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A quantitative program for Hadwiger's covering conjecture

Dedicated to Professor Wang Yuan on the Occasion of his 80th Birthday

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Abstract In 1957, Hadwiger made a conjecture that every n-dimensional convex body can be covered by 2^n translates of its interior. Up to now, this conjecture is still open for all $n \geqslant 3$. In 1933, Borsuk made a conjecture that every n-dimensional bounded set can be divided into $n+1$ subsets of smaller diameters. Up to now, this conjecture is open for $4 \leqslant n \leqslant 297$. In this article we encode the two conjectures into continuous functions defined on the spaces of convex bodies, propose a four-step program to attack them, and obtain some partial results.

Keywords convex body, Hadwiger's conjecture, Banach-Mazur metric, β-net, Borsuk's conjecture

MSC(2000): 52C17, 11H31

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1 Introduction

In n-dimensional Euclidean space E^n, let K be a convex body with boundary $\partial(K)$, interior $\text{int}(K)$ and volume $v(K)$, and let $c(K)$ denote the smallest number of translates of $\text{int}(K)$ that their union can cover K. In 1955, Levi [23] studied $c(K)$ for the two-dimensional convex domains and proved that

$$c(K) = \begin{cases} 4, & \text{if } K \text{ is a parallelogram,} \\ 3, & \text{otherwise.} \end{cases}$$

Let P denote an n-dimensional parallelopiped. Clearly, any translate of $\text{int}(P)$ cannot cover two vertices of P. Therefore, it can be deduced that $c(P) = 2^n$. Let B denote an n-dimensional ball. It is easy to see that $c(B) = n+1$. In fact, this is true for all n-dimensional convex bodies with smooth boundaries.

Based on these results and some other observations, in 1957 Hadwiger [16] made the following conjecture: For every n-dimensional convex body K we have

$$c(K) \leqslant 2^n, \tag{1}$$

where the equality holds if and only if K is a parallelopiped.

This conjecture has been studied by many authors. In the course, many partial results have been achieved and several connections with other important problems such as the illumination problem and the separation problem have been discovered (see [3,7,10,32] for general references). For example, Lassak [21] proved this conjecture for the three-dimensional centrally symmetric case, Rogers and Zong [27] obtained

$$c(K) \leqslant \binom{2n}{n}(n\log n + n\log\log n + 5n)$$

for general n-dimensional convex bodies, and

$$c(K) \leqslant 2^n(n\log n + n\log\log n + 5n)$$

for centrally symmetric ones. Nevertheless, we are still far away from the solution of the conjecture, even the three-dimensional case.

Let m be a positive integer and let $\gamma_m(K)$ be the smallest positive number r such that K can be covered by m translates of rK. Clearly, we have $\gamma_m(K) = 1$ for all $m \leqslant n$, and $\gamma_m(K) \geqslant \gamma_{m+1}(K)$ for all positive integers m and all n-dimensional convex bodies K.

Let $\mathcal{T}^n$ denote the set of all non-singular linear transformations in E^n and let $\mathcal{K}^n$ denote the space of all n-dimensional convex bodies with the Banach-Mazur metric defined by

$$\|K_1, K_2\| = \log\,\min\{r:\ K_1 \subseteq T(K_2) \subseteq rK_1 + \boldsymbol{x};\ \boldsymbol{x} \in E^n;\ T \in \mathcal{T}^n\}.$$

By John's theorem (see Section 3) and Blaschke's selection theorem (see [14]) it follows that $\mathcal{K}^n$ is bounded, connected and compact. On the other hand, for any given positive integer m, it can be shown that $\gamma_m(K)$ as a function of K defined on $\mathcal{K}^n$ is continuous (see Theorem A in Section 3). In addition, we have $\gamma_m(K_1) = \gamma_m(K_2)$ whenever $\|K_1, K_2\| = 0$. Then we define

$$\Gamma(n,m) = \max_{K\in\mathcal{K}^n}\{\gamma_m(K)\} \quad \text{and} \quad \gamma(n,m) = \min_{K\in\mathcal{K}^n}\{\gamma_m(K)\}.$$

Note that $\gamma_m(K)$ is continuous on $\mathcal{K}^n$ and $\mathcal{K}^n$ is compact. It is easy to see that (1) holds for all n-dimensional convex bodies K if and only if $\gamma_{2^n}(K) < 1$ holds for all $K \in \mathcal{K}^n$. Therefore, it is equivalent to $\Gamma(n, 2^n) < 1$. Thus, Hadwiger's covering conjecture can be encoded into the functions $\gamma_m(K)$ defined on the space $\mathcal{K}^n$.

