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Approximation algorithms for indefinite complex quadratic maximization problems

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Abstract In this paper, we consider the following indefinite complex quadratic maximization problem: maximize $z^{\mathrm{H}}Qz$, subject to $z_k \in \mathbb{C}$ and $z_k^m = 1$, $k = 1, \ldots, n$, where Q is a Hermitian matrix with $\operatorname{tr} Q = 0$, $z \in \mathbb{C}^n$ is the decision vector, and $m \geqslant 3$. An $\Omega(1/\log n)$ approximation algorithm is presented for such problem. Furthermore, we consider the above problem where the objective matrix Q is in bilinear form, in which case a $0.7118 \left(\cos\frac{\pi}{m}\right)^2$approximation algorithm can be constructed. In the context of quadratic optimization, various extensions and connections of the model are discussed.

Keywords indefinite Hermitian matrix, randomized algorithms, approximation ratio, semidefinite programming relaxation

MSC(2000): 90C20, 90C22

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1 Introduction

Polynomial-time approximation algorithms for NP-hard problems via semidefinite programming (SDP) have received much attention in the last decade, since, in several important cases, this approach leads to significant improvements on the worst-case approximation ratios. The pioneering work along this direction is the famous 0.87856 approximation algorithm of Goemans and Williamson [8] for the Max-Cut problem. Nesterov [18] and Ye [23] obtained $\frac{2}{\pi}$ approximation algorithm for quadratic maximization problem with a positive semidefinite objective matrix. Actually, the $\frac{2}{\pi}$ approximation result can also be derived from the so-called real matrix cube theorem developed by Ben-Tal and Nemirovski in [3]. Alternatively, Alon and Naor [1] demonstrated the $\frac{2}{\pi}$ approximation result via Rietz's identity [19]. Moreover, the bound $\frac{2}{\pi}$ was proved to be essentially tight by Grothendieck [10], and Ben-Tal and Nemirovski [3] in different settings.

Recently, Goemans and Williamson [9] proposed a randomized approximation algorithm, via complex SDP relaxation, for solving the Max-3-Cut problem, which is formulated as a quadratic maximization problem with complex-valued decision variables. In particular, they considered the following model: maximize $z^{\mathrm{H}}Qz$ subject to $z_k^3 = 1$, $k = 1, \ldots, n$, where Q is the Laplacian of the graph (hence positive

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semidefinite for a nonnegatively weighted graph). By a SDP relaxation and random hyperplane method, Goemans and Williamson showed that the algorithm achieves a 0.836 approximation ratio. Recently, Zhang and Huang [25] extended Goemans and Williamson's model [9], where they first developed a closed-form formula for computing the probability of a complex-valued normally distributed bivariate random vector to be in a given angular region, and then applying this formula they obtained an approximation ratio $\frac{\pi}{4}$ for the problem: maximize $z^{\mathrm{H}}Qz$ subject to $|z_k| = 1$, $k = 1, \ldots, n$, where Q is Hermitian positive semidefinite. Similar to that with its real-case counterpart, in fact this $\frac{\pi}{4}$ approximation ratio can also be obtained in two other ways: either by the so-called complex matrix cube theorem developed by Ben-Tal et al. [4], or by the complex Grothendieck's inequality approach developed by Haagerup [11]. However, these approaches shed lights on the problem from very different angles. For the discrete version of the model: maximize $z^{\mathrm{H}}Qz$ subject to $z_k^m = 1$, $k = 1, \ldots, n$, So et al. [21] obtained the $m^2(1 - \cos\frac{2\pi}{m})/8\pi$ approximation ratio based on an identity similar to Rietz's identity. We obtained the same approximation ratio in a later version of [25], by using the probability formula that we developed earlier.

So et al. [21] also considered the following problem: maximize $z^{\mathrm{H}}Qz$ subject to $|z_k| = 1$, $k = 1, \ldots, n$, where Q is a symmetric matrix with zero diagonal elements. They presented an $\Omega(1/\log n)$-approximation algorithm for such problems, and their result provides an alternative analysis of the algorithm in Charikar and Wirth [6] for the (real) quadratic optimization problem. In fact, the $\Omega(1/\log n)$ also follows directly from results of Nemirovski et al. [17] and Luo et al. [16]. However, algorithms and analysis in [21], [17] and [16] are very different.

In this paper, we consider approximation algorithms for indefinite complex quadratic programming with m-point constellation constraint. Specifically, we consider the following indefinite complex quadratic maximization problem: maximize $z^{\mathrm{H}}Qz$, subject to $z_k \in \mathbb{C}$ and $z_k^m = 1$, $k = 1, \ldots, n$, where Q is a Hermitian matrix with $\operatorname{tr} Q = 0$, $z \in \mathbb{C}^n$ is the decision vector, and $m \geqslant 3$. An $\Omega(1/\log n)$ approximation algorithm is presented for such a problem in general. Furthermore, we consider the above problem where the objective matrix Q is the bilinear form. We show that with the bilinear form of Q, a $0.7118\left(\cos\frac{\pi}{m}\right)^2$ approximation algorithm can be constructed. Various extensions and connections of the model, in the context of quadratic optimization, are discussed.

This paper is organized as follows. In Section 2 we study the indefinite complex quadratic maximization model with discrete decision variables. In Section 3, we introduce and study the bilinear maximization problems.

