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Smooth solutions of non-linear stochastic partial differential equations driven by multiplicative noises

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Abstract In this paper, we study the regularity of solutions of nonlinear stochastic partial differential equations (SPDEs) with multiplicative noises in the framework of Hilbert scales. Then we apply our abstract result to several typical nonlinear SPDEs such as stochastic Burgers and Ginzburg-Landau equations on the real line, stochastic 2D Navier-Stokes equations (SNSEs) in the whole space and a stochastic tamed 3D Navier-Stokes equation in the whole space, and obtain the existence of their smooth solutions respectively. In particular, we also get the existence of local smooth solutions for 3D SNSEs.

Keywords smooth solutions, stochastic partial differential equations, stochastic Navier-Stokes equations

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1 Introduction

Consider the following stochastic Burgers and Ginzburg-Landau equation on the real line:

$$\begin{cases} \mathrm{d}u(t,x) = [\nu\partial_x^2 u(t,x) + c_0 \cdot \partial_x u(t,x)^2 + c_1 \cdot u(t,x) - c_2 \cdot u(t,x)^3]\mathrm{d}t \\ \qquad\qquad + \sum_k \sigma_k(t,x,u(t,x))\mathrm{d}W^k(t), \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $c_0, c_1 \in \mathbb{R}$, $\nu, c_2 > 0$, $\{W^k(t), k \in \mathbb{N}\}$ is a sequence of independent standard one-dimensional Brownian motions, and the coefficients $\{\sigma_k, k \in \mathbb{N}\}$ satisfy some smoothness conditions. Up to now, there are many papers devoted to the studies of stochastic Burgers equation and stochastic Ginzburg-Landau equation (cf. [1, 6, 7, 19] and references therein). In [6], using heat kernel estimates, Gyöngy and Nualart proved the existence and uniqueness of $L^2(\mathbb{R})$-solution to stochastic Burgers equation on the real line. By solving an infinite-dimensional Kolmogorov equation, Röckner and Sobol [19] developed a new method to solve the generalized stochastic Burgers and reaction diffusion equations. More recently, Kim [7] studied the stochastic Burgers type equation with the first order term having polynomial growth, as well as the existence of the associated invariant measures.

Since all of these works are concerned with stochastic Burgers equation driven by space-time white noises, they had to consider weak or mild solutions rather than strong or classical solutions. A natural question is: do there exist smooth or classical solutions in x to equation (1.1) if all the data are smooth?

Of course, for this question, we can only consider it driven by white noises in time and coloured smooth noises in space. We remark that for the deterministic Burgers equation, i.e., $\sigma_k = c_1 = c_2 = 0$ and $c_0 = 1, \nu > 0$, it is well-known that there exists a unique smooth solution if the initial data are smooth (cf. [8]).

Let us also consider the following stochastic 2D Navier-Stokes equation in $\mathbb{R}^2$:

$$\begin{cases} \partial_t u_1 = \nu\Delta u_1 - u_1\partial_{x_1}u_1 - u_2\partial_{x_2}u_1 - \partial_{x_1}p + f_1, \\ \partial_t u_2 = \nu\Delta u_2 - u_1\partial_{x_1}u_2 - u_2\partial_{x_2}u_2 - \partial_{x_2}p + f_2, \\ \partial_{x_1}u_1 + \partial_{x_2}u_2 = 0, \end{cases} \tag{1.2}$$

where ν is the viscosity constant, $\mathbf{u}(t,x) = (u_1, u_2)$ is the velocity field, p is the pressure function, and $f = (f_1, f_2)$ is white in time and additive stochastic forcing. In [17], Mikulevicius and Rozovskii studied the existence of martingale solutions for arbitrary dimensional stochastic Navier-Stokes equations in the whole space. In particular, they obtained the existence of a unique weak solution for the above two-dimensional equation. In the periodic boundary case, using Galerkin's approximation and Fourier's transformation, Mattingly [15] proved the spatial analyticity for the solution to the above stochastically forced 2D Navier-Stokes equation. However, using his method, it seems hard to consider the multiplicative noise force.

As for the stochastic 3D Navier-Stokes equation, Röckner and the author [20, 21] recently studied the following tamed or modified scheme in the whole space $\mathbb{R}^3$ and periodic cases:

$$\begin{cases} \partial_t u_j = \nu\Delta u_j - \sum_{i=1}^{3} u_i\partial_{x_i}u_j - \partial_{x_j}p - g_N\left(\sum_{i=1}^{3}|u_i|^2\right)u_j + f_j, \quad j = 1,2,3, \\ \sum_{i=1}^{3}\partial_{x_i}u_i = 0, \end{cases} \tag{1.3}$$

where the taming function $g_N : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is smooth and satisfies that

$$g_N(r) = 0 \quad \text{on } r \leqslant N \quad \text{ and } \quad g_N(r) = (r-N)/\nu \quad \text{on } r \geqslant N+2.$$

In [21], we proved the existence of a unique strong solution as well as the ergodicity of the associated invariant measure under periodic boundary conditions. The proof of the existence is mainly based on Galerkin's approximation, and the smooth solution of (1.3) is not obtained therein.

Our main purpose in this paper is to present a unified setting for proving the existence of smooth solutions to the above three typical nonlinear stochastic partial differential equations. We first consider an abstract semilinear stochastic evolution equation in the scope of Hilbert scales determined by a sectorial operator. Here, the analytic semigroup generated by the sectorial operator plays a mollifying role, and will be used to construct a regularized approximation sequence of stochastic ordinary differential equations in Hilbert spaces. After obtaining some uniform estimates of the approximating solutions in the spaces of Hilbert scales, we can prove that the solutions of approximating equations strongly converge to a smooth solution. Our approach is much inspired by the energy method used in the deterministic case (cf. [14]), and is different from Galerkin's approximation and semigroup methods which were extensively used in the well-known studies of SPDEs (cf. [4, 11], etc.). We remark that the regularity of solutions will be decreasing when we use the semigroup method to deal with SPDEs (cf. [2, Sections 5 and 8]). The main advantage of our method is that we can obtain better regularity unlike the semigroup method.

In [26], using the semigroup method and nonlinear interpolations, we have already proved the existence of smooth solutions to a large class of semilinear SPDEs when the coefficients are smooth and have all bounded derivatives. However, the result in [26] cannot be applied to the above mentioned equations. It should be emphasized that the existence of smooth solutions for nonlinear partial differential equations, for example, Navier-Stokes equations, usually depends on the spatial dimensions. Thus, it is not expected to use our general result (see Theorem 2.2 below) to treat high-dimensional nonlinear SPDEs and obtain smooth solutions. Nevertheless, we may still apply our general result to achieving the existence of strong solutions for a class of semilinear SPDEs with Lipschitz nonlinear coefficients in Euclidean space (cf. [9, 10, 16]). We also want to say that although our main attention concentrates on the above three typical nonlinear SPDEs, our result can also be applied to the stochastic Kuramoto-Sivashinsky equation and stochastic Cahn-Hilliard equation (cf. [23]), as well as the stochastic partial differential equation in the abstract Wiener space (cf. [25]).

This paper is organized as follows: in Section 2, we shall give the general framework and state our main result. In Section 3, we devote to the proof of our main result. In Section 4, we investigate a class of semilinear SPDEs in the whole space and in bounded smooth domains of Euclidean space, and obtain the existence of unique strong solutions. In Section 5, we study stochastic Burgers and Ginzburg-Landau equations on the real line, and get the existence of smooth solutions. In Section 6, we prove the existence of smooth solutions to stochastic tamed 3D Navier-Stokes equations. In particular, we obtain the existence of maximal smooth solutions for the genuine 3D SNSE. In Section 7, stochastic 2D Navier-Stokes equations with multiplicative noises in the whole space are considered.

Convention: The letter C with or without subscripts will denote a positive constant, which is unimportant and may change from one line to another.

2 General settings and main result

Let $(\mathbb{H}, \|\cdot\|_{\mathbb{H}})$ be a separable Hilbert space, $\mathfrak{L}$ a symmetric and non-positive sectorial operator in $\mathbb{H}$, which generates a symmetric analytic semigroup $(\mathfrak{T}_\epsilon)_{\epsilon\geqslant 0}$ in $\mathbb{H}$ (cf. [18]). It will play a mollifying role in the sequel. For $\alpha \geqslant 0$, we define the Sobolev space $\mathbb{H}^\alpha$ by

$$\mathbb{H}^\alpha := \mathscr{D}((I-\mathfrak{L})^{\alpha/2})$$

together with the norm

$$\|u\|_\alpha := \|(I-\mathfrak{L})^{\alpha/2}u\|_{\mathbb{H}}.$$

The inner product in $\mathbb{H}^\alpha$ is denoted by $\langle\cdot,\cdot\rangle_\alpha$. The dual space of $\mathbb{H}^\alpha$ is denoted by $\mathbb{H}^{-\alpha}$ with the norm

$$\|u\|_{-\alpha} := \|(I-\mathfrak{L})^{-\alpha/2}u\|_{\mathbb{H}}.$$

Then $(\mathbb{H}^\alpha)_{\alpha\in\mathbb{R}}$ forms a Hilbert scale (cf. [12, 22]), i.e.,

(i) for any $\alpha < \beta$, $\mathbb{H}^\beta \subset \mathbb{H}^\alpha$;

(ii) for any $\alpha < \gamma < \beta$ and $u \in \mathbb{H}^\beta$,

$$\|u\|_\gamma \leqslant C_{\alpha,\beta,\gamma}\|u\|_\alpha^{\frac{\beta-\gamma}{\beta-\alpha}} \cdot \|u\|_\beta^{\frac{\gamma-\alpha}{\beta-\alpha}}. \tag{2.1}$$

Set $\mathbb{H}^\infty := \bigcap_{m\in\mathbb{N}} \mathbb{H}^m$. Then (cf. [18])

Proposition 2.1. *For all integer $m \in \mathbb{Z}$, we have*

(i) *$\mathbb{H}^\infty$ is dense in $\mathbb{H}^m$, and for every $\epsilon > 0$ and $u \in \mathbb{H}^m$, $\mathfrak{T}_\epsilon u \in \mathbb{H}^\infty$;*

(ii) *for every $\epsilon > 0$ and $u \in \mathbb{H}^m$,*

$$(I-\mathfrak{L})^{m/2}\mathfrak{T}_\epsilon u = \mathfrak{T}_\epsilon(I-\mathfrak{L})^{m/2}u;$$

(iii) *for every $\epsilon > 0$, $k = 1, 2, \ldots$ and $u \in \mathbb{H}^{m+k}$,*

$$\|\mathfrak{T}_\epsilon u - u\|_m \leqslant C_{m,k} \cdot \epsilon^{k/2}\|u\|_{m+k};$$

(iv) *for every $\epsilon > 0$ and $u \in \mathbb{H}^m$, $k = 0, 1, 2, \ldots,$*

$$\|\mathfrak{T}_\epsilon u\|_{m+k} \leqslant \frac{C_{m,k}}{\epsilon^{k/2}}\|u\|_m.$$

Let l^2 be the usual Hilbert space of square summable sequences of real numbers. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$ be a stochastic basis. A family of independent standard one-dimensional $(\mathcal{F}_t)$-adapted Brownian motions $\{W^k(t); t \geqslant 0, k = 1, 2, \ldots\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ are given. Then $\{W(t), t \geqslant 0\}$ can be regarded as a cylindrical Brownian motion in l^2 (cf. [4]).

Consider the following stochastic evolution equation

$$\mathrm{d}u(t) = [\mathfrak{L}u(t) + F(t, u(t))]\mathrm{d}t + \sum_k B_k(t, u(t))\mathrm{d}W^k(t), \quad u(0) = u_0, \tag{2.2}$$

where the stochastic integral is understood as Itô's integral, and for some $K \in \mathbb{N}$ the coefficients

$$F(t, \omega, u) : \mathbb{R}_+ \times \Omega \times \mathbb{H}^K \to \mathbb{H}^{-1} \quad \text{and} \quad B(t, \omega, u) : \mathbb{R}_+ \times \Omega \times \mathbb{H}^0 \to \mathbb{H}^0 \otimes l^2$$

are two measurable functions, and for every $t \geqslant 0$ and $u \in \mathbb{H}^K$,

$$F(t, \cdot, u) \in \mathcal{F}_t/\mathscr{B}(\mathbb{H}^{-1}), \quad B(t, \cdot, u) \in \mathcal{F}_t/\mathscr{B}(\mathbb{H}^0 \otimes l^2).$$

We also require that $F(t, \omega, u) \in \mathbb{H}^0$ for $u \in \mathbb{H}^{K+1}$ and $B_k(t, \omega, u) \in \mathbb{H}^m$ for any $m \in \mathbb{N}$ and $u \in \mathbb{H}^{m+1}$.

