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On singular Lüroth quartics

Dedicated to Fabrizio Catanese on the Occasion of his 60th Birthday

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Abstract Plane quartics containing the ten vertices of a complete pentalateral and limits of them are called Lüroth quartics. The locus of singular Lüroth quartics has two irreducible components, both of codimension two in $\mathbb{P}^{14}$. We compute the degree of them and discuss the consequences of this computation on the explicit form of the Lüroth invariant. One important tool is the Cremona hexahedral equations of the cubic surface. We also compute the class in $\overline{M}_3$ of the closure of the locus of nonsingular Lüroth quartics.

Keywords plane quartics, Lueroth quartics, invariant, moduli space of curves

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0 Introduction

All schemes and varieties will be assumed to be defined over an algebraically closed field $\mathbf{k}$ of characteristic zero. We recall that a *complete pentalateral* in $\mathbb{P}^2$ is a configuration consisting of five lines, three by three linearly independent, together with the ten double points of their union, which are called *vertices* of the pentalateral. A *nonsingular Lüroth quartic* is a nonsingular quartic plane curve containing the ten vertices of a complete pentalateral. Such curves fill an open set of an irreducible, SL(3)-invariant, hypersurface $\mathcal{L} \subset \mathbb{P}^{14}$. The (possibly singular) quartic curves parametrized by the points of $\mathcal{L}$ will be called *Lüroth quartics*. In [16] we have computed that $\mathcal{L}$ has degree 54, by reconstructing a proof published by Morley in 1919 [14]. Another proof has been given by Le Potier and Tikhomirov in [13].

In this paper we put together the projective techniques of [14] and [16] with the cohomological techniques in [13], and we prove some new results about the Lüroth hypersurface. We refer to the introduction of [16] for an explanation of the connection of this topic with moduli of vector bundles on $\mathbb{P}^2$.

The locus of singular Lüroth quartics has been considered in [14] and [13]. It is obtained as the intersection between the Lüroth hypersurface of degree 54 and the discriminant of degree 27. It has two irreducible components $\mathcal{L}_1$ and $\mathcal{L}_2$, both of codimension 2 in $\mathbb{P}^{14}$, and it is known that $\deg(\mathcal{L}_2)_{\mathrm{red}} = 27\cdot 15$ [13, Corollary 9.4]. We compute the degree of $\mathcal{L}_1$, a question left open in [13] (end of 9.2). Indeed we prove the following theorem.

Theorem 0.1. *The intersection between the Lüroth hypersurface $\mathcal{L}$ and the discriminant $\mathcal{D}$ is transverse along $\mathcal{L}_1$, and*

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(i) $\deg \mathcal{L}_1 = 27 \cdot 24$;

(ii) $\mathcal{L}_2$ *is non-reduced of degree* $27 \cdot 30$.

We warn the reader that our $(\mathcal{L}_2)_{\text{red}}$ corresponds to $\mathcal{L}_2$ in [13].

An interesting aspect of the geometrical construction in [16] is that any smooth cubic surface S defines in a natural way 36 planes, which we called Cremona planes, one for each of the 36 double-six configurations of lines on S. Their main property is that the ramification locus of the projection π_P centered at $p \in S$ is a Lüroth quartic if and only if p belongs to any of the Cremona planes.

We describe the Cremona planes on a nonsingular cubic surface by pure projective geometry. To give the flavour of this construction we state the following result.

Theorem 0.2. *Let S be a nonsingular cubic surface. Fix a double-six on S. Let ℓ_s, $s = 1, \dots, 15$, be the 15 remaining lines. For each $1 \leqslant s \leqslant 15$ consider the three planes $\Pi_{s,h}$, $h = 1, 2, 3$, containing ℓ_s such that $S \cap \Pi_{s,h}$ consists of ℓ_s and of two residual lines not belonging to the double-six. Let $P_{s,h}$ be the intersection point of the two residual lines. Then the 15 points $\ell_s \cap \langle P_{s,1}, P_{s,2}, P_{s,3} \rangle$, $s = 1, \dots, 15$, lie on a plane, which is the Cremona plane associated with the double-six.*

Theorem 3.6 contains the statement of this theorem with additional informations on the *involutory* and *non-involutory* points. They give a geometrical explanation of the reducibility of $\mathcal{L} \cap \mathcal{D}$ (see Proposition 3.1).

We conclude with the statement of non-existence of an invariant of degree 15 (Proposition 3.7) vanishing on $(\mathcal{L}_2)_{\text{red}}$, which we have obtained by a computer computation. This means that $(\mathcal{L}_2)_{\text{red}}$ is not a complete intersection, and we relate this fact with the last sentence in Morley's paper [14]. This leads to a reconstruction of some speculations of Morley about the (still unknown) explicit form of the Lüroth hypersurface. Our result implies that these speculations are partially wrong, but with a slight correction they might become true.

In 1967 Shioda [17] found the Hilbert series for the invariant ring of plane quartics. From his formula it follows that the space of invariants of degree 54 has dimension 1165. This shows the difficulty to find the explicit expression (or the symbolic expression) of the Lüroth invariant, which, to the best of our knowledge, is still unknown. In the last section we compute the class in $\overline{M}_3$ of the divisor of Lüroth quartics.

