SCIENCE CHINA
Mathematics

• ARTICLES •

September 2010 Vol. 53 No. 9: 2417–2424
doi: 10.1007/s11425-010-4091-7

Note on the number of integral ideals in Galois extensions

Dedicated to Professor Wang Yuan on the Occasion of his 80th Birthday

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Received March 1, 2010; accepted July 7, 2010; published online August 18, 2010

Abstract Let K be an algebraic number field of finite degree over the rational filed $\mathbb{Q}$. Let a_k be the number of integral ideals in K with norm k. In this paper we study the l-th integral power sum of a_k, i.e., $\sum_{k\leqslant x} a_k^l$ ($l = 2, 3, \ldots$). We are able to improve the classical result of Chandrasekharan and Good. As an application we consider the number of solutions of polynomial congruences.

Keywords Dedekind zeta-function, integral ideal, congruences

MSC(2000): 11N37, 11F70

Citation: Lü G S, Wang Y H. Note on the number of integral ideals in Galois extensions. Sci China Math, 2010, 53(9): 2417–2424, doi: 10.1007/s11425-010-4091-7

1 Introduction and main results

Let K be an algebraic number field of finite degree n over the rational filed $\mathbb{Q}$. The Dedekind zeta-function $\zeta_K(s)$ of the field K is defined by, for $\sigma > 1$,

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{\mathfrak{N}(\mathfrak{a})^s}, \qquad s = \sigma + it,$$

where $\mathfrak{a}$ varies over the integral ideals of K, and $\mathfrak{N}(\mathfrak{a})$ denotes its norm. If a_k denotes the number of integral ideals in K with norm k, then we have $\zeta_K(s) = \sum_{k=1}^{\infty} a_k k^{-s}$. It is known that a_k is a multiplicative function and satisfies

$$a_k \leqslant d(k)^n, \tag{1.1}$$

where $d(k)$ is the divisor function, and n is the degree of $K/\mathbb{Q}$, see e.g. [1].

It is a classical problem to study the l-th integral power sum of a_k, i.e.,

$$\sum_{k\leqslant x} a_k^l, \qquad l = 1, 2, 3, \ldots \tag{1.2}$$

Let K be an arbitrary algebraic number field of degree $n \geqslant 2$ over $\mathbb{Q}$. Landau [12] proved that

$$\sum_{k\leqslant x} a_k = cx + O(x^{1-\frac{2}{n+1}+\varepsilon}),$$

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where c is the residue of $\zeta_K(s)$ at its simple pole $s = 1$. It is a hard problem to improve Laudau's result. Later, Huxley and Watt [6] and Müller [14] improved the results for the quadratic and cubic fields, respectively. For any algebraic number field of degree $n \geqslant 3$, Nowak [15] established the best result hitherto

$$\sum_{k\leqslant x} a_k = cx + \begin{cases} O(x^{1-\frac{2}{n}+\frac{8}{n(5n+2)}}(\log x)^{\frac{10}{5n+2}}), & \text{for} \quad 3 \leqslant n \leqslant 6, \\ O(x^{1-\frac{2}{n}+\frac{3}{2n^2}}(\log x)^{\frac{2}{n}}), & \text{for} \quad n \geqslant 7. \end{cases} \tag{1.3}$$

The case $l = 2$ of the sum (1.2) was first considered in [2], where it was shown that if K is a Galois extension of $\mathbb{Q}$ of degree n, then

$$\sum_{k\leqslant x} a_k^2 \sim c_1 x(\log x)^{n-1}, \qquad \text{as } x \to \infty, \tag{1.4}$$

for a suitable constant $c_1 = c_1(K)$.

If $l = 2$ and K is a quadratic field with discriminant $D = -4$, then a_k denotes the number of integral solutions of $k = x^2 + y^2$. In this case, Ramanujan [17] gave Formula (1.4) with the error term $O(x^{\frac{3}{5}+\varepsilon})$. Its proof can be found in [20].

In [1], Chandraseknaran and Good showed that if K is a Galois extension of $\mathbb{Q}$ of degree n, then for any $\varepsilon > 0$ and any integer $l \geqslant 2$, we have

$$\sum_{k\leqslant x} a_k^l = xP_l(\log x) + O(x^{1-\frac{2}{n^l}+\varepsilon}), \tag{1.5}$$

where $P_l(t)$ denotes a suitable polynomial in t of degree $n^{l-1} - 1$.