We make the following assumptions on F and B:

(H1$_{\boldsymbol{K}}$) There exist $q_1, q_2 \geqslant 1$ and constants $\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4 > 0$ such that for all $(t, \omega) \in \mathbb{R}_+ \times \Omega$ and $u, v \in \mathbb{H}^K$,

$$\|F(t, \omega, u) - F(t, \omega, v)\|_{-1} \leqslant \lambda_0 \cdot (\|u\|_K^{q_1} + \|v\|_K^{q_1} + 1) \cdot \|u - v\|_0, \tag{2.3}$$

$$\sum_k \|B_k(t, \omega, u) - B_k(t, \omega, v)\|_0^2 \leqslant \lambda_1 \|u - v\|_0^2, \tag{2.4}$$

and for $u \in \mathbb{H}^{K+1}$,

$$\langle u, F(t, \omega, u)\rangle_0 \leqslant -\frac{1}{2}\|u\|_1^2 + \lambda_2 \cdot (\|u\|_0^2 + 1), \tag{2.5}$$

$$\|F(t, \omega, u)\|_0 \leqslant \lambda_3 \cdot (\|u\|_{K+1} + \|u\|_K^{q_2} + 1), \tag{2.6}$$

$$\sum_k \|B_k(t, \omega, u)\|_0^2 \leqslant \lambda_4 \cdot (\|u\|_0^2 + 1). \tag{2.7}$$

(H2$_{\mathcal{K}}$) For some integer $\mathcal{K} \geqslant K$, and each $m = 1, \ldots, \mathcal{K}$, and any $\delta \in (0, 1)$, there exist $\alpha_m, \beta_m \geqslant 1$ and constants $\lambda_{1m}, \lambda_{2m} > 0$ such that for all $u \in \mathbb{H}^\infty$ and $(t, \omega) \in \mathbb{R}_+ \times \Omega$,

$$\begin{aligned}\langle u, F(t, \omega, u)\rangle_m &= \langle (I - \mathfrak{L})^{m+\frac{1}{2}}u, (I - \mathfrak{L})^{-\frac{1}{2}}F(t, \omega, u)\rangle_0 \\ &\leqslant \frac{1}{2}\|u\|_{m+1}^2 + \lambda_{1m} \cdot (\|u\|_{m-1}^{\alpha_m} + 1)\end{aligned} \tag{2.8}$$

and

$$\sum_k \|B_k(t, \omega, u)\|_m^2 \leqslant \delta\|u\|_{m+1}^2 + \lambda_{2m} \cdot (\|u\|_{m-1}^{\beta_m} + 1). \tag{2.9}$$

Our main result is that

Theorem 2.2. *Under* (H1$_K$) *and* (H2$_{\mathcal{K}}$) *with* $\mathcal{K} \geqslant K \geqslant 1$, *for any* $u_0 \in \mathbb{H}^{\mathcal{K}}$, *there exists a unique process* $u(t) \in \mathbb{H}^{\mathcal{K}}$ *such that*

(i) *the process* $\mathbb{R}^+ \ni t \mapsto u(t) \in \mathbb{H}^0$ *is* $(\mathcal{F}_t)$*-adapted and continuous, and for any* $T > 0$ *and* $p \geqslant 2$,

$$\mathbb{E}\Big(\sup_{s\in[0,T]} \|u(s)\|_{\mathcal{K}}^p\Big) + \int_0^T \mathbb{E}\|u(s)\|_{\mathcal{K}+1}^2 \mathrm{d}s < +\infty;$$

(ii) $u(t)$ *satisfies the following equation in* $\mathbb{H}^0$: *for all* $t \geqslant 0$,

$$u(t) = u_0 + \int_0^t [\mathfrak{L}u(s) + F(s, u(s))]\mathrm{d}s + \sum_k \int_0^t B_k(s, u(s))\mathrm{d}W^k(s), \quad \mathbb{P}\text{-}a.s.$$

Remark 2.3. By (2.6), (2.7) and (i), one knows that all the integrals appearing in (ii) are well defined. Moreover, if for some $C > 0$, $p \geqslant 1$ and any $u \in \mathbb{H}^{\mathcal{K}+1}$,

$$\|F(s,u)\|_{\mathcal{K}-1} \leqslant C(\|u\|_{\mathcal{K}+1} + \|u\|_{\mathcal{K}}^p + 1),$$

then we can find an $\mathbb{H}^{\mathcal{K}}$-valued continuous version of u (cf. [22]).

Remark 2.4. The solution $u(t)$ also satisfies the following integral equation via the analytic semigroup $(\mathfrak{T}_t)_{t\geqslant 0}$:

$$u(t) = \mathfrak{T}_t u_0 + \int_0^t \mathfrak{T}_{t-s} F(s,u(s))\mathrm{d}s + \sum_k \int_0^t \mathfrak{T}_{t-s} B_k(s,u(s))\mathrm{d}W^k(s).$$

Using this representation, we can further prove the Hölder continuity of $u(t)$ in t (cf. [25]).

3 Proof of Theorem 2.2

3.1 Regularized stochastic differential equations

For $m = 0,\ldots,\mathcal{K}$, consider the following regularized stochastic ordinary differential equation in $\mathbb{H}^m$:

$$\mathrm{d}u^\epsilon(t) = A^\epsilon(t,u^\epsilon(t))\mathrm{d}t + \sum_k B_k^\epsilon(t,u^\epsilon(t))\mathrm{d}W^k(t), \quad u^\epsilon(0) = u_0, \tag{3.1}$$

where the regularized operators are defined by:

$$A^\epsilon(t,\omega,u) := \mathfrak{T}_\epsilon \mathfrak{L} \mathfrak{T}_\epsilon u + \mathfrak{T}_\epsilon F(t,\omega,\mathfrak{T}_\epsilon u), \quad B_k^\epsilon(t,\omega,u) := \mathfrak{T}_\epsilon B_k(t,\omega,\mathfrak{T}_\epsilon u).$$

The following two lemmas are direct from ($\mathrm{H1}_K$) and ($\mathrm{H2}_{\mathcal{K}}$). We omit the proof.

Lemma 3.1. *There exists a constant $C > 0$ such that for any $\epsilon > 0$ and all $(t,\omega) \in \mathbb{R}_+ \times \Omega$, $u \in \mathbb{H}^0$,*

$$\langle u, A^\epsilon(t,\omega,u)\rangle_0 \leqslant -\frac{1}{2}\|\mathfrak{T}_\epsilon u\|_1^2 + C(\|u\|_0^2 + 1),$$
$$\sum_k \|B_k^\epsilon(t,\omega,u)\|_0^2 \leqslant C(\|u\|_0^2 + 1).$$

Lemma 3.2. *For any $m = 1,\ldots,\mathcal{K}$ and $\delta \in (0,1)$, there exist two constants $C_m, C_{m,\delta} > 0$ such that for any $\epsilon > 0$ and all $(t,\omega) \in \mathbb{R}_+ \times \Omega$ and $u \in \mathbb{H}^m$,*

$$\langle u, A^\epsilon(t,\omega,u)\rangle_m \leqslant -\frac{1}{2}\|\mathfrak{T}_\epsilon u\|_{m+1}^2 + C_m(\|u\|_{m-1}^{\alpha_m} + 1),$$
$$\sum_k \|B_k^\epsilon(t,\omega,u)\|_m^2 \leqslant \delta\|\mathfrak{T}_\epsilon u\|_{m+1}^2 + C_{m,\delta}(\|u\|_{m-1}^{\beta_m} + 1),$$

where α_m and β_m are the same as in ($\mathrm{H2}_{\mathcal{K}}$).

We also have

Lemma 3.3. *For any $m = 0,\ldots,\mathcal{K}$, the functions A^ϵ and B^ϵ are locally Lipschitz continuous in $\mathbb{H}^m$ with respect to u. More precisely, for any $R > 0$ there are $C_{R,\epsilon}, C'_{R,\epsilon} > 0$ such that for all $(t,\omega) \in \mathbb{R}_+ \times \Omega$ and $u, v \in \mathbb{H}^m$ with $\|u\|_m, \|v\|_m \leqslant R$,*

$$\|A^\epsilon(t,\omega,u) - A^\epsilon(t,\omega,v)\|_m \leqslant C_{R,\epsilon}\|u-v\|_m,$$
$$\sum_k \|B_k^\epsilon(t,\omega,u) - B_k^\epsilon(t,\omega,v)\|_m^2 \leqslant C'_{R,\epsilon}\|u-v\|_m^2.$$

Proof. By (ii) and (iv) of Proposition 2.1, we have

$$\|\mathfrak{T}_\epsilon \mathfrak{L}(\mathfrak{T}_\epsilon u) - \mathfrak{T}_\epsilon \mathfrak{L}(\mathfrak{T}_\epsilon v)\|_m = \|\mathfrak{L}\mathfrak{T}_{2\epsilon}(u-v)\|_m \leqslant C_\epsilon \|u-v\|_m,$$

and by (H1$_K$),

$$\begin{aligned}&\|\mathfrak{T}_\epsilon F(t,\mathfrak{T}_\epsilon u)-\mathfrak{T}_\epsilon F(t,\mathfrak{T}_\epsilon v)\|_m^2+\sum_k\|\mathfrak{T}_\epsilon B_k(t,\mathfrak{T}_\epsilon u)-\mathfrak{T}_\epsilon B_k(t,\mathfrak{T}_\epsilon v)\|_m^2\\&\quad\leqslant C_\epsilon\|F(t,\mathfrak{T}_\epsilon u)-F(t,\mathfrak{T}_\epsilon v)\|_{-1}^2+C_\epsilon\sum_k\|B_k(t,\mathfrak{T}_\epsilon u)-B_k(t,\mathfrak{T}_\epsilon v)\|_0^2\\&\quad\leqslant C_{R,\epsilon}\|\mathfrak{T}_\epsilon u-\mathfrak{T}_\epsilon v\|_0^2\leqslant C_{R,\epsilon}\|u-v\|_m^2.\end{aligned}$$

The proof is complete.

We now prove the following key estimate about the solution of regularized stochastic differential equation (3.1).

Lemma 3.4. *For any $u_0\in\mathbb{H}^{\mathcal{K}}$, there exists a unique continuous $(\mathcal{F}_t)$-adapted solution u^ϵ to* (3.1) *in $\mathbb{H}^{\mathcal{K}}$ such that for any $p\geqslant 1$ and $T>0$,*

$$\sup_{\epsilon\in(0,1)}\mathbb{E}\Big(\sup_{t\in[0,T]}\|u^\epsilon(t)\|_{\mathcal{K}}^{2p}\Big)+\sup_{\epsilon\in(0,1)}\int_0^T\mathbb{E}\|\mathfrak{T}_\epsilon u^\epsilon(s)\|_{\mathcal{K}+1}^2\mathrm{d}s\leqslant C_{p,T}. \tag{3.2}$$

Proof. First of all, it is a standard fact by Lemma 3.3 that there exists a unique continuous $(\mathcal{F}_t)$-adapted local solution $u^\epsilon(t)$ in $\mathbb{H}^m$ for any $m=0,\dots,\mathcal{K}$ (e.g., [4]). We now use the induction method to prove that

$$(\mathscr{P}_m):\begin{cases}u^\epsilon(t)\text{ is non-explosive in }\mathbb{H}^m\text{ and, for any }p\geqslant1\text{ and }T>0,\\ \sup\limits_{\epsilon\in(0,1)}\mathbb{E}\Big(\sup\limits_{t\in[0,T]}\|u^\epsilon(t)\|_m^{2p}\Big)\leqslant C_{m,p,T},\quad m=0,\dots,\mathcal{K}.\end{cases}$$

By the standard stopping times technique, it suffices to prove the estimate in $(\mathscr{P}_m)$. In the following, we fix $T>0$. By Itô's formula, we have for any $p\geqslant1$,

$$\|u^\epsilon(t)\|_m^{2p}=\|u_0\|_m^{2p}+I_{m1}(t)+I_{m2}(t)+I_{m3}(t)+I_{m4}(t), \tag{3.3}$$

where

$$\begin{aligned}I_{m1}(t)&:=2p\int_0^t\|u^\epsilon(s)\|_m^{2(p-1)}\langle u^\epsilon(s),A^\epsilon(u^\epsilon(s))\rangle_m\mathrm{d}s,\\I_{m2}(t)&:=2p\sum_{k=1}^\infty\int_0^t\|u^\epsilon(s)\|_m^{2(p-1)}\langle u^\epsilon(s),B_k^\epsilon(s,u^\epsilon(s))\rangle_m\mathrm{d}W_s^k,\\I_{m3}(t)&:=p\sum_{k=1}^\infty\int_0^t\|u^\epsilon(s)\|_m^{2(p-1)}\|B_k^\epsilon(s,u^\epsilon(s))\|_m^2\mathrm{d}s,\\I_{m4}(t)&:=2p(p-1)\sum_{k=1}^\infty\int_0^t\|u^\epsilon(s)\|_m^{2(p-2)}|\langle u^\epsilon(s),B_k^\epsilon(s,u^\epsilon(s))\rangle_m|^2\mathrm{d}s.\end{aligned}$$

Let us first look at $(\mathscr{P}_0)$. By Lemma 3.1 and Young's inequality, we have

$$I_{01}(t)+I_{03}(t)+I_{04}(t)\leqslant C_p\int_0^t(\|u^\epsilon(s)\|_0^{2p}+1)\mathrm{d}s. \tag{3.4}$$

Taking expectations for (3.3) with $m=0$ gives that

$$\mathbb{E}\|u^\epsilon(t)\|_0^{2p}\leqslant\|u_0\|_0^{2p}+C_p\int_0^t(\mathbb{E}\|u^\epsilon(s)\|_0^{2p}+1)\mathrm{d}s.$$

By Gronwall's inequality, we obtain that for any $p\geqslant1$,

$$\sup_{t\in[0,T]}\mathbb{E}\|u^\epsilon(t)\|_0^{2p}\leqslant C_{p,T}(\|u_0\|_0^{2p}+1). \tag{3.5}$$

On the other hand, by Burkholder's inequality and Lemma 3.1 again, we have

$$\begin{aligned}\mathbb{E}\Big(\sup_{s\in[0,T]}|I_{02}(s)|\Big) &\leqslant C_p\mathbb{E}\bigg(\int_0^T \|u^\epsilon(s)\|_0^{4(p-1)}\|\langle u^\epsilon(s), B^\epsilon_\cdot(s,u^\epsilon(s))\rangle_0\|^2_{l^2}\mathrm{d}s\bigg)^{1/2}\\ &\leqslant C_p\mathbb{E}\bigg(\int_0^T \|u^\epsilon(s)\|_0^{4p-2}\cdot\bigg(\sum_k\|B^\epsilon_k(s,u^\epsilon(s))\|_0^2\bigg)\mathrm{d}s\bigg)1/2\\ &\leqslant C_p\mathbb{E}\bigg(\int_0^T(\|u^\epsilon(s)\|_0^{4p}+1)\mathrm{d}s\bigg)^{1/2}\\ &\leqslant C_p\bigg(\int_0^T(\mathbb{E}\|u^\epsilon(s)\|_0^{4p}+1)\mathrm{d}s\bigg)^{1/2}\\ &\leqslant C_{p,T}(\|u_0\|_0^{2p}+1),\end{aligned}$$

where the last step is due to (3.5). Thus, by (3.4) and (3.5), it is easy to see that $(\mathscr{P}_0)$ holds.