In this article, based on some nice properties of $\gamma_m(K)$ and $\mathcal{K}^n$, we suggest a four-step program (see Section 3) to attack Hadwiger's conjecture as well as Borsuk's conjecture. In addition, we study the values of $\gamma_m(K)$ for some particular m and K. Among other things, the following results are proved:

Theorem 1. *Let K be a bounded three-dimensional convex cone (the convex hull of a convex domain and a point which is not in the plane of the domain), then we have $\gamma_8(K) \leqslant \frac{2}{3}$.*

Theorem 2. *Let K_p be the unit ball of the three-dimensional ℓ_p norm, in other words,*

$$K_p = \{(x,y,z):\ |x|^p + |y|^p + |z|^p \leqslant 1\}.$$

For all p satisfying $1 \leqslant p \leqslant +\infty$ we have $\gamma_8(K_p) \leqslant \sqrt{\frac{2}{3}}$.

2 The two-dimensional case, a brief review

The values of $\gamma(2,m)$ and $\Gamma(2,m)$ have been studied by several authors. Clearly, we have $\gamma(2,2) = \Gamma(2,2) = \Gamma(2,3) = 1$ and, by considering the area measures, $\gamma(2,m) \geqslant \frac{1}{\sqrt{m}}$.

However, for the nontrivial cases, it is not easy to determine the exact values of $\gamma(2,m)$ and $\Gamma(2,m)$. We list the known results in Table 1 and Table 2.

Table 1

m	3	4	5
$\gamma(2,m)$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{2}$
Authors	Belousov [1]	Krotoszynski [20]	Krotoszynski [20]

Table 2

m	3	4	5	6	7	8
$\Gamma(2,m)$	1	$\frac{\sqrt{2}}{2}$	??	??	$\frac{1}{2}$	$\frac{1}{2}$
Authors		Lassak [22]	??	??	Levi [23]	Levi [23]

Remark 1. By $\Gamma(2,2^2)=\sqrt{2}/2$ it follows that every two-dimensional convex domain K can be covered by four translates of $\frac{\sqrt{2}}{2}K$. As shown in Table 2, the values of $\Gamma(2,5)$ and $\Gamma(2,6)$ have not been determined yet.

3 A four-step program for Hadwiger's conjecture

First, let us introduce a basic result about $\gamma_m(K)$.

Theorem A. *For any pair of positive integers m and n, the function $\gamma_m(K)$ is continuous on $\mathcal{K}^n$.*

Proof. Let λ and r be positive numbers, $r\leqslant 1$, let K be an n-dimensional convex body, and let $\boldsymbol{x}_i$ be m suitable points. Assume that $\gamma_m(K)=r$ and $K\subseteq\bigcup_{i=1}^m(rK+\boldsymbol{x}_i)$.

For each point $\boldsymbol{x}\in K$, there are a corresponding point $\boldsymbol{y}\in K$ and an index i satisfying $\boldsymbol{x}=r\boldsymbol{y}+\boldsymbol{x}_i$. Thus we can deduce $\lambda\boldsymbol{x}=\lambda r\boldsymbol{y}+\lambda\boldsymbol{x}_i$ and

$$\lambda K\subseteq\bigcup_{i=1}^m(\lambda rK+\lambda\boldsymbol{x}_i). \tag{2}$$

Let ϵ be a small positive number and let K' be any n-dimensional convex body satisfying $\|K,K'\|\leqslant\log(1+\epsilon)$. In other words, without loss of generality, we may take $K\subseteq K'\subseteq(1+\epsilon)K$. Then, by (2) we have

$$K'\subseteq(1+\epsilon)K\subseteq\bigcup_{i=1}^m((1+\epsilon)rK+(1+\epsilon)\boldsymbol{x}_i)\subseteq\bigcup_{i=1}^m((1+\epsilon)rK'+(1+\epsilon)\boldsymbol{x}_i),$$

which implies that

$$\gamma_m(K')\leqslant r+\epsilon r\leqslant\gamma_m(K)+\epsilon.$$

Similarly, we can get $\gamma_m(K)\leqslant\gamma_m(K')+\epsilon$ and therefore,

$$|\gamma_m(K')-\gamma_m(K)|\leqslant\epsilon,$$

which means that the function $\gamma_m(K)$ is continuous at K. The theorem is proved. □

Remark 2. In fact, it follows from the proof that $\gamma_m(K)$ is uniformly continuous on $\mathcal{K}^n$.