Notation. We denote by $\bar{a}$ the conjugate of a complex number a, by $\operatorname{Arg} z$ the argument of z, by $|z|$ the modulus of z, and by $\mathbb{C}^n$ the space of n-dimensional complex vectors. As a convention we assume $\operatorname{Arg} z = 0$ if $z = 0$. For a given vector $z \in \mathbb{C}^n$, we denote z^{H} the conjugate transpose of z, and $\operatorname{Diag}(z)$ the $n \times n$ diagonal matrix with diagonal entries taken from z, and if Z is an $n \times n$ matrix, then $\operatorname{diag}(Z)$ denotes an n-dimensional vector formed by the diagonal elements of Z. The space of $n \times n$ real symmetric and the space of complex Hermitian matrices are denoted by $\mathcal{S}^n$ and $\mathcal{H}^n$, respectively. For a matrix $Z \in \mathcal{H}^n$, we write $\operatorname{Re} Z$ and $\operatorname{Im} Z$ for the real part and imaginary part of Z, respectively. Matrix Z being Hermitian implies that $\operatorname{Re} Z$ is symmetric and $\operatorname{Im} Z$ is skew-symmetric. We denote by $\mathcal{S}^n_+$ ($\mathcal{S}^n_{++}$) and $\mathcal{H}^n_+$ ($\mathcal{H}^n_{++}$) the cones of real symmetric positive semidefinite (positive definite) and complex Hermitian positive semidefinite (positive definite) matrices, respectively. The notation $Z \succeq 0$ ($\succ 0$) means that Z is positive semidefinite (positive definite). For two complex matrices Y and Z, their inner product $Y \bullet Z$ is defined to be $\operatorname{Re}(\operatorname{tr} Y^{\mathrm{H}} Z) = \operatorname{tr}\left[(\operatorname{Re} Y)^{\mathrm{T}}(\operatorname{Re} Z) + (\operatorname{Im} Y)^{\mathrm{T}}(\operatorname{Im} Z)\right]$, where "tr" denotes the trace of a matrix and "$^{\mathrm{T}}$" denotes the transpose of a matrix.

2 Indefinite complex quadratic maximization

In this section, we consider the following (indefinite) complex quadratic maximization problem

$$
\begin{aligned}
\text{(DQP)} \quad & \max \quad z^{\mathrm{H}}Qz \\
& \text{s.t.} \quad z_k \in \mathbb{C} \text{ and } z_k^m = 1, \quad k = 1, \ldots, n,
\end{aligned}
$$

where $Q \neq 0$ is an indefinite Hermitian matrix with $\mathrm{diag}(Q) = 0$, and $m \geqslant 2$ is an integer which is a part of the input parameter of the problem. Clearly, the problem can be more explicitly written as

$$\begin{array}{lll} \text{(DQP)} & \max & z^{\mathrm{H}} Q z \\ & \text{s.t.} & z_k \in \{1, \omega, \ldots, \omega^{m-1}\}, \quad k = 1, \ldots, n, \end{array}$$

where $\omega = e^{\boldsymbol{i}\frac{2\pi}{m}} = \cos\frac{2\pi}{m} + \boldsymbol{i}\sin\frac{2\pi}{m}$.

We remark that, as $|z_k| = 1$, the objective function value remains unchanged if we replace the condition $\mathrm{diag}(Q) = 0$ by $\mathrm{tr}\, Q = 0$. Zhang and Huang [25] considered the approximation algorithms for the same problem where Q is assumed to be Hermitian positive semidefinite. In that case, applications of such models arise from solving the Max-3-Cut problem [9], the signal processing for wireless communications [12, 15] and the radar signal processing [7].

We solve the following semidefinite programming as a relaxation of (DQP):

$$\begin{array}{lll} \text{(SDP)} & \max & Q \bullet Z \\ & \text{s.t.} & Z_{kk} = 1, \quad k = 1, \ldots, n, \\ & & Z \succeq 0. \end{array}$$

This relaxed problem (complex SDP) can be solved in polynomial time up to any prescribed precision. For practical solution methods, see e.g. [22].

In [21], So et al. considered approximation algorithms for solving the continuous version of (DQP):

$$\begin{array}{lll} \text{(CQP)} & \max & z^{\mathrm{H}} Q z \\ & \text{s.t.} & |z_k| = 1, \quad k = 1, \ldots, n. \end{array}$$

They proposed the following rounding scheme. Draw a random complex vector $\xi \in \mathcal{N}_c(0, Z^*)$, where $\mathcal{N}_c(0, Z^*)$ stands for the n-dimensional complex-valued normal distribution with mean vector zero and covariance matrix Z^*, and Z^* is an optimal solution of (SDP). For $k = 1, 2, \ldots, n$, let

$$y_k := \begin{cases} \xi_k/|\xi_k|, & \text{if } |\xi_k| > T, \\ \xi_k/T, & \text{if } |\xi_k| \leqslant T, \end{cases} \tag{1}$$

where $T > 0$ is an appropriately chosen parameter. Then, $x_k \in \mathbb{C}^n$ is generated as follows:

$$x_k = \begin{cases} e^{\boldsymbol{i}\mathrm{Arg}\, y_k}, & \text{with probability } (1 + |y_k|)/2, \\ -e^{\boldsymbol{i}\mathrm{Arg}\, y_k}, & \text{with probability } (1 - |y_k|)/2, \end{cases} \tag{2}$$

for $k = 1, \ldots, n$.

This process produces a randomized feasible solution $x \in \mathbb{C}^n$ for (CQP). So et al. [21] established that

$$E[x^{\mathrm{H}} Q x] \geqslant \Omega(1/\log n) v(\mathrm{SDP}).$$

Now our scheme is to generate a feasible solution z for (DQP), based on the solution x for (CQP) as generated by the algorithm of So et al. [21]. Below we shall prove that the objective value of z is a constant proportion of that of x, in expectation.

Let x be any feasible solution for (CQP). We now use x_k to further randomly generate z_k (independently) as follows:

$$z_k = \begin{cases} 1, & \text{with probability } (1 + \mathrm{Re}\, x_k)/m, \\ \vdots & \\ \omega^j, & \text{with probability } (1 + \mathrm{Re}\,(\omega^{-j} x_k))/m, \\ \vdots & \\ \omega^{m-1}, & \text{with probability } (1 + \mathrm{Re}\,(\omega^{-(m-1)} x_k))/m, \end{cases} \tag{3}$$

where $k=1,\dots,n$. Indeed we note that $(1+\mathrm{Re}\,(\omega^{-j}x_k))/m \geqslant 0$ for all $j=0,1,\dots,m-1$, and that

$$\sum_{j=0}^{m-1}\frac{1+\mathrm{Re}\,(\omega^{-j}x_k)}{m}=1+\frac{1}{m}\mathrm{Re}\left(\left(\sum_{j=0}^{m-1}\omega^{-j}\right)x_k\right)=1.$$

With regard to this second randomization process (from x to z), we have the following general result.

