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The existence theorem of absolute equilibrium about games on connected graph with state payoff vector

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Abstract By introducing state payoff vector to every state node on the connected graph in this paper, dynamic game is researched on finite graphs. The concept of simple strategy about games on graph defined by Berge is introduced to prove the existence theorem of absolute equilibrium about games on the connected graph with state payoff vector. The complete algorithm and an example in the three-dimensional connected mesh-like graph are given in this paper.

Keywords connected graph, state payoff vector, simple strategy, absolute equilibrium, three-dimensional mesh-like graph

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1 Introduction

The game tree which mainly describes the process of dynamic games is a kind of graph with simple structure. Therefore, the research results of games on graph could be generally extended to the category of dynamic games. The original type of games on graph, whose definition was given by Berge [1], is the games on finite tree [7]. Rozen [6] discussed games on graph whose target structure is defined by coherent relation of terminal state set. Rozen [6] extended Berge's concept of simple strategy, namely, on every given state of graph, the choice for the next state is determined by the former experienced state, rather than only determined by the last state that the player had just reached. The results in [1, 6] are both given on the two-dimensional graph for games with terminal payoff. The state payoff vector that is introduced to every state node on finite graph is expected in this paper. The absolute equilibrium of dynamic games is researched by applying the concept of strategy of games on the graph defined by Berge. The related tasks finished by the authors contain the following:

(1) By establishing the corresponding relations between plays and routes of game tree on two-dimensional directed graph, games on directed graph are transformed to game tree. Also, the algorithm of characteristic function is given, and the Shapley vector is chosen as the cooperative solution of the two-dimensional directed graph.

(2) Partial cooperative dynamic games are studied on the two-dimensional mesh-like finite graph [3]. Players adopt partial cooperative behaviors rather than completed cooperative behaviors. The main

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feature of partial cooperation is that behaviors of each player are the combination of cooperative behaviors and individual behaviors. Also, algorithms of the solution of partial cooperative games and the optimal path are given on the two-dimensional directed graph.

Symbol system, strategy of plays and state payoff of games on graph are given in Section 2. In Section 3, the existence theorem of absolute equilibrium about games on connected graph with state payoff vector is proved. The complete algorithm of absolute equilibrium is constructed in Section 4. Finally in Section 5, we give an example of absolute equilibrium about games on the three-dimensional connected mesh-like graph with state payoff vector.

2 Symbol and definition

Connected graph with state set A is written as $\langle A, \gamma \rangle$, where $\gamma \subseteq A^2$ (γ is the arc set of the connected graph) and state set is $A = \{a_0, a_1, \dots\}$. Let $\gamma\langle a \rangle$ be all of the states after state a, and $\gamma'\langle a \rangle$ be the immediate subsequent state set of state a. A_f is written as the state set which has no subsequent states. $\langle A, \gamma \rangle$ is called n state graph, if subdivision of $A \backslash A_f$ is given, which is $\{A_1, \dots, A_n\}$. Using the terminology of games, the set of players is $N = \{1, 2, \dots, n\}$, A_i is the state set of Player i, $i \in N$, the set of decision-making nodes is written as $\tilde{A} = \{A_1, A_2, \dots, A_n\}$ and A_f is the set of terminal states. The path (orbit) of graph $\langle A, \gamma \rangle$ is the sequence (finite or infinite) of states $a_0, a_1, \dots, a_k$. For $k = 1, \dots, t, \dots,$ we have $a_k \in \gamma\langle a_{k-1} \rangle$. An orbit is called a play, if it is infinite or it contains terminal state $a_l \in A_f$. All of the rest orbits are called opening play. Every mapping $s_i : A_i \to A$ satisfying the condition $s_i \subseteq \gamma$ is called the simple strategy of Player i. The set of all the simple strategies of Player i is written as S_i. State $a \in A$ and a situation $(s_1, \dots, s_n) \in S_1 \times \cdots \times S_n$ under simple strategy define the play $\langle a; s_1, \dots, s_n \rangle = a, s(a), s^2(a), \dots$ If each path on graph $\langle A, \gamma \rangle$ is finite, then every state $a \in A$ has a relation with a mapping $F_a : S_1 \times \cdots \times S_n \to A_f$, which maps the situation $(s_1, \dots, s_n)$ under a simple strategy to a terminal state of the play $\langle a; s_1, \dots, s_n \rangle$.