The content of the paper is as follows. In the first section we summarize some results from [16] on Cremona hexaedral equations and Cremona planes on a cubic surface. We recall the purely geometric construction of the involutory points. In the second section we summarize the well-known facts about the description of plane quartics as symmetric determinants with linear entries. We recall how Lüroth quartics can be found in this description (they are the image of a pfaffian hypersurface Λ, which is an invariant of degree 6) and, following [13] and [9], we also describe how the two components of singular Lüroth quartics can be found. The third section contains our new results, the main ones being described above, and their proofs. The last section is devoted to the computation of the class $[L]$ of the divisor in $\overline{M}_3$ parametrizing Lüroth quartics, as well as to some related remarks.

We thank Igor Dolgachev for calling to our attention the reference [9] and Carel Faber for a helpful conversation with the second author about the topics of the last section.

1 Cremona hexahedral equations and Cremona planes

Recall that a *double-six* of lines on a nonsingular cubic surface $S \subset \mathbb{P}^3$ consists of two sets of six lines $\Delta = (A_1, \dots, A_6; B_1, \dots, B_6)$ such that the lines A_j are mutually skew as well as the lines B_j; moreover each A_i meets each B_j except when $i = j$.

In $\mathbb{P}^5$ with coordinates $(Z_0, \dots, Z_5)$ consider the following equations:

$$\begin{cases} Z_0^3 + Z_1^3 + Z_2^3 + Z_3^3 + Z_4^3 + Z_5^3 = 0, \\ Z_0 + Z_1 + Z_2 + Z_3 + Z_4 + Z_5 = 0, \\ \beta_0 Z_0 + \beta_1 Z_1 + \beta_2 Z_2 + \beta_3 Z_3 + \beta_4 Z_4 + \beta_5 Z_5 = 0, \end{cases} \tag{1}$$

where the β_s's are general constants. These equations define a nonsingular cubic surface S in a $\mathbb{P}^3$ contained in $\mathbb{P}^5$ and are called *Cremona hexahedral equations* of S, after [6].

For any choice of two disjoint pairs of indices $\{i,j\} \cup \{k,l\} \subset \{0,\dots,5\}$, the equations $Z_i + Z_j = Z_k + Z_l = 0$ define a line contained in S. There are 15 such lines and the remaining 12 determine a double-six of lines on S. Therefore the equations (1) define a double six on S. More precisely we have the following:

Theorem 1.1. *Each system of Cremona hexahedral equations of a nonsingular cubic surface S defines a double-six of lines on S. Conversely, the choice of a double-six of lines on S defines a system of Cremona hexahedral equations* (1) *of S, which is uniquely determined up to replacing the coefficients $(\beta_0,\dots,\beta_5)$ by $(a+b\beta_0,\dots,a+b\beta_5)$ for some $a,b \in \mathbf{k}$, $b \neq 0$.*

We refer to [8, Subsection 9.4], for the proof. We need to point out from [16, Corollary 4.2], the following:

Corollary 1.2. *To a pair (S,Δ) consisting of a nonsingular cubic surface $S \subset \mathbb{P}^3$ and a double-six of lines Δ on S, there is canonically associated a plane $\Xi \subset \mathbb{P}^3$ which is given by the equations*

$$\begin{cases} Z_0 + Z_1 + Z_2 + Z_3 + Z_4 + Z_5 = 0, \\ \beta_0 Z_0 + \beta_1 Z_1 + \beta_2 Z_2 + \beta_3 Z_3 + \beta_4 Z_4 + \beta_5 Z_5 = 0, \\ \beta_0^2 Z_0 + \beta_1^2 Z_1 + \beta_2^2 Z_2 + \beta_3^2 Z_3 + \beta_4^2 Z_4 + \beta_5^2 Z_5 = 0, \end{cases}$$

where the coefficients $\beta_0,\dots,\beta_5$ are those appearing in the Cremona equations of (S,Δ).

Definition 1.3. *The plane $\Xi \subset \mathbb{P}^3$ will be called the* Cremona plane *associated with the pair (S,Δ).*

The link with the Lüroth quartics is given by the following.

Theorem 1.4. *Let $p \in S$ be a point. The projection from p defines a rational double covering $\pi_P \colon S \dashrightarrow \mathbb{P}^2$ ramified over a plane quartic. The ramification curve is a Lüroth quartic if and only if p belongs to any of the Cremona planes.*

Proof. See the Remark 10.7 and Theorem 6.1 of [16]. □

There is a second description of the Cremona planes, by means of the involutory points. In order to state it, we give, following [16], a geometric construction of the involutory points. Consider two skew lines $A, B \subset S$. Denote by $f\colon A \to B$ the double cover associating with $p \in A$ the point $f(p) := T_pS \cap B$, where T_pS is the tangent plane to S at p. Define $g\colon B \to A$ similarly. Let $p_1, p_2 \in A$ (resp. $q_1, q_2 \in B$) be the ramification points of f (resp. g). Consider the pairs of branch points $f(p_1), f(p_2) \in B$, $g(q_1), g(q_2) \in A$, and the new morphisms

$$f'\colon A \to \mathbb{P}^1, \qquad g'\colon B \to \mathbb{P}^1$$

defined by the conditions that $g(q_1), g(q_2)$ are ramification points of f' and $f(p_1), f(p_2)$ are ramification points of g'. Let $Q_1 + Q_2$ (resp. $P_1 + P_2$) be the common divisor of the two g_2^1's on A (resp. on B) defined by f and f' (resp. by g and g'). The points

$$\bar{P} = g(P_1) = g(P_2) \in A, \qquad \bar{Q} = f(Q_1) = f(Q_2) \in B$$

are called the *involutory points* (relative to the pair of lines A and B).