In this paper, we are able to prove the following results.

Theorem 1.1. *If K is a Galois extension of $\mathbb{Q}$ of degree n, then for any $\varepsilon > 0$ and any integer $l \geqslant 2$, we have $\sum_{k\leqslant x} a_k^l = xP_l(\log x) + O(x^{1-\frac{3}{n^l+6}+\varepsilon})$, where $P_l(t)$ denotes a suitable polynomial in t of degree $n^{l-1} - 1$.*

For some special fields, we are able to further improve Theorem 1.1.

Theorem 1.2. *If K is an abelian extension of $\mathbb{Q}$ of degree $n \geqslant 4$, then for any $\varepsilon > 0$ and any integer $l \geqslant 2$, we have $\sum_{k\leqslant x} a_k^l = xP_l(\log x) + O(x^{1-\frac{3}{n^l}+\varepsilon})$, where $P_l(t)$ denotes a suitable polynomial in t of degree $n^{l-1} - 1$.*

Similarly for cubic fields, we have the following result.

Theorem 1.3. *If K is an abelian cubic field, then for any $\varepsilon > 0$ and any integer $l \geqslant 3$, we have $\sum_{k\leqslant x} a_k^l = xP_l(\log x) + O(x^{1-\frac{3}{3^l}+\varepsilon})$, where $P_l(t)$ denotes a suitable polynomial in t of degree $3^{l-1} - 1$.*

If $l = 2$, we have $\sum_{k\leqslant x} a_k^2 = xP_2(\log x) + O(x^{\frac{5}{7}+\varepsilon})$.

For quadratic fields, we have the following result.

Theorem 1.4. *If K is a quadratic field, then for any $\varepsilon > 0$ and any integer $l \geqslant 4$, we have $\sum_{k\leqslant x} a_k^l = xP_l(\log x) + O(x^{1-\frac{3}{2^l}+\varepsilon})$, where $P_l(t)$ denotes a suitable polynomial in t of degree $2^{l-1} - 1$.*

If $l = 2$, we have $\sum_{k\leqslant x} a_k^2 = xP_2(\log x) + O(x^{\frac{1}{2}+\varepsilon})$. If $l = 3$, we have $\sum_{k\leqslant x} a_k^3 = xP_3(\log x) + O(x^{\frac{2}{3}+\varepsilon})$.

As an application of Theorems 1.1–1.4, we consider the number of solutions of certain congruences. Let $f(x) = x^d + c_1x^{d-1} + \cdots + c_d$, $c_1, \ldots, c_d \in \mathbb{Z}$, $d \geqslant 2$ be an irreducible polynomial, and $N_f(n)$ the number of solutions of $f(x) \equiv 0 \pmod n$. It is an important problem to study the function $N_f(n)$. Let L be the splitting field of f with the Galois group $G = \mathrm{Gal}(L/\mathbb{Q})$. If the Galois group G is abelian, the field L is called an abelian field. In this case we also call $f(x)$ an abelian polynomial.

In 1952, Erdös [3] proved the asymptotic formulae

$$\sum_{p\leqslant x} N_f(p) = \frac{x}{\log x} + O\bigg(\frac{x}{(\log x)^2}\bigg), \qquad \sum_{p\leqslant x} \frac{N_f(p)}{p} = \log\log x + c(f) + o(1),$$

and the lower estimate $\sum_{n\leqslant x} N_f(n) \gg x$. In 2001, Fomenko [9] showed that

$$\sum_{n\leqslant x} N_f(n) = C(f)x + O\bigg(\frac{x}{(\log x)^{\frac{1}{2}-\varepsilon}}\bigg),$$

where $C(f)$ is a positive constant depending on f. And for abelian polynomials $f(x)$,

$$\sum_{n\leqslant x} N_f(n) = C(f)x + O(x\exp(-B(\log x)^{\beta}))$$

holds for a certain positive constant B and an arbitrary fixed $\beta < \frac{3}{5}$. Recently, Kim [8] proved that unconditionally we have

$$\sum_{n\leqslant x} N_f(n) = C(f)x + O(x^{\frac{d+2}{d+4}+\varepsilon});$$

for abelian polynomials $f(x)$, we have $\sum_{n\leqslant x} N_f(n) = C(f)x + O(x^{\frac{d+3}{d+6}+\varepsilon})$.