Lemma 2.1. *For $k\neq l$, it holds that*

$$E[z_k\bar z_l]=\begin{cases}E\left[\mathrm{Re}\,x_k\mathrm{Re}\,\bar x_l\right], & \text{for } m=2,\\ \dfrac{1}{4}E[x_k\bar x_l], & \text{for } m\geqslant 3.\end{cases}$$

Proof. The case $m=2$ is easy to see, and is actually used in [20] and [12]. Now we consider the case $m\geqslant 3$. Since for the random variables z and x it holds that

$$E[z_k\bar z_l]=E[E[z_k\bar z_l\mid (x_k,x_l)]],\quad \text{for } k\neq l,$$

we shall first compute $E[z_k\bar z_l\mid (x_k=x_k^0,x_l=x_l^0)]$. For simplicity, we drop the superscript naughts of x_k^0 and x_l^0, and denote the expectation $E[z_k\bar z_l\mid (x_k=x_k^0,x_l=x_l^0)]$ simply by $E[z_k\bar z_l\mid (x_k,x_l)]$, and $\mathrm{Prob}\,\{z_k=\omega^j,z_l=\omega^{j-i}\mid (x_k=x_k^0,x_l=x_l^0)\}$ by $\mathrm{Prob}\,\{z_k=\omega^j,z_l=\omega^{j-i}\mid (x_k,x_l)\}$. We have

$$\begin{aligned}E[z_k\bar z_l\mid (x_k,x_l)]=\,&1\times\sum_{j=0}^{m-1}\mathrm{Prob}\,\{z_k=\omega^j,z_l=\omega^j\mid (x_k,x_l)\}+\cdots\\&+\omega^i\times\sum_{j=0}^{m-1}\mathrm{Prob}\,\{z_k=\omega^j,z_l=\omega^{j-i}\mid (x_k,x_l)\}+\cdots\\&+\omega^{m-1}\times\sum_{j=0}^{m-1}\mathrm{Prob}\,\{z_k=\omega^j,z_l=\omega^{j-m+1}\mid (x_k,x_l)\}.\end{aligned}$$

Obviously,

$$\begin{aligned}\sum_{j=0}^{m-1}\mathrm{Prob}\,\{z_k=\omega^j,z_l=\omega^{j-i}\mid (x_k,x_l)\}&=\sum_{j=0}^{m-1}\frac{1+\mathrm{Re}\,(\omega^{-j}x_k)}{m}\times\frac{1+\mathrm{Re}\,(\omega^{-j+i}x_l)}{m}\\&=\frac{1}{m}+\frac{1}{m^2}\sum_{j=0}^{m-1}\mathrm{Re}\,(\omega^{-j}x_k)\mathrm{Re}\,(\omega^{-j+i}x_l).\end{aligned}$$

Thus we further have

$$\begin{aligned}E[z_k\bar z_l\mid (x_k,x_l)]&=\frac{1}{m}\sum_{j=0}^{m-1}\omega^j+\frac{1}{m^2}\sum_{i=0}^{m-1}\omega^i\left(\sum_{j=0}^{m-1}\mathrm{Re}\,(\omega^{-j}x_k)\mathrm{Re}\,(\omega^{-j+i}x_l)\right)\\&=\frac{1}{m^2}\sum_{j=0}^{m-1}\mathrm{Re}\,(\omega^{-j}x_k)\left(\sum_{i=0}^{m-1}\omega^i\mathrm{Re}\,(\omega^{-j+i}x_l)\right)\\&=\frac{1}{m^2}\sum_{j=0}^{m-1}\frac{\omega^{-j}x_k+\omega^j\bar x_k}{2}\left(\sum_{i=0}^{m-1}\frac{\omega^{-j+2i}x_l+\omega^j\bar x_l}{2}\right)\\&=\frac{1}{4m^2}\sum_{j=0}^{m-1}\sum_{i=0}^{m-1}x_k\bar x_l=\frac{1}{4}x_k\bar x_l,\end{aligned}$$

where we used the fact that $\sum_{i=0}^{m-1}\omega^{2i}=\sum_{j=0}^{m-1}\omega^{-2j}=0$ for $m\geqslant 3$. That is, $E\left[z_k\bar z_l\mid (x_k=x_k^0,x_l=x_l^0)\right]=\frac{1}{4}x_k^0\overline{x_l^0}$. Therefore

$$E\left[z_k\bar z_l\right]=E\left[E\left[z_k\bar z_l\mid (x_k,x_l)\right]\right]=\frac{1}{4}E\left[x_k\bar x_l\right].$$

The lemma is proven. □

Lemma 2.1 implies that $E[z^{\mathrm{H}}Qz] = \frac{1}{4}E[x^{\mathrm{H}}Qx]$ since the trace of Q is zero. Therefore we have the following result.

Theorem 2.2. *There is an approximation algorithm for* (DQP) *with approximation ratio* $\Omega(1/\log n)$.

There is an immediate consequence of Lemma 2.1 regarding the relationship between the optimal Max-2-Cut value and the optimal Max-3-Cut value for the same weighted graph. To be specific, consider a weighted graph (undirected) with n nodes, and the weight on the edge (k,l) being w_{kl} $(k \neq l)$. Let Q be the Laplacian matrix of a weighted graph, i.e., $Q_{kl} = -w_{kl}$ for $k \neq l$, and $Q_{kk} = \sum_{l \neq k} w_{kl}$, $k = 1, \dots, n$. Let $x_k \in \{1, -1\}$, $k = 1, \dots, n$, and $z_k \in \{1, e^{\boldsymbol{i}\frac{2\pi}{3}}, e^{\boldsymbol{i}\frac{4\pi}{3}}\}$. Let $X = xx^{\mathrm{T}}$ and $Z = zz^{\mathrm{H}}$. It is easy to verify that the corresponding 2-Cut value associated with x is $\frac{1}{4}Q \bullet X$, and the corresponding 3-Cut value associated with z is $\frac{1}{3}Q \bullet Z$. Let us denote the sum of all weights by $W^* := \sum_{k<l} w_{kl}$. Now let $x \in \{1, -1\}^n$ correspond to the optimal Max-2-Cut solution. Based on x we again generate $z \in \{1, e^{\boldsymbol{i}\frac{2\pi}{3}}, e^{\boldsymbol{i}\frac{4\pi}{3}}\}$ as described in the above procedure. Then we have