Note that each line $A \subset S$ contains 16 involutory points, which correspond to the 16 lines $B \subset S$ which are skew with A, and they are distinct (see the proof of Proposition 6.3 of [16]).

Let $\bar{P}_i \in A_i$, $\bar{Q}_i \in B_i$ be the involutory points relative to the pair A_i and B_i. We obtain twelve points

$$\bar{P}_1,\dots,\bar{P}_6,\bar{Q}_1,\dots,\bar{Q}_6 \in S,$$

which are canonically associated with the double-six Δ.

Theorem 1.5. *For any double-six Δ there is a unique plane $\Xi \subset \mathbb{P}^3$ containing the involutory points*

$$\bar{P}_1,\dots,\bar{P}_6,\bar{Q}_1,\dots,\bar{Q}_6.$$

Moreover Ξ *coincides with the Cremona plane associated with the pair* (S, Δ).

The 36 *Cremona planes obtained in this way are distinct.*

Proof. See [16, Theorem 6.1 and Proposition 6.3]. □

2 The symmetric representation of Lüroth quartics

Let Q_0, Q_1, Q_2 be three linearly independent quadrics in $\mathbb{P}^3 = \mathbb{P}(W)$. They generate a net of quadrics $\langle Q_0, Q_1, Q_2 \rangle$ whose base locus, in general, consists of eight points in general position. We can parametrize any net of quadrics by the points of $\mathbb{P}^2 = \mathbb{P}(V)$, and as such it can be seen as an element $f \in \mathbb{P}(V \otimes S^2 W)$. The symmetric determinantal representation of the quadrics of the net gives a dominant rational map $\delta\colon \mathbb{P}(V \otimes S^2 W) \dashrightarrow \mathbb{P}(S^4 V)$ (see [8]). In [20] Wall studied the map δ in the setting of invariant theory. He proved that the non-semistable points for the action of $SL(W)$ on $\mathbb{P}(V \otimes S^2 W)$ are exactly given by the locus $Z(\delta)$ where δ is not defined.

There is a factorization through the GIT quotient

$$\begin{array}{ccc} \mathbb{P}(V \otimes S^2 W)^{ss} & & \\ \downarrow \pi & \searrow \delta & \\ \mathbb{P}(V \otimes S^2 W)//SL(W) & \xrightarrow{g} & \mathbb{P}(S^4 V), \end{array}$$

where g is generically finite of degree 36 and $\mathbb{P}(V \otimes S^2 W)//SL(W)$ parametrizes pairs (B, t) consisting of a plane quartic B and an even theta-characteristic t on it.

Consider the hypersurface $\Lambda \subset \mathbb{P}(V \otimes S^2 W)$ of degree 6 consisting of the nets $\langle Q_0, Q_1, Q_2 \rangle$ satisfying the equation:

$$\mathrm{Pf} \begin{bmatrix} 0 & Q_0 & -Q_1 \\ -Q_0 & 0 & Q_2 \\ Q_1 & -Q_2 & 0 \end{bmatrix} = 0,$$

where we identify each quadric Q_i with its corresponding symmetric matrix. It can be shown [3, 19] that a net belongs to Λ if and only if Q_0, Q_1, Q_2 are the polar quadrics of three points with respect to a cubic surface in $\mathbb{P}^3$. It follows (see [3, Section 4]) the classical fact that all Lüroth quartics can be obtained as plane sections of the Hessian of a cubic surface, and conversely every such plane section is a Lüroth quartic.

In [15, Section 4], it is shown that Λ is the 5-secant variety of $\mathbb{P}(V) \times \mathbb{P}(W)$ embedded with $\mathcal{O}(1, 2)$, and this fact is used to give a new proof of the Lüroth theorem.

Let $\mathcal{L} \subset S^4(V)$ be the Lüroth invariant of degree 54. The map δ is constructed as the determinant of a 4×4 symmetric matrix, hence the entries of δ have degree four in the 30 indeterminates of $V \otimes S^2 W$, so that $\delta^* \mathcal{L}$ has degree 216 as a hypersurface in $\mathbb{P}(V \otimes S^2 W)$. The crucial fact, for our purposes, is that $\delta^* \mathcal{L}$ contains Λ as an irreducible component (see for example [15, Proposition 6.3(ii)]).

Then $g^{-1}(\mathcal{L})$ decomposes into two irreducible components P and $\widetilde{P}$, both dominating $\mathcal{L}$, with degree 1 and 35 respectively, and P generically parametrizes the pairs (B, t) where t is the pentalateral theta-characteristic on B (see Remark 10.7 of [16]). In particular, g has a rational section over $\mathcal{L}$.

There are two other classically known invariants of nets of quadrics with respect to $SL(V) \times SL(W)$, nicely reviewed by Gizatullin in [9]. The *tact-invariant* J has degree 48 and vanishes if and only if two of the eight base points of the net coincide. The invariant I of degree 30 vanishes when the net contains a quadric of rank less than or equal to 2.

