4^t \in \ker\sigma$. Write $f = f_1 + f_2 + \cdots + f_r$, where $f_i = \sum_{0 \leqslant d \leqslant \min\{s_i,m_i\}} a_{i,d} y_1^{t_i} y_2^{s_i-d} y_3^{m_i-d} y_4^{n_i+2d}$. Then we have that

$$\begin{aligned}\sigma(f(y_1,y_2,y_3,y_4)) &= \sigma(f_1) + \cdots + \sigma(f_r)\\ &= \sum_{0 \leqslant d \leqslant \min\{s_1,m_1\}} a_{1,d} x_1^{t_1} x_2^{s_1-d} x_3^{m_1-d} x_4^{n_1+2d} + \cdots\\ &\quad + \sum_{0 \leqslant d \leqslant \min\{s_r,m_r\}} a_{r,d} x_1^{t_r} x_2^{s_r-d} x_3^{m_r-d} x_4^{n_r+2d}\\ &= 0.\end{aligned}$$

In the following we will prove the theorem by induction on $\deg_{y_4} f$.

When $\deg_{y_4} f = 0, 1$ or 2, since $\deg_{y_4} f_i = n_i + 2d_i$, $d_i = 0, \forall\, i = 1,\ldots,r$. Thus we have that for every f_i, and there is only one term $a_{i,0} y_1^{t_i} y_2^{s_i} y_3^{m_i} y_4^{n_i}$. It follows that we have $\sigma(f_i) = a_{i,0} x_1^{t_i} x_2^{s_i} x_3^{m_i} x_4^{n_i}$. From the definition of f and Lemma 3.5 we get the highest term of $\sigma(f_i)$, different from each other, where $i \in \{1,\ldots,r\}$. Applying the lexicographic order to the highest term of $\sigma(f_i)$, we get $a_{i,0} = 0$, $\forall\, i = 1,\ldots,r$. Thus $f(y_1,y_2,y_3,y_4) = 0$, and then $f(y_1,y_2,y_3,y_4) \in \ker\sigma$.

Now suppose that for any $f \in k[y_1,y_2,y_3,y_4]$ with $2 < \deg_{y_4} f < n$ and $f \in \ker\sigma$, we have $f(y_1,y_2,y_3,y_4) \in I$. Let $f \in k[y_1,y_2,y_3,y_4]$, $\deg_{y_4} f = n$ and $f \in \ker\sigma$. Noting that $f = f_1 + f_2 + \cdots + f_r$, we can suppose that

$$\deg_{y_4} f_i = n, \; i = 1,2,\ldots,l; \quad \deg_{y_4} f_i < n, \; i = l+1,\ldots,n.$$

Moveover, $\deg_{y_4} f_i = n_i + 2d_i$, so $n_i + 2d_i = n,\ i = 1, 2, \dots, l$. Thus when $1 \leqslant i \leqslant l$, we have

$$f_i = a_{i,d_i} y_1^{t_i} y_2^{s_i-d_i} y_3^{m_i-d_i} y_4^{n} + \sum_{0\leqslant d<d_i} a_{i,d} y_1^{t_i} y_2^{s_i-d} y_3^{m_i-d} y_4^{n_i+2d}.$$

Thus we have

$$f = \sum_{j=1}^{l}\left(a_{j,d_j} y_1^{t_j} y_2^{s_j-d} y_3^{m_j-d_j} y_4^{n} + \sum_{0\leqslant d<d_j} a_{j,d} y_1^{t_j} y_2^{s_j-d} y_3^{m_j-d} y_4^{n_j+2d}\right) + \sum_{i=l+1}^{r} f_i.$$