Definition 1. *Giving every state a an n-dimensional real vector $f_a = (f_a^1, \dots, f_a^n)^{\mathrm{T}}$ (where* T *stands for matrix transposition), it is called a state payoff vector of state a and the i-th component f_a^i is called Player i's state payoff of state a.*

Definition 2. *The n-dimensional vector $h(a_r, \dots, a_l) = \sum_{k=r}^{l} f_{a_k} = (h_1(a_r, \dots, a_l), \dots, h_n(a_r, \dots, a_l))^{\mathrm{T}}$ is called situation payoff vector corresponding to play $a_r, \dots, a_l$ ($a_l \in A_f$) on graph $\langle A, \gamma \rangle$. The i-th component $h_i(a_r, \dots, a_l)$, $i = 1, \dots, n$ is called Player i's play payoff corresponding to play $a_r, \dots, a_l$.*

According to the definition of simple strategy, different situations may lead to different plays all from some initial state a_r. Suppose that the play corresponding with situation $(s_1, \dots, s_i, \dots, s_n)$ is $a_r, \dots, a_l$ ($a_l \in A_f$), and the situation corresponding with $(s_1, \dots, s_i', \dots, s_n)$ is $a_r, \dots, a_k$ ($a_k \in A_f$). Notation $\leqslant^j$ is defined as

$$h_j(a_r, \dots, a_l) \leqslant h_j(a_r, \dots, a_k) \Longleftrightarrow F_{a_r}(s_1, \dots, s_i, \dots, s_n) \leqslant^j F_{a_r}(s_1, \dots, s_i', \dots, s_n),$$

where $i = 1, \dots, n, j = 1, \dots, n$.

Choosing $a_0 \in A$ as an initial state, non-cooperative simple games $\Gamma_{a_0}(T)$ are achieved on graph with state payoff vector $T = \langle A, \gamma; A_1, \dots, A_n, A_f; f_{a \in A} \rangle$, where the strategy set of Player i is S_i, the set of terminal state is A_f, and f_a is the state payoff vector of state a on graph $\langle A, \gamma \rangle$.

Definition 3. *The situation $s^* = (s_1^*, \dots, s_n^*)$ that is independent of the initial state is called absolute equilibrium, if s^* is Nash equilibrium about any simple game $\Gamma_{a_0}(T), a_0 \in A$.*

3 Existence theorem of absolute equilibrium

Theorem. *Absolute equilibrium situation exists in games on finite connected graph with state payoff vector.*

Proof. First, the subset of state set $C_\alpha \subseteq A$ is defined for α by induction as follows:

a) $C_0 = A_f$,

b) suppose that for all the $\beta < \alpha$, subset $C_\beta \subseteq A$ have been defined.

If α is finite, then $C_\alpha = C_{\alpha-1} \cup \{a \in A \backslash A_f : \gamma\langle a\rangle \subseteq C_{\alpha-1}\}$ is defined. If α is infinite, then $C_\alpha = \bigcup_{\beta<\alpha} C_\beta$. The smallest ordinal number $p(a)$ is called the rank of state $a \in A$, then we have $a \in C_{p(a)}$. If there is not infinite path on graph $\langle A, \gamma\rangle$, then every state has a rank and the rank is finite.

Second, the mapping $s^* : A \to A$ is defined according to the rank of state by induction as follows. For each 0 rank state a ($a \in A_f$), we define $s^*(a) = a$. If the rank of state $a \in A\backslash A_f$ is $p(a) = \alpha$, and mapping $s^*(a')$ is defined by state $a' \in A$ with rank $p(a') < \alpha$. If $\gamma\langle a'\rangle \neq \emptyset$, then $s^*(a') \in \gamma\langle a'\rangle$. Therefore, s^* has been defined in $C_{\alpha-1}$, and it satisfies condition $s^* \subseteq \gamma$. $s_j^{*\alpha-1}$ is noted as s^*'s restriction which is established in the subset $C_{\alpha-1} \cap A_j$ ($j = 1,\dots,n$). Since $C_{\alpha-1}$ is γ-steady, that is to say, the play whose initial state has emerged into $C_{\alpha-1}$ is completely located in $C_{\alpha-1}$, and $s_j^{*\alpha-1}$ is regarded as a simple strategy of Player j in the subgame of $C_{\alpha-1}$. Considering $a \in C_\alpha \backslash C_{\alpha-1}$, we have $\emptyset \neq \gamma\langle a\rangle \subseteq C_{\alpha-1}$. As stated above, the function $F_x(s_1^{*\alpha-1},\dots,s_n^{*\alpha-1})$ is defined on each $x \in \gamma\langle a\rangle$. We denote