Let $\mathcal{D}$ be the discriminant invariant, which is irreducible of degree 27; then $\delta^* \mathcal{D}$ is an invariant of degree 108 of the nets of quadrics. Salmon proved the beautiful identity (up to scalar constants)

$$\delta^* \mathcal{D} = I^2 J. \tag{3}$$

It can be interpreted as saying that there are two ways to get a singular quartic as a symmetric determinant. This is interesting when applied to Lüroth quartics. The singular Lüroth quartics are the elements of $\mathcal{L} \cap \mathcal{D}$. In [13, Section 9], it is shown that this locus has two irreducible components $\mathcal{L}_1$ and $\mathcal{L}_2$, so that we have necessarily

$$\mathcal{L}_1 = \delta(\Lambda \cap \{J = 0\}) \qquad (\mathcal{L}_2)_{\mathrm{red}} = \delta(\Lambda \cap \{I = 0\})$$

(see Proposition 3.1). To connect our description with the setting of [13] it is enough to note that the geometric quotient of Λ by $SL(W)$ is isomorphic to a compactification P of the moduli space $M(0,4)$ of rank 2 stable bundles on $\mathbb{P}^2$ with $c_1 = 0$ and $c_2 = 4$ and the restriction of g to M can be identified with the Barth map (see [15, Section 8]).

The fact that the locus of singular Lüroth quartics consists of two irreducible components was known also to Morley.

The 36 elements of the g-fiber over a general point of $\mathcal{D}$ are of two types: there are 16 points in $\pi(\{J = 0\})$ and 10 double points in $\pi(\{I = 0\})$. This decomposition corresponds to the two types of even theta-characteristics on a quartic nodal curve: 16 of them are represented by invertible sheaves, and 10 by torsion-free non-invertible sheaves, each counted with multiplicity two (see [9, Remark 10.1] and [10]).

3 The main results and their proofs

Consider the projective bundle $\pi\colon \mathbb{P}(\mathbf{Q}) \to \mathbb{P}^3$, where $\mathbf{Q} = T_{\mathbb{P}^3}(-1)$ is the tautological quotient bundle. For each $z \in \mathbb{P}^3$ the fibre $\pi^{-1}(z)$ is the projective plane of lines through z. Also consider the projective bundle $\beta\colon \mathbb{P}(S^4\mathbf{Q}^\vee) \to \mathbb{P}^3$. For each $z \in \mathbb{P}^3$ the fibre $\beta^{-1}(z)$ is the linear system of quartics in $\pi^{-1}(z)$. The Picard group of $\mathbb{P} = \mathbb{P}(S^4\mathbf{Q}^\vee)$ is generated by $H = \mathcal{O}_{\mathbb{P}}(1)$ and by the pullback F of a plane in $\mathbb{P}^3$. Let $\widetilde{\mathcal{L}} \subset \mathbb{P}(S^4\mathbf{Q}^\vee)$ be the β-relative hypersurface of Lüroth quartics. It is invariant under the natural action of SL(4) on $\mathbb{P}(S^4\mathbf{Q}^\vee)$, and in [16] we showed that $\widetilde{\mathcal{L}} = 54H - 72F$. Moreover every invariant of a plane quartic of degree d gives a covariant of the cubic surface of degree $\frac{2d}{3}$ (see [16, Remark 8.2]).

Consider also the relative invariant subvarieties $\widetilde{\mathcal{L}}_1$, $\widetilde{\mathcal{L}}_2$ and $\widetilde{\mathcal{D}}$ in the projective bundle $\mathbb{P}(S^4\mathbf{Q}^\vee)$ on $\mathbb{P}^3$. For every smooth cubic surface S we have the projection $\mathbb{P}(S^4\mathbf{Q}^\vee)_{|S} \xrightarrow{\beta} S$ and have defined the section $s : S \to \mathbb{P}(S^4\mathbf{Q}^\vee)_{|S}$ associating with $p \in S$ the branch curve of the projection from p. It is well-known that such branch curve is singular if and only if p belongs to one of the twenty-seven lines. Hence $s^*(\widetilde{\mathcal{D}})$ consists of the divisor of the twenty-seven lines, with multiplicity two, cut indeed by a covariant of S of degree 18, which is the square of the classical covariant of degree 9 cutting the lines.

In Theorem 1.4 we have proved that $s^*(\widetilde{\mathcal{D}} \cap \widetilde{\mathcal{L}})$ consists of the intersection of the divisor $s^*(\widetilde{\mathcal{D}})$ with the 36 Cremona planes. This is a zero dimensional scheme, consisting of two parts: the involutory points (see Theorem 1.5) and the non-involutory points. By the above, its length is given by

$$\deg \mathcal{D} \cdot \deg \mathcal{L} \cdot \left(\frac{2}{3}\right)^2 \cdot 3 = 27 \cdot 54 \cdot \frac{4}{3} = 27 \cdot 72.$$

Therefore on every line on S the scheme $s^*(\widetilde{\mathcal{D}} \cap \widetilde{\mathcal{L}})$ has length 72, and multiplicity greater than or equal to 2 at each point. It is supported on the 16 involutory points and on less than or equal to 20 non-involutory points.