$$
\begin{aligned}
v(\mathrm{M3C}) \geqslant \frac{1}{3}E[Q \bullet Z] &= \frac{1}{3}\left(\sum_{k=1}^{n} Q_{kk} + 2\sum_{k<l} Q_{kl} \mathrm{Re}\, E[z_k \bar{z}_l]\right) \\
&= \frac{1}{3}\left(2W^* + \frac{1}{2}\sum_{k<l} Q_{kl} x_k x_l\right) \\
&= \frac{1}{2}W^* + \frac{1}{3}v(\mathrm{M2C}),
\end{aligned} \tag{4}
$$

where $v(\mathrm{M3C})$ is the optimal Max-3-Cut value and $v(\mathrm{M2C})$ is the optimal Max-2-Cut value. Note that in the relation (4), no assumption is made regarding the signs of the weights.

We remark that the main ingredient to achieve an $\Omega(1/\log n)$-approximation algorithm for (DQP) is the new rounding procedure (3), and we believe that it could be of independent interest. Also we note that as in the analysis of [18, 23, 24] for real continuous case, we can extend all the above results to the following more general setting:

$$
\begin{aligned}
\max \quad & z^{\mathrm{H}}Qz \\
\text{s.t.} \quad & \mathrm{Arg}\, z_k \in \left\{0, \frac{1}{m}2\pi, \dots, \frac{m-1}{m}2\pi\right\}, \quad k = 1, \dots, n, \\
& (|z_1|^2, \dots, |z_n|^2)^{\mathrm{T}} \in \mathcal{F},
\end{aligned}
$$

where $Q \in \mathcal{H}^n$ with $\mathrm{diag}(Q) = 0$, and $\mathcal{F}$ is a closed convex set in $\mathbb{R}^n$. The corresponding convex SDP relaxation is

$$
\begin{aligned}
\max \quad & Q \bullet Z \\
\text{s.t.} \quad & \mathrm{diag}(Z) \in \mathcal{F}, \\
& Z \succeq 0.
\end{aligned}
$$

3 Discrete complex bilinear maximization

In this section, we shall consider the discrete complex bilinear maximization problem

$$
\begin{aligned}
\max \quad & \mathrm{Re}\, y^{\mathrm{H}}Qz \\
\text{s.t.} \quad & y_k, z_l \in \{1, \omega, \dots, \omega^{m-1}\}, \quad k = 1, \dots, p, \ l = 1, \dots, q,
\end{aligned}
$$

or, equivalently, the following complex discrete quadratic program (DBLP):

$$
\begin{aligned}
(\mathrm{DBLP}) \quad \max \quad & \frac{1}{2}(y^{\mathrm{H}}, z^{\mathrm{H}}) \begin{pmatrix} 0 & Q \\ Q^{\mathrm{H}} & 0 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} \\
\text{s.t.} \quad & y_k, z_l \in \{1, \omega, \dots, \omega^{m-1}\}, \quad k = 1, \dots, p, \ l = 1, \dots, q,
\end{aligned}
$$

where Q is a $p \times q$ complex matrix, and $\omega = \cos(2\pi/m) + \boldsymbol{i}\sin(2\pi/m)$, $m \geqslant 3$. We also consider the following continuous version of (DBLP),

$$\begin{aligned}
\text{(CBLP)} \quad & \max \quad \frac{1}{2}(y^{\mathrm{H}}, z^{\mathrm{H}}) \begin{pmatrix} 0 & Q \\ Q^{\mathrm{H}} & 0 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} \\
& \text{s.t.} \quad |y_k| = |z_l| = 1, \quad k = 1, \dots, p, \ l = 1, \dots, q.
\end{aligned}$$

Clearly, the objective matrix of (DBLP) is Hermitian with zero diagonals. As its real counter-part, (DBLP) is a subclass of (DQP) and (CBLP) is a subclass of (CQP). The SDP relaxation for (DBLP) and (CBLP) is

$$\begin{aligned}
\text{(SDBLP)} \quad & \max \quad \frac{1}{2} \begin{pmatrix} 0 & Q \\ Q^{\mathrm{H}} & 0 \end{pmatrix} \bullet W \\
& \text{s.t.} \quad W_{kk} = 1, \quad k = 1, \dots, p+q, \\
& \qquad\ \ W \succeq 0.
\end{aligned}$$

Lemma 4 of [21] implies that if $Q \neq 0$ then $v(\text{SDBLP}) > 0$. Also, we note that in the summation form, the objective function of (SDBLP) is $\sum_{k=1}^{p} \sum_{l=1}^{q} \mathrm{Re}\,(\bar{Q}_{kl} W_{k,p+l})$, and the objective function of (DBLP) and (CBLP) is $\sum_{k=1}^{p} \sum_{l=1}^{q} \mathrm{Re}\,(\bar{Q}_{kl} y_k \bar{z}_l)$.

3.1 An approximation algorithm for the discrete complex bilinear program

Let W^* be an optimal solution for (SDBLP). We draw a random complex vector as

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} \in \mathcal{N}_c(0, W^*),$$

and generate complex vectors $y \in \mathbb{C}^p$ and $z \in \mathbb{C}^q$ as follows:

$$\text{for } k = 1, \dots, p, \text{ assign } y_k := \omega^j \text{ if } \mathrm{Arg}\,\xi_k \in [\tfrac{j}{m}2\pi, \tfrac{j+1}{m}2\pi) \text{ with } j \in \{0, 1, \dots, m-1\};$$

and

$$\text{for } l = 1, \dots, q, \text{ assign } z_l := \omega^j \text{ if } \mathrm{Arg}\,\eta_l \in [\tfrac{j}{m}2\pi, \tfrac{j+1}{m}2\pi) \text{ with } j \in \{0, 1, \dots, m-1\}.$$