It seems to us that no result about $\sum_{n\leqslant x} N_f^l(n)$, $l \geqslant 2$ has been established. Following the same method as Theorems 1.1–1.4, we are able to show similar results. For example, corresponding to Theorem 1.2, we have the following result.

Corollary 1.5. *For any Abelian polynomial $f(x)$ with degree $d \geqslant 4$, we have that for $l \geqslant 2$,*

$$\sum_{n\leqslant x} N_f^l(n) = xQ_l(\log x) + O(x^{1-\frac{3}{d^l}+\varepsilon}),$$

where $Q_l(t)$ denotes a suitable polynomial in t of degree $d^{l-1} - 1$.

After we completed this work, we found that one result about generalized divisor problems of Kanemitsu, Sankaranarayanan and Tanigawa [7]. If we apply their results to our problem directly, a slightly weaker result can be established: If K is a Galois extension of $\mathbb{Q}$ of degree n, then for any $\varepsilon > 0$ and any integer $l \geqslant 2$, we have $\sum_{k\leqslant x} a_k^l = xP_l(\log x) + O(x^{1-\frac{3}{n^l+n}+\varepsilon})$, where P_l denotes a suitable polynomial of degree $n^{l-1} - 1$.

2 Proof of Theorem 1.1

To prove Theorem 1.1, we need the following lemmas.

Lemma 2.1. *Let K be a Galois extension of degree n over $\mathbb{Q}$, and a_k be defined in* (1.1)*. Define*

$$D_l(s) = \sum_{k=1}^{\infty} a_k^l k^{-s}, \quad \sigma > 1. \tag{2.1}$$

Then we have $D_l(s) = \zeta_K^{n^{l-1}}(s)U_l(s)$, where $U_l(s)$ denotes a Dirichlet series, which is absolutely convergent for $\sigma > \frac{1}{2}$.

Proof. This is Lemma 1 in [1], whose proof is based on the decomposition law for prime ideals in Galois extension. See, e.g., Section 1.5 in [19]. □

Lemma 2.2. *Let K be an algebraic number field of degree n, Then $\zeta_K(1/2+it) \ll_K t^{\frac{n}{6}+\varepsilon} (t \geqslant 1)$ for any fixed $\varepsilon > 0$.*

Proof. Heath-Brown [5] applied an n-dimensional variant of van der Corput's method to establish this subconvexity bound for the Dedekind zeta-function. □

Now we begin to complete the proof of Theorem 1.1. By (1.1), (2.1) and Perron's formula (see Proposition 5.54 in [11]), we have

$$\sum_{k\leqslant x} a_k^l = \frac{1}{2\pi i}\int_{b-iT}^{b+iT} D_l(s)\frac{x^s}{s}ds + O\bigg(\frac{x^{1+\varepsilon}}{T}\bigg), \tag{2.2}$$

where $b = 1 + \varepsilon$ and $1 \leqslant T \leqslant x$ is a parameter to be chosen later.

Next we move the integration to the parallel segment with $\mathrm{Re}\, s = \frac{1}{2} + \varepsilon$. By Cauchy's residue theorem, we have

$$\sum_{k \leqslant x} a_k^l = \mathrm{Res}_{s=1} D_l(s) \frac{x^s}{s} + \frac{1}{2\pi i} \bigg\{ \int_{\frac{1}{2}+\varepsilon-iT}^{\frac{1}{2}+\varepsilon+iT} + \int_{\frac{1}{2}+\varepsilon+iT}^{b+iT} + \int_{b-iT}^{\frac{1}{2}+\varepsilon-iT} \bigg\} D_l(s) \frac{x^s}{s} ds + O\bigg(\frac{x^{1+\varepsilon}}{T} \bigg)$$
$$:= x P_l(\log x) + J_1 + J_2 + J_3 + O\bigg(\frac{x^{1+\varepsilon}}{T} \bigg), \tag{2.3}$$

where $P_l(t)$ denotes a suitable polynomial in t of degree $n^{l-1} - 1$.