Let B^n denote the n-dimensional unit ball centered at the origin. In 1948, John [18] proved the following result:

John's theorem. *For each n-dimensional convex body K there is an non-singular linear transformation $T\in\mathcal{T}^n$ such that*

$$B^n\subseteq T(K)\subseteq nB^n.$$

Let $\overline{\mathcal{K}^n}$ denote the set of all convex bodies K satisfying

$$B^n\subseteq K\subseteq nB^n. \tag{3}$$

By John's theorem we have

$$\Gamma(n,m)=\max_{K\in\overline{\mathcal{K}^n}}\gamma_m(K).$$

Definition 1. *Let β be a positive number, and let $K_1, K_2,\ldots,K_{l(n,\beta)}$ be $l(n,\beta)$ convex bodies in $\mathcal{K}^n$, where $l(n,\beta)$ is an integer depending on n and β. If for any $K\in\mathcal{K}^n$ there is a corresponding K_i satisfying $\|K,K_i\|\leqslant\beta$, then we call $\mathcal{N}=\{K_1,K_2,\ldots,K_{l(n,\beta)}\}$ a β-net in $\mathcal{K}^n$.*

Remark 3. Defining $\mathcal{B}(K_i,\beta)=\{K\in\mathcal{K}^n:\ \|K,K_i\|\leqslant\beta\}$, it is easy to see that $\mathcal{N}=\{K_1,K_2,\ldots,K_{l(n,\beta)}\}$ is a β-net in $\mathcal{K}^n$ if and only if

$$\bigcup_{i=1}^{l(n,\beta)}\mathcal{B}(K_i,\beta)=\mathcal{K}^n.$$

The existence of the β-nets in $\mathcal{K}^n$ is guaranteed by the following lemma which seems important in the study of $\mathcal{K}^n$.

Fundamental lemma. *For each pair $\{n,\beta\}$ of positive integer n and positive number β there is a corresponding integer $l(n,\beta)$ such that $\mathcal{K}^n$ has a β-net of $l(n,\beta)$ elements.*

Proof. Let θ be a small positive number and assume that $X=\{\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots,\boldsymbol{x}_{c(n,\theta)}\}$ is a subset of $\partial(nB^n)$ such that the $c(n,\theta)$ caps $(\theta B^n+\boldsymbol{x}_i)\cap\partial(nB^n)$ form a covering on $\partial(nB^n)$. For convenience, we use θ' to denote the spherical radii of the caps.

Let m be a large integer and define

$$X_{i,m}=\left\{\frac{1}{n}\boldsymbol{x}_i+\frac{j(n-1)}{mn}\boldsymbol{x}_i:\ j=0,1,\ldots,m\right\},$$

and

$$\mathcal{P}=\{\mathrm{conv}\{\boldsymbol{y}_1,\boldsymbol{y}_2,\ldots,\boldsymbol{y}_{c(n,\theta)}\}:\ \boldsymbol{y}_i\in X_{i,m}\}.$$

Note that $\frac{1}{n}\boldsymbol{x}_i\in\partial(B^n)$. We proceed to show that $\mathcal{P}$ is a β-net in $\mathcal{K}^n$ provided both $1/\theta$ and m are large enough. In fact, guaranteed by John's theorem, it is sufficient to prove that $\mathcal{P}$ is a β-net in $\overline{\mathcal{K}^n}$.

Assume that K is an n-dimensional convex body satisfying $B^n\subseteq K\subseteq nB^n$. Let $\boldsymbol{p}_i$ be the point in $X_{i,m}\cap K$ which is the furthest to the origin. Then we define $P=\mathrm{conv}\{\boldsymbol{p}_1,\boldsymbol{p}_2,\ldots,\boldsymbol{p}_{c(n,\theta)}\}$ and

$$\boldsymbol{q}_i=\begin{cases}\boldsymbol{p}_i, & \text{if } \boldsymbol{p}_i=\boldsymbol{x}_i,\\ \boldsymbol{p}_i+\frac{n-1}{mn}\boldsymbol{x}_i, & \text{otherwise.}\end{cases}$$

Clearly the $c(n,\theta)$ caps $(\frac{\theta}{n}B^n+\frac{1}{n}\boldsymbol{x}_i)\cap\partial(B^n)$ form a covering on $\partial(B^n)$ and therefore we have

$$\left(1-\frac{\theta}{n}\right)B^n\subseteq P\subseteq K. \tag{4}$$

Let C_i denote the cone with vertex $\boldsymbol{o}$ over the cap $(\theta B^n+\boldsymbol{x}_i)\cap\partial(nB^n)$, let $\boldsymbol{x}$ be a point which is on the boundary of K and belongs to C_i, and let H be a two-dimensional plane passing $\boldsymbol{o}$, $\boldsymbol{p}_i$ and $\boldsymbol{x}$. As shown in Figure 1, T_1 is tangent to $(1-\frac{\theta}{n})B^n\cap H$ at $\boldsymbol{w}_1$ and passing $\boldsymbol{q}_i$, T_2 is tangent to $(1-\frac{\theta}{n})B^n\cap H$ at $\boldsymbol{w}_2$ and passing $\boldsymbol{p}_i$, the straight line determined by $\boldsymbol{o}$ and $\boldsymbol{x}$ intersects T_i at $\boldsymbol{v}_i$. It follows by convexity that the point $\boldsymbol{x}$ is between $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. By elementary geometry, letting $d(\boldsymbol{u}_1,\boldsymbol{u}_2)$ denote the distance between $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$, we get