$$T_a = \{F_x(s_1^{*\alpha-1},\dots,s_n^{*\alpha-1}) : x \in \gamma\langle a\rangle\},$$

i.e., T_a is the terminal state set of play whose initial state passes set $\gamma\langle a\rangle$ by Player j ($j = 1,\dots,n$) adopting strategy $s_j^{*\alpha-1}$. Obviously, T_a is the nonempty subset of the set of terminal state which can be reached from state a. If $a \in A_i$, we can calculate Player i's play payoff, the maximum of which is h_i^*, in every state of subgame T_a. When Player i chooses maximal play payoff h_i^* on state a, and the chosen state is x^*. We denote $s^*(a) = x^*$. Hence, the following relation is satisfied for each $x \in \gamma\langle a\rangle$:

$$F_x(s_1^{*\alpha-1},\dots,s_n^{*\alpha-1}) \leqslant^j F_{x^*}(s_1^{*\alpha-1},\dots,s_n^{*\alpha-1}). \tag{1}$$

By induction, the mapping s^* has been defined for each $a \in A$. If $\gamma\langle a\rangle \neq \emptyset$, then $s^*(a) \in \gamma\langle a\rangle$, i.e., $s^* \subseteq \gamma$. If s_j^* is the restriction of mapping s^* in A_j, then s_j^* is a simple strategy of Player j.

Finally, it is proved by induction for the rank of a_0 that situation $s^* = (s_1^*,\dots,s_n^*)$ is Nash equilibrium of every game $\Gamma_{a_0}(T)$, where $a_0 \in A$. Situation $s^* = (s_1^*,\dots,s_n^*)$ is called absolute equilibrium of games on graph $\langle A, \gamma\rangle$.

Step 1. Suppose $p(a_0) = 0$, i.e. $a_0 \in C_0 = A_f$. Now the value a_0 of function F_{a_0} is independent of the situation. Therefore, every situation is Nash equilibrium.

Step 2. If α is given, suppose that $p(a_0) = \alpha$ and the situation $(s_1^*,\dots,s_n^*)$ is Nash equilibrium of every game $\Gamma_x(T)$, where $p(x) < \alpha$. The initial state $a_0 \in A$ is chosen, we will prove that $(s_1^*,\dots,s_n^*)$ is Nash equilibrium of game $\Gamma_{a_0}(T)$. In fact, if $a_0 \in A_i$, suppose that Player j ($j = 1,\dots,n$) adopts the simple strategy s_j' instead of s_j^*. If $j \neq i$, considering $\gamma\langle a_0\rangle \subseteq C_{\alpha-1}$, we get the result. We only need to consider $j = i$. Denote $s_i^*(a_0) = a_1, s_i'(a_0) = a_1'$. By Formula (1), we have

$$F_{a_1'}(s_1^*,\dots,s_n^*) \leqslant^i F_{a_1}(s_1^*,\dots,s_n^*).$$

Because $a_1' \in \gamma\langle a_0\rangle \subseteq C_{\alpha-1}$, then according to the assumption mentioned above,

$$F_{a_1'}(s_1^*,\dots,s_{i-1}^*,s_i',s_{i+1}^*,\dots,s_n^*) \leqslant^i F_{a_1'}(s_1^*,\dots,s_n^*).$$

By the relation between the two formulas above, we have

$$F_{a_1'}(s_1^*,\dots,s_{i-1}^*,s_i',s_{i+1}^*,\dots,s_n^*) \leqslant^i F_{a_1}(s_1^*,\dots,s_n^*). \tag{2}$$

According to the definition of F_a and s^*, the following equations hold:

$$\begin{aligned}
&F_{a_0}(s_1^*,\dots,s_n^*) = F_{a_1}(s_1^*,\dots,s_n^*),\\
&F_{a_0}(s_1^*,\dots,s_{i-1}^*,s_i',s_{i+1}^*,\dots,s_n^*) = F_{a_1'}(s_1^*,\dots,s_{i-1}^*,s_i',s_{i+1}^*,\dots,s_n^*).
\end{aligned}$$

Considering Formula (2), we have

$$F_{a_0}(s_1^*, \dots, s_{i-1}^*, s_i', s_{i+1}^*, \dots, s_n^*) \leqslant^i F_{a_0}(s_1^*, \dots, s_n^*),$$

i.e., situation $(s_1^*, \dots, s_n^*)$ is Nash equilibrium in game $F_{a_0}(T)$. Because the initial state $a_0 \in A$ is chosen arbitrarily, situation $s^* = (s_1^*, \dots, s_n^*)$ is absolute equilibrium of games on graph $\langle A, \gamma\rangle$. The proof of the theorem is finished. □