In [25] we have shown that

$$E[y_k \bar{z}_l] = \frac{m(2 - \omega - \omega^{-1})}{8\pi^2} \sum_{j=0}^{m-1} \omega^j (\arccos(-\mathrm{Re}\,(\omega^{-j} W^*_{k,p+l})))^2 =: F_m(W^*_{k,p+l}), \quad \forall\, k, l,$$

and $F_m(z) = z$ for $z \in \{1, \omega, \dots, \omega^{m-1}\}$. Furthermore, in Appendix of [25] we established that

$$F_m(z) = \frac{m^2(1 - \cos\frac{2\pi}{m})}{8\pi} z + \phi_1(z) + \phi_2(z)$$

with

$$\phi_1(z) = \frac{m^2(1 - \cos\frac{2\pi}{m})}{4\pi} \sum_{r=1}^{\infty} a_r \sum_{i=0}^{2r+1} b_{2r+2-2i} z^i (\bar{z})^{2r+1-i}$$

and

$$\phi_2(z) = \frac{m^2(1 - \cos\frac{2\pi}{m})}{4\pi^2} \sum_{s=0,t=0}^{\infty} a_s a_t \sum_{i=0}^{2s+2t+2} b_{2s+2t+3-2i} z^i (\bar{z})^{2s+2t+2-i},$$

where

$$a_r = \frac{(2r)!}{2^{4r+1}(r!)^2(2r+1)}, \quad b_{k+1-2i} = \begin{pmatrix} k \\ i \end{pmatrix} \sum_{j=0}^{m-1} e^{\boldsymbol{i}(\frac{2\pi}{m} ji)}.$$

Note that $\sum_{j=0}^{m-1} e^{\boldsymbol{i}(\frac{2\pi}{m} ji)}$ is either 0 or m.

Let $\phi(z) := \phi_1(z) + \phi_2(z)$. If $Z \succeq 0$ then $\bar{Z} \succeq 0$. Moreover, the Hadamard product of Hermitian positive semidefinite matrices remain positive semidefinite. This implies that if $Z \succeq 0$ then $\phi(Z) \succeq 0$, where $\phi(Z) := (\phi(Z_{kl}))_{n\times n}$.

On the other hand, since $F_m(1) = 1$, we have $1 = \frac{m^2(1-\cos\frac{2\pi}{m})}{8\pi} + \phi(1)$. Let $\beta_m := \frac{m^2(1-\cos\frac{2\pi}{m})}{8\pi}$. We conclude that $(\phi(W^*))_{kk}/(1-\beta_m) = 1$, for $k = 1, \ldots, p+q$, and so, $\phi(W^*)/(1-\beta_m)$ is itself a feasible solution for (SDBLP). Now observe that for any feasible solution of (SDBLP), say W, it necessarily follows that

$$-v(\text{SDBLP}) \leqslant \frac{1}{2}\begin{pmatrix} 0 & Q \\ Q^{\mathrm{H}} & 0 \end{pmatrix} \bullet W \leqslant v(\text{SDBLP}). \tag{5}$$

The second inequality is obvious, by definition of the feasibility. To argue that the first inequality also holds, we consider a decomposition of $W \succeq 0$, i.e.,

$$W = \begin{pmatrix} U^{\mathrm{H}} \\ V^{\mathrm{H}} \end{pmatrix} \cdot (U, V),$$

where the number of rows in U^{H} is p, and the number of rows in V^{H} is q. Let us now consider another solution,

$$\tilde{W} := \begin{pmatrix} U^{\mathrm{H}} \\ -V^{\mathrm{H}} \end{pmatrix} \cdot (U, -V) \succeq 0.$$

Since the diagonal of $\tilde{W}$ remains the all-one vector, it is also a feasible solution for (SDBLP). Therefore

$$v(\text{SDBLP}) \geqslant \frac{1}{2}\begin{pmatrix} 0 & Q \\ Q^{\mathrm{H}} & 0 \end{pmatrix} \bullet \tilde{W} = -\frac{1}{2}\begin{pmatrix} 0 & Q \\ Q^{\mathrm{H}} & 0 \end{pmatrix} \bullet W,$$

and so the first inequality in (5) follows. Therefore,

$$\frac{1}{2}\begin{pmatrix} 0 & Q \\ Q^{\mathrm{H}} & 0 \end{pmatrix} \bullet \frac{\phi(W^*)}{1-\beta_m} \geqslant -v(\text{SDBLP}).$$

Now we are in a position to calculate the expected value of the randomized solution

$$\begin{aligned} E\Bigg[\sum_{k=1}^{p}\sum_{l=1}^{q} \mathrm{Re}\,(\bar{Q}_{kl} y_k \bar{z}_l)\Bigg] &= \sum_{k=1}^{p}\sum_{l=1}^{q} \mathrm{Re}\,(\bar{Q}_{kl} F_m(W^*_{k,p+l})) = \sum_{k=1}^{p}\sum_{l=1}^{q} \mathrm{Re}\,(\bar{Q}_{kl}(\beta_m W^*_{k,p+l} + \phi(W^*_{k,p+l}))) \\ &= \beta_m \times v(\text{SDBLP}) + (1-\beta_m) \times \frac{1}{2}\begin{pmatrix} 0 & Q \\ Q^{\mathrm{H}} & 0 \end{pmatrix} \bullet \frac{\phi(W^*)}{1-\beta_m} \\ &\geqslant \beta_m \times v(\text{SDBLP}) + (\beta_m - 1) \times v(\text{SDBLP}) \\ &= (2\beta_m - 1) \times v(\text{SDBLP}) = \left(\frac{m^2(1-\cos\frac{2\pi}{m})}{4\pi} - 1\right) \times v(\text{SDBLP}). \end{aligned}$$

This leads to the following result.

Theorem 3.1. *There is an approximation algorithm for* (DBLP) *with the ratio* $\alpha_m := \frac{m^2(1-\cos\frac{2\pi}{m})}{4\pi} - 1$ *for* $m \geqslant 3$. *In particular,* $\alpha_3 \geqslant 0.0742, \alpha_4 \geqslant 0.2732, \alpha_5 \geqslant 0.3746, \alpha_{10} \geqslant 0.5198$, *and* $\alpha_{100} \geqslant 0.5702$.