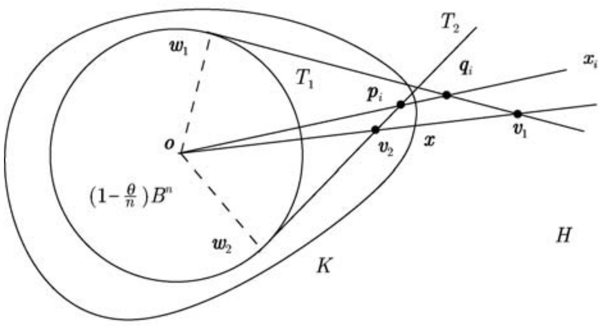

Figure 1

$$\begin{aligned} d(\boldsymbol{v}_1,\boldsymbol{v}_2) &\leqslant d(\boldsymbol{q}_i,\boldsymbol{v}_1)+d(\boldsymbol{p}_i,\boldsymbol{q}_i)+d(\boldsymbol{v}_2,\boldsymbol{p}_i)\\ &=(d(\boldsymbol{w}_1,\boldsymbol{v}_1)-d(\boldsymbol{w}_1,\boldsymbol{q}_i))+d(\boldsymbol{p}_i,\boldsymbol{q}_i)+(d(\boldsymbol{w}_2,\boldsymbol{p}_i)-d(\boldsymbol{w}_2,\boldsymbol{v}_2))\\ &\leqslant\left(1-\frac{\theta}{n}\right)\cdot\tan\left(\arccos\frac{n-\theta}{n^2}+\theta'\right)-\sqrt{n^2-\left(1-\frac{\theta}{n}\right)^2}+\frac{n-1}{m}\\ &\quad+\sqrt{\left(n-\frac{n-1}{m}\right)^2-\left(1-\frac{\theta}{n}\right)^2}-\left(1-\frac{\theta}{n}\right)\cdot\tan\left(\arccos\frac{1-\frac{\theta}{n}}{n-\frac{n-1}{m}}-\theta'\right), \end{aligned}$$

where

$$\theta'=2\arcsin\frac{\theta}{2n}.$$

For convenience, we abbreviate the final complicated function as $f(n,m,\theta)$. Let $\boldsymbol{p}$ denote the point on the boundary of P and in the direction of $\boldsymbol{x}$. Then we have

$$\frac{d(\boldsymbol{o},\boldsymbol{x})}{d(\boldsymbol{o},\boldsymbol{p})}\leqslant\frac{d(\boldsymbol{o},\boldsymbol{p})+d(\boldsymbol{v}_1,\boldsymbol{v}_2)}{d(\boldsymbol{o},\boldsymbol{p})}\leqslant 1+\frac{f(n,m,\theta)}{1-\theta/n},$$

and therefore,

$$K\subseteq\left(1+\frac{f(n,m,\theta)}{1-\theta/n}\right)P. \tag{5}$$

In addition, for any fixed n, it can be shown by a routine argument that

$$\lim_{\substack{m\to\infty\\ \theta\to 0}}\frac{f(n,m,\theta)}{1-\theta/n}=0. \tag{6}$$

As a conclusion of (4), (5) and (6), for any given pair of n and β there is a set $\mathcal{P}$ of $l(n,\beta)$ polytopes such that for each $K\in\mathcal{K}^n$ there is a $P\in\mathcal{P}$ satisfying $\|K,P\|<\beta$. The lemma is proved. □

Since

$$\log(1+x)\leqslant x$$

holds for all $x\geqslant 0$, it follows by (4) and (5) that $\mathcal{P}$ will be a β-net in $\mathcal{K}^n$ if

$$\frac{f(n,m,\theta)}{1-\theta/n}\leqslant\beta. \tag{7}$$

By a routine computation one can deduce that, when both m and $1/\theta$ are sufficiently large and n is fixed,

$$f(n,m,\theta)\sim 2n\theta+\frac{n-1}{m}<3n\left(\theta+\frac{1}{m}\right).$$

Therefore, when β is small, (7) can be guaranteed by taking $\theta=\frac{\beta}{7n}$ and $m=\lfloor\frac{7n}{\beta}\rfloor$. In this case, we have $\theta'=2\arcsin\frac{\beta}{14n^2}$. According to Böröczky and Wintsche [8] there is a cap covering satisfying

$$c(n,\theta)\leqslant c\cdot n^{\frac{3}{2}}\cos\theta'\sin^{-n}\theta'\cdot\log(2+n\cos^2\theta')\leqslant c\cdot 14^n\cdot n^{2n+3}\cdot\beta^{-n},$$

where c is a suitable constant. Thus, there is a β-net $\mathcal{N}=\{K_1,K_2,\dots,K_{l(n,\beta)}\}$ in $\mathcal{K}^n$ satisfying

$$l(n,\beta)\leqslant m^{c(n,\theta)}\leqslant\left\lfloor\frac{7n}{\beta}\right\rfloor^{c\cdot 14^n\cdot n^{2n+3}\cdot\beta^{-n}}.$$

Remark 4. Clearly, to estimate the minimal cardinality of the β-nets in $\mathcal{K}^n$ and to construct the corresponding good β-nets are challenging problems.