4 Algorithm about absolute equilibrium in games with state payoff vector on connected graph

First, calculate the rank $p(a)$ of the state $a \in A$ on graph $\langle A, \gamma\rangle$ according to the definition of rank. Assume $\max_{a\in A} p(a) = K$. The set of state A on graph $\langle A, \gamma\rangle$ is split into $K+1$ subsets $P_0, P_1, \dots, P_K$, where P_k is the set of state whose rank equals k, $\bigcup_{k=0}^{K} P_k = A, P_l \cap P_m = \emptyset, l \neq m, 0 \leqslant l, m \leqslant K$. In the following, we will use backward induction according to the ranks of the state.

Step 0. Consider each state a_0 whose rank equals 0, i.e., $a_0 \in P_0 = C_0 = A_f$. Since nobody makes move here, by Definition 2, we have $h(a_0) = f_{a_0}$. Denote Bellman function [5] by $r_i^0 : P_0 \to \mathbb{R}$, where $r_i^0(a_0) = h_i(a_0)$, let $r^0(a_0) = (r_1^0(a_0), \dots, r_n^0(a_0))^{\mathrm{T}} = h(a_0)$, we denote $s^*(a_0) = a_0$.

Step 1. Consider each state $a_1 \in P_1$. Since $\gamma'\langle a_1\rangle \subseteq P_0$ for a_1, by Definition 2, we have $h(a_1, a_0) = f_{a_1} + r^0(a_0)$. Assuming $a_1 \in A_i$, then Player i chooses $\bar{a}_0 \in \gamma'\langle a_1\rangle$ satisfying $\max_{a_0\in\gamma'\langle a_1\rangle} h_i(a_1, a_0) = h_i(a_1, \bar{a}_0)$. Denote function by $r_i^1 : P_1 \to \mathbb{R}$, where $r_i^1(a_1) = h_i(a_1, \bar{a}_0)$ at a_1, and let $r^1(a_1) = (r_1^1(a_1), \dots, r_n^1(a_1))^{\mathrm{T}} = h(a_1, \bar{a}_0)$. Now we get $s_i^*(a_1) = \bar{a}_0$ at $a_1 \in P_1$.

Step 2. Consider each state $a_2 \in P_2$. Denote $\gamma'\langle a_2\rangle = Z_0\langle a_2\rangle \cup Z_1\langle a_2\rangle$, where $Z_0\langle a_2\rangle \subseteq P_0$ prescribes the set of state next to a_2 of rank 0, $Z_1\langle a_2\rangle \subseteq P_1$ prescribes the set of state next to a_2 of rank 1. In the following part, we use the similar prescription.

1) For the state in $Z_0\langle a_2\rangle \subseteq P_0$, when $a_0 \in \gamma'\langle a_2\rangle$, by Definition 2 we have $h(a_2, a_0) = f_{a_2} + r^0(a_0)$.

2) For the state in $Z_1\langle a_2\rangle \subseteq P_1$, when $a_1 \in \gamma'\langle a_2\rangle$ and $\bar{a}_0$ has been chosen on Step 1, by Definition 2 we have $h(a_2, a_1, \bar{a}_0) = f_{a_2} + r^1(a_1)$. Assuming that $a_2 \in A_i$, then Player i chooses the state $\bar{a}_1 \in \gamma'\langle a_2\rangle$ which can reach $\max\{\max_{a_0\in Z_0\langle a_2\rangle} h_i(a_2, a_0),\ \max_{a_1\in Z_1\langle a_2\rangle} h_i(a_2, a_1, \bar{a}_0)\}$. Also we denote function by $r_i^2 : P_2 \to \mathbb{R}$, where

$$r_i^2(a_2) = \begin{cases} h_i(a_2, \bar{a}_1), & \text{if} \quad \bar{a}_1 \in Z_0\langle a_2\rangle \subseteq P_0, \\ h_i(a_2, \bar{a}_1, \bar{a}_0), & \text{if} \quad \bar{a}_1 \in Z_1\langle a_2\rangle \subseteq P_1. \end{cases}$$

At a_2, let $r^2(a_2) = (r_1^2(a_2), \dots, r_n^2(a_2))^{\mathrm{T}}$. Now we get $s_i^*(a_2) = \bar{a}_1$ at $a_2 \in P_2$.