Since (CBLP) is the limit of (DBLP) with $m \to \infty$, it is clear that if we let $y_k = e^{\boldsymbol{i}\mathrm{Arg}\,\xi_k}$ and $z_l = e^{\boldsymbol{i}\mathrm{Arg}\,\eta_l}$ where $(\xi^{\mathrm{H}}, \eta^{\mathrm{H}})^{\mathrm{H}}$ is generated from $\mathcal{N}_c(0, W^*)$ and W^* is an optimal solution of (SDBLP), then we will get an approximation algorithm with an approximation ratio of $\lim_{m\to\infty} \frac{m^2(1-\cos\frac{2\pi}{m})}{4\pi} - 1 = \frac{\pi}{2} - 1 \approx 0.5708$.

3.2 An improved approximation algorithm for the continuous bilinear problem

In this subsection, we shall show that this ratio 0.5708 can be improved if we further exploit particular structures of Problem (CBLP). Our analysis below makes use of some of the results in our previous paper [25], and also important insights presented in a paper by Haagerup [11].

According to the analysis in Subsections 3.1 and 3.3 of [25], if we generate $y_k = e^{\boldsymbol{i}\operatorname{Arg}\xi_k}$ and $z_l = e^{\boldsymbol{i}\operatorname{Arg}\eta_l}$ with $(\xi^{\mathrm{H}}, \eta^{\mathrm{H}})^{\mathrm{H}}$, then we have

$$E[y_k \bar{z}_l] = \lim_{m\to\infty} F_m(W^*_{k,p+l}) \ (=: F(W^*_{k,p+l})) = \frac{1}{4\pi}\int_0^{2\pi} e^{\boldsymbol{i}\theta}(\arccos(-\gamma\cos(\theta-\alpha)))^2 d\theta$$
$$= e^{\boldsymbol{i}\alpha}\int_0^{\pi/2} \arcsin(\gamma\sin\theta)\sin\theta d\theta = \frac{\pi}{4}e^{\boldsymbol{i}\alpha}\sum_{r=0}^{\infty} c_r\gamma^{2r+1},$$

where we denoted $W^*_{k,p+l}$ as $\gamma e^{\boldsymbol{i}\alpha}$ and $c_r = \frac{((2r)!)^2}{2^{4r}(r!)^4(r+1)}$. For $\gamma \in [-1,1]$, let

$$\psi(\gamma) := \int_0^{\pi/2} \arcsin(\gamma\sin\theta)\sin\theta d\theta \ \left(= \gamma\int_0^{\pi/2} \frac{(\cos\theta)^2}{\sqrt{1-(\gamma\sin\theta)^2}} d\theta \right).$$

In that notation, the transformation function F can be rewritten as

$$F(z) = e^{\boldsymbol{i}\operatorname{Arg} z}\psi(|z|).$$

We remark here that the equation (3.9) in [25] coincides with Lemma 3.2 in [11], although we were not aware of [11] at the time when we derived that equation.

Important properties of the function $\psi(\gamma)$ are discussed in [11]. In particular, Theorem 2.1 of [11] states that the inverse function $\psi^{-1} : [-1,1] \to [-1,1]$ of ψ exists and it can be expanded into an absolutely convergent power series:

$$\psi^{-1}(s) = \sum_{r=0}^{\infty} b_{2r+1}s^{2r+1}, \quad s \in [-1,1],$$

with $b_1 = \frac{4}{\pi}$ and $b_{2r+1} \leqslant 0$ for all $r \leqslant 1$. Specifically, $b_3 = -8/\pi^3$, $b_5 = 0$, $b_7 = -16/\pi^7$, $b_9 = -80/\pi^9$, $b_{11} = -480/\pi^{11}$, $b_{13} = -3136/\pi^{13}$ and $b_{2r+1} \sim -4/((2r+1)\log(2r+1))^2$ for $r \to \infty$. Moreover, the following result is shown in [11], which was used to bound the complex Grothendieck constant.

Lemma 3.2. *There is a unique $\beta \in (0,1)$ for which $\sum_{r=0}^{\infty} |b_{2r+1}|\beta^{2r+1} = 1$ (i.e., $\psi^{-1}(\beta) = \frac{8}{\pi}\beta - 1$), with $\beta \approx 0.7118$.*

Now the inverse function of $F(z)$ can be written as

$$F^{-1}(z) = e^{\boldsymbol{i}\operatorname{Arg} z}\psi^{-1}(|z|) = \sum_{r=0}^{\infty} b_{2r+1}z^{r+1}\bar{z}^r.$$

For a given $W \in \mathcal{H}^{p+q}_+$ with all-one diagonal elements, let us construct another Hermitian matrix $G(W) \in \mathcal{H}^{p+q}$ as follows:

$$G_{k,p+l}(W) := \frac{4}{\pi}\beta W_{k,p+l} - \sum_{r=1}^{\infty} |b_{2r+1}|\beta^{2r+1}(W_{k,p+l})^{r+1}(\bar{W}_{k,p+l})^r, \quad k=1,\dots,p,\ l=1,\dots,q,$$
$$G_{k_1,k_2}(W) := \frac{4}{\pi}\beta W_{k_1,k_2} + \sum_{r=1}^{\infty} |b_{2r+1}|\beta^{2r+1}(W_{k_1,k_2})^{r+1}(\bar{W}_{k_1,k_2})^r, \quad k_1,k_2=1,\dots,p,$$
$$G_{p+l_1,p+l_2}(W) := \frac{4}{\pi}\beta W_{p+l_1,p+l_2} + \sum_{r=1}^{\infty} |b_{2r+1}|\beta^{2r+1}(W_{p+l_1,p+l_2})^{r+1}(\bar{W}_{p+l_1,p+l_2})^r, \quad l_1,l_2=1,\dots,q.$$