By Lemma 2.2, and the Phragmen-Lindelöf principle for a strip (see, e.g., Theorem 5.53 in [11]), we have that for $\frac{1}{2} \leqslant \sigma \leqslant 1 + \varepsilon$,

$$\zeta_K(\sigma + it) \ll (1 + |t|)^{\frac{n}{3}(1-\sigma)+\varepsilon}. \tag{2.4}$$

Therefore, we have for $\frac{1}{2} \leqslant \sigma \leqslant 1 + \varepsilon$,

$$|\zeta_K^{n^{l-1}}(\sigma + it)| \ll (|t| + 1)^{\frac{n^l}{3}(1-\sigma)+\varepsilon}. \tag{2.5}$$

For J_1, we have

$$J_1 \ll x^{\frac{1}{2}+\varepsilon} + x^{\frac{1}{2}+\varepsilon} \int_1^T |D_l(1/2 + \varepsilon + it)| t^{-1} dt \ll x^{\frac{1}{2}+\varepsilon} + x^{\frac{1}{2}+\varepsilon} \int_1^T |\zeta_K^{n^{l-1}}(1/2 + \varepsilon + it)| t^{-1} dt.$$

Then by (2.5) we have

$$J_1 \ll x^{\frac{1}{2}+\varepsilon} + x^{\frac{1}{2}+\varepsilon} \log T \max_{T_1 \leqslant T} \bigg\{ T_1^{-1} \int_{T_1/2}^{T_1} |\zeta_K(1/2 + \varepsilon + it)|^{n^{l-1}} dt \bigg\}$$
$$\ll x^{\frac{1}{2}+\varepsilon} + x^{\frac{1}{2}+\varepsilon} \log T \max_{T_1 \leqslant T} \bigg\{ T_1^{-1} \int_{T_1/2}^{T_1} t^{\frac{n^l}{6}+\varepsilon} dt \bigg\} \ll x^{\frac{1}{2}+\varepsilon} + x^{\frac{1}{2}+\varepsilon} T^{\frac{n^l}{6}+\varepsilon}. \tag{2.6}$$

For the integrals over the horizontal segments, we have

$$J_2 + J_3 \ll \int_{\frac{1}{2}+\varepsilon}^{b} x^\sigma |\zeta_K^{n^{l-1}}(\sigma + iT)| T^{-1} d\sigma \ll \max_{\frac{1}{2}+\varepsilon \leqslant \sigma \leqslant b} x^\sigma T^{\frac{n^l}{3}(1-\sigma)+\varepsilon} T^{-1}$$
$$= \max_{\frac{1}{2}+\varepsilon \leqslant \sigma \leqslant b} \bigg(\frac{x}{T^{\frac{n^l}{3}}} \bigg)^\sigma T^{\frac{n^l}{3}-1+\varepsilon} \ll \frac{x^{1+\varepsilon}}{T} + x^{\frac{1}{2}+\varepsilon} T^{\frac{n^l}{6}-1+\varepsilon}. \tag{2.7}$$

From (2.3), (2.6) and (2.7), we have

$$\sum_{k \leqslant x} a_k^l = x P_l(\log x) + O(x^{1+\varepsilon} T^{-1}) + O(x^{\frac{1}{2}+\varepsilon} T^{\frac{n^l}{6}+\varepsilon}). \tag{2.8}$$

By taking $T = x^{\frac{3}{n^l+6}}$ in (2.8), we have $\sum_{k \leqslant x} a_k^l = x P_l(\log x) + O(x^{1-\frac{3}{n^l+6}+\varepsilon})$. This completes the proof of Theorem 1.1. □

3 Proof of Theorem 1.2

Let $\mathbb{Q}(\zeta_m)$, $\zeta_m = e^{\frac{2\pi i}{m}}$, be the least cyclotomic fields which contains the abelian field K. Then we call m the conductor of the abelian field K. We have $\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q}) \cong (\mathbb{Z}/m\mathbb{Z})^*$, and then $H = \mathrm{Gal}(\mathbb{Q}(\zeta_m)/K)$ can be regarded as a subgroup of $(\mathbb{Z}/m\mathbb{Z})^*$. The characters of finite abelian group $\mathrm{Gal}(K/\mathbb{Q}) \cong (\mathbb{Z}/m\mathbb{Z})^*/H$ are also called the characters of field K. We denote the character group of K by $\widehat{K}$. Therefore $\widehat{K}$ consists of Dirichlet characters modulo m that are trivial on H.