Suppose now that $(\mathscr{P}_{m-1})$ holds. By Lemma 3.2 and Young's inequality, we have

$$\begin{aligned}I_{m1}(t) &\leqslant p\int_0^t[-\|u^\epsilon(s)\|_m^{2(p-1)}\|\mathfrak{T}_\epsilon u^\epsilon(s)\|^2_{m+1}+2C_m\|u^\epsilon(s)\|_m^{2(p-1)}\cdot(\|u^\epsilon(s)\|^{\alpha_m}_{m-1}+1)]\mathrm{d}s\\ &\leqslant -p\int_0^t\|u^\epsilon(s)\|_m^{2(p-1)}\|\mathfrak{T}_\epsilon u^\epsilon(s)\|^2_{m+1}\mathrm{d}s+C_{m,p}\int_0^t\|u^\epsilon(s)\|_m^{2p}\mathrm{d}s\\ &\quad+C_{m,p}\int_0^t(\|u^\epsilon(s)\|^{p\cdot\alpha_m}_{m-1}+1)\mathrm{d}s,\end{aligned}\tag{3.6}$$

and for any $\delta\in(0,1)$,

$$\begin{aligned}I_{m3}(t)+I_{m4}(t) &\leqslant p(2p-1)\int_0^t\|u^\epsilon(s)\|_m^{2(p-1)}\cdot\bigg(\sum_k\|B^\epsilon_k(s,u^\epsilon(s))\|^2_m\bigg)\mathrm{d}s\\ &\leqslant p(2p-1)\delta\int_0^t\|u^\epsilon(s)\|_m^{2(p-1)}\|\mathfrak{T}_\epsilon u^\epsilon(s)\|^2_{m+1}\mathrm{d}s\\ &\quad+C_{m,p}\int_0^t\|u^\epsilon(s)\|_m^{2p}\mathrm{d}s+C_{m,p}\int_0^t(\|u^\epsilon(s)\|^{p\cdot\beta_m}_{m-1}+1)\mathrm{d}s.\end{aligned}\tag{3.7}$$

Choosing $\delta=\frac{1}{2(2p-1)}$ and taking expectations for (3.3), one finds that

$$\begin{aligned}&\mathbb{E}\|u^\epsilon(t)\|_m^{2p}+\frac{p}{2}\int_0^t\mathbb{E}(\|u^\epsilon(s)\|_m^{2(p-1)}\cdot\|\mathfrak{T}_\epsilon u^\epsilon(s)\|^2_{m+1})\mathrm{d}s\\ &\qquad\leqslant\|u_0\|_m^{2p}+C_{m,p}\int_0^t\mathbb{E}\|u^\epsilon(s)\|_m^{2p}\mathrm{d}s+C_{m,p}\int_0^t\mathbb{E}(\|u^\epsilon(s)\|^{p_m}_{m-1}+1)\mathrm{d}s,\end{aligned}$$

where $p_m:=p\cdot(\alpha_m\vee\beta_m)$.

By Gronwall's inequality again and $(\mathscr{P}_{m-1})$, we get for any $p\geqslant 1$,

$$\sup_{t\in[0,T]}\mathbb{E}\|u^\epsilon(t)\|_m^{2p}+\int_0^T\mathbb{E}(\|u^\epsilon(s)\|_m^{2(p-1)}\|\mathfrak{T}_\epsilon u^\epsilon(s)\|^2_{m+1})\mathrm{d}s\leqslant C_{m,p,T}.\tag{3.8}$$

Furthermore, by Burkholder's inequality and (3.8), we have

$$\begin{aligned}\mathbb{E}\Big(\sup_{s\in[0,T]}|I_{m2}(s)|\Big) &\leqslant C_p\mathbb{E}\bigg(\int_0^T\|u^\epsilon(s)\|_m^{4(p-1)}\cdot\|\langle u^\epsilon(s),B^\epsilon_\cdot(s,u^\epsilon(s))\rangle_m\|^2_{l^2}\mathrm{d}s\bigg)^{1/2}\\ &\leqslant C_p\mathbb{E}\bigg(\int_0^T\|u^\epsilon(s)\|_m^{4p-2}\bigg(\sum_k\|B^\epsilon_k(s,u^\epsilon(s))\|^2_m\bigg)\mathrm{d}s\bigg)^{1/2}\\ &\leqslant C_p\mathbb{E}\bigg(\sup_{s\in[0,T]}\|u^\epsilon(s)\|_m^p\cdot\bigg[\int_0^T\|u^\epsilon(s)\|_m^{2(p-1)}\end{aligned}$$

$$\times\,(\delta\|\mathfrak{T}_\epsilon u^\epsilon(s)\|_{m+1}^2+C_{m,\delta}(\|u^\epsilon(s)\|_{m-1}^{\beta_m}+1))\mathrm{d}s\bigg]^{1/2}\bigg)$$
$$\leqslant\frac{1}{2}\mathbb{E}\Big(\sup_{t\in[0,T]}\|u^\epsilon(t)\|_m^{2p}\Big)+C_p\delta\int_0^T\mathbb{E}(\|u^\epsilon(s)\|_m^{2(p-1)}\|\mathfrak{T}_\epsilon u^\epsilon(s)\|_{m+1}^2)\mathrm{d}s$$
$$+\,C_{m,p}\int_0^T\mathbb{E}(\|u^\epsilon(s)\|_m^{2(p-1)}(\|u^\epsilon(s)\|_{m-1}^{\beta_m}+1))\mathrm{d}s$$
$$\leqslant\frac{1}{2}\mathbb{E}\Big(\sup_{t\in[0,T]}\|u^\epsilon(t)\|_m^{2p}\Big)+C_{m,p,T},$$

which together with (3.3) and (3.6)–(3.8) yields

$$\frac{1}{2}\mathbb{E}\Big(\sup_{t\in[0,T]}\|u^\epsilon(t)\|_m^{2p}\Big)\leqslant C_{m,p,T}+C_{m,p}\int_0^T\mathbb{E}(\|u^\epsilon(s)\|_{m-1}^{p_m}+1)\mathrm{d}s\leqslant C_{m,p,T}.$$

So, $(\mathscr{P}_m)$ holds. The proof is complete.

3.2 Convergence of $u^\epsilon(t)$

Lemma 3.5. *For any $R>0$, there exists a constant $C_R>0$ such that for any $0<\epsilon'<\epsilon<1$, $(t,\omega)\in\mathbb{R}_+\times\Omega$ and $u,v\in\mathbb{H}^{K+1}$ with $\|u\|_K,\|v\|_K\leqslant R$,*

$$\langle u-v,A^\epsilon(t,\omega,u)-A^{\epsilon'}(t,\omega,v)\rangle_0\leqslant C_R\cdot\sqrt{\epsilon}\cdot(1+\|\mathfrak{T}_{\epsilon'}v\|_{K+1})+C_R\cdot\|u-v\|_0^2,$$
$$\sum_k\|B_k^\epsilon(t,\omega,u)-B_k^{\epsilon'}(t,\omega,v)\|_0^2\leqslant C_R\cdot\sqrt{\epsilon}\cdot(1+\|\mathfrak{T}_{\epsilon'}v\|_2^2)+C_R\cdot\|u-v\|_0^2.$$

Proof. We only prove the first estimate. The second is analogous.

First of all, by (ii) and (iii) of Proposition 2.1, we have

$$\begin{aligned}\langle u-v,\mathfrak{T}_\epsilon\mathfrak{L}(\mathfrak{T}_\epsilon u)-\mathfrak{T}_{\epsilon'}\mathfrak{L}(\mathfrak{T}_{\epsilon'}v)\rangle_0&=\langle\mathfrak{T}_\epsilon(u-v),\mathfrak{L}\mathfrak{T}_\epsilon(u-v)\rangle_0+\langle u-v,(\mathfrak{T}_{2\epsilon}-\mathfrak{T}_{2\epsilon'})\mathfrak{L}v\rangle_0\\&\leqslant-\|\mathfrak{T}_\epsilon(u-v)\|_1^2+\|\mathfrak{T}_\epsilon(u-v)\|_0^2+C_R\cdot\|(\mathfrak{T}_{2\epsilon}-\mathfrak{T}_{2\epsilon'})v\|_1\\&\leqslant-\|\mathfrak{T}_\epsilon(u-v)\|_1^2+C\|u-v\|_0^2+C_R\cdot\sqrt{\epsilon}\cdot\|\mathfrak{T}_{\epsilon'}v\|_2.\end{aligned}\tag{3.9}$$

Second, we decompose the term involving F in A^ϵ as:

$$\begin{aligned}\langle u-v,\mathfrak{T}_\epsilon F(t,\mathfrak{T}_\epsilon u)-\mathfrak{T}_{\epsilon'}F(t,\mathfrak{T}_{\epsilon'}v)\rangle_0&=\langle\mathfrak{T}_\epsilon(u-v),F(t,\mathfrak{T}_\epsilon u)-F(t,\mathfrak{T}_{\epsilon'}v)\rangle_0\\&\quad+\langle(\mathfrak{T}_\epsilon-\mathfrak{T}_{\epsilon'})(u-v),F(t,\mathfrak{T}_{\epsilon'}v)\rangle_0\\&=:I_1+I_2.\end{aligned}$$

By (2.3) and (iii) of Proposition 2.1, we have for I_1,

$$\begin{aligned}I_1&\leqslant\frac{1}{4}\|\mathfrak{T}_\epsilon(u-v)\|_1^2+\|F(t,\mathfrak{T}_\epsilon u)-F(t,\mathfrak{T}_{\epsilon'}v)\|_{-1}^2\\&\leqslant\frac{1}{4}\|\mathfrak{T}_\epsilon(u-v)\|_1^2+C_R\cdot\|\mathfrak{T}_\epsilon u-\mathfrak{T}_{\epsilon'}v\|_0^2\\&\leqslant\frac{1}{4}\|\mathfrak{T}_\epsilon(u-v)\|_1^2+C_R\cdot\sqrt{\epsilon}+C_R\cdot\|u-v\|_0^2,\end{aligned}$$

and for I_2, by (2.6),

$$I_2\leqslant\|(\mathfrak{T}_\epsilon-\mathfrak{T}_{\epsilon'})(u-v)\|_0\cdot\|F(t,\mathfrak{T}_{\epsilon'}v)\|_0\leqslant C_R\cdot\sqrt{\epsilon}\cdot(\|\mathfrak{T}_{\epsilon'}v\|_{K+1}+1).$$

Combining the above calculations yields the first one.

We now prove that

Lemma 3.6. *For any $T > 0$, we have*

$$\lim_{\epsilon,\epsilon'\downarrow 0} \mathbb{E}\Big(\sup_{t\in[0,T]} \|u^\epsilon(t) - u^{\epsilon'}(t)\|_0^2\Big) = 0.$$

Proof. For any $R > 0$ and $1 > \epsilon > \epsilon' > 0$, define the stopping times

$$\tau_R^{\epsilon,\epsilon'} := \inf\{t > 0 : \|u^\epsilon(t)\|_K \wedge \|u^{\epsilon'}(t)\|_K \geqslant R\}.$$

Then, by Lemma 3.4, we have

$$\mathbb{P}(\tau_R^{\epsilon,\epsilon'} < T) \leqslant \frac{\mathbb{E}(\sup_{t\in[0,T]}(\|u^\epsilon(t)\|_K^2 \wedge \|u^{\epsilon'}(t)\|_K^2))}{R^2} \leqslant \frac{C_T}{R^2}. \tag{3.10}$$

Set

$$v(t) := u^\epsilon(t) - u^{\epsilon'}(t).$$

By Itô's formula, we have

$$\|v(t)\|_0^2 = J_1(t) + J_2(t) + J_3(t), \tag{3.11}$$

where

$$J_1(t) := 2\int_0^t \langle v(s), A^\epsilon(s, u^\epsilon(s)) - A^{\epsilon'}(s, u^{\epsilon'}(s))\rangle_0 \mathrm{d}s,$$

$$J_2(t) := \sum_k \int_0^t \|B_k^\epsilon(s, u^\epsilon(s)) - B_k^{\epsilon'}(s, u^{\epsilon'}(s))\|_0^2 \mathrm{d}s,$$

$$J_3(t) := 2\sum_k \int_0^t \langle v(s), B_k^\epsilon(s, u^\epsilon(s)) - B_k^{\epsilon'}(s, u^{\epsilon'}(s))\rangle_0 \mathrm{d}W_s^k.$$

By Lemma 3.5, we have

$$\begin{aligned} J_1(t\wedge\tau_R^{\epsilon,\epsilon'}) + J_2(t\wedge\tau_R^{\epsilon,\epsilon'}) &\leqslant C_R\cdot\sqrt{\epsilon}\cdot\left(1 + \int_0^T \|\mathfrak{T}_{\epsilon'}u^{\epsilon'}(s)\|_{K+1}^2\mathrm{d}s\right) + C_R\int_0^{t\wedge\tau_R^{\epsilon,\epsilon'}} \|v(s)\|_0^2\mathrm{d}s \\ &\leqslant C_R\cdot\sqrt{\epsilon}\cdot\left(1 + \int_0^T \|\mathfrak{T}_{\epsilon'}u^{\epsilon'}(s)\|_{K+1}^2\mathrm{d}s\right) + C_R\int_0^t \|v(s\wedge\tau_R^{\epsilon,\epsilon'})\|_0^2\mathrm{d}s. \end{aligned} \tag{3.12}$$

Hence, by Lemma 3.4 and taking expectations for (3.11), we obtain

$$\mathbb{E}\|v(t\wedge\tau_R^{\epsilon,\epsilon'})\|_0^2 \leqslant C_{R,T}\cdot\sqrt{\epsilon} + C_R\int_0^t \mathbb{E}\|v(s\wedge\tau_R^{\epsilon,\epsilon'})\|_0^2\mathrm{d}s.$$

By Gronwall's inequality, we get

$$\sup_{t\in[0,T]} \mathbb{E}\|v(t\wedge\tau_R^{\epsilon,\epsilon'})\|_0^2 \leqslant C_{R,T}\cdot\sqrt{\epsilon}. \tag{3.13}$$

On the other hand, setting

$$\mathcal{B}(s,\epsilon,\epsilon') := \sum_k \|B_k^\epsilon(s, u^\epsilon(s)) - B_k^{\epsilon'}(s, u^{\epsilon'}(s))\|_0^2,$$

by Burkholder's inequality and Young's inequality, we have

$$\begin{aligned} \mathbb{E}\Big(\sup_{s\in[0,T\wedge\tau_R^{\epsilon,\epsilon'}]} |J_3(s)|\Big) &\leqslant C\mathbb{E}\left(\int_0^{T\wedge\tau_R^{\epsilon,\epsilon'}} \|v(s)\|_0^2\cdot\mathcal{B}(s,\epsilon,\epsilon')\mathrm{d}s\right)^{1/2} \\ &\leqslant \frac{1}{2}\mathbb{E}\Big(\sup_{s\in[0,T\wedge\tau_R^{\epsilon,\epsilon'}]} \|v(s)\|_0^2\Big) + C\mathbb{E}\Big(\int_0^{T\wedge\tau_R^{\epsilon,\epsilon'}} \mathcal{B}(s,\epsilon,\epsilon')\mathrm{d}s\Big). \end{aligned}$$

Thus, by (3.11)–(3.13) and Lemma 3.5, we obtain

$$\mathbb{E}\Big(\sup_{s\in[0,T\wedge\tau_R^{\epsilon,\epsilon'}]}\|v(s)\|_0^2\Big)\leqslant C_{R,T}\cdot\sqrt{\epsilon}.$$

Therefore, by Lemma 3.4 and (3.10), we get

$$\begin{aligned}\mathbb{E}\Big(\sup_{s\in[0,T]}\|v(s)\|_0^2\Big)&=\mathbb{E}\Big(\sup_{s\in[0,T]}\|v(s)\|_0^2\cdot 1_{\{\tau_R^{\epsilon,\epsilon'}<T\}}\Big)+\mathbb{E}\Big(\sup_{s\in[0,T]}\|v(s)\|_0^2\cdot 1_{\{\tau_R^{\epsilon,\epsilon'}\geqslant T\}}\Big)\\&\leqslant\Big[\mathbb{E}\Big(\sup_{s\in[0,T]}\|v(s)\|_0^4\Big)\Big]^{1/2}\cdot[\mathbb{P}(\tau_R^{\epsilon,\epsilon'}<T)]^{1/2}+\mathbb{E}\Big(\sup_{s\in[0,T\wedge\tau_R^{\epsilon,\epsilon'}]}\|v(s)\|_0^2\Big)\\&\leqslant C_{R,T}\cdot\sqrt{\epsilon}+C_T/R.\end{aligned}$$

Last, letting $\epsilon\downarrow 0$ and $R\to\infty$ yields the desired limit.