That is,

$$G_{k,p+l}(W) = F^{-1}(\beta W_{k,p+l}), \quad k=1,\dots,p,\ l=1,\dots,q,$$

$$G_{k_1,k_2}(W) = \frac{8}{\pi}\beta W_{k_1,k_2} - F^{-1}(\beta W_{k_1,k_2}), \quad k_1, k_2 = 1, \dots, p,$$
$$G_{p+l_1,p+l_2}(W) = \frac{8}{\pi}\beta W_{p+l_1,p+l_2} - F^{-1}(\beta W_{p+l_1,p+l_2}), \quad l_1, l_2 = 1, \dots, q.$$

By the choice of β (see Lemma 3.2), we see that if W has all-one diagonal elements, then so is true for the Hermitian matrix $G(W)$. Denote $E := (e_p^{\mathrm{T}}, -e_q^{\mathrm{T}})^{\mathrm{T}}(e_p^{\mathrm{T}}, -e_q^{\mathrm{T}})\,(\succeq 0)$, where e_p and e_q are the all-one vectors in $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively. We can now write $G(W)$ in a uniform fashion as follows,

$$G(W) = \frac{4}{\pi}\beta W + \sum_{r=1}^{\infty} |b_{2r+1}|\beta^{2r+1} E \circ (W)^{(r+1)} \circ (W^{\mathrm{T}})^{(r)},$$

where '$A \circ B$' stands for the Hadamard product between A and B, and $A^{(r)}$ is the r-th power in the Hadamard sense, i.e.,

$$A^{(r)} = \overbrace{A \circ A \circ \cdots \circ A}^{r}.$$

If $W \succeq 0$, then, by the fact that the Hadamard product of positive semidefinite matrices remains positive semidefinite, we have $G(W) \succeq 0$. As a remark, we note here that combining this and the previous observation (regarding the diagonals of $G(W)$) leads to the conclusion that if W is a feasible solution for (SDBLP) then so is true for $G(W)$.

Suppose that W^* is an optimal solution of (SDBLP). Let us take $y_k = e^{\boldsymbol{i}\operatorname{Arg}\xi_k}$ and $z_l = e^{\boldsymbol{i}\operatorname{Arg}\eta_l}$ with $(\xi^{\mathrm{H}}, \eta^{\mathrm{H}})^{\mathrm{H}}$ is randomly generated from $\mathcal{N}_c(0, G(W^*))$. In that case, the expected objective value is

$$\begin{aligned} E\Bigg[\sum_{k,l} \mathrm{Re}\,(\bar{Q}_{kl} y_k \bar{z}_l)\Bigg] &= \sum_{k,l} \mathrm{Re}\,(\bar{Q}_{kl} E[y_k \bar{z}_l]) = \sum_{k,l} \mathrm{Re}\,(\bar{Q}_{kl} F(G(W^*_{k,p+l}))) \\ &= \sum_{k,l} \mathrm{Re}\,(\bar{Q}_{kl} F(F^{-1}(\beta W^*_{k,p+l}))) = \sum_{k,l} \mathrm{Re}\,(\bar{Q}_{kl} \beta W^*_{k,p+l}) \\ &= \beta \sum_{k,l} \mathrm{Re}\,(\bar{Q}_{kl} W^*_{k,p+l}) = \beta \times v(\mathrm{SDBLP}) \approx 0.7118 \times v(\mathrm{SDBLP}). \end{aligned}$$

This proves the following theorem.

Theorem 3.3. *The above randomized algorithm has an approximation ratio* 0.7118 *for the continuous complex bilinear maximization problem* (CBLP)*.*

We remark that the 0.7118 approximation ratio should be in contrast to the 0.56 approximation ratio for the real analog of the continuous bilinear maximization problem (CBLP), obtained by Alon and Naor [1] where their proof was based on Krivine's proof for the real Grothendick's constant [13, 14].

3.3 Improved approximation algorithms for discrete complex bilinear programs

In this subsection we should note that the improved bound which we developed in Subsection 3.2 for the continuous problem (CBLP) further helps to improve the approximation bound for the discrete problem (DBLP). A first approximation algorithm for (DBLP) was proposed in Subsection 3.1.

Let us first consider a general problem

$$\begin{array}{lll} (\mathrm{GDQP}) & \max & z^{\mathrm{H}} Q z \\ & \text{s.t.} & z_k \in P_k, \quad k = 1, \dots, n, \end{array}$$

where P_k is a finite set contained in the complex plane, $k = 1, \dots, n$. Connect the points in P_k clockwise, resulting a polygon to be denoted by $\bar{P}_k$, $k = 1, \dots, n$. Now, let us assume that the origin is contained in the interior of $\bar{P}_k$, and moreover, assume that there is a pair of circles C_{in} and C_{out} in the complex plane, centered at the origin, such that

$$C_{\mathrm{in}} \subseteq \bar{P}_k \subseteq C_{\mathrm{out}}, \quad k = 1, \dots, n.$$

Furthermore, let us denote the radius of $C_{\rm in}$ to be $R_{\rm in}$, the radius of $C_{\rm out}$ to be $R_{\rm out}$, and $r := R_{\rm in}/R_{\rm out} \leqslant 1$. For instance, if $P_k = \{1, \omega, \ldots, \omega^m\}$ with $\omega = e^{\boldsymbol{i}\frac{2\pi}{m}}$, then $R_{\rm in} = \cos\frac{\pi}{m}$, $R_{\rm out} = 1$, and $r = \cos\frac{\pi}{m}$. However, (GDQP) is a general model allowing for an irregular constellation of discrete sets in the constraints.