As a simple corollary of abelian class filed theory we can write $\zeta_K(s)$ as a product of the Riemann zeta-function and Dirichlet L-functions. More precisely, we have

$$\zeta_K(s)=\prod_{\chi\in\widehat{K}}L(s,\chi^*)=\zeta(s)\prod_{\substack{\chi\in\widehat{K}\\ \chi\neq\chi_0}}L(s,\chi^*),$$

where χ^* is a primitive character modulo m' with $m'|m$, which induces $\chi \mod m$. For simplicity, we shall write

$$\zeta_K(s)=\zeta(s)\prod_{j=1}^{n-1}L(s,\chi_j), \tag{3.1}$$

where $L(s,\chi_j)$ are primitive Dirichlet L-functions.

Therefore from Lemma 2.1, we have

$$D_l(s)=\zeta_K^{n^{l-1}}(s)U(s)=\zeta^{n^{l-1}}(s)\prod_{j=1}^{n-1}L^{n^{l-1}}(s,\chi_j)U(s), \tag{3.2}$$

which admits a meromorphic continuation to the half-plane $\mathrm{Re}\, s>\frac{1}{2}$, and only has a pole $s=1$ of order n^{l-1} in this region.

Now we begin to complete the proof of Theorem 1.2. By (3.2) and Perron's formula (see Proposition 5.54 in [11]), we have

$$\sum_{k\leqslant x}a_k^l=\frac{1}{2\pi i}\int_{b-iT}^{b+iT}D_l(s)\frac{x^s}{s}ds+O\bigg(\frac{x^{1+\varepsilon}}{T}\bigg), \tag{3.3}$$

where $b=1+\varepsilon$ and $1\leqslant T\leqslant x$ is a parameter to be chosen later.

Next we move the integration to the parallel segment with $\mathrm{Re}\, s=\frac{1}{2}+\varepsilon$. By Cauchy's residue theorem, we have

$$\begin{aligned}\sum_{k\leqslant x}a_k^l&=\mathrm{Res}_{s=1}D_l(s)\frac{x^s}{s}+\frac{1}{2\pi i}\bigg\{\int_{\frac{1}{2}+\varepsilon-iT}^{\frac{1}{2}+\varepsilon+iT}+\int_{\frac{1}{2}+\varepsilon+iT}^{b+iT}+\int_{b-iT}^{\frac{1}{2}+\varepsilon-iT}\bigg\}D_l(s)\frac{x^s}{s}ds+O\bigg(\frac{x^{1+\varepsilon}}{T}\bigg)\\&:=xP_l(\log x)+I_1+I_2+I_3+O\bigg(\frac{x^{1+\varepsilon}}{T}\bigg).\end{aligned} \tag{3.4}$$

For I_1, by (3.2) we have (note that $n\geqslant 4$ and $l\geqslant 2$)

$$\begin{aligned}I_1&\ll x^{\frac{1}{2}+\varepsilon}+x^{\frac{1}{2}+\varepsilon}\int_1^T|D_l(1/2+\varepsilon+it)|t^{-1}dt\\&\ll x^{\frac{1}{2}+\varepsilon}+x^{\frac{1}{2}+\varepsilon}\int_1^T|\zeta_K^{n^{l-1}}(1/2+\varepsilon+it)|t^{-1}dt\\&\ll x^{\frac{1}{2}+\varepsilon}+x^{\frac{1}{2}+\varepsilon}\int_1^T\bigg|\zeta^3(1/2+\varepsilon+it)\zeta^{n^{l-1}-3}(1/2+\varepsilon+it)\prod_{j=1}^{3}L(1/2+\varepsilon+it,\chi_j)^3\\&\quad\times\prod_{j=1}^{3}L(1/2+\varepsilon+it,\chi_j)^{n^{l-1}-3}\prod_{j=4}^{n-1}L(1/2+\varepsilon+it,\chi_j)^{n^{l-1}}\bigg|t^{-1}dt\\&\ll x^{\frac{1}{2}+\varepsilon}+x^{\frac{1}{2}+\varepsilon}\int_1^T\bigg|\zeta^3(1/2+\varepsilon+it)\prod_{j=1}^{3}L(1/2+\varepsilon+it,\chi_j)^3\bigg|t^{\frac{n^l-12}{6}-1}dt,\end{aligned}$$