It follows that

$$\begin{aligned}\sigma(f) &= \sum_{j=1}^{l}\left(a_{j,d_j} x_1^{t_j} x_2^{s_j-d_j} x_3^{m_j-d_j} x_4^{n} + \sum_{0\leqslant d<d_j} a_{j,d} x_1^{t_j} x_2^{s_j-d} x_3^{m_j-d} x_4^{n}\right) + \sum_{i=l+1}^{r} \sigma(f_i)\\ &= \sum_{j=1}^{l}\left(a_{j,d_j} x_1^{t_j} x_2^{s_j-d_j} x_3^{m_j-d_j} x_4^{n-2} x_4^{2} + \sum_{0\leqslant d<d_j} a_{j,d} x_1^{t_j} x_2^{s_j-d} x_3^{m_j-d} x_4^{n}\right) + \sum_{i=l+1}^{r} \sigma(f_i)\\ &= \sum_{j=1}^{l}\left(a_{j,d_j} x_1^{t_j} x_2^{s_j-d_j} x_3^{m_j-d_j} x_4^{n-2}(4x_1^2 + 2x_1x_2 + 2x_2x_3 + 2x_1x_3 - 8x_4\right.\\ &\quad \left. - 16) + \sum_{0\leqslant d<d_j} a_{j,d} x_1^{t_j} x_2^{s_j-d} x_3^{m_j-d} x_4^{n}\right) + \sum_{i=l+1}^{r} \sigma(f_i)\\ &= 0.\end{aligned}$$

Let

$$\begin{aligned} g &= g(y_1, y_2, y_3, y_4)\\ &= \sum_{j=1}^{l}\left(a_{j,d_j} y_1^{t_j} y_2^{s_j-d_j} y_3^{m_j-d_j} y_4^{n-2}(4y_1^2 + 2y_1y_2 + 2y_2y_3 + 2y_1y_3 - 8y_4 - 16)\right.\\ &\quad \left. + \sum_{0\leqslant d<d_j} a_{j,d} y_1^{t_j} y_2^{s_j-d} y_3^{m_j-d} y_4^{n}\right) + \sum_{i=l+1}^{r} f_i.\end{aligned}$$

Then we have

$$\begin{aligned}&\sigma(g(y_1, y_2, y_3, y_4))\\ &= \sum_{j=1}^{l}\left(a_{j,d_j} x_1^{t_j} x_1^{t_j} x_2^{s_j-d_j} x_3^{m_j-d_j} x_4^{n-2}(4x_1^2 + 2x_1x_2 + 2x_2x_3 + 2x_1x_3 - 8x_4 - 16)\right.\\ &\quad \left. + \sum_{0\leqslant d<d_j} a_{j,d} x_1^{t_j} x_2^{s_j-d} x_3^{m_j-d} x_4^{n}\right) + \sum_{i=l+1}^{r} f_i,\end{aligned}$$

i.e., $g(y_1, y_2, y_3, y_4) \in \ker\sigma$. Notice that $2 < \deg_{y_4} g < n$, so by the inductive hypothesis, we have $g = g(y_1, y_2, y_3, y_4) \in I$. Moveover, we have

$$\begin{aligned}&f(y_1, y_2, y_3, y_4) - g(y_1, y_2, y_3, y_4)\\ &= \sum_{j=1}^{l}[a_{j,d_j} y_1^{t_j} y_2^{s_j-d_j} y_3^{m_j-d_j} y_4^{n-2}(4y_1^2 + 2y_1y_2 + 2y_2y_3 + 2y_1y_3 - 8y_4 - 16)]\\ &\in I,\end{aligned}$$

it follows that $f(y_1, y_2, y_3, y_4) \in I$.

Thus for any $f(y_1, y_2, y_3, y_4) = \sum_{i,j,s\in\mathbb{N}} a_{i,j,s,t} y_1^i y_2^j y_3^s y_4^t \in \ker\sigma$, we have $f(y_1, y_2, y_3, y_4) \in I$, it follows that $\ker\sigma \subseteq I$. Hence the assertion holds.

Based on the main results of this paper and [17], we formulate the following.

Problem 3.7. Describe the minimal generators in general for $(U_{\mathrm{ev}}^0)^W$ and the relation for the minimal generators.

We will treat the case $\mathfrak{g} = \mathfrak{sl}_{n+1}$ in a forthcoming paper.

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