The philosophy of our program. If Hadwiger's conjecture is true in E^n, since $\gamma_{2^n}(K)$ is continuous on $\mathcal{K}^n$ and $\mathcal{K}^n$ is compact, then there is a positive number $c_n<1$ such that

$$\gamma_{2^n}(K)\leqslant c_n \tag{8}$$

holds for all $K \in \mathcal{K}^n$. On the other hand, if (8) holds with certain $c_n < 1$ for all convex bodies $K \in \mathcal{K}^n$, then Hadwiger's conjecture is true in E^n. Since $\gamma_{2^n}(K)$ is continuous on $\mathcal{K}^n$, there is a positive number β such that

$$|\gamma_{2^n}(K) - \gamma_{2^n}(K')| \leqslant \frac{1}{2}(1 - c_n) \tag{9}$$

holds whenever $\|K, K'\| \leqslant \beta$. If we can construct a β-net

$$\mathcal{N} = \{K_1, K_2, \ldots, K_{l(n,\beta)}\}$$

for this particular β and can verify (with the assistance of a computer if necessary) $\gamma_{2^n}(K_i) \leqslant c_n$ for all $K_i \in \mathcal{N}$, then by (9) we get $\gamma_{2^n}(K) \leqslant \frac{1}{2}(1 + c_n)$ for all $K \in \mathcal{K}^n$ and therefore Hadwiger's conjecture.

A four-step program for Hadwiger's conjecture.

Step 1. In the considered dimension, for example $n = 3$, study the values of $\gamma_{2^n}(K)$ for some particular convex bodies K and therefore choose a possible candidate constant c_n for (8).

Step 2. Choose a suitable positive number β to guarantee (9), based on a close study on the function $\gamma_{2^n}(K)$.

Step 3. Construct a suitable β-net $\mathcal{N}$ based on the fundamental lemma.

Step 4. Verify that $\gamma_{2^n}(K_i) \leqslant c_n$ holds for all $K_i \in \mathcal{N}$ (with the assistance of a computer if necessary).

Remark 5. In principle, the conjecture can be proved in E^n by our program if it is true in this particular dimension and if the computing facility is fast enough. Clearly the set $\mathcal{P}$ can be much reduced in cardinality.

Let $d(X)$ denote the diameter of a bounded set X in E^n. In other words,

$$d(X) = \sup\{d(\boldsymbol{x}_1, \boldsymbol{x}_2) : \ \boldsymbol{x}_i \in X\},$$

where $d(\boldsymbol{x}_1, \boldsymbol{x}_2)$ denotes the Euclidean distance between $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. In 1933, Borsuk [9] made a conjecture that each bounded n-dimensional set X can be partitioned into $n+1$ parts $X_1, X_2, \ldots, X_{n+1}$ such that

$$d(X_i) < d(X), \qquad i = 1, 2, \ldots, n+1.$$

This conjecture has attracted much attention. The two-dimensional case was proved by Borsuk himself, the three-dimensional case was first proved by Perkal [26] in 1947, and the case of sufficiently high dimensions was disproved by Kahn and Kalai [19] in 1993. For a survey on this conjecture we refer to [32]. Up to now, it is still open for $4 \leqslant n \leqslant 297$.

It is known in convexity that for each bounded set X there is a convex body $\widehat{X}$ of constant width satisfying both $X \subseteq \widehat{X}$ and $d(X) = d(\widehat{X})$. Therefore, to study Borsuk's conjecture it is sufficient to consider all convex bodies of constant width 1.

Let $r(K)$ and $R(K)$ denote the radii of the maximal insphere and the minimal circumsphere of an n-dimensional convex body K of constant width 1, respectively. It is known in convexity (see [11]) that

$$1 - \sqrt{\frac{n}{2n+2}} \leqslant r(K) \leqslant R(K) \leqslant \sqrt{\frac{n}{2n+2}}.$$

For convenience, we take

$$\mu_n = \sqrt{\frac{n}{2n+2}} \Big/ \left(1 - \sqrt{\frac{n}{2n+2}}\right) = \frac{\sqrt{2n^2+2n} + n}{n+2},$$

and define $\widehat{\mathcal{K}^n}$ to be the set of all n-dimensional convex bodies K satisfying $B^n \subseteq K \subseteq \mu_n B^n$ associated with the Hausdorff metric $\|\cdot\|^*$, where

$$\|K_1, K_2\|^* = \min\{r : \ K_1 \subseteq K_2 + rB^n, \ K_2 \subseteq K_1 + rB^n\}.$$

Then, we define $\varphi_m(K)$ to be the minimal number r such that K can be partitioned into m parts K_1, $K_2, \dots, K_m$ such that $d(K_i) \leqslant r \cdot d(K)$ holds for all $i = 1, 2, \dots, m$.