Step k. Consider each state $a_k \in P_k,\ k \leqslant K$. Now $r_i^0(a_0), \dots, r_i^{k-1}(a_{k-1})$ and $r^0(a_0), \dots, r^{k-1}(a_{k-1})$ have been defined. Denote

$$\gamma'\langle a_k\rangle = Z_0\langle a_k\rangle \cup Z_1\langle a_k\rangle \cup \cdots \cup Z_{k-1}\langle a_k\rangle,$$

where $Z_0\langle a_k\rangle \subseteq P_0 = A_f$, $Z_1\langle a_k\rangle \subseteq P_1, \dots, Z_{k-1}\langle a_k\rangle \subseteq P_{k-1}$.

1) For the state in $Z_0\langle a_k\rangle \subseteq P_0$, when $a_0 \in \gamma'\langle a_k\rangle$, by Definition 2 we have $h(a_k, a_0) = f_{a_k} + r^0(a_0)$.

2) For the state in $Z_1\langle a_k\rangle \subseteq P_1$, when $a_1 \in \gamma'\langle a_k\rangle$ and $\bar{a}_0$ has been chosen on Step 1. By Definition 2, we have $h(a_k, a_1, \bar{a}_0) = f_{a_k} + r^1(a_1)$.

$\vdots$

k) For the state in $Z_{k-1}\langle a_k\rangle \subseteq P_{k-1}$, when $a_{k-1} \in \gamma'\langle a_k\rangle$ and $\bar{a}_0, \bar{a}_1, \dots, \bar{a}_{k-2}$ have been chosen respectively on Steps $1, 2, \dots, k-1$, by Definition 2 we have $h(a_k, a_{k-1}, \bar{a}_{k-2}, \dots, \bar{a}_0) = f_{a_k} + r^{k-1}(a_{k-1})$. Assuming that $a_k \in A_i$, then Player i chooses $\bar{a}_{k-1} \in \gamma'\langle a_k\rangle$ which can reach

$$\max\Big\{\max_{a_0\in Z_0\langle a_k\rangle} h_i(a_k, a_0), \max_{a_1\in Z_1\langle a_k\rangle} h(a_k, a_1, \bar{a}_0), \dots, \max_{a_{k-1}\in Z_{k-1}\langle a_k\rangle} h(a_k, a_{k-1}, \bar{a}_{k-2}, \dots, \bar{a}_0)\Big\}.$$

Denote Bellman function by $r_i^k : P_k \to \mathbb{R}$, where

$$r_i^k(a_k) = \begin{cases} h_i(a_k, \bar{a}_{k-1}), & \text{if} \quad \bar{a}_{k-1} \in Z_0\langle a_k\rangle \subseteq P_0, \\ h_i(a_k, \bar{a}_{k-1}, \bar{a}_0), & \text{if} \quad \bar{a}_{k-1} \in Z_1\langle a_k\rangle \subseteq P_1, \\ \quad \vdots & \\ h_i(a_k, \bar{a}_{k-1}, \bar{a}_{k-2}, \dots, \bar{a}_0), & \text{if} \quad \bar{a}_{k-1} \in Z_{k-1}\langle a_k\rangle \subseteq P_{k-1}. \end{cases}$$

At a_k, let $r^k(a_k) = (r_1^k(a_k), \dots, r_n^k(a_k))^{\mathrm{T}}$. Now we get $s_i^*(a_k) = \bar{a}_{k-1}$ at $a_k \in P_k$.

Continue the process till $k = K$. For each K rank state $a_K \in P_K$, suppose $a_k \in A_i$, similarly, we get $s_i^*(a_K) = \bar{a}_{K-1}$.

Given all above, by the algorithm we get the player's choice for each state on connected graph with state payoff vector. According to the proof of Theorem in Section 3, when we arbitrarily choose the state $a_0 \in A$ to be the initial state on connected graph $\langle A, \gamma\rangle$ with state payoff vector, strategy $(s_1^*, \dots, s_n^*)$ which is independent of a_0 is Nash equilibrium of simple game $\Gamma_{a_0}(T)$, i.e., situation $s^* = (s_1^*, \dots, s_n^*)$ is the absolute equilibrium of the game. The equilibrium route is related to the initial state. When we choose the state $a_0 \in P_L, 0 \leqslant L \leqslant K$ and $a_0 \in A_i$ as the initial state, the absolute equilibrium s^* can define the equilibrium route of the simple game $\Gamma_{a_0}(T)$, and the payoff on equilibrium orbit (play) is $r^L(a_0) = (r_1^L(a_0), \dots, r_n^L(a_0))^{\mathrm{T}}$. We need to point out that, in the algorithm above, the definition of function $r^k(a)$ on some state $a \in P_k, 0 < k \leqslant K$ maybe more than one, so choose one of them randomly as defined. If this case does not happen, the absolute equilibrium of the game will be unique.