where we have used

$$\zeta(1/2+it)\ll(1+|t|)^{\frac{1}{6}+\varepsilon}, \tag{3.5}$$

and

$$|L(1/2+\varepsilon+it,\chi)|\ll(1+|t|)^{\frac{1}{6}+\varepsilon}. \tag{3.6}$$

Then we have

$$I_1\ll x^{\frac{1}{2}+\varepsilon}+x^{\frac{1}{2}+\varepsilon}\log T\max_{T_1\leqslant T}\bigg\{T_1^{\frac{n^l-12}{6}-1}\bigg(\int_{T_1/2}^{T_1}|\zeta(1/2+\varepsilon+it)|^{12}dt\bigg)^{\frac{1}{4}}\prod_{j=1}^{3}\bigg(\int_{T_1/2}^{T_1}|L(1/2+\varepsilon+it,\chi_j)|^{12}dt\bigg)^{\frac{1}{4}}\bigg\}$$

$$\ll x^{\frac{1}{2}+\varepsilon}T^{\frac{n^l}{6}-1+\varepsilon}+x^{\frac{1}{2}+\varepsilon}, \tag{3.7}$$

where we have used

$$\int_{T_1/2}^{T_1}|\zeta(1/2+\varepsilon+it)|^{12}dt \ll T_1^{2+\varepsilon} \quad \text{and} \quad \int_{T_1/2}^{T_1}|L(1/2+\varepsilon+it,\chi)|^{12}dt \ll T_1^{2+\varepsilon}.$$

These results can be established by Gabriel's convexity theorem (see e.g. Lemma 8.3 in [10]), and the results of Heath-Brown [4] and Meurman [13], respectively, which state that

$$\int_{T_1/2}^{T_1}|\zeta(1/2+it)|^{12}dt \ll T_1^{2}(\log T_1)^{17} \quad \text{and} \quad \int_{T_1/2}^{T_1}|L(1/2+it,\chi)|^{12}dt \ll T_1^{2+\varepsilon}.$$

By (2.4), for the integrals over the horizontal segments we have

$$\begin{aligned} I_2+I_3 &\ll \int_{\frac{1}{2}+\varepsilon}^{b} x^{\sigma}|\zeta_K^{n^{l-1}}(\sigma+iT)|T^{-1}d\sigma \ll \max_{\frac{1}{2}+\varepsilon\leqslant\sigma\leqslant b} x^{\sigma}T^{\frac{n^l}{3}(1-\sigma)+\varepsilon}T^{-1} \\ &= \max_{\frac{1}{2}+\varepsilon\leqslant\sigma\leqslant b}\left(\frac{x}{T^{\frac{n^l}{3}}}\right)^{\sigma}T^{\frac{n^l}{3}-1+\varepsilon} \ll \frac{x^{1+\varepsilon}}{T}+x^{\frac{1}{2}+\varepsilon}T^{\frac{n^l}{6}-1+\varepsilon}. \end{aligned} \tag{3.8}$$

From (3.4), (3.7) and (3.8), we have

$$\sum_{k\leqslant x}a_k^l = xP_l(\log x)+O\bigg(\frac{x^{1+\varepsilon}}{T}\bigg)+O(x^{\frac{1}{2}+\varepsilon}T^{\frac{n^l}{6}-1+\varepsilon}). \tag{3.9}$$

By taking $T=x^{\frac{3}{n^l}}$ in (3.9), we have $\sum_{k\leqslant x}a_k^l = xP_l(\log x)+O(x^{1-\frac{3}{n^l}+\varepsilon})$. □

4 Proof of Corollary 1.5

Let $f(x)=x^d+c_1x^{d-1}+\cdots+c_d$, $c_1,\ldots,c_d\in\mathbb{Z}$, $d\geqslant 2$ be an irreducible polynomial, and $N_f(n)$ the number of solutions of $f(x)\equiv 0 \pmod n$. L is the splitting field of f with the Abelian Galois group $G=\mathrm{Gal}(L/\mathbb{Q})$.