It is easy to see that $\widehat{\mathcal{K}^n}$ is compact and, for any positive integer m, $\varphi_m(K)$ is continuous on $\widehat{\mathcal{K}^n}$. To prove Borsuk's conjecture, it is sufficient to show

$$\varphi_{n+1}(K) \leqslant c_n < 1, \qquad K \in \widehat{\mathcal{K}^n}.$$

Therefore, in a given dimension, for instance $n = 4$, Borsuk's conjecture should be approachable by a four-step program similar to that for Hadwiger's conjecture.

Remark 6. It was shown (see [15]) that $\varphi_3(K) \leqslant \frac{\sqrt{3}}{2}$ holds for all two-dimensional K, and $\varphi_4(K) \leqslant 0.9887$ holds for all three-dimensional K.

4 The covering functions on $\mathcal{K}^3$

In this section, among other things, we will prove Theorem 1 and Theorem 2. As a consequence, we give some insight to a reasonable estimate for the constant c_3 defined in the previous section. First, let us introduce two lemmas.

Lemma 1 [2]**.** *Each two-dimensional convex domain has an inscribed affine regular hexagon.*

Remark 7. Affine regular hexagons are the imagines of a regular hexagon under non-singular linear transformations.

Lemma 2. *Let K be a two-dimensional convex domain, let λ be a real number satisfying $0 < \lambda < 1$, and let $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3\}$ be an ordered triple on the boundary of K. If $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3\} \subset \lambda K + \boldsymbol{y}$, then the whole curve from $\boldsymbol{x}_1$ to $\boldsymbol{x}_3$ passing $\boldsymbol{x}_2$ belongs to $\lambda K + \boldsymbol{y}$.*

Proof. For convenience, we assume that $\boldsymbol{o} \in \mathrm{int}(K)$ and $\boldsymbol{y} = (0, a)$. It is well known in convexity (see [12]) that the set of regular convex domains (each tangent touches K at exactly one point and there is one and only one tangent at each boundary point) is dense in $\mathcal{K}^2$. Therefore, without loss of generality, we assume that K is regular.

Let $\boldsymbol{x}_1 = (x_1, y_1)$ and $\boldsymbol{x}_3 = (x_3, y_3)$ denote the points of $\partial(K) \cap (\lambda K + \boldsymbol{y})$ with maximal and minimal x-coordinates, respectively. Let $y = f(x)$ denote the curve of $\partial(K)$ from $\boldsymbol{x}_3$ to $\boldsymbol{x}_1$, and let $y = g(x)$ denote the above part of $\partial(\lambda K) + \boldsymbol{y}$ in the strip of $x_3 \leqslant x \leqslant x_1$. By convexity, as shown in Figure 2, we have

$$g(x_3) \geqslant f(x_3), \quad g(x_1) \geqslant f(x_1), \quad g'(x) = f'\left(\frac{1}{\lambda}x\right) \geqslant f'(x)$$

for $x_3 \leqslant x \leqslant 0$, and

$$g'(x) = f'\left(\frac{1}{\lambda}x\right) \leqslant f'(x)$$

for $0 \leqslant x \leqslant x_1$. Thus, we get

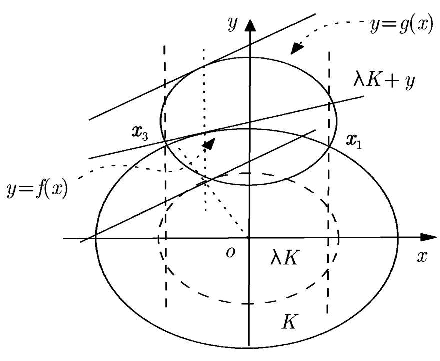

Figure 2

$$g(x) - f(x) = g(x_3) - f(x_3) + \int_{x_3}^{x} (g'(t) - f'(t))dt \geqslant 0$$

when $x_3 \leqslant x \leqslant 0$, and

$$g(x) - f(x) = g(x_1) - f(x_1) + \int_{0}^{x} (f'(t) - g'(t))dt \geqslant 0$$

when $0 \leqslant x \leqslant x_1$. Therefore, by convexity, the whole curve $y = f(x)$ belongs to $\lambda K + \boldsymbol{y}$. The lemma is proved. □

Corollary 1. *Let K be an n-dimensional convex body, λ be a real number satisfying $0 < \lambda < 1$, R be a closed region on $\partial(K)$ with boundary Γ and a relatively interior point $\boldsymbol{p}$. If $\Gamma \cup \{\boldsymbol{p}\} \subset \lambda K + \boldsymbol{y}$ holds for some point $\boldsymbol{y}$, then we have $R \subset \lambda K + \boldsymbol{y}$.*

Proof of Theorem 1. Let K be a three-dimensional cone over a convex domain D. By Lemma 1, there is an affine regular hexagon H inscribed in D. Without loss of generality, we assume that $\boldsymbol{v} = (0, 0, 1)$ is the vertex of K, H is perpendicular to $\boldsymbol{v}$ and centered at the origin $\boldsymbol{o} = (0, 0, 0)$.

Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_6$ be the six vertices of H and let $\boldsymbol{m}_1, \boldsymbol{m}_2, \dots, \boldsymbol{m}_6$ denote the midpoints of $\boldsymbol{v}_1\boldsymbol{v}_2, \boldsymbol{v}_2\boldsymbol{v}_3, \dots, \boldsymbol{v}_6\boldsymbol{v}_1$, respectively. By elementary argument, as shown in Figure 3, we have

$$\left\{\frac{2}{3}\boldsymbol{v}_1, \frac{2}{3}\boldsymbol{v}_2\right\} \subset \frac{1}{3}D + \frac{2}{3}\boldsymbol{m}_1.$$

Thus, by Lemma 2 we get

$$\frac{2}{3}D + \frac{1}{3}\boldsymbol{v} \subseteq \bigcup_{i=0}^{6} \left(\frac{1}{3}D + \frac{2}{3}\boldsymbol{m}_i + \frac{1}{3}\boldsymbol{v}\right), \tag{10}$$

where $\boldsymbol{m}_0 = (0, 0, 0)$. Similarly, as shown in Figure 4, we have

$$\{\boldsymbol{v}_1, \boldsymbol{v}_2\} \subset \frac{2}{3}D + \frac{2}{3}\boldsymbol{m}_1,$$

and therefore

$$D \subseteq \bigcup_{i=0}^{6} \left(\frac{2}{3}D + \frac{2}{3}\boldsymbol{m}_i\right). \tag{11}$$

On the other hand, we have

$$\frac{1}{3}D + \frac{2}{3}\boldsymbol{m}_i + \frac{1}{3}\boldsymbol{v} \subset \frac{2}{3}K + \frac{2}{3}\boldsymbol{m}_i, \tag{12}$$

and

$$\frac{2}{3}D + \frac{2}{3}\boldsymbol{m}_i \subset \frac{2}{3}K + \frac{2}{3}\boldsymbol{m}_i. \tag{13}$$

Therefore, by (10)–(13) and convexity we get

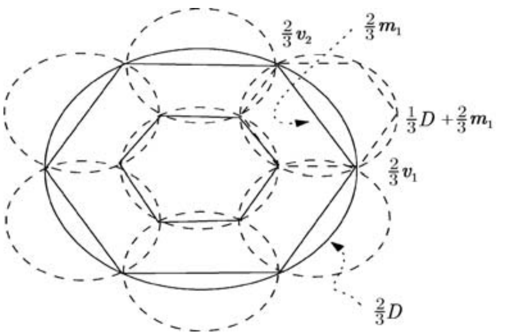

Figure 3

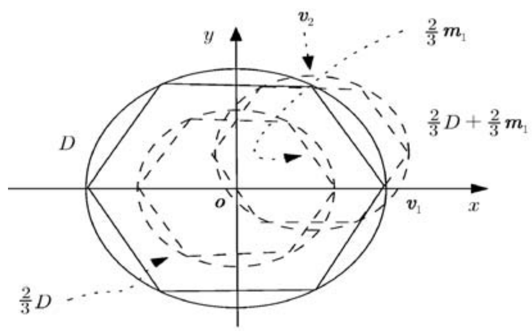

Figure 4

$$K \subseteq \bigcup_{i=0}^{7} \left(\frac{2}{3} K + \frac{2}{3} \boldsymbol{m}_i \right),$$

where $\boldsymbol{m}_7 = \frac{1}{2}\boldsymbol{v}$. Theorem 1 is proved. □

Proof of Theorem 2. **Case 1.** $1 \leqslant p \leqslant 2$. In this case, we take

$$\Gamma = \left\{ \boldsymbol{x} = (x_1, x_2, x_3) : \ x_1 = \left(\frac{1}{3}\right)^{\frac{1}{p}}, \ \boldsymbol{x} \in \partial(K_p) \right\},$$

$\lambda = \sqrt{\frac{2}{3}}$, $\boldsymbol{y} = ((\frac{1}{3})^{\frac{1}{p}}, 0, 0)$, and let R denote the part of $\partial(K_p)$ bounded by Γ and containing $(1,0,0)$.