Proposition 3.1. (i) *Projecting S from an involutory point of $s^*(\widetilde{\mathcal{D}} \cap \widetilde{\mathcal{L}})$ we get a branch quartic in $\mathcal{L}_1$ (corresponding to the tact-invariant J).*

(ii) *Projecting S from a non-involutory point of $s^*(\widetilde{\mathcal{D}} \cap \widetilde{\mathcal{L}})$ we get a branch quartic in $\mathcal{L}_2$ (corresponding to the invariant I).*

(iii) *On each line of S there are exactly* 10 *non-involutory points, each counts with multiplicity four in $s^*(\widetilde{\mathcal{D}} \cap \widetilde{\mathcal{L}})$.*

Proof. By the Salmon identity (3), the two types of points correspond to the vanishing of the two invariants I and J. We have just to distinguish which is the type obtained by each invariant. A check on the degrees ($\frac{60}{48} = \frac{20}{16}$) suffices to prove (i) and (ii). In order to prove (iii), consider that in the Salmon identity (3) the invariant I appears with exponent two, and this implies that the non-involutory points have to be double ones, that is there are 10 distinct non-involutory points on each line, and at each of these points two Cremona planes meet. □

Proposition 3.1 explains why $\mathcal{L}_2$ is non-reduced. We will construct directly the ten non-involutory points in Theorem 3.6.

Remark 3.2. It appears that the 36 Cremona planes carry an interesting combinatorial configuration. Each of them has 27 marked points given by the intersection with the lines on the cubic surface, 12 of these points are involutory points (corresponding to the twelve lines of the corresponding double-six, see Theorem 1.5) and the other 15 belong respectively to other 15 Cremona planes. It is natural to expect that this configuration of 36 planes can be obtained by cutting with a linear space the 36 hyperplanes in $\mathbb{P}^5$ considered at 6.1.5.1 of [12], related to the Weyl group of the exceptional group E_6, which have exactly the same properties.

Remark 3.3. The two components of the locus of singular Lüroth quartics can be also interpreted analytically as follows. Consider the general equation

$$\sum_{k=0}^{4} \lambda_k \ell_0 \cdots \hat{\ell}_k \cdots \ell_4 = 0$$

of a Lüroth quartic with inscribed pentalateral $\{\ell_0, \ldots, \ell_4\}$, considered in (21) of [16]. Quartics in $\mathcal{L}_2$ have three of the five lines ℓ_k which are concurrent at the same point. This follows easily from the identity:

$$\begin{vmatrix} \ell'_1 + \ell'_0 & \ell'_0 & \ell'_0 & \ell'_0 \\ \ell'_0 & \ell'_2 + \ell'_0 & \ell'_0 & \ell'_0 \\ \ell'_0 & \ell'_0 & \ell'_3 + \ell'_0 & \ell'_0 \\ \ell'_0 & \ell'_0 & \ell'_0 & \ell'_4 + \ell'_0 \end{vmatrix} = \frac{1}{\lambda_0\lambda_1\lambda_2\lambda_3\lambda_4} \sum_{k=0}^{4} \lambda_k \ell_0 \cdots \hat{\ell}_k \cdots \ell_4,$$

where $\ell'_k := \frac{\ell_k}{\lambda_k}$. Indeed, if ℓ_0, ℓ_1, ℓ_2 are concurrent at the same point, then the matrix evaluated at this point has rank two. This condition corresponds to the vanishing of the invariant I, as already remarked in Section 2. Quartics in $\mathcal{L}_1$ can be obtained for any given pentalateral, by a convenient choice of constants λ_i. *Summarizing*: *by specializing the* ℓ_k*'s we obtain quartics in* $\mathcal{L}_2$, *and by specializing the* λ_k*'s we obtain quartics in* $\mathcal{L}_1$.

Remark 3.4. The referee pointed out another geometrical interpretation of the two components of the locus of singular Lüroth quartics. We saw in Section 2 that Lüroth quartics can be obtained as plane sections of the Hessian of a cubic surface. The tangent plane sections at nonsingular points of the Hessian cut quartics in $\mathcal{L}_1$, while the plane sections meeting the Hessian at one of its 10 singular points cut quartics in $\mathcal{L}_2$. The referee also asked if the fact that the dual of the Hessian has degree 16 has anything to do with the existence of 16 involutory points on each line of the cubic surface. We do not know the answer to this intriguing question.

Proof of Theorem 0.1. We recall that $s^*(\widetilde{\mathcal{D}})$ consists of the divisor of the 27 lines, with multiplicity two. The scheme $s^*(\widetilde{\mathcal{L}}_1)$ is supported on the involutory points, and its length (on each line) can be computed as the difference between the length of $s^*(\widetilde{\mathcal{D}} \cap \widetilde{\mathcal{L}})$ and the length of $s^*(\widetilde{\mathcal{L}}_2)$ (both on a line); precisely it is equal to $72 - 40 = 32$. Since the 16 involutory points on each line are distinct (see the proof of Proposition 6.3 in [16]), it follows that $s^*(\widetilde{\mathcal{L}}_1)$ consists of 16 points of length 2, in particular $s^*(\widetilde{\mathcal{L}})$ is transversal to the lines of S at the involutory points, which implies formula (i). Then (ii) follows easily from the computation $27 \cdot 54 - 27 \cdot 24 = 27 \cdot 30$. □

Remark 3.5. From Theorem 0.1 it follows that the Barth map is not ramified over $\mathcal{L}_1$, which answers a question posed originally by Peskine, see [13, 6.3 and 9.2].

Part (ii) of the following theorem contains Theorem 0.2 of the introduction, with additional information.