To prove Corollary 1.5, we firstly introduce the L-function associated with $N_f^l(n)$,

$$L(s)=\sum_{n=1}^{\infty}\frac{N_f^l(n)}{n^s}. \tag{4.1}$$

Since $N_f(n)$ is multiplicative, for Re $s>1$ we can write

$$L(s)=\prod_p\bigg(1+\frac{N_f^l(p)}{p^s}+\frac{N_f^l(p^2)}{p^{2s}}+\cdots\bigg). \tag{4.2}$$

Essentially, Kim [8] showed that except for finitely many primes,

$$a_p = N_f(p), \tag{4.3}$$

where a_p denotes the number of integral ideas of L of norm p. See [3] for a detailed argument.

From Lemma 2.1, we have

$$D_l(s)=\sum_{k=1}^{\infty}\frac{a_k^l}{k^s}=\zeta_L^{d^{l-1}}(s)U_1(s),$$

where $U_1(s)$ denotes a Dirichlet series, which is absolutely convergent for $\sigma>\frac{1}{2}$. Then we have

$$\zeta_L^{d^{l-1}}(s)U_1(s)=\prod_p\bigg(1+\frac{a_p^l}{p^s}+\frac{a_{p^2}^l}{p^{2s}}+\cdots\bigg). \tag{4.4}$$

From (4.2), (4.3) and (4.4), we have that for $\mathrm{Re}\, s > 1$,

$$L(s) = \sum_{n=1}^{\infty} \frac{N_f^l(n)}{n^s} = \zeta_L^{d^{l-1}}(s) U_1(s) U_2(s) = D_l(s) U_2(s), \tag{4.5}$$

where $U_j(s)$, $j = 1, 2$ denote two Dirichlet series, which are absolutely convergent for $\sigma > \frac{1}{2}$. Therefore, $L(s)$ admits a meromorphic continuation to the half-plane $\mathrm{Re}\, s > \frac{1}{2}$, and only has a pole $s = 1$ of order d^{l-1} in this region.

By (4.5) and the same arguments in Theorem 1.2, we obtain Corollary 1.5. □

5 Proofs of Theorems 1.3 and 1.4

The proofs of Theorems 1.3 and 1.4 are similar to that of Theorem 1.2. As an example, we give a sketch proof of Theorem 1.4.

To this end, let us recall one result of Sankaranarayanan.

Lemma 5.1. *Let K be a quadratic field. Then we have that for $T \geqslant 1$ and any $\varepsilon > 0$, $\int_T^{2T} |\zeta_K(1/2 + it)|^6 dt \ll T^{2+\varepsilon}$.*

Proof. See the main result in [18]. □

Now we begin to prove Theorem 1.4. For the case $l = 2$, for I_1 in (3.4) we have

$$\begin{aligned} I_1 &\ll x^{\frac{1}{2}+\varepsilon} + x^{\frac{1}{2}+\varepsilon} \int_1^T |\zeta_K^2(1/2 + \varepsilon + it)| t^{-1} dt \\ &\ll x^{\frac{1}{2}+\varepsilon} + x^{\frac{1}{2}+\varepsilon} \int_1^T |\zeta(1/2 + \varepsilon + it)^2 L(1/2 + \varepsilon + it, \chi)^2| t^{-1} dt \\ &\ll x^{\frac{1}{2}+\varepsilon} + x^{\frac{1}{2}+\varepsilon} \log T \max_{T_1 \leqslant T} \left\{ T_1^{-1} \left(\int_{T_1/2}^{T_1} |\zeta(1/2 + \varepsilon + it)|^4 dt \right)^{\frac{1}{2}} \left(\int_{T_1/2}^{T_1} |L(1/2 + \varepsilon + it, \chi)|^4 dt \right)^{\frac{1}{2}} \right\} \\ &\ll x^{\frac{1}{2}+\varepsilon}, \end{aligned} \tag{5.1}$$

where we have used

$$\int_{T_1/2}^{T_1} |\zeta(1/2 + \varepsilon + it)|^4 dt \ll T_1^{1+\varepsilon} \quad \text{and} \quad \int_{T_1/2}^{T_1} |L(1/2 + \varepsilon + it, \chi)|^4 dt \ll T_1^{1+\varepsilon}.$$