We are now in a position to give

Proof of Theorem 2.2. First of all, by Lemma 3.6, there is a $u(\cdot)\in L^2(\Omega,\mathcal{F},\mathbb{P};C([0,T];\mathbb{H}^0))$ such that

$$\lim_{\epsilon\to 0}\mathbb{E}\Big(\sup_{s\in[0,T]}\|u^\epsilon(s)-u(s)\|_0^2\Big)=0,\tag{3.14}$$

which together with Lemma 3.4 yields that for any $p\geqslant 1$,

$$\mathbb{E}\Big(\sup_{t\in[0,T]}\|u(t)\|_{\mathcal{K}}^{2p}\Big)+\int_0^T\mathbb{E}\|u(s)\|_{\mathcal{K}+1}^2\mathrm{d}s\leqslant C_{p,T}.$$

We now show that $u(t)$ is a solution of (2.2) and satisfies (ii) of Theorem 2.2. It suffices to prove that for any $v\in\mathbb{H}^\infty$,

$$\begin{aligned}\langle v,u(t)\rangle_0=\langle v,u_0\rangle_0&+\int_0^t\langle v,\mathfrak{L}u(s)\rangle_0\mathrm{d}s+\int_0^t\langle v,F(s,u(s))\rangle_0\mathrm{d}s\\&+\sum_k\int_0^t\langle v,B_k(s,u(s))\rangle_0\mathrm{d}W^k(s),\quad\mathbb{P}\text{-a.s.}\end{aligned}$$

Note that

$$\begin{aligned}\langle v,u^\epsilon(t)\rangle_0=\langle v,u_0\rangle_0&+\int_0^t\langle v,\mathfrak{T}_\epsilon\mathfrak{L}\mathfrak{T}_\epsilon u^\epsilon(s)\rangle_0\mathrm{d}s+\int_0^t\langle v,\mathfrak{T}_\epsilon F(s,\mathfrak{T}_\epsilon u^\epsilon(s))\rangle_0\mathrm{d}s\\&+\sum_k\int_0^t\langle v,\mathfrak{T}_\epsilon B_k(s,\mathfrak{T}_\epsilon u^\epsilon(s))\rangle_0\mathrm{d}W^k(s),\quad\mathbb{P}\text{-a.s.}\end{aligned}$$

We only prove that the third term on the right-hand side converges to the corresponding term, that is, as $\epsilon\downarrow 0$,

$$\mathcal{P}(t,\epsilon):=\int_0^t|\langle v,\mathfrak{T}_\epsilon F(s,\mathfrak{T}_\epsilon u^\epsilon(s))-F(s,u(s))\rangle_0|\mathrm{d}s\overset{L^1(\Omega;\mathbb{P})}{\longrightarrow}0.\tag{3.15}$$

For any $R>0$, define the stopping times

$$\tau_R^\epsilon:=\inf\{t>0:\|u^\epsilon(t)\|_K\wedge\|u(t)\|_K\geqslant R\}.$$

Thus,

$$\mathcal{P}(t,\epsilon)=\mathcal{P}(t,\epsilon)\cdot 1_{\{\tau_R^\epsilon\geqslant t\}}+\mathcal{P}(t,\epsilon)\cdot 1_{\{\tau_R^\epsilon<t\}}.\tag{3.16}$$

For the first term of (3.16), we have by ($\mathrm{H1}_K$),

$$\mathbb{E}\big(\mathcal{P}(t,\epsilon)\cdot 1_{\{\tau_R^\epsilon\geqslant t\}}\big)\leqslant\mathbb{E}(\mathcal{P}(t\wedge\tau_R^\epsilon,\epsilon))$$

$$
\begin{aligned}
&\leqslant \mathbb{E}\bigg(\int_0^{t\wedge\tau_R^\epsilon} |\langle \mathfrak{T}_\epsilon v, F(s,\mathfrak{T}_\epsilon u^\epsilon(s)) - F(s,u(s))\rangle_0| \mathrm{d}s\bigg) \\
&\quad + \mathbb{E}\bigg(\int_0^{t\wedge\tau_R^\epsilon} |\langle \mathfrak{T}_\epsilon v - v, F(s,u(s))\rangle_0| \mathrm{d}s\bigg) \\
&\leqslant \|\mathfrak{T}_\epsilon v\|_1 \cdot \mathbb{E}\bigg(\int_0^{t\wedge\tau_R^\epsilon} \|F(s,\mathfrak{T}_\epsilon u^\epsilon(s)) - F(s,u(s))\|_{-1} \mathrm{d}s\bigg) \\
&\quad + \|\mathfrak{T}_\epsilon v - v\|_1 \cdot \mathbb{E}\bigg(\int_0^{t\wedge\tau_R^\epsilon} \|F(s,u(s))\|_{-1} \mathrm{d}s\bigg) \\
&\overset{(2.3)}{\leqslant} C_R \cdot \|v\|_1 \cdot \int_0^t \mathbb{E}\|\mathfrak{T}_\epsilon u^\epsilon(s) - u(s)\|_0 \mathrm{d}s + C_{R,T}\cdot \epsilon \cdot \|v\|_3.
\end{aligned}
$$

For the second term of (3.16), as above it is easy to see by (2.6) and Lemma 3.4 that

$$\mathbb{E}(\mathcal{P}(t,\epsilon)\cdot 1_{\{\tau_R^\epsilon < t\}}) \leqslant C_T \cdot \mathbb{P}(\tau_R^\epsilon < t)^{1/2} \leqslant C_T/R.$$

Therefore, letting $\epsilon \downarrow 0$ and $R \to \infty$ for (3.16) gives that

$$\lim_{\epsilon\downarrow 0} \mathbb{E}(\mathcal{P}(t,\epsilon)) = 0.$$

The uniqueness follows from similar calculations as in proving Lemma 3.6. The proof is thus complete.

Remark 3.7. By (3.14) and Lemma 3.4, using interpolation inequality (2.1) and Hölder's inequality, we in fact have for any $0 < \alpha < \mathcal{K}$,

$$\lim_{\epsilon\downarrow 0} \mathbb{E}\Big(\sup_{s\in[0,T]} \|u^\epsilon(s) - u(s)\|_\alpha^2\Big) = 0.$$

4 Strong solutions of semilinear SPDEs in Euclidean space

Consider the following Cauchy problem of SPDE in $\mathbb{R}^d$:

$$
\begin{cases}
\mathrm{d}u(t,x) = \Big[\Delta u(t,x) + \sum_{i=1}^d \partial_{x_i} f_i(t,\omega,x,u(t,x)) + g(t,\omega,x,u(t,x))\Big]\mathrm{d}t \\
\qquad\qquad + \sum_k \sigma_k(t,\omega,x,u(t,x))\mathrm{d}W^k(t), \\
u(0,x) = u_0(x),
\end{cases}
$$

where $\Delta := \sum_{i=1}^d \partial_{x_i}^2$ is the Laplace operator, and the other coefficients are respectively measurable with respect to their variables:

$$
\begin{aligned}
&f : \mathbb{R}_+ \times \Omega \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d, \\
&g : \mathbb{R}_+ \times \Omega \times \mathbb{R}^d \to \mathbb{R}, \\
&\sigma : \mathbb{R}_+ \times \Omega \times \mathbb{R}^d \to l^2.
\end{aligned}
$$

We impose the following conditions on f, g and σ:

(A1) For each $x \in \mathbb{R}^d$, $z \in \mathbb{R}$ and $t \geqslant 0$, $f(t,\cdot,x,z), g(t,\cdot,x,z)$ and $\sigma(t,\cdot,x,z)$ are $\mathcal{F}_t$-measurable.

(A2) There exist $\kappa_1, \kappa_2 > 0$ and $h_0, h_1 \in L^2(\mathbb{R})$ such that for all $(t,\omega) \in \mathbb{R}_+ \times \Omega$, $x \in \mathbb{R}^d$ and $z \in \mathbb{R}$,

$$\|\partial_z f(t,\omega,x,z)\|_{\mathbb{R}^d} + |\partial_z g(t,\omega,x,z)| + \|\partial_z \sigma(t,\omega,x,z)\|_{l^2} \leqslant \kappa_1,$$

and for $j = 0, 1$,

$$\|\nabla_x^j f(t,\omega,x,z)\|_{\mathbb{R}^d} + |\nabla_x^j g(t,\omega,x,z)| + \|\nabla_x^j \sigma(t,\omega,x,z)\|_{l^2} \leqslant \kappa_2|z| + h_j(x),$$

where $\nabla_x = (\partial_{x_1}, \dots, \partial_{x_d})$ is the gradient operator.

For $m \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$, let $\mathbb{W}_2^m(\mathbb{R}^d)$ be the usual Sobolev space in $\mathbb{R}^d$, i.e., the completion of the space $C_0^\infty(\mathbb{R}^d)$ of smooth functions with compact supports with respect to the norm

$$\|u\|_m := \left(\int_{\mathbb{R}^d} |u(x)|^2 \mathrm{d}x + \int_{\mathbb{R}^d} |\nabla^m u(x)|^2 \mathrm{d}x\right)^{1/2} = \left(\int_{\mathbb{R}^d} |(I-\Delta)^{\frac{m}{2}} u(x)|^2 \mathrm{d}x\right)^{1/2}.$$

Set $\mathbb{H}^m := \mathbb{W}_2^m(\mathbb{R}^d)$ and $\mathfrak{L} := \Delta$ and define for $u \in \mathbb{W}^{2,1}(\mathbb{R}^d)$,

$$F(t,\omega,u) := \sum_{i=1}^d \partial_{x_i} f_i(t,\omega,\cdot,u(\cdot)) + g(t,\omega,\cdot,u(\cdot)), \tag{4.1}$$

$$B_k(t,\omega,u) := \sigma_k(t,\omega,\cdot,u(\cdot)), \quad k \in \mathbb{N}. \tag{4.2}$$

For the simplicity of notations, the variables t and ω in F and B will be dropped below.

Lemma 4.1. *Assume* (A1) *and* (A2). *Then Namiki's operators F and B defined by* (4.1) *and* (4.2) *satisfy* (H1$_1$) *and* (H2$_1$).

Proof. Notice that $(I-\Delta)^{-\frac{1}{2}}\partial_{x_i}$ and $(I-\Delta)^{-\frac{1}{2}}$ are bounded linear operators from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$. It is clear by (A2) that for any $u, v \in \mathbb{H}^1$,

$$\|F(u) - F(v)\|_{-1}^2 + \sum_k \|B_k(u) - B_k(v)\|_0^2 \leqslant C\|u - v\|_0^2,$$

$$\|F(u)\|_0^2 + \sum_k \|B_k(u)\|_0^2 \leqslant C(\|u\|_1^2 + 1),$$

and by the integration by parts formula and Young's inequality,

$$\langle u, F(u)\rangle_0 \leqslant \frac{1}{2}\|u\|_1^2 + C\|f(\cdot,u(\cdot))\|_0^2 + C\|g(\cdot,u(\cdot))\|_0^2 \leqslant \frac{1}{2}\|u\|_1^2 + C(\|u\|_0^2 + 1).$$

Hence, (H1$_1$) holds.

For (H2$_1$), we have

$$\begin{aligned}\langle u, F(u)\rangle_1 &\leqslant \frac{1}{4}\|u\|_2^2 + C\|F(u)\|_0^2 \leqslant \frac{1}{4}\|u\|_2^2 + C(\|u\|_1^2 + 1) \\ &\leqslant \frac{1}{4}\|u\|_2^2 + C(\|u\|_2 \cdot \|u\|_0 + 1) \leqslant \frac{1}{2}\|u\|_2^2 + C(\|u\|_0^2 + 1),\end{aligned}$$

and by (A2)

$$\sum_k \|B_k(u)\|_1^2 \leqslant C(\|u\|_1^2 + 1) \leqslant \delta\|u\|_2^2 + C_\delta(\|u\|_0^2 + 1).$$

The proof is complete.

By Lemma 4.1 and Theorem 2.2, we obtain the following result:

Theorem 4.2. *Assume* (A1) *and* (A2). *For any $u_0 \in \mathbb{H}^1$, there exists a unique $\mathbb{H}^1$-valued continuous and $(\mathcal{F}_t)$-adapted process $u(t)$ such that for any $T > 0$ and $p \geqslant 1$,*

$$\mathbb{E}\Big(\sup_{s\in[0,T]} \|u(s)\|_1^p\Big) + \int_0^T \mathbb{E}\|u(s)\|_2^2 \mathrm{d}s < +\infty,$$

and the following equation holds in $\mathbb{H}^0$: for all $t \geqslant 0$,

$$\begin{aligned}u(t,\cdot) = u_0(\cdot) &+ \int_0^t \left[\Delta u(s,\cdot) + \sum_{i=1}^d \partial_{x_i} f_i(s,\cdot,u(s,\cdot)) + g(s,\cdot,u(s,\cdot))\right]\mathrm{d}s \\ &+ \sum_k \int_0^t \sigma_k(s,\cdot,u(s,\cdot))\mathrm{d}W^k(s), \quad \mathbb{P}\text{-}a.s.\end{aligned}$$

Remark 4.3. This result is not new (cf. [16]). However, our general result can be used to treat the initial-boundary values problem as follows.

We now turn to the initial-boundary values problem. Let $\mathcal{O}$ be a bounded smooth domain in $\mathbb{R}^d$. Consider the following SPDE with Dirichlet boundary conditions:

$$\begin{cases} \mathrm{d}u(t,x) = \Big[\Delta u(t,x) + \sum_{i=1}^d \partial_{x_i} f_i(t,\omega,x,u(t,x)) + g(t,\omega,x,u(t,x))\Big]\mathrm{d}t \\ \qquad\qquad + \sum_k \sigma_k(t,\omega,x,u(t,x))\mathrm{d}W^k(t), \\ u(t,x) = 0, \quad (t,x) \in \mathbb{R}_+ \times \partial\mathcal{O}, \\ u(0,x) = u_0(x). \end{cases}$$

For $m \in \mathbb{N}_0$, let $\mathbb{W}_2^m(\mathcal{O})$ and $\mathbb{W}_2^{m,0}(\mathcal{O})$ be the usual Soblev spaces on $\mathcal{O}$, which are the respective completions of smooth functions spaces $C^\infty(\mathcal{O})$ and $C_0^\infty(\mathcal{O})$ (with compact supports in $\mathcal{O}$) with respect to the norm

$$\|f\|_{m,2} := \left(\sum_{j=0}^m \int_{\mathcal{O}} |\nabla^j f(x)|^2 \mathrm{d}x\right)^{1/2}.$$

Set $\mathfrak{L} := \Delta$ and $\mathscr{D}(\mathfrak{L}) := \mathbb{W}_2^2(\mathcal{O}) \cap \mathbb{W}_2^{1,0}(\mathcal{O})$. Then $(\mathfrak{L},\mathscr{D})$ forms a sectorial operator in $L^2(\mathcal{O})$ (cf. [18]), and we have the corresponding $\mathbb{H}^m$. We remark that $\mathbb{H}^1 = \mathbb{W}_2^{1,0}(\mathcal{O})$ and $(I-\mathfrak{L})^{-1/2}\partial_{x_i}$ and $\partial_{x_i}(I-\mathfrak{L})^{-1/2}$ are bounded linear operators in $L^2(\mathcal{O})$.