For any point $\boldsymbol{x} \in \Gamma$, we have

$$\left(\left(\frac{1}{3}\right)^{\frac{1}{p}} \right)^p + |x_2|^p + |x_3|^p = 1, \quad |x_2|^p + |x_3|^p = \frac{2}{3} \leqslant \left(\sqrt{\frac{2}{3}} \right)^p,$$

and therefore $\Gamma \subset \lambda K_p + \boldsymbol{y}$. On the other hand, it can be verified that $(1,0,0) \in \lambda K_p + \boldsymbol{y}$. By Corollary 1 we have $R \subset \lambda K_p + \boldsymbol{y}$. Therefore, in this case K_p can be covered by the union of $\lambda K_p \pm ((\frac{1}{3})^{\frac{1}{p}}, 0, 0)$, $\lambda K_p \pm (0, (\frac{1}{3})^{\frac{1}{p}}, 0)$ and $\lambda K_p \pm (0, 0, (\frac{1}{3})^{\frac{1}{p}})$, and thus $\gamma_8(K_p) \leqslant \gamma_6(K_p) \leqslant \sqrt{\frac{2}{3}}$.

Case 2. $2 \leqslant p \leqslant \infty$. In this case we define

$$\Gamma_i = \{ \boldsymbol{x} = (x_1, x_2, x_3) : \ x_i = 0, \ x_j \geqslant 0, \ j \neq i, \ \boldsymbol{x} \in \partial(K_p) \}, \quad \Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3,$$

$\lambda = \sqrt{\frac{2}{3}}$, $\boldsymbol{y} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, and let R denote the part of $\partial(K_p)$ bounded by Γ and containing the point $((\frac{1}{3})^{\frac{1}{p}}, (\frac{1}{3})^{\frac{1}{p}}, (\frac{1}{3})^{\frac{1}{p}})$.

Let J denote the intersection of K_p with the plane $x_1 = 0$, and let J' denote the intersection of $\lambda K_p + \boldsymbol{y}$ with the plane. It is easy to see that J' is homothetic to J. By a routine computation, for all $2 \leqslant p \leqslant +\infty$, it can be shown that $(\frac{2}{3})^p + 2(\frac{1}{3})^p \leqslant (\frac{2}{3})^{\frac{p}{2}}$. Thus, both $(0,1,0)$ and $(0,0,1)$ belong to J'. Consequently, we also have $(0, (\frac{1}{2})^{\frac{1}{p}}, (\frac{1}{2})^{\frac{1}{p}}) \in J'$. By Lemma 2, we get $\Gamma_1 \subset J' \subset \lambda K_p + \boldsymbol{y}$ and therefore $\Gamma \subset \lambda K_p + \boldsymbol{y}$.

On the other hand, it can be verified that

$$\boldsymbol{o} \in \lambda K_p + \boldsymbol{y}, \quad 3\left(\left(\frac{1}{3}\right)^{\frac{1}{p}} - \frac{1}{3} \right)^p \leqslant \left(\frac{2}{3}\right)^{\frac{p}{2}},$$

and therefore,

$$\left(\left(\frac{1}{3}\right)^{\frac{1}{p}}, \left(\frac{1}{3}\right)^{\frac{1}{p}}, \left(\frac{1}{3}\right)^{\frac{1}{p}} \right) \in \lambda K_p + \boldsymbol{y}.$$

By Corollary 1 we get $R \subset \lambda K_p + \boldsymbol{y}$. Thus, in this case K_p can be covered by the union of the eight translates $\sqrt{\frac{2}{3}} K_p + (\pm\frac{1}{3}, \pm\frac{1}{3}, \pm\frac{1}{3})$ and hence $\gamma_8(K_p) \leqslant \sqrt{\frac{2}{3}}$.

As a conclusion of the two cases, Theorem 2 is proved. □

Remark 8. It was shown by Schütte [29] that $\gamma_8(K_2) \leqslant \sin 48°9' = 0.744894\cdots < \sqrt{\frac{2}{3}}$. Thus, it follows from the proof of Theorem 2 that $\sqrt{\frac{2}{3}}$ is not the optimal upper bound for $\gamma_8(K_p)$. However, perhaps one can take $c_3 = \sqrt{\frac{2}{3}}$.

Remark 9. Let T denote a regular tetrahedron. The values of $\gamma_m(K)$ for some small m and some particular K are listed in Table 3. The values of $\gamma_4(K_2)$ and $\gamma_6(K_2)$ were discovered by Fejes Tóth [13] and the values of $\gamma_5(K_2)$ and $\gamma_7(K_2)$ were determined by Schütte [29].

Table 3

m	4	5	6	7	8
$\gamma_m(T)$	$\frac{3}{4}$	$\frac{9}{13}$	?	?	?
$\gamma_m(K_1)$	1	1	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$
$\gamma_m(K_2)$	$0.9428\cdots$	$0.8944\cdots$	$0.8164\cdots$	$0.7775\cdots$	?

Remark 10. If Hadwiger's conjecture is true for all dimensions, then we have $\Gamma(n, 2^n) < 1$ for all n. Nevertheless, it seems that $\lim_{n\to\infty} \Gamma(n, 2^n) = 1$.

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