5 The calculation model of absolute equilibrium about games on three-dimensional connected graph with state payoff vector

Consider the three-dimensional connected graph $\langle A, \gamma\rangle$ (Figure 1), where the set of players is $N = \{1, 2, 3\}$. The set of terminal states is $A_f = \{b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}, b_{11}, b_{12}, b_{13}, b_{14}\}$. The decision state sets of Player 1, Player 2, Player 3 are respectively $A_1 = \{a_{000}, a_{110}, a_{020}, a_{121}, a_{011}, a_{002}\}$, $A_2 = \{a_{100}, a_{012}, a_{021}, a_{111}, a_{122}, a_{102}\}$, $A_3 = \{a_{010}, a_{120}, a_{001}, a_{101}, a_{112}, a_{022}\}$.

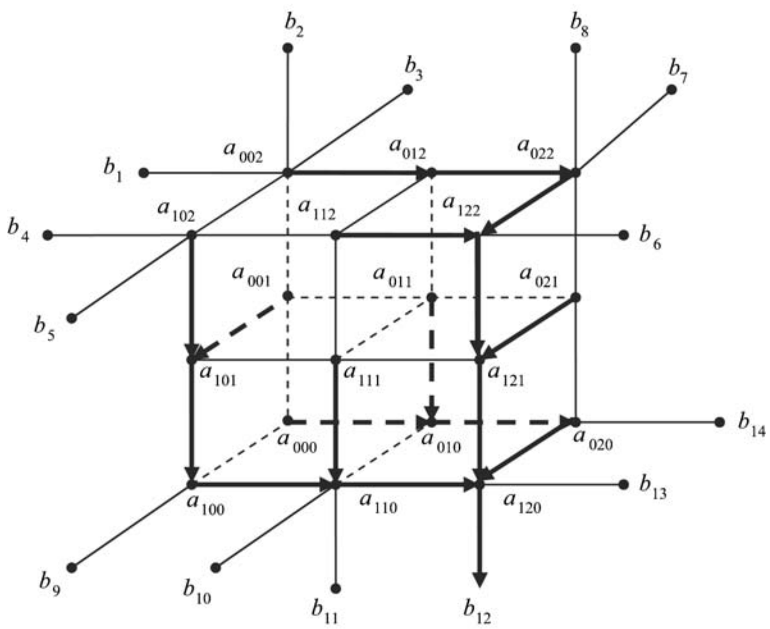

Figure 1 The three-dimensional connected graph $\langle A, \gamma\rangle$

Players' strategy. On graph $\langle A, \gamma\rangle$, we define that the players' strategies in horizontal and two-dimensional state are along the mesh to the right, the left or terminal state; the strategies of players in right-and-left and two-dimensional state are along the mesh to downward, the front, or terminal state; the strategies of players in fore-and-aft and two-dimensional state are along the mesh to the right, downward or terminal state.

By Definition 1, we give the state payoff vector on every state on graph $\langle A, \gamma\rangle$. Then we get the game on the three-dimensional mesh-like and connected graph with state payoff vector $T = \langle A, \gamma; A_1, A_2, A_3, A_f; f_{a\in A}\rangle$. This example gives the state payoff vector as follows:

$$\begin{aligned}
&f_{a_{000}} = (1,2,2)^{\mathrm{T}}, \quad f_{a_{010}} = (2,1,3)^{\mathrm{T}}, \quad f_{a_{020}} = (4,2,1)^{\mathrm{T}}, \quad f_{a_{001}} = (5,3,2)^{\mathrm{T}},\\
&f_{a_{011}} = (2,2,2)^{\mathrm{T}}, \quad f_{a_{021}} = (1,3,2)^{\mathrm{T}}, \quad f_{a_{002}} = (2,4,5)^{\mathrm{T}}, \quad f_{a_{012}} = (6,5,2)^{\mathrm{T}},\\
&f_{a_{022}} = (1,5,3)^{\mathrm{T}}, \quad f_{a_{100}} = (3,6,7)^{\mathrm{T}}, \quad f_{a_{110}} = (2,5,1)^{\mathrm{T}}, \quad f_{a_{120}} = (5,7,4)^{\mathrm{T}},\\
&f_{a_{101}} = (2,5,4)^{\mathrm{T}}, \quad f_{a_{111}} = (4,0,2)^{\mathrm{T}}, \quad f_{a_{121}} = (3,3,4)^{\mathrm{T}}, \quad f_{a_{102}} = (4,6,3)^{\mathrm{T}},\\
&f_{a_{112}} = (0,5,3)^{\mathrm{T}}, \quad f_{a_{122}} = (3,2,4)^{\mathrm{T}}, \quad f_{b_1} = (4,3,2)^{\mathrm{T}}, \quad f_{b_2} = (2,5,4)^{\mathrm{T}},\\
&f_{b_3} = (3,4,5)^{\mathrm{T}}, \quad f_{b_4} = (4,6,3)^{\mathrm{T}}, \quad f_{b_5} = (2,5,3)^{\mathrm{T}}, \quad f_{b_6} = (4,2,3)^{\mathrm{T}}, \quad f_{b_7} = (3,2,6)^{\mathrm{T}},\\
&f_{b_8} = (4,3,1)^{\mathrm{T}}, \quad f_{b_9} = (3,5,7)^{\mathrm{T}}, \quad f_{b_{10}} = (4,3,5)^{\mathrm{T}}, \quad f_{b_{11}} = (2,6,5)^{\mathrm{T}}, \quad f_{b_{12}} = (3,2,4)^{\mathrm{T}},\\
&f_{b_{13}} = (2,4,1)^{\mathrm{T}}, \quad f_{b_{14}} = (5,3,2)^{\mathrm{T}}.
\end{aligned}$$

First, compute the ranks of all states. We get the rank-subdivision of the state set A on the graph $\langle A, \gamma\rangle$:

$$\begin{aligned}
&P_0 = \{b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}, b_{11}, b_{12}, b_{13}, b_{14}\}, \quad P_1 = \{a_{120}\},\\
&P_2 = \{a_{110}, a_{121}, a_{020}\}, \quad P_3 = \{a_{100}, a_{111}, a_{010}, a_{122}, a_{021}\},\\
&P_4 = \{a_{101}, a_{112}, a_{000}, a_{011}, a_{022}\}, \quad P_5 = \{a_{102}, a_{001}, a_{012}\}, \quad P_6 = \{a_{002}\}.
\end{aligned}$$

Step 0. Consider every state in P_0. According to the algorithm, we denote

$$r^0(b_j) = h(b_j), \quad j = 1, \dots, 14,$$

and define $s^*(b_j) = b_j, j = 1, \dots, 14$.

Step 1. Consider every state in $P_1 = \{a_{120}\}$. According to the algorithm, we get $h(a_{120}, b_{12}) = f_{a_{120}} + r^0(b_{12}) = (8,9,8)^{\mathrm{T}}$, $h(a_{120}, b_{13}) = (7,11,5)^{\mathrm{T}}$. Since $a_{120} \in A_3$, $h_3(a_{120}, b_{12}) = 8 > 5 = h_3(a_{120}, b_{13})$, Player 3 will choose b_{12}. We denote

$$r^1(a_{120}) = (r_1^1(a_{120}), r_2^1(a_{120}), r_3^1(a_{120}))^{\mathrm{T}} = h(a_{120}, b_{12}) = (8,9,8)^{\mathrm{T}}.$$

Similarly we get $s_3^*(a_{120}) = b_{12}$.

Step 2. Consider every state in $P_2 = \{a_{110}, a_{121}, a_{020}\}$. For $a_{110} \in P_2$, we have $\gamma'\langle a_{110}\rangle = Z_0\langle a_{110}\rangle \cup Z_1\langle a_{110}\rangle$, where $Z_0\langle a_{110}\rangle = \{b_{10}, b_{11}\}$, $Z_1\langle a_{110}\rangle = \{a_{120}\}$. We get

$$\begin{aligned}
&h(a_{110}, b_{10}) = (6,8,6)^{\mathrm{T}}, \quad h(a_{110}, b_{11}) = (4,11,6)^{\mathrm{T}},\\
&h(a_{110}, a_{120}, b_{12}) = f_{a_{110}} + r^1(a_{120}) = (10,14,9)^{\mathrm{T}}.
\end{aligned}$$