We assume that:

(A1)′ For each $x \in \mathcal{O}$, $z \in \mathbb{R}$ and $t \geqslant 0$, $f(t,\cdot,x,z), g(t,\cdot,x,z)$ and $\sigma(t,\cdot,x,z)$ are $\mathcal{F}_t$-measurable.

(A2)′ There exist $\kappa_1,\kappa_2 > 0$ and $h_0,h_1 \in L^2(\mathcal{O})$ such that for all $(t,\omega) \in \mathbb{R}_+ \times \Omega$, $x \in \mathcal{O}$ and $z \in \mathbb{R}$,

$$\|\partial_z f(t,\omega,x,z)\|_{\mathbb{R}^d} + |\partial_z g(t,\omega,x,z)| + \|\partial_z \sigma(t,\omega,x,z)\|_{l^2} \leqslant \kappa_1,$$

and for $j = 0,1$,

$$\|\nabla_x^j f(t,\omega,x,z)\|_{\mathbb{R}^d} + |\nabla_x^j g(t,\omega,x,z)| + \|\nabla_x^j \sigma(t,\omega,x,z)\|_{l^2} \leqslant \kappa_2|z| + h_j(x).$$

(A3)′ One of the following conditions holds:

$$\sigma(t,\omega,x,0) = 0 \quad \text{or} \quad \sigma(t,\omega,x,z) = 0 \quad \text{for all } x \in \partial\mathcal{O}.$$

Remark 4.4. Notice the following characterization of $\mathbb{W}_2^{1,0}(\mathcal{O})$ (cf. [24]):

$$u \in \mathbb{W}_2^{1,0}(\mathcal{O}) \text{ if and only if } u \in \mathbb{W}_2^1(\mathcal{O}) \text{ and } u(x) = 0 \text{ for almost all } x \in \partial\mathcal{O}.$$

Thus, (A2)′ and (A3)′ imply that if $u \in \mathbb{H}^1$, then $\sigma_k(t,\omega,\cdot,u(\cdot)) \in \mathbb{H}^1$.

Basing on the similar calculations as above, we have

Theorem 4.5. *Assume* (A1)′–(A3)′. *For any* $u_0 \in \mathbb{H}^1$, *the same conclusions as in Theorem* 4.2 *hold. Moreover,* $u(t,x) = 0$ *for almost all* $x \in \partial\mathcal{O}$ *and any* $t \geqslant 0$.

5 Stochastic Burgers and Ginzburg-Landau equations on the real line

In this section, we consider the following generalized stochastic Burgers and Ginzburg-Landau equation on the real line $\mathbb{R}$:

$$\mathrm{d}u(t,x) = [\partial_x^2 u(t,x) + \partial_x f(t,\omega,u(t,x)) + g(t,\omega,x,u(t,x))]\mathrm{d}t + \sum_k \sigma_k(t,\omega,x,u(t,x))\mathrm{d}W^k(t),$$

where the coefficients f, g and $\sigma_k, k \in \mathbb{N}$, are measurable with respect to their variables, and satisfy the following assumptions:

(B1) For each $t \geqslant 0$ and $x, z \in \mathbb{R}$, $f(t, \cdot, z), g(t, \cdot, x, z)$ and $\sigma_k(t, \cdot, x, z), k \in \mathbb{N}$, are $\mathcal{F}_t$-measurable.

(B2) For every $(t, \omega) \in \mathbb{R}_+ \times \Omega$, $f(t, \omega, \cdot) \in C^\infty(\mathbb{R})$, and for each $m \in \mathbb{N}$, there exist $q_m \geqslant 0$ and $\kappa^f_m > 0$ such that for all $(t, \omega, z) \in \mathbb{R}_+ \times \Omega \times \mathbb{R}$,

$$|\partial^m_z f(t, \omega, z)| \leqslant \kappa^f_m \cdot (|z|^{q_m} + 1),$$

where $q_1 < 2$.

(B3) For every $(t, \omega) \in \mathbb{R}_+ \times \Omega$, $g(t, \omega, \cdot, \cdot) \in C^\infty(\mathbb{R}^2)$, and for each $n \in \mathbb{N}_0$ and $m \in \mathbb{N}$, there exist $l_{nm}, l_n \geqslant 1$, $\kappa^g_{nm}, \kappa^g_n > 0$ and $h^g_n \in L^2(\mathbb{R})$ such that for all $(t, \omega, x, z) \in \mathbb{R}_+ \times \Omega \times \mathbb{R} \times \mathbb{R}$,

$$|\partial^n_x \partial^m_z g(t, \omega, x, z)| \leqslant \kappa^g_{nm} \cdot (|z|^{l_{nm}} + 1),$$
$$|\partial^n_x g(t, \omega, x, z)| \leqslant \kappa^g_n \cdot |z|^{l_n} + h^g_n(x),$$

where $1 \leqslant l_1 < 7$, and for some $\kappa^g > 0$, $\partial_z g(t, \omega, x, z) \leqslant \kappa^g$.

(B4) For every $(t, \omega) \in \mathbb{R}_+ \times \Omega$ and $k \in \mathbb{N}$, $\sigma_k(t, \omega, \cdot, \cdot) \in C^\infty(\mathbb{R}^2)$, and for each $n \in \mathbb{N}_0$ and $m \in \mathbb{N}$, there exist $p_{nm} \geqslant 0, p_n \geqslant 1$, $\kappa^\sigma_{nm}, \kappa^\sigma_n > 0$ and $h^\sigma_n \in L^1(\mathbb{R})$ such that for all $(t, \omega, x, z) \in \mathbb{R}_+ \times \Omega \times \mathbb{R} \times \mathbb{R}$,

$$\sum_k |\partial^n_x \partial^m_z \sigma_k(t, \omega, x, z)|^2 \leqslant \kappa^\sigma_{nm} \cdot (|z|^{p_{nm}} + 1),$$
$$\sum_k |\partial^n_x \sigma_k(t, \omega, x, z)|^2 \leqslant \kappa^\sigma_n \cdot |z|^{2p_n} + h^\sigma_n(x),$$

where $p_{01} = 0$ and $1 \leqslant p_1 < 5$.

Remark 5.1. The growth requirements in (B2)–(B4) are mainly due to the Sobolev embedding. The condition $\partial_z g(t, \omega, x, z) \leqslant \kappa^g$ implies that

$$z \cdot g(t, \omega, x, z) \leqslant \kappa^g \cdot z^2 + g(t, \omega, x, 0) \cdot |z|, \quad \forall z \in \mathbb{R}.$$

In particular, $f(z) = z^2$ and $g(z) = z - z^{2n-1}$ for some $n \in \mathbb{N}$ satisfy (B2) and (B3).

Let $\mathbb{W}^m_2(\mathbb{R})$ be the usual Sobolev spaces on $\mathbb{R}$. We need the following Gagliardo-Nirenberg inequality: for any $p \in [2, +\infty]$, $m \in \mathbb{N}$ and $u \in \mathbb{W}^m_2(\mathbb{R})$ (cf. [5, p. 24, Theorem 9.3]),

$$\|u\|_{L^p} \leqslant C\|u\|_m^{\frac{p-2}{2mp}} \|u\|_0^{\frac{2mp-p+2}{2mp}} \leqslant C\|u\|_m. \tag{5.1}$$

Below, we take $\mathbb{H}^m = \mathbb{W}^m_2(\mathbb{R})$ and $\mathfrak{L} = \partial^2_x$, and define for $u \in L^2(\mathbb{R})$,

$$F(t, \omega, u) := \partial_x f(t, \omega, u(\cdot)) + g(t, \omega, \cdot, u(\cdot)), \tag{5.2}$$
$$B_k(t, \omega, u) := \sigma_k(t, \omega, \cdot, u(\cdot)), \quad k \in \mathbb{N}. \tag{5.3}$$

As in the previous section, the variables t and ω will be dropped below. We have

Lemma 5.2. *Assume* (B1)–(B4). *Then F and B defined by* (5.2) *and* (5.3) *satisfy* (H1$_1$).

Proof. Noting that $(I - \partial^2_x)^{-\frac{1}{2}}\partial_x$ and $(I - \partial^2_x)^{-\frac{1}{2}}$ are bounded linear operators on $L^2(\mathbb{R})$ and the following elementary formula

$$\phi(u) - \phi(v) = (u - v)\int_0^1 \phi'(s(u - v) + v)\mathrm{d}s, \tag{5.4}$$

by (B2), (B3) and (5.1) we have for any $u, v \in \mathbb{H}^1$,

$$\begin{aligned}\|F(u) - F(v)\|_{-1} &\leqslant C\|f(u(\cdot)) - f(v(\cdot))\|_0 + \|g(\cdot, u(\cdot)) - g(\cdot, v(\cdot))\|_0 \\ &\leqslant C(\|u\|^{q_1\vee l_{01}}_{L^\infty} + \|v\|^{q_1\vee l_{01}}_{L^\infty} + 1) \cdot \|u - v\|_0 \\ &\leqslant C(\|u\|^{q_1\vee l_{01}}_1 + \|v\|^{q_1\vee l_{01}}_1 + 1) \cdot \|u - v\|_0.\end{aligned}$$

It is obvious by (B4) with $p_{01}=0$ and (5.4) that

$$\sum_k \|B_k(u)-B_k(v)\|_0^2 \leqslant 2\kappa_{01}^\sigma \cdot \|u-v\|_0^2.$$

Moreover, by the integration by parts formula, we have for $u\in\mathbb{H}^1$,

$$\int_{\mathbb{R}} u(x)\partial_x f(u(\cdot))\mathrm{d}x = -\int_{\mathbb{R}} \partial_x u(x) f(u(x))\mathrm{d}x = -\int_{\mathbb{R}} \partial_x\left(\int_0^{u(\cdot)} f(r)\mathrm{d}r\right)\mathrm{d}x = 0.$$

By Remark 5.1, we have

$$\int_{\mathbb{R}} u(x)g(x,u(x))\mathrm{d}x \leqslant C(\|u\|_0^2+1).$$

Hence,

$$\langle u, F(u)\rangle_0 \leqslant C(\|u\|_0^2+1).$$

On the other hand, as above, it is easy to see that for some $p>1$,

$$\|F(u)\|_0 \leqslant C\|u\|_{L^\infty}^{q_1}(\|u\|_1+1) + C\|u\|_{L^\infty}^{l_0-1}(\|u\|_0+1) \leqslant C(\|u\|_1^p+1)$$

and

$$\sum_k \|B_k(u)\|_0^2 \leqslant C(\|u\|_0^2+1).$$

The proof is complete.

To verify (H2$_{\mathcal{K}}$), we need the following elementary differential formula, which can be proved by induction.

Lemma 5.3. *Let $\phi\in C^\infty(\mathbb{R}^2)$. For $m\geqslant 3$ and $u\in\mathbb{H}^m$, we have*

$$\partial_x^m\phi(\cdot,u) = (\partial_u\phi)(\cdot,u)\partial_x^m u + m[(\partial_u^2\phi)(\cdot,u)\partial_x u + (\partial_x\partial_u\phi)(\cdot,u)]\cdot\partial_x^{m-1}u$$
$$+ P(\partial_x^{m-2}u,\ldots,\partial_x u) + (\partial_x^m\phi)(\cdot,u),$$

where P is a polynomial function.

Lemma 5.4. *For any $m\in\mathbb{N}$ and $u\in\mathbb{H}^m$, we have for some $\alpha_m\geqslant 1$,*

$$\langle \partial_x^m u, \partial_x^m g(\cdot,u)\rangle_0 + \|\partial_x^m f(u)\|_0^2 \leqslant \frac{1}{4}\|u\|_{m+1}^2 + C(\|u\|_{m-1}^{\alpha_m}+1).$$

Proof. For $m=1$, by (B3) and (5.1), we have

$$\begin{aligned}\langle \partial_x u, \partial_x g(\cdot,u)\rangle_0 &= \langle \partial_x u, (\partial_x g)(x,u)\rangle_0 + \langle \partial_x u, (\partial_u g)(x,u)\partial_x u\rangle_0\\ &\leqslant \|\partial_x u\|_0\cdot(\kappa_1^g\cdot\||u|^{l_1}\|_0 + \|h_1^g\|_0) + \kappa^g\cdot\|\partial_x u\|_0^2\\ &\leqslant C(\|u\|_1^2+1) + \kappa_1^g\cdot\|u\|_1\cdot\|u\|_{L^{2l_1}}^{l_1}\\ &\leqslant C\cdot(\|u\|_2\cdot\|u\|_0+1) + C\|u\|_2^{\frac{l_1+1}{4}}\cdot\|u\|_0^{\frac{3l_1+3}{4}},\end{aligned}$$

and by (B2),

$$\begin{aligned}\|\partial_x f(u)\|_0^2 &\leqslant (\kappa_1^f)^2\cdot\int_{\mathbb{R}}(|u|^{q_1}+1)^2\cdot|\partial_x u|^2\mathrm{d}x\\ &\leqslant (\kappa_1^f)^2\cdot(\|u\|_{L^\infty}^{2q_1}+1)\cdot\|u\|_1^2\\ &\leqslant C\|u\|_2^{\frac{q_1}{2}+1}\|u\|_0^{\frac{3q_1}{2}+1} + C\cdot\|u\|_2\cdot\|u\|_0.\end{aligned}$$

Since $l_1<7$ and $q_1<2$, by Young's inequality, we get for some $\alpha_1>1$,

$$\langle \partial_x u, \partial_x g(u)\rangle_0 + \|\partial_x f(u)\|_0^2 \leqslant \frac{1}{4}\|u\|_2^2 + C(\|u\|_0^{\alpha_1}+1).$$