These results can be established by Gabriel's convexity theorem (see, e.g., Lemma 8.3 in [10]), and the two following classical results (see, e.g., Theorems 29.3.1 and 29.3.4 in [16]),

$$\int_{T_1/2}^{T_1} |\zeta(1/2 + it)|^4 dt \ll T_1 (\log T_1)^4 \quad \text{and} \quad \int_{T_1/2}^{T_1} |L(1/2 + it, \chi_j)|^4 dt \ll T_1 (\log T_1)^4.$$

Inserting this estimate for I_1 into (3.9), we obtain the expected result.

For the case $l = 3$, for I_1 in (3.4) we have

$$\begin{aligned} I_1 &\ll x^{\frac{1}{2}+\varepsilon} + x^{\frac{1}{2}+\varepsilon} \int_1^T |\zeta_K^4(1/2 + \varepsilon + it)| t^{-1} dt \\ &\ll x^{\frac{1}{2}+\varepsilon} + x^{\frac{1}{2}+\varepsilon} \log T \max_{T_1 \leqslant T} \left\{ T_1^{-1} \left(\int_{T_1/2}^{T_1} |\zeta_K(1/2 + \varepsilon + it)|^6 dt \right)^{\frac{1}{2}} \left(\int_{T_1/2}^{T_1} |\zeta_K(1/2 + \varepsilon + it)|^2 dt \right)^{\frac{1}{2}} \right\} \\ &\ll x^{\frac{1}{2}+\varepsilon} + x^{\frac{1}{2}+\varepsilon} T^{\frac{1}{2}+\varepsilon}, \end{aligned} \tag{5.2}$$

where we have used

$$\int_{T_1}^{2T_1} |\zeta_K(1/2 + \varepsilon + it)|^6 dt \ll T^{2+\varepsilon}. \tag{5.3}$$

These results can be established by Gabriel's convexity theorem and Lemma 5.1. Inserting this into (3.9) and choosing $T = x^{\frac{1}{3}}$, we obtain the expected result.

For the case $l \geqslant 4$, for I_1 in (3.4) we have

$$\begin{aligned} I_1 &\ll x^{\frac{1}{2}+\varepsilon} + x^{\frac{1}{2}+\varepsilon} \int_1^T |\zeta_K^{2^{l-1}}(1/2+\varepsilon+it)|t^{-1}dt \\ &\ll x^{\frac{1}{2}+\varepsilon} + x^{\frac{1}{2}+\varepsilon} \log T \max_{T_1 \leqslant T} \left\{ T_1^{\frac{2^l-12}{6}-1} \int_{T_1/2}^{T_1} |\zeta_K(1/2+\varepsilon+it)|^6 dt \right\} \\ &\ll x^{\frac{1}{2}+\varepsilon} + x^{\frac{1}{2}+\varepsilon} T^{\frac{2^l}{6}-1+\varepsilon}, \end{aligned} \tag{5.4}$$

where we have used (5.3), and Lemma 2.2 with $n = 2$. Inserting this into (3.9) and choosing $T = x^{\frac{3}{2^l}}$, we obtain the expected result.

Acknowledgements The first author is supported in part by National Natural Science Foundation of China (Grant No. 10701048), Natural Science Foundation of Shandong Province (Grant No. ZR2009AM007), and Independent Innovation Foundation of Shandong University. The second author is supported in part by National Basic Research Program of China (973 Program) (Grant No. 2007CB807902) and National Natural Science Foundation of China (Grant No. 10601034). The authors are grateful to Professor Jianya Liu for his encouragement. This work was done while we were CSC visiting scholars in Department of Mathematics, Stanford University. We are grateful to Professor Kannan Soundararajan for his generous hospitality and for providing a stimulating environment.

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