Theorem 3.6. *Let* S *be a nonsingular cubic surface.*

(i) *Given a line* ℓ *on* S*, consider the five planes* Π_i*,* $i = 1, \dots, 5$*, containing* ℓ *and such that* $S \cap \Pi_i$ *consists of* ℓ *and of two residual lines. Let* P_i *be the intersection point of the two residual lines. For every choice of distinct points* $P_i, P_j, P_k \in \{P_1, \dots, P_5\}$ *let* $\Pi_{ijk} = \langle P_i, P_j, P_k \rangle$ *be the plane they span. Then*

$$\{\ell \cap \Pi_{ijk} : 1 \leqslant i < j < k \leqslant 5\}$$

are the ten non-involutory points on ℓ*.*

(ii) *Fix a double-six on* S*. Let* ℓ_s*,* $s = 1, \dots, 15$*, be the* 15 *remaining lines. For each* $1 \leqslant s \leqslant 15$ *consider the three planes* $\Pi_{s,h}$*,* $h = 1, 2, 3$*, containing* ℓ_s *such that* $S \cap \Pi_{s,h}$ *consists of* ℓ_s *and of two residual lines not belonging to the double-six. Let* $P_{s,h}$ *be the intersection point of the two residual lines. Then the* 15 *points* $\ell_s \cap \langle P_{s,1}, P_{s,2}, P_{s,3} \rangle$*,* $s = 1, \dots, 15$*, lie on a plane, which is the Cremona plane associated to the double-six. In particular, the points* $\ell_s \cap \langle P_{s,1}, P_{s,2}, P_{s,3} \rangle$*,* $s = 1, \dots, 15$*, are non-involutory points on* ℓ_s*.*

Proof. It is enough to prove (ii). Write the equation of S in Cremona form (1) and take ℓ_s to be $Z_0 + Z_1 = Z_2 + Z_3 = Z_4 + Z_5 = 0$. The three pairs of lines coplanar with ℓ_s are:

$$\begin{aligned}
&Z_0 + Z_1 = Z_2 + Z_4 = Z_3 + Z_5 = 0, \quad Z_0 + Z_1 = Z_2 + Z_5 = Z_3 + Z_4 = 0,\\
&Z_2 + Z_3 = Z_0 + Z_4 = Z_1 + Z_5 = 0, \quad Z_2 + Z_3 = Z_0 + Z_5 = Z_1 + Z_4 = 0,\\
&Z_4 + Z_5 = Z_0 + Z_2 = Z_1 + Z_4 = 0, \quad Z_4 + Z_5 = Z_0 + Z_4 = Z_1 + Z_2 = 0.
\end{aligned}$$

The intersection points of the three pairs are respectively

$$\begin{aligned}
P_{s,1} &= (-(\beta_2 + \beta_3 - \beta_4 - \beta_5), \beta_2 + \beta_3 - \beta_4 - \beta_5, \beta_0 - \beta_1, \beta_0 - \beta_1, -(\beta_0 - \beta_1), -(\beta_0 - \beta_1)),\\
P_{s,2} &= (\beta_2 - \beta_3, \beta_2 - \beta_3, -(\beta_0 + \beta_1 - \beta_4 - \beta_5), \beta_0 + \beta_1 - \beta_4 - \beta_5, -(\beta_2 - \beta_3), -(\beta_2 - \beta_3)),\\
P_{s,3} &= (\beta_4 - \beta_5, \beta_4 - \beta_5, -(\beta_4 - \beta_5), -(\beta_4 - \beta_5), -(\beta_0 + \beta_1 - \beta_2 - \beta_3), \beta_0 + \beta_1 - \beta_2 - \beta_3).
\end{aligned}$$

The plane spanned by the three points cuts ℓ_s at the point

$$\begin{aligned}
&-(\beta_2 - \beta_3)(\beta_4 - \beta_5)P_{s,1} + (\beta_0 - \beta_1)(\beta_4 - \beta_5)P_{s,2} - (\beta_0 - \beta_1)(\beta_2 - \beta_3)P_{s,3}\\
&\quad = \left(\frac{\beta_2 + \beta_3 - \beta_4 - \beta_5}{\beta_0 - \beta_1}, -\frac{\beta_2 + \beta_3 - \beta_4 - \beta_5}{\beta_0 - \beta_1}, -\frac{\beta_0 + \beta_1 - \beta_4 - \beta_5}{\beta_2 - \beta_3},\right.\\
&\qquad \left.\frac{\beta_0 + \beta_1 - \beta_4 - \beta_5}{\beta_2 - \beta_3}, \frac{\beta_0 + \beta_1 - \beta_2 - \beta_3}{\beta_4 - \beta_5}, -\frac{\beta_0 + \beta_1 - \beta_2 - \beta_3}{\beta_4 - \beta_5}\right)
\end{aligned}$$

and a direct computation shows that this point belongs to the Cremona plane $\beta_0^2 Z_0 + \beta_1^2 Z_1 + \beta_2^2 Z_2 + \beta_3^2 Z_3 + \beta_4^2 Z_4 + \beta_5^2 Z_5 = 0$. Since this computation can be repeated for all the 15 lines not belonging to the double six corresponding to the given Cremona equations, we obtain the conclusion. □

There is a more conceptual proof of the above theorem obtained by rephrasing an argument of Le Potier-Tikhomirov in the geometry of the cubic surface, and indeed we discovered it in this way. In [13], Proposition 7.1, they consider a nodal quartic with its six tangent lines passing through the node. It is a classical fact (see e.g., [2, p. 279]) that the six contact points lie on a conic. If the quartic is in $\mathcal{L}_2$ then this conic is singular (and also the converse holds). On the cubic surface S, when P belongs to a line ℓ in S, the ramification quartic of the projection centered at P has the six bitangents passing through the node corresponding to the five planes Π_i and to the tangent plane at P itself. The contact points of the six bitangents correspond to the lines $\langle PP_i \rangle$ and the sixth one corresponds to the tangent of the residual conic cut by the tangent plane at P. Hence these six lines lie in a quadric cone with vertex at P. This quadric cone splits into two planes when three contact points lie on a line and this in turn corresponds to the fact that P, P_i, P_j, P_k lie on the same plane. So $P = \ell \cap \langle P_i, P_j, P_k \rangle$ is a non-involutory point, as in the proof given above.