For $m=2$, noticing that

$$\partial_x^2 g(\cdot,u)=(\partial_u g)(x,u)\partial_x^2 u+(\partial_u^2 g)(x,u)(\partial_x u)^2+2(\partial_x\partial_u g)(x,u)\partial_x u+(\partial_x^2 g)(x,u),$$

by (B3) and (5.1), we have

$$\begin{aligned}\langle\partial_x^2 u,\partial_x^2 g(\cdot,u)\rangle_0 &\leqslant \kappa^g\cdot\|\partial_x^2 u\|_0^2+\|\partial_x^2 u\|_0\cdot[\|(\partial_u^2 g)(\cdot,u)(\partial_x u)^2\|_0\\ &\quad+2\|(\partial_x\partial_u g)(\cdot,u)\partial_x u\|_0+\|(\partial_x^2 g)(\cdot,u)\|_0]\\ &\leqslant \kappa^g\cdot\|u\|_2^2+\|u\|_2\cdot[\kappa_{02}^g(\|u\|_{L^\infty}^{l_{02}}+1)\|\partial_x u\|_{L^4}^4\\ &\quad+2\kappa_{11}^g(\|u\|_{L^\infty}^{l_{11}}+1)\|\partial_x u\|_0+\kappa_2^g\|u\|_{L^{2l_2}}^{l_2}+\|h_2^g\|_0]\\ &\leqslant \kappa^g\cdot\|u\|_2^2+\|u\|_2\cdot[C(\|u\|_1^{l_{02}}+1)\|u\|_2\cdot\|u\|_1^3\\ &\quad+C(\|u\|_1^{l_{11}}+1)\|u\|_1+\kappa_2^g\|u\|_1^{l_2}+\|h_2^g\|_0],\\ &\leqslant C\|u\|_3\cdot\|u\|_1\cdot[1+C(\|u\|_1^{l_{02}}+1)\cdot\|u\|_1^3]\\ &\quad+C(\|u\|_1^{2(l_{11}+1)}+\|u\|_1^{2l_2}+1),\end{aligned}$$

and by (B2),

$$\begin{aligned}\|\partial_x^2 f(u)\|_0^2 &\leqslant 2\int_{\mathbb{R}}|(\partial_u f)(u)\partial_x^2 u|^2\mathrm{d}x+2\int_{\mathbb{R}}|(\partial_u^2 f)(u)(\partial_x u)^2|^2\mathrm{d}x\\ &\leqslant C(\|u\|_{L^\infty}^{q_1}+1)\cdot\|\partial_x^2 u\|_0^2+C(\|u\|_{L^\infty}^{q_2}+1)\cdot\|\partial_x u\|_{L^4}^4\\ &\leqslant C(\|u\|_1^{q_1+1}+1)\cdot\|u\|_3+C(\|u\|_1^{q_2+7/2}+1)\cdot\|u\|_3^{1/2}.\end{aligned}$$

Hence, for some $\alpha_2>1$,

$$\langle\partial_x^2 u,\partial_x^2 g(\cdot,u)\rangle_0+\|\partial_x^2 f(u)\|_0^2\leqslant\frac{1}{4}\|u\|_3^2+C(\|u\|_1^{\alpha_2}+1).$$

The higher derivatives can be estimated similarly by Lemma 5.3.

Lemma 5.5. *Under* (B1)–(B4)*, F and B defined by* (5.2) *and* (5.3) *satisfy* (H2$_{\mathcal{K}}$) *for any* $\mathcal{K}\in\mathbb{N}$.

Proof. For any $m\in\mathbb{N}$, we have by Lemma 5.4,

$$\begin{aligned}\langle u,F(u)\rangle_m &=\langle u,F(u)\rangle_0+\langle\partial_x^m u,\partial_x^m F(u)\rangle_0\\ &\leqslant C(\|u\|_0^2+1)+\langle\partial_x^m u,\partial_x^{m+1}f(u)\rangle_0+\langle\partial_x^m u,\partial_x^m g(\cdot,u)\rangle_0\\ &\leqslant C(\|u\|_0^2+1)+\frac{1}{4}\|u\|_{m+1}^2+\|\partial_x^m f(u)\|_0^2+\langle\partial_x^m u,\partial_x^m g(\cdot,u)\rangle_0\\ &\leqslant\frac{1}{2}\|u\|_{m+1}^2+C(\|u\|_{m-1}^{\alpha_m}+1).\end{aligned}$$

Similar to the proof of Lemma 5.4, we also have that for any $\delta>0$ and some $\beta_m\geqslant1$,

$$\sum_k\|B_k(u)\|_m^2\leqslant\delta\|u\|_{m+1}^2+C(\|u\|_{m-1}^{\beta_m}+1).$$

The proof is complete.

Summarizing the above calculations, by Theorem 2.2 we obtain

Theorem 5.6. *Under* (B1)–(B4)*, for any $u_0\in\mathbb{H}^\infty$, there exists a unique process $u(t)\in\mathbb{H}^\infty$ such that*

(i) *for any $m\in\mathbb{N}$, the process $t\mapsto u(t)\in\mathbb{H}^m$ is $(\mathcal{F}_t)$-adapted and continuous, and for any $T>0$ and $p\geqslant2$,*

$$\mathbb{E}\Big(\sup_{s\in[0,T]}\|u(s)\|_m^p\Big)<+\infty;$$

(ii) *for almost all ω and all $t\geqslant0$, $x\in\mathbb{R}$,*

$$\begin{aligned}u(t,x)=u_0(x)&+\int_0^t[\partial_x^2 u(s,x)+\partial_x f(s,u(s,x))+g(s,x,u(s,x))]\mathrm{d}s\\ &+\sum_k\int_0^t\sigma_k(s,x,u(s,x))\mathrm{d}W^k(s).\end{aligned}$$

6 Stochastic tamed 3D Navier-Stokes equations in $\mathbb{R}^3$

In this and next sections, we shall use bold-face letters $\mathbf{u} = (u_1, u_2, u_3), \dots$ to denote the velocity fields in $\mathbb{R}^3$ (or $\mathbb{R}^2$).

Consider the following stochastic tamed 3D Navier-Stokes equation with viscosity constant $\nu = 1$ in $\mathbb{R}^3$:

$$\begin{aligned} \mathrm{d}\mathbf{u}(t) &= [\Delta \mathbf{u}(t) - (\mathbf{u}(t)\cdot\nabla)\mathbf{u}(t) + \nabla p(t) - g_N(|\mathbf{u}(t)|^2)\mathbf{u}(t)]\mathrm{d}t \\ &\quad + \sum_{k=1}^{\infty}[\nabla \tilde{p}_k(t) + \mathbf{h}_k(t,\omega,x,\mathbf{u}(t))]\mathrm{d}W_t^k \end{aligned} \tag{6.1}$$

subject to the incompressibility condition

$$\mathrm{div}(\mathbf{u}(t)) = 0, \tag{6.2}$$

and the initial condition

$$\mathbf{u}(0) = \mathbf{u}_0, \tag{6.3}$$

where $p(t,x)$ and $\tilde{p}_k(t,x)$ are unknown scalar functions, $N > 0$ and the taming function $g_N : \mathbb{R}^+ \to \mathbb{R}^+$ is a smooth function with

$$\begin{cases} g_N(r) := 0, \quad r \in [0, N], \\ g_N(r) := r - N, \quad r \geqslant N + 2, \\ 0 \leqslant g_N'(r) \leqslant C, \quad r \geqslant 0, \\ |g_N^{(k)}(r)| \leqslant C_k, \quad r \geqslant 0, \quad k \in \mathbb{N}, \end{cases} \tag{6.4}$$

and $\mathbf{h}_k, k \in \mathbb{N}$ satisfy that

(C1) for each $k \in \mathbb{N}$ and $t \geqslant 0$, $x, z \in \mathbb{R}^3$, $\mathbf{h}_k(t,\cdot,x,z)$ are $\mathcal{F}_t$-measurable;

(C2) for every $(t,\omega) \in \mathbb{R}_+ \times \Omega$ and $k \in \mathbb{N}$, $\mathbf{h}_k(t,\omega,\cdot,\cdot) \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^3; \mathbb{R}^3)$, and for each $n \in \mathbb{N}_0$ and $m \in \mathbb{N}$, there exist $\kappa_{nm}, \kappa_n > 0$ and $b_n \in L^1(\mathbb{R}^3)$ such that for all $(t,\omega,x,z) \in \mathbb{R}_+ \times \Omega \times \mathbb{R}^3 \times \mathbb{R}^3$,

$$\sum_k |\partial_x^n \partial_z^m \mathbf{h}_k(t,\omega,x,z)|^2 \leqslant \kappa_{nm}$$

and

$$\sum_k |\partial_x^n \mathbf{h}_k(t,\omega,x,z)|^2 \leqslant \kappa_n \cdot |z|^2 + b_n(x).$$

For $m \in \mathbb{N}_0$, set

$$\mathbb{H}^m := \{\mathbf{u} \in \mathbb{W}_2^m(\mathbb{R}^3)^3 : \mathrm{div}(\mathbf{u}) = 0\}, \tag{6.5}$$

where div is taken in the sense of Schwartz distributions. The following Gagliardo-Nirenberg inequality will be used frequently below (cf. [5, p. 24, Theorem 9.3]): for $r \in [2, +\infty]$ and $3(\frac{1}{2} - \frac{1}{r}) \leqslant m \in \mathbb{N}$,

$$\|\mathbf{u}\|_{L^r}^r \leqslant C\|\mathbf{u}\|_m^{\frac{3(r-2)}{2m}}\|\mathbf{u}\|_0^{r-\frac{3(r-2)}{2m}}. \tag{6.6}$$

Let $\mathscr{P}$ be the orthogonal projection operator from $L^2(\mathbb{R}^3)^3$ to $\mathbb{H}^0$. It is well-known that $\mathscr{P}$ can be restricted to a bounded linear operator from $\mathbb{W}_2^m(\mathbb{R}^3)^3$ to $\mathbb{H}^m$, and that $\mathscr{P}$ commutes with the derivative operators (cf. [13]). For any $\mathbf{u} \in \mathbb{H}^0$ and $\mathbf{v} \in L^2(\mathbb{R}^3)^3$, we have

$$\langle \mathbf{u}, \mathbf{v} \rangle_{\mathbb{H}^0} := \langle \mathbf{u}, \mathscr{P}\mathbf{v} \rangle_{\mathbb{H}^0} = \langle \mathbf{u}, \mathbf{v} \rangle_{L^2}.$$

For $\mathbf{u} \in \mathbb{H}^2$, define

$$\mathfrak{L}\mathbf{u} := \mathscr{P}\Delta\mathbf{u} = \Delta\mathbf{u}, \tag{6.7}$$

$$F(\mathbf{u}) := -\mathscr{P}((\mathbf{u}\cdot\nabla)\mathbf{u}) - \mathscr{P}(g_N(|\mathbf{u}|^2)\mathbf{u}), \tag{6.8}$$
$$B_k(t,\omega,\mathbf{u}) := \mathscr{P}(\mathbf{h}_k(t,\omega,\cdot,\mathbf{u})). \tag{6.9}$$

Using $\mathscr{P}$ to act on both sides of (6.1), we may consider the following equivalent equation of (6.1)–(6.3):

$$\mathrm{d}\mathbf{u}(t) = [\mathfrak{L}\mathbf{u}(t) + F(\mathbf{u}(t))]\mathrm{d}t + \sum_{k=1}^{\infty} B_k(t,\mathbf{u}(t))\mathrm{d}W_t^k, \quad \mathbf{u}(0) = \mathbf{u}_0.$$

Lemma 6.1. *Under* (C2)*, the operators* F *and* B *defined by* (6.8) *and* (6.9) *satisfy* (H1$_2$)*.*

Proof. Noting that $(I-\mathfrak{L})^{-\frac{1}{2}}\nabla\mathscr{P}$ and $(I-\mathfrak{L})^{-\frac{1}{2}}\mathscr{P}$ are bounded linear operators on $\mathbb{H}^0$ and

$$\mathscr{P}((\mathbf{u}\cdot\nabla)\mathbf{u}) = \sum_{i=1}^{3}\partial_{x_i}\mathscr{P}(u_i\cdot\mathbf{u}),$$

we have by (6.6),

$$\begin{aligned}\|F(\mathbf{u})-F(\mathbf{v})\|_{-1} &\leqslant C\|\mathbf{u}\otimes\mathbf{u}-\mathbf{v}\otimes\mathbf{v}\|_0 + C\|g_N(|\mathbf{u}|^2)\mathbf{u} - g_N(|\mathbf{v}|^2)\mathbf{v}\|_0\\ &\leqslant C(\|\mathbf{u}\|_{L^\infty}+\|\mathbf{v}\|_{L^\infty})\|\mathbf{u}-\mathbf{v}\|_0 + C(\|\mathbf{u}\|_{L^\infty}^2+\|\mathbf{v}\|_{L^\infty}^2)\|\mathbf{u}-\mathbf{v}\|_0\\ &\leqslant C(\|\mathbf{u}\|_2^2+\|\mathbf{v}\|_2^2+1)\|\mathbf{u}-\mathbf{v}\|_0.\end{aligned}$$

Moreover, it is easy to see by (C2) that

$$\sum_k \|B_k(t,\mathbf{u})-B_k(t,\mathbf{u})\|_0^2 \leqslant C\|\mathbf{u}-\mathbf{v}\|_0^2$$

and

$$\sum_k \|B_k(t,\mathbf{u})\|_0^2 \leqslant C(\|\mathbf{u}\|_0^2+1).$$

On the other hand, observing that

$$\langle\mathbf{u},(\mathbf{u}\cdot\nabla)\mathbf{u}\rangle_0 = \frac{1}{2}\langle\mathbf{u},\nabla|\mathbf{u}|^2\rangle_0 = 0$$

and

$$\langle\mathbf{u},g_N(|\mathbf{u}|^2)\mathbf{u}\rangle_0 \geqslant 0,$$

we have

$$\langle\mathbf{u},F(\mathbf{u})\rangle_0 \leqslant 0. \tag{6.10}$$

Last, by (6.6) we also have

$$\begin{aligned}\|F(\mathbf{u})\|_0 &\leqslant \|\mathbf{u}\|_{L^\infty}\|\mathbf{u}\|_1 + \|\mathbf{u}\|_{L^6}^3\\ &\leqslant C\|\mathbf{u}\|_2^{3/4}\cdot\|\mathbf{u}\|_0^{1/4}\cdot\|\mathbf{u}\|_1 + C\|\mathbf{u}\|_1^3\\ &\leqslant C(\|\mathbf{u}\|_2 + \|\mathbf{u}\|_1^5 + 1).\end{aligned}$$

The proof is complete.