Let us now focus on (GDQP) where Q is either positive semidefinite, or it is in the form of

$$Q = \begin{pmatrix} 0 & Q_{12} \\ Q_{12}^{\rm H} & 0 \end{pmatrix}.$$

In both cases, maximizing $z^{\rm H}Qz$ where $z^{\rm T} = (z_1, \ldots, z_n) \in P_1 \times P_2 \times \cdots \times P_n$ will have at least one optimal solution $\hat{z}$ such that $\hat{z}_k$ is an extremal point of conv $(\bar{P}_k)$ (hence in $\bar{P}_k$), $1 \leqslant k \leqslant n$. Let z^* be the optimal solution of the following problem,

$$\begin{array}{lll} \text{(GCQP)} & \max & z^{\rm H}Qz \\ & \text{s.t.} & |z_k| = 1, \quad k = 1, \ldots, n. \end{array}$$

Suppose now that we have a feasible solution of (GCQP), say $\hat{z}$. Let $\tilde{z} := R_{\rm in}\hat{z}$ and implement the following simple rounding procedure:

For $k = 1, 2, \ldots, n$ sequentially, let

$$\tilde{z}'_k := \arg\max\,\{z^{\rm H}Qz \mid z_k \in P_k, \text{ while } (z_1, \ldots, z_{k-1}, z_{k+1}, \ldots, z_n) \equiv (\tilde{z}'_1, \ldots, \tilde{z}'_{k-1}, \tilde{z}_{k+1}, \ldots, \tilde{z}_n)\}.$$

Clearly this procedure runs in polynomial-time of the input data. The solution so obtained is feasible for (GDQP) and its objective value is never worse than that of $R_{\rm in}\hat{z}$, i.e., $\tilde{z}'^{\rm H}Q\tilde{z}' \geqslant R_{\rm in}^2\hat{z}^{\rm H}Q\hat{z}$. If $\hat{z}$ is chosen to be optimal for (GCQP), then we have $v(\text{GDQP}) \geqslant R_{\rm in}^2 v(\text{GCQP})$. For convenience, let us denote C to be the unit circle in the complex plane, and D to be the unit disk in the complex plane (thus $D = \text{conv}(C)$). Since $R_{\rm out}D$ contains P_k and any $z_k \in P_k$ can be expressed as a convex combination of points in $R_{\rm out}C$, we have $v(\text{GDQP}) \leqslant R_{\rm out}^2 v(\text{GCQP})$.

Suppose that there is $\rho \in (0, 1)$ such that $\hat{z}^{\rm H}Q\hat{z} \geqslant \rho \times v(\text{GCQP})$. Then we have

$$\tilde{z}'^{\rm H}Q\tilde{z}' \geqslant R_{\rm in}^2\hat{z}^{\rm H}Q\hat{z} \geqslant R_{\rm in}^2 \times \rho \times v(\text{GCQP}) \geqslant \rho\left(\frac{R_{\rm in}}{R_{\rm out}}\right)^2 v(\text{GDQP}). \tag{6}$$

Consider a generalized form of (DBLP):

$$\begin{array}{lll} \text{(GDBLP)} & \max & \frac{1}{2}(y^{\rm H}, z^{\rm H})\begin{pmatrix} 0 & Q \\ Q^{\rm H} & 0 \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix} \\ & \text{s.t.} & |y_k|^{m_k} = 1, \quad k = 1, \ldots, p, \\ & & |z_l|^{m_l} = 1, \quad l = 1, \ldots, q. \end{array}$$

By Theorem 3.3 and the above observation, the next result follows.

Theorem 3.4. *Let $m_0 = \min\{m_1, \ldots, m_{p+q}\}$. There is a polynomial-time approximation algorithm for* (GDBLP) *with approximation ratio* $0.7118(\cos\frac{\pi}{m_0})^2$.

In the case $m_k \equiv m$ for all $k = 1, \ldots, p+q$, the above result yields the approximation ratios $\alpha'_m = 0.7118(\cos\frac{\pi}{m})^2$ for (DBLP): $\alpha'_3 \approx 0.1780$, $\alpha'_4 \approx 0.3559$, $\alpha'_5 \approx 0.4659$, $\alpha'_{10} \approx 0.6438$, $\alpha'_{100} \approx 0.7111$, while, as a comparison, in Theorem 3.1 the approximation ratios for the previous approximation algorithm were $\alpha_3 \approx 0.0742$, $\alpha_4 \approx 0.2732$, $\alpha_5 \approx 0.3746$, $\alpha_{10} \approx 0.5198$, and $\alpha_{100} \approx 0.5702$. The improvements are considerable.

A general form of (DQP) is

$$\begin{array}{lll} \text{(GDQP)} & \max & z^{\rm H}Qz \\ & \text{s.t.} & z_k \in P_k, \quad k = 1, \ldots, n. \end{array}$$

The next result follows from combining (6) with the well-known $\frac{\pi}{4}$ approximation ratio for the continuous complex quadratic program [25].

Theorem 3.5. *Consider* (GDQP) *with* $Q \succeq 0$. *Let* $\mathrm{conv}\, P_k$ *be the convex hull of* P_k, $k = 1, \ldots, n$. *Suppose that there are* $0 < R_{\mathrm{in}} \leqslant R_{\mathrm{out}} < \infty$ *such that* $R_{\mathrm{in}} D \subseteq \mathrm{conv}(\bar{P}_k) \subseteq R_{\mathrm{out}} D$, *where* D *is the unit disk in the complex plane. Then, there is a polynomial-time approximation algorithm for* (GDQP) *with approximation ratio* $\frac{\pi}{4}(\frac{R_{\mathrm{in}}}{R_{\mathrm{out}}})^2$.

As an example of application for (GDQP), consider the following obnoxious facility location problem. There are p possible locations for n ($n < p$) obnoxious facilities in the plane, say $\{a_1, \ldots, a_p\}$, and there are q given locations, say $\{b_1, \ldots, b_q\}$, on which the obnoxious facilities would have adverse effects. Moreover, the obnoxious facilities also have adverse effect on each other. Suppose that the harm caused by each pair of facilities diminishes affine linearly in their square distances. Let z_k be the location of obnoxious facility k, $k = 1, \ldots, n$. Then the problem becomes

$$\begin{aligned} \max \quad & \sum_{1 \leqslant k < l \leqslant n} c_{kl} |z_k - z_l|^2 + \sum_{k=1}^{n} \sum_{j=1}^{q} d_{kj} |z_k - b_j|^2 \\ \text{s.t.} \quad & z_k \in \{a_1, \ldots, a_p\}, \quad k = 1, \ldots, n. \end{aligned}$$

If the topology of the constellation $\{a_1, \ldots, a_p\}$ is reasonable, in the sense that they are spread more or less evenly over a circular region, then Theorem 3.3 suggests that such problems can be solved with a worst case guarantee.

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