We now come to the question of giving an explicit expression of the Lüroth invariant $\mathcal{L}$. We have used the description of invariants of plane quartics given in [4]. From this description we have shown, with a brute force computer computation, that

Proposition 3.7. *There is no invariant of degree less than or equal to* 15 *which vanishes on* $(\mathcal{L}_2)_{\mathrm{red}}$*.*

The existence of such an invariant was asked by Morley in the last paragraph of [14]. This is interesting because from Proposition 3.7 it follows that $(\mathcal{L}_2)_{\mathrm{red}}$ is a divisor on the Lüroth hypersurface which is not cut by hypersurfaces. This makes it conceivable that there is an invariant I_{30} of degree 30 which vanishes (doubly) on $(\mathcal{L}_2)_{\mathrm{red}}$, and that there is an invariant I_{24} of degree 24 which vanishes on $\mathcal{L}_1$.

Morley gives an explicit form of the restriction of the invariant of degree 24 to a nodal cubic surface, assuming that it exists. But he does not show that his formula can be extended to all cubic surfaces, and therefore he does not prove the existence of the invariant. If this invariant exists, then $\mathcal{L}_1$ is a complete intersection. This might be true, but it has not yet been proved.

These speculations are important because they give a hint about a possible explicit description of the Lüroth invariant. Morley suggested the form $\mathcal{L} = I_{27} \cdot I'_{27} + I_{24} I_{15}^2$ (up to scalar constants in the invariants appearing in the formula) where $I_{27} = \mathcal{D}$ is the discriminant. By Proposition 3.7 this is not possible, but still the expression $\mathcal{L} = I_{27} \cdot I'_{27} + I_{24} I_{30}$ could be possible, and we leave its existence as a question.

This expression means that, modulo the discriminant, the Lüroth invariant is the product of the two invariants I_{24} and I_{30}. The computation of $\mathcal{L}$ could reduce in this case to smaller degree invariants. Let us recall that the main result of Dixmier [7] states that every invariant of a plane quartics is algebraically dependent on an explicit system of invariants of degrees up to 27.

Moreover, Morley asks if I'_{27} is the discriminant too. This can be probably answered by a computational analysis, which does not seem easy and we do not pursue here.

The following result says something about the singularities of the Lüroth hypersurface $\mathcal{L}$:

Proposition 3.8. *The Lüroth hypersurface $\mathcal{L}$ is not normal.*

Proof. Consider the incidence relation:

$$\widetilde{\mathcal{L}} := \left\{ (B, \{\ell_0, \dots, \ell_4\}) : \begin{array}{l} \{\ell_0, \dots, \ell_4\} \text{ is a complete 5-lateral and } B \text{ is an n.s.} \\ \text{quartic circumscribed to it} \end{array} \right\} \subset \mathbb{P}^{14} \times (\mathbb{P}^{2\vee})^{(5)}$$

and the projections:

$$\begin{array}{ccc} & \widetilde{\mathcal{L}} & \\ {}^{q_1}\swarrow & & \searrow^{q_2} \\ \mathbb{P}^{14} & & (\mathbb{P}^{2\vee})^{(5)}. \end{array}$$

Clearly $\overline{q_1(\widetilde{\mathcal{L}})} = \mathcal{L} \subset \mathbb{P}^{14}$. $\widetilde{\mathcal{L}}$ is irreducible of dimension 14, and its nonsingular locus contains $q_2^{-1}(U)$, where $U \subset (\mathbb{P}^{2\vee})^{(5)}$ is the locus of strict pentalaterals (i.e., those having 10 distinct vertices). The fibre $q_1^{-1}(C)$ over a general $[C] \in \mathcal{L}$ is one-dimensional and irreducible, consisting of the 5-laterals which are inscribed in C. But there is a class of nonsingular Lüroth quartics, the *desmic quartics*, such that $q_1^{-1}(C)$ is disconnected and each one of its component intersects $q_2^{-1}(U)$ (see for example [3]). Consider the morphism

$$q_2^{-1}(U) \longrightarrow \mathbb{P}^{14}$$

and its Stein factorization:

$$q_2^{-1}(U) \longrightarrow \mathcal{S} \xrightarrow{\ \upsilon\ } \mathbb{P}^{14}\ .$$

From the above description it follows that υ maps $\mathcal{S}$ birationally and dominantly to $\mathcal{L}$, but it has disconnected fibres over the locus of desmic quartics. From Zariski's main theorem it follows that $\mathcal{L}$ is not normal along the locus of desmic quartics. In particular the singular locus of $\mathcal{L}$ has codimension 1 in $\mathcal{L}$. □

4 The class of the Lüroth divisor in $\overline{M}_3$

Let $\overline{M}_3$ be the coarse moduli space of stable curves of genus 3. We will compute certain rational divisor classes in $\mathrm{Pic}_{\mathrm{fun}}(\overline{M}_3) \otimes \mathbb{Q}$ in terms of the Hodge class λ and of δ_0, δ_1, the classes of the boundary components Δ_0, Δ_1. Precisely, Δ_0 generically parametrizes irreducible singular stable curves, and Δ_1 generically parametrizes reducible stable curves.