In order to check (H2$_\mathcal{K}$), we need the following Moser-type calculus inequality (cf. [24, p. 294, Proposition 21.77]):

Lemma 6.2. *For any* $m\in\mathbb{N}_0$*, there exists* $C_m>0$ *such that for any* $\mathbf{u},\mathbf{v}\in L^\infty\cap\mathbb{H}^m$*,*

$$\|\mathbf{u}\cdot\mathbf{v}\|_m \leqslant C_m[\|\mathbf{u}\|_{L^\infty}\cdot\|\mathbf{v}\|_m + \|\mathbf{v}\|_{L^\infty}\cdot\|\mathbf{u}\|_m]. \tag{6.11}$$

We now prove the following key estimate.

Lemma 6.3. *Under* (C2)*, the operators* F *and* B *defined by* (6.8) *and* (6.9) *satisfy* (H2$_{\mathcal{K}}$) *for any* $\mathcal{K} \in \mathbb{N}$.

Proof. Let us first verify (2.8) for $m = 1$. By Young's inequality, we have

$$-\langle \mathbf{u}, \mathscr{P}((\mathbf{u}\cdot\nabla)\mathbf{u})\rangle_1 \leqslant \frac{1}{4}\|\mathbf{u}\|_2^2 + \|(\mathbf{u}\cdot\nabla)\mathbf{u}\|_0^2 \leqslant \frac{1}{4}\|\mathbf{u}\|_2^2 + \||\mathbf{u}|\cdot|\nabla\mathbf{u}|\|_0^2,$$

where

$$|\mathbf{u}|^2 = \sum_{k=1}^3 |u_k|^2, \quad |\nabla\mathbf{u}|^2 = \sum_{k,i=1}^3 |\partial_{x_i} u_k|^2.$$

From the expression of g_N, we also have

$$\begin{aligned}
-\langle \mathbf{u}, \mathscr{P}(g_N(|\mathbf{u}|^2)\mathbf{u})\rangle_1 &= -\langle \nabla(g_N(|\mathbf{u}|^2)\mathbf{u}), \nabla\mathbf{u}\rangle_0 - \langle g_N(|\mathbf{u}|^2)\mathbf{u}, \mathbf{u}\rangle_0 \\
&= -\sum_{k,i=1}^3 \int_{\mathbb{R}^3} \partial_{x_i} u_k \cdot \partial_{x_i}(g_N(|\mathbf{u}|^2)u_k)\mathrm{d}x - \int_{\mathbb{R}^3} |\mathbf{u}|^2 \cdot g_N(|\mathbf{u}|^2)\mathrm{d}x \\
&\leqslant -\sum_{k,i=1}^3 \int_{\mathbb{R}^3} \partial_{x_i} u_k \cdot (g_N(|\mathbf{u}|^2)\cdot\partial_{x_i} u_k - g_N'(|\mathbf{u}|^2)\partial_{x_i}|\mathbf{u}|^2 \cdot u_k)\mathrm{d}x \\
&= -\int_{\mathbb{R}^3} |\nabla\mathbf{u}|^2 \cdot g_N(|\mathbf{u}|^2)\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^3} g_N'(|\mathbf{u}|^2)|\nabla|\mathbf{u}|^2|^2\mathrm{d}x \\
&\leqslant -\int_{\mathbb{R}^3} |\nabla\mathbf{u}|^2 \cdot |\mathbf{u}|^2\mathrm{d}x + N\|\nabla\mathbf{u}\|_0^2.
\end{aligned}$$

Hence,

$$\langle \mathbf{u}, F(\mathbf{u})\rangle_1 \leqslant \frac{1}{4}\|\mathbf{u}\|_2^2 + N\|\mathbf{u}\|_1^2 \leqslant \frac{1}{2}\|\mathbf{u}\|_2^2 + C_N\|\mathbf{u}\|_0^2. \tag{6.12}$$

For $m \geqslant 2$, by calculus inequality (6.11),

$$\begin{aligned}
-\langle \mathbf{u}, \mathscr{P}((\mathbf{u}\cdot\nabla)\mathbf{u})\rangle_m &= -\langle (I-\Delta)^{m/2}\mathbf{u}, (I-\Delta)^{m/2}((\mathbf{u}\cdot\nabla)\mathbf{u})\rangle_0 \\
&\leqslant \frac{1}{8}\|\mathbf{u}\|_{m+1}^2 + 2\|(\mathbf{u}\cdot\nabla)\mathbf{u}\|_{m-1}^2 \\
&\leqslant \frac{1}{8}\|\mathbf{u}\|_{m+1}^2 + C_m(\|\mathbf{u}\|_{L^\infty}^2\|\mathbf{u}\|_m^2 + \|\nabla\mathbf{u}\|_{L^\infty}^2\|\mathbf{u}\|_{m-1}^2).
\end{aligned}$$

Noting that by Agmon's inequality (cf. [8]),

$$\|\mathbf{u}\|_{L^\infty}^2 \leqslant C\|\mathbf{u}\|_2 \cdot \|\mathbf{u}\|_1,$$

we have

$$\|\mathbf{u}\|_{L^\infty}^2\|\mathbf{u}\|_m^2 \leqslant C\|\mathbf{u}\|_m^3 \cdot \|\mathbf{u}\|_1 \leqslant C_m\|\mathbf{u}\|_{m+1}^{\frac{3}{2}} \cdot \|\mathbf{u}\|_{m-1}^{\frac{3}{2}} \cdot \|\mathbf{u}\|_1$$

and

$$\|\nabla\mathbf{u}\|_{L^\infty}^2\|\mathbf{u}\|_{m-1}^2 \leqslant C\|\mathbf{u}\|_3 \cdot \|\mathbf{u}\|_2 \cdot \|\mathbf{u}\|_{m-1}^2 \leqslant C\|\mathbf{u}\|_3^{\frac{3}{2}} \cdot \|\mathbf{u}\|_1^{\frac{1}{2}} \cdot \|\mathbf{u}\|_{m-1}^2 \leqslant C\|\mathbf{u}\|_{m+1}^{\frac{3}{2}} \cdot \|\mathbf{u}\|_{m-1}^{\frac{5}{2}}.$$

Thus, by Young's inequality, we get

$$-\langle \mathbf{u}, \mathscr{P}((\mathbf{u}\cdot\nabla)\mathbf{u})\rangle_m \leqslant \frac{1}{4}\|\mathbf{u}\|_{m+1}^2 + C_m\|\mathbf{u}\|_{m-1}^{10}.$$

Let us now look at the term $-\langle \mathbf{u}, \mathscr{P}(g_N(|\mathbf{u}|^2)\mathbf{u})\rangle_m$. By the calculus inequality (6.11) again, we have

$$\|g_N(|\mathbf{u}|^2)\mathbf{u}\|_{m-1}^2 \leqslant C_m(\|g_N(|\mathbf{u}|^2)\|_{L^\infty}^2 \cdot \|\mathbf{u}\|_{m-1}^2 + \|g_N(|\mathbf{u}|^2)\|_{m-1}^2 \cdot \|\mathbf{u}\|_{L^\infty}^2)$$

and

$$\||\mathbf{u}|^2\|_{m-1}^2 \leqslant C_m \|\mathbf{u}\|_{m-1}^2 \cdot \|\mathbf{u}\|_{L^\infty}^2.$$

Noting that for any $k \geqslant 2$,

$$|g_N^{(k)}(r)| = 0 \quad \text{on } r < N \text{ and } r > N+2,$$

we have

$$\|g_N(|\mathbf{u}|^2)\|_{m-1} \leqslant C_m \||\mathbf{u}|^2\|_{m-1} + C_{N,m}(\|\mathbf{u}\|_{m-2}^{\gamma_m} + 1), \quad \gamma_m > 2.$$

As above, we obtain

$$\begin{aligned}
-\langle \mathbf{u}, \mathscr{P}(g_N(|\mathbf{u}|^2)\mathbf{u})\rangle_m &\leqslant \frac{1}{8}\|\mathbf{u}\|_{m+1}^2 + 2\|g_N(|\mathbf{u}|^2)\mathbf{u}\|_{m-1}^2 \\
&\leqslant \frac{1}{8}\|\mathbf{u}\|_{m+1}^2 + C_m\|\mathbf{u}\|_{L^\infty}^4 \cdot \|\mathbf{u}\|_{m-1}^2 + C_{N,m}\|\mathbf{u}\|_{L^\infty}^2 \cdot (\|\mathbf{u}\|_{m-2}^{\gamma_m} + 1) \\
&\leqslant \frac{1}{4}\|\mathbf{u}\|_{m+1}^2 + C_m(\|\mathbf{u}\|_{m-1}^{\gamma_m} + 1), \quad \gamma_m > 2.
\end{aligned}$$

Combining the above calculations yields that for some $\alpha_m \geqslant 1$,

$$\langle \mathbf{u}, F(\mathbf{u})\rangle_m \leqslant \frac{1}{2}\|\mathbf{u}\|_{m+1}^2 + C_m(\|\mathbf{u}\|_{m-1}^{\alpha_m} + 1).$$

We now check (2.9). For $m = 1$, by (C2) and (2.1), we have

$$\begin{aligned}
\sum_{k=1}^{\infty} \|\nabla B_k(t, \mathbf{u})\|_0^2 &\leqslant \sum_{k=1}^{\infty} \|(\nabla_x \mathbf{h}_k)(t, \mathbf{u})\|_0^2 + \sum_{k=1}^{\infty} \|(\partial_u \mathbf{h}_k)(t, \mathbf{u})\nabla \mathbf{u}\|_0^2 \\
&\leqslant C(\|\mathbf{u}\|_0^2 + 1) + C\|\nabla \mathbf{u}\|_0^2 \\
&\leqslant C(\|\mathbf{u}\|_0^2 + 1) + C\|\mathbf{u}\|_2 \cdot \|\mathbf{u}\|_0 \\
&\leqslant \delta\|\mathbf{u}\|_2^2 + C(\|\mathbf{u}\|_0^2 + 1).
\end{aligned} \tag{6.13}$$

The higher derivatives can be calculated similarly. The proof is complete.

Thus, we obtain the following main result in this section.

Theorem 6.4. *Under* (C1) *and* (C2)*, for any* $\mathbf{u}_0 \in \mathbb{H}^\infty$ *there exists a unique solution* $\mathbf{u}(t) \in \mathbb{H}^\infty$ *to* (6.1) *such that*

(i) *for any* $m \in \mathbb{N}$*, the process* $t \mapsto u(t) \in \mathbb{H}^m$ *is* $(\mathcal{F}_t)$*-adapted and continuous, and for any* $T > 0$ *and* $p \geqslant 2$,

$$\mathbb{E}\Big(\sup_{s\in[0,T]} \|\mathbf{u}(s)\|_m^p\Big) < +\infty;$$

(ii) *for almost all* ω *and all* $t \geqslant 0$,

$$\begin{aligned}
\mathbf{u}(t) = \mathbf{u}_0 &+ \int_0^t [\Delta \mathbf{u}(s) - \mathscr{P}((\mathbf{u}(s)\cdot\nabla)\mathbf{u}(s))]\mathrm{d}s \\
&- \int_0^t \mathscr{P}(g_N(|\mathbf{u}(s)|^2)\mathbf{u}(s))\mathrm{d}s + \sum_k \int_0^t \mathscr{P}(\mathbf{h}_k(s, \mathbf{u}(s)))\mathrm{d}W^k(s).
\end{aligned}$$

Remark 6.5. Below, the solution in Theorem 2.2 corresponding to the taming function g_N will be denoted by $\mathbf{u}_N(t)$. Set

$$\tau_N := \inf\Big\{t \geqslant 0 : \sup_{x\in\mathbb{R}^3} |\mathbf{u}_N(t,x)| \geqslant \sqrt{N}\Big\}.$$

By the definition of g_N, it is clear that $\mathbf{u}_N$ satisfies the classical SNSE on $[0, \tau_N]$. Moreover, it is easy to see by the uniqueness that

$$\mathbf{u}_{N+1} = \mathbf{u}_N \quad \text{on } [0, \tau_N],\ \mathbb{P}\text{-a.s.}$$

Thus, we have $\tau_N \leqslant \tau_{N+1}, \mathbb{P}$-a.s. Define $\tau := \lim_{N\to\infty} \tau_N$ and $\mathbf{u}(t) = \mathbf{u}_N(t), t \leqslant \tau_N$. Then $(\mathbf{u}, \tau)$ is the unique maximal smooth solution of the classical 3D SNSE. Here, the maximal smooth solution means that $\mathbf{u}(t,\cdot) \in \mathbb{H}^\infty$ for any $t < \tau$ and

$$\limsup_{t\uparrow\tau} \sup_{x\in\mathbb{R}^3} |\mathbf{u}(t,x)| = +\infty.$$

7 Stochastic 2D Navier-Stokes equations in $\mathbb{R}^2$

Consider the following stochastic 2D Navier-Stokes equation in $\mathbb{R}^2$:

$$\mathrm{d}\mathbf{u}(t) = [\Delta \mathbf{u}(t) - (\mathbf{u}(t)\cdot\nabla)\mathbf{u}(t) + \nabla p(t)]\mathrm{d}t + \sum_{k=1}^{\infty}[\nabla \tilde{p}_k(t) + \mathbf{h}_k(t,\omega,x,\mathbf{u}(t))]\mathrm{d}W_t^k, \tag{7.1}$$

subject to the incompressibility condition $\mathrm{div}(\mathbf{u}(t)) = 0$, and the initial condition $\mathbf{u}(0) = \mathbf{u}_0$. As in Section 6, the functions $p(t,x)$ and $\tilde{p}_k(t,x)$ are unknown scalar functions, $\mathbf{u}(t,x)$ is the velocity field in $\mathbb{R}^2$, and $\mathbf{h}_k, k \in \mathbb{N}$ satisfy (C1) and (C2) in Section 6.

For $m \in \mathbb{N}_0$, set

$$\mathbb{H}^m := \{\mathbf{u} \in \mathbb{W}_2^m(\mathbb{R}^2)^2 : \mathrm{div}(\mathbf{u}) = 0\}.$$

We also have the projection operator $\mathscr{P}$ from $L^2(\mathbb{R}^2)^2$ to $\mathbb{H}^0$.

Our main result in this section is

Theorem 7.1. *Assume* (C1) *and* (C2). *For any* $\mathbf{u}_0 \in \mathbb{H}^\infty$, *there exists a unique solution* $\mathbf{u}(t) \in \mathbb{H}^\infty$ *to* (7.1) *such that*

(i) *for any* $m \in \mathbb{N}$, *the process* $t \mapsto u(t) \in \mathbb{H}^m$ *is* $(\mathcal{F}_t)$*-adapted and continuous, and there exists a sequence of stopping times* $\tau_n \uparrow \infty$ *such that for any* $T > 0$, $p \geqslant 2$ *and* $m \in \mathbb{N}$,

$$\mathbb{E}\Big(\sup_{s\in[0,T\wedge\tau_n]} \|\mathbf{u}(s)\|_m^p\Big) \leqslant C_{n,m,p,T};$$

(ii) *for almost all* ω *and all* $t \geqslant 0$,

$$\mathbf{u}(t) = \mathbf{u}_0 + \int_0^t [\Delta\mathbf{u}(s) - \mathscr{P}((\mathbf{u}(s)\cdot\nabla)\mathbf{u}(s))]\mathrm{d}s + \sum_k \int_0^t \mathscr{P}(\mathbf{h}_k(s,\mathbf{u}(s)))\mathrm{d}W^k(s).$$

Here, one cannot directly use Theorem 2.2 to prove this result because $\mathbb{H}^1$ does not embed in $L^\infty(\mathbb{R}^2)^2$. However, we can first consider the modified equation like (6.1), and then use the stopping time technique to obtain the existence of smooth solutions for equation (7.1).