The hyperelliptic locus $\overline{H} \subset \overline{M}_3$ is a divisor whose class is

$$\overline{h} = 9\lambda - \delta_0 - 3\delta_1.$$

For the proof see [11] and [18].

The discriminant $\mathcal{D} \subset \mathbb{P}^{14}$, being an SL(3)-invariant hypersurface of degree 27, admits a rational map $\Phi : \mathcal{D} - - \succ \overline{M}_3$. The closure of the image is a divisor $D := \overline{\mathrm{Im}(\Phi)} \subset \overline{M}_3$. Since $\mathcal{D}$ contains the double conics and the cuspidal curves, D contains $\overline{H}$, Δ_0 and Δ_1. In [1] it is proved that $\mathcal{D}$ vanishes with multiplicity 14 on double conics and with multiplicity 2 on cuspidal quartics. If we consider any test curve intersecting double conics and cuspidal quartics transversely, we will have to perform a base change in order to get stable reduction. Standard facts about stable reduction (see [11, Chapter 3.C]) imply that the degrees of the base changes needed are respectively 2 and 6. It follows that $[D]$ contains $\overline{H}$ and Δ_1 with multiplicity $2 \cdot 14$ and $6 \cdot 2$ respectively. Therefore the class $[D]$ is computed as follows:

$$\begin{aligned}[D] &= 2 \cdot 14[H] + 6 \cdot 2\delta_1 + \delta_0 \\ &= 28(9\lambda - \delta_0 - 3\delta_1) + 12\delta_1 + \delta_0 \\ &= 9(28\lambda - 3\delta_0 - 8\delta_1).\end{aligned}$$

This formula can be easily tested and confirmed using the pencils considered in Exercise (3.166) of [11].

We define *the divisor $L \subset \overline{M}_3$ of Luroth quartics*, to be the closure of the locus of nonsingular Luroth quartics.

Proposition 4.1. *Let $\mathcal{L} \subset \mathbb{P}^{14}$ be the hypersurface of Luroth quartics. Then $\mathcal{L}$ does not contain the loci of double conics and of cuspidal quartics. Moreover we have*

$$[L] = 18(28\lambda - 3\delta_0 - 8\delta_1).$$

Proof. From Proposition 3.1 it follows that general quartics in $\mathcal{L}_1$ and in $\mathcal{L}_2$ can be obtained as branch curves of projections from points belonging to a line ℓ of a general cubic surface S. It is well-known that such branch curves have one node, which is the projection of the line ℓ, with principal tangents the projections of the two planes containing ℓ and such that the residual conic is tangent to ℓ. Therefore both $\mathcal{L}_1$ and $\mathcal{L}_2$ contain a nodal quartic. Since $\mathcal{L} \cap \mathcal{D} = \mathcal{L}_1 \cup \mathcal{L}_2$ has pure dimension 12, as well as the locus of cuspidal quartics, it follows that this locus is not entirely contained in $\mathcal{L}$.

For double conics we argue as follows. Consider an irreducible quartic C with a node O but otherwise nonsingular. Then the six contact points of C with the tangents from O (which are the intersections different from O of C with the first polar of O with respect to C) are on a conic θ, and this conic is reducible if and only if $C \in \mathcal{L}_2$ (see [13, Section 7]). If the quartic C degenerates to a double conic 2ϑ then ϑ is a component of the first polar of any of its points. This implies that ϑ is a degeneration of θ. Then, since θ is reducible if $C \in \mathcal{L}_2$, it follows that C cannot degenerate to a double irreducible conic. Therefore $\mathcal{L}_2$ does not contain the locus of double conics. Consider now a quartic $C \in \mathcal{L}_1$, such that $C \notin \mathcal{L}_1 \cap \mathcal{L}_2$. Then C is circumscribed to a 5-lateral having 10 distinct vertices (Remark 3.3). Therefore C cannot be a double irreducible conic. In conclusion, $\mathcal{L}$ does not contain double irreducible conics.

Being SL(3)-invariant, $\mathcal{L}$ defines a divisor in $\overline{M}_3$, containing L, whose class is

$$[\mathcal{L}] = 2[D],$$

because $\deg(\mathcal{L}) = 54 = 2 \deg(\mathcal{D})$. On the other hand, we have

$$[L] = [\mathcal{L}] - a\overline{h} - b\delta_1 = 2[D] - a\overline{h} - b\delta_1,$$

where a, b are the multiplicities of $\mathcal{L}$ along the loci of double conics and of cuspidal quartics respectively. From the first part of the proof it follows that $a = b = 0$. Therefore

$$[L] = 2[D] = 18(28\lambda - 3\delta_0 - 8\delta_1).$$

□

In a similar vein one computes the class [**Cat**] of the catalecticant hypersurface to be:

$$[\mathbf{Cat}] = 2(28\lambda - 3\delta_0 - 8\delta_1).$$

Note that $28\lambda - 3\delta_0 - 8\delta_1$ is necessarily the class of the divisor defined by the SL(3) invariant of (smallest) degree 3.

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