Note that the result in Section 6 also holds for 2D Navier-Stokes equation. In what follows, we shall use the same notations as in Section 6. As in Remark 6.5, set

$$\tau_N := \inf\Big\{t \geqslant 0 : \sup_{x\in\mathbb{R}^2} |\mathbf{u}_N(t,x)| \geqslant \sqrt{N}\Big\}. \tag{7.2}$$

Then τ_N is increasing. For proving Theorem 7.1, it suffices to prove that

$$\tau_N \uparrow \infty, \quad \mathbb{P}\text{-a.s.}$$

In particular, τ_N is the desired stopping time sequence in (i) of Theorem 7.1.

We first prepare the following lemma.

Lemma 7.2. *There exists a constant* $C > 0$ *independent of* N *such that for any* $\mathbf{u} \in \mathbb{H}^3$,

$$2\langle \mathbf{u}, F(\mathbf{u})\rangle_2 + \sum_k \|B_k(s,\mathbf{u})\|_2^2 \leqslant \|\mathbf{u}\|_3^2 + C\|\mathbf{u}\|_2^2\cdot(1+\|\mathbf{u}\|_1^2)\cdot(1+\|\mathbf{u}\|_0^2) + C.$$

Proof. By Young's inequality, we have

$$\langle \mathbf{u}, F(\mathbf{u})\rangle_2 \leqslant \frac{1}{4}\|\mathbf{u}\|_3^2 + 2\|(\mathbf{u}\cdot\nabla)\mathbf{u}\|_1^2 + 2\|g_N(|\mathbf{u}|^2)\mathbf{u}\|_1^2.$$

Noticing that

$$\|\mathbf{u}\|_{L^4}^4 \leqslant C\|\mathbf{u}\|_1^2\cdot\|\mathbf{u}\|_0^2, \quad \|\mathbf{u}\|_{L^\infty}^4 \leqslant C\|\mathbf{u}\|_2^2\cdot\|\mathbf{u}\|_0^2,$$

we have

$$\|\nabla((\mathbf{u}\cdot\nabla)\mathbf{u})\|_0^2 \leqslant 2\||\mathbf{u}|\cdot|\nabla^2\mathbf{u}|\|_0^2 + 2\|\nabla\mathbf{u}\|_{L^4}^4$$

$$\begin{aligned}&\leqslant 2\|\mathbf{u}\|_{L^4}^2\cdot\|\nabla^2\mathbf{u}\|_{L^4}^2+C\|\mathbf{u}\|_2^2\cdot\|\mathbf{u}\|_1^2\\&\leqslant C\|\mathbf{u}\|_0\cdot\|\mathbf{u}\|_1\cdot\|\mathbf{u}\|_2\cdot\|\mathbf{u}\|_3+C\|\mathbf{u}\|_2^2\cdot\|\mathbf{u}\|_1^2\\&\leqslant \frac{1}{4}\|\mathbf{u}\|_3^2+C\|\mathbf{u}\|_2^2\cdot\|\mathbf{u}\|_1^2\cdot(1+\|\mathbf{u}\|_0^2)\end{aligned}$$

and

$$\|\nabla(g_N(|\mathbf{u}|^2)\mathbf{u})\|_0^2\leqslant C\|\mathbf{u}\|_{L^\infty}^4\cdot\|\mathbf{u}\|_1^2\leqslant C\|\mathbf{u}\|_2^2\cdot\|\mathbf{u}\|_0^2\cdot\|\mathbf{u}\|_1^2.$$

Moreover, it is easy to see by (C2) that

$$\sum_k\|B_k(s,\mathbf{u})\|_2^2\leqslant C\|\mathbf{u}\|_2^2\cdot(\|\mathbf{u}\|_1^2+1)+C.$$

The desired estimate now follows.

Now we can give the proof of Theorem 7.1.

Proof of Theorem 7.1. By Ito's formula, we have for any $p\geqslant 1$,

$$\|\mathbf{u}_N(t)\|_m^{2p}=\|\mathbf{u}_0\|_m^{2p}+\sum_{j=1}^5 I_{mj}^{(p)}(t), \tag{7.3}$$

where

$$\begin{aligned}I_{m1}^{(p)}(t)&:=2p\int_0^t\|\mathbf{u}_N(s)\|_m^{2(p-1)}\langle\mathbf{u}_N(s),\Delta\mathbf{u}_N(s)\rangle_m\mathrm{d}s,\\I_{m2}^{(p)}(t)&:=2p\int_0^t\|\mathbf{u}_N(s)\|_m^{2(p-1)}\langle\mathbf{u}_N(s),F(\mathbf{u}_N(s))\rangle_m\mathrm{d}s,\\I_{m3}^{(p)}(t)&:=p\sum_k\int_0^t\|\mathbf{u}_N(s)\|_m^{2(p-1)}\|B_k(s,\mathbf{u}_N(s))\|_m^2\mathrm{d}s,\\I_{m4}^{(p)}(t)&:=2p\sum_{k=1}^\infty\int_0^t\|\mathbf{u}_N(s)\|_m^{2(p-1)}\langle\mathbf{u}_N(s),B_k(s,\mathbf{u}_N(s))\rangle_m\mathrm{d}W_s^k,\\I_{m5}^{(p)}(t)&:=2p(p-1)\sum_{k=1}^\infty\int_0^t\|\mathbf{u}_N(s)\|_m^{2(p-2)}|\langle\mathbf{u}_N(s),B_k(s,\mathbf{u}_N(s))\rangle_m|^2\mathrm{d}s.\end{aligned}$$

For $m=0$, by (6.10) and taking expectations for (7.3), it is easy to see that for any $T>0$ and $p\geqslant 1$,

$$\sup_{t\in[0,T]}\mathbb{E}\|\mathbf{u}_N(t)\|_0^{2p}+\int_0^T\mathbb{E}(\|\mathbf{u}_N(s)\|_0^{2(p-1)}\cdot\|\mathbf{u}_N(s)\|_1^2)\mathrm{d}s\leqslant C_{T,p}. \tag{7.4}$$

Here and after, the constant $C_{T,p}$ is independent of N.

Define

$$\lambda_N(t):=\int_0^t(1+\|\mathbf{u}_N(s)\|_1^2)\cdot(1+\|\mathbf{u}_N(s)\|_0^2)\mathrm{d}s$$

and the stopping times

$$\theta_t^N:=\inf\{s\geqslant 0:\lambda_N(s)\geqslant t\}.$$

Then $\lambda_N^{-1}(t)=\theta_t^N$ and $\theta_t^N\leqslant t$. Moreover, by (7.4) we have for any $M>0$,

$$\lim_{t\to\infty}\sup_N\mathbb{P}(\theta_t^N<M)=0. \tag{7.5}$$

From (7.3) and using Lemma 7.2, we have for $m=2$ and $p=1$,

$$\|\mathbf{u}_N(\theta_t^N)\|_2^2\leqslant\|\mathbf{u}_0\|_2^2-\int_0^{\theta_t^N}\|\mathbf{u}_N(s)\|_3^2\mathrm{d}s+C\int_0^{\theta_t^N}\|\mathbf{u}_N(s)\|_2^2\,\mathrm{d}\lambda_N(s)+C\theta_t^N+I_{24}^{(1)}(\theta_t^N)$$

$$\leqslant \|\mathbf{u}_0\|_2^2 - \int_0^{\theta_t^N} \|\mathbf{u}_N(s)\|_3^2 \mathrm{d}s + C\int_0^t \|\mathbf{u}_N(\theta_s^N)\|_2^2 \,\mathrm{d}s + Ct + I_{24}^{(1)}(\theta_t^N).$$

Taking expectations and using Gronwall's inequality, we find that for any $T > 0$,

$$\sup_{t\in[0,T]} \mathbb{E}\|\mathbf{u}_N(\theta_t^N)\|_2^2 + \mathbb{E}\bigg(\int_0^{\theta_T^N} \|\mathbf{u}_N(s)\|_3^2 \mathrm{d}s\bigg) \leqslant C_T(\|\mathbf{u}_0\|_2^2 + 1).$$

Using the same trick as in proving (3.2), we further have

$$\mathbb{E}\Big(\sup_{t\in[0,\theta_T^N]} \|\mathbf{u}_N(t)\|_2^2\Big) = \mathbb{E}\Big(\sup_{t\in[0,T]} \|\mathbf{u}_N(\theta_t^N)\|_2^2\Big) \leqslant C_T(\|\mathbf{u}_0\|_2^2 + 1).$$

Hence, by (7.2) and Sobolev embedding theorem, we have for any $T > 0$,

$$\begin{aligned}\lim_{N\to\infty} \mathbb{P}(\tau_N \leqslant \theta_T^N) &= \lim_{N\to\infty} \mathbb{P}\Big(\sup_{t\in[0,\theta_T^N]}\sup_{x\in\mathbb{R}^2} |\mathbf{u}_N(t,x)| \geqslant \sqrt{N}\Big)\\ &\leqslant \lim_{N\to\infty} \mathbb{P}\Big(C\sup_{t\in[0,\theta_T^N]} \|\mathbf{u}_N(t)\|_2 \geqslant \sqrt{N}\Big)\\ &\leqslant \lim_{N\to\infty} C\mathbb{E}\Big(\sup_{t\in[0,\theta_T^N]} \|\mathbf{u}_N(t)\|_2^2\Big)/N\\ &\leqslant \lim_{N\to\infty} C_T(\|\mathbf{u}_0\|_2^2 + 1)/N = 0. \end{aligned} \tag{7.6}$$

Noting that for any $M, N, T > 0$,

$$\mathbb{P}(\tau_N < M) = \mathbb{P}(\tau_N < M; \theta_T^N < M) + \mathbb{P}(\tau_N < M; \theta_T^N \geqslant M) \leqslant \mathbb{P}(\theta_T^N < M) + \mathbb{P}(\tau_N \leqslant \theta_T^N),$$

by (7.5) and (7.6), letting $N \to \infty$ and $T \to \infty$ one finds $\lim_{N\to\infty} \mathbb{P}(\tau_N < M) = 0$, which means that $\lim_{N\to\infty} \tau_N = \infty, \mathbb{P}$-a.s. The whole proof is complete.

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References

1 Bertini L, Cancrini N, Jona-Lasinio G. The stochastic Burgers equation. Comm Math Phys, 1984, 165: 211–232

2 Brzeźniak Z. Stochastic partial differential equations in M-type 2 Banach spaces. Potential Anal, 1995, 4: 1–45

3 Da Prato G. Kolmogorov Equations for Stochastic PDEs. Advanced Courses in Mathematics CRM Barcelona. Berlin: Birkhäuser, 2000

4 Da Prato G, Zabczyk J. Stochastic Equations in Infinite Dimensions. Cambridge: Cambridge University Press, 1992

5 Friedman A. Partial Differential Equations. New York: Holt, Rinehart and Winston, 1969

6 Gyöngy I, Nualart D. On the stochastic Burgers' equation in the real line. Ann Probab, 1999, 27: 782–802

7 Kim J U. On the stochastic Burgers equation with a polynomial nonlinearity in the real line. Discrete Contin Dyn Syst Ser B, 2006, 6: 835–866

8 Kreiss H O. Initial boundary value problems and the Navier-Stokes equations. Boston: Academic Press, 1989

9 Krylov N V. An analytic approach to SPDEs. In: Stochastic Partial Differential Equations: Six Perspectives, Mathematical Surveys and Monographs, Vol. 64. Providence: AMS, 1999, 185–242

10 Krylov N V. A W_2^n-theory of the Dirichlet problem for SPDEs in general smooth domains. Probab Theory Related Fields, 1994, 98: 389–421

11 Krylov N V, Rozovskii B L. Stochastic evolution equations (in Russian). J Soviet Math, 1979, 16: 71–147; Transl, 1981, 16: 1233–1277

12 Krein S G, Retunin Y I, Semenov E M. Interpolation of Linear Operators. Moscow: Nauka, 1978

13 Lions P L. Mathematical Topics in Fluid Mechanics, Volume 1, Incompressible Models. Oxford Lect. Series in Math. and its App. 3, 1996

14 Majda A J, Bertozzi A L. Vorticity and Incompressible Flow. Cambridge Texts in Applied Mathematics. Cambridge: Cambridge University Press, 2002

15 Mattingly J C. The dissipative scale of the stochastics Navier-Stokes equation: regularization and analyticity. J Statist Phys, 2002, 108: 1157–1179

16 Mikulevicius R, Rozovskii B L. A note of Krylov's L^p-theory for systems of SPDEs. Electron J Probab, 2001, 6: 1–35

17 Mikulevicius R, Rozovskii B L. Global L_2-solution of stochastic Navier-Stokes equations. Ann Probab, 2005, 33: 137–176

18 Pazy A. Semi-Groups of Linear Operators and Applications. Berlin: Springer-Verlag, 1985

19 Röckner M, Sobol Z. Kolmogorov equations in infinite dimensions: well-posedness and regularity of solutions, with applications to stochastic generalized Burgers equations. Ann Probab, 2006, 34: 663–727

20 Röckner M, Zhang X. Tamed 3D Navier-Stokes equation: existence, uniqueness and regularity. Infin Dimens Anal Quantum Probab, 2009, 12: 525–549

21 Röckner M, Zhang X. Stochastic tamed 3D Navier-Stokes equations: existence, uniqueness and ergodicity. Probab Theory Related Fields, 2009, 145: 211–267

22 Rozovskii B L. Stochastic Evolution Systems, Linear Theory and Applications to Non-linear Filtering. Dordrecht-Boston-London: Kluwer Academic Publishers, 1990

23 Sell G R, You Y C. Dynamics of Evolutionary Equations. New York: Springer, 2002

24 Zeidler E. Nonlinear Functional Analysis and its Applications, Vol. II (A, B). New York: Springer-Verlag, 1990

25 Zhang X. L^p-theory of semi-linear SPDEs on general measure spaces and applications. J Funct Anal, 2006, 239: 44–75

26 Zhang X. Regularities for semilinear stochastic partial differential equations. J Funct Anal, 2007, 249: 454–476