For $a_{110} \in A_1$, and

$$\max\{h_1(a_{110}, b_{10}), h_1(a_{110}, b_{11}), h_1(a_{110}, a_{120}, b_{12})\} = 10 = h_1(a_{110}, a_{120}, b_{12}),$$

so Player 1 will choose a_{120}, denote

$$r^2(a_{110}) = (r_1^2(a_{110}), r_2^2(a_{110}), r_3^2(a_{110}))^{\mathrm{T}} = h(a_{110}, a_{120}, b_{12}) = (10,14,9)^{\mathrm{T}}.$$

Then we get $s_1^*(a_{110}) = a_{120}$.

Similarly, for $a_{121} \in P_2$, since $a_{121} \in A_1$, and Player 1 has the only choice a_{120}, we denote

$$r^2(a_{121}) = (r_1^2(a_{121}), r_2^2(a_{121}), r_3^2(a_{121}))^{\mathrm{T}} = h(a_{121}, a_{120}, b_{12}) = (11, 12, 12)^{\mathrm{T}},$$

and get $s_1^*(a_{121}) = a_{120}$.

For $a_{020} \in P_2$, we have $\gamma'\langle a_{020}\rangle = Z_0\langle a_{020}\rangle \cup Z_1\langle a_{020}\rangle$, where $Z_0\langle a_{020}\rangle = \{b_{14}\}$, $Z_1\langle a_{020}\rangle = \{a_{120}\}$. Since $a_{020} \in A_1, h_1(a_{020}, b_{14}) = 9 < 12 = h_1(a_{020}, a_{120}, b_{12})$, Player 1 will also choose a_{120}, denote

$$r^2(a_{020}) = (r_1^2(a_{020}), r_2^2(a_{020}), r_3^2(a_{020}))^{\mathrm{T}} = h(a_{020}, a_{120}, b_{12}) = (12, 11, 9)^{\mathrm{T}}.$$

Then we get $s_1^*(a_{020}) = a_{120}$.

Step 3. Consider every state in $P_3 = \{a_{100}, a_{111}, a_{010}, a_{122}, a_{021}\}$. The results are given as follows:

$$r^3(a_{100}) = (13, 20, 16)^{\mathrm{T}}, \quad s_2^*(a_{100}) = a_{110}, \quad r^3(a_{111}) = (14, 14, 11)^{\mathrm{T}}, \quad s_2^*(a_{111}) = a_{110},$$

$$r^3(a_{122}) = (14, 14, 16)^{\mathrm{T}}, \quad s_2^*(a_{122}) = a_{121}, \quad r^3(a_{021}) = (12, 15, 14)^{\mathrm{T}}, \quad s_2^*(a_{021}) = a_{121}.$$

For $a_{010} \in P_3$, we have $\gamma'\langle a_{010}\rangle = Z_0\langle a_{010}\rangle \cup Z_1\langle a_{010}\rangle \cup Z_2\langle a_{010}\rangle$, where $Z_0\langle a_{010}\rangle = Z_1\langle a_{010}\rangle = \emptyset$, $Z_2\langle a_{010}\rangle = \{a_{020}, a_{110}\}$, and $h_3(a_{010}, a_{020}, a_{120}, b_{12}) = 12 = h_3(a_{010}, a_{110}, a_{120}, b_{12})$, by the assumption, Player 3 can arbitrarily choose a_{110} or a_{020}. Then if Player 3 chooses state a_{020}, denote $r^3(a_{010}) = (14, 12, 12)^{\mathrm{T}}$. Then $s_3^*(a_{010}) = a_{020}$.

Step 4. Consider every state in $P_4 = \{a_{101}, a_{112}, a_{000}, a_{011}, a_{022}\}$. The results are

$$r^4(a_{101}) = (15, 25, 20)^{\mathrm{T}}, \quad s_3^*(a_{101}) = a_{100}, \quad r^4(a_{112}) = (14, 19, 19)^{\mathrm{T}}, \quad s_3^*(a_{112}) = a_{122},$$
$$r^4(a_{000}) = (15, 14, 14)^{\mathrm{T}}, \quad s_1^*(a_{000}) = a_{010}, \quad r^4(a_{011}) = (16, 14, 14)^{\mathrm{T}}, \quad s_1^*(a_{011}) = a_{010}$$
$$r^4(a_{022}) = (15, 19, 19)^{\mathrm{T}}, \quad s_3^*(a_{022}) = a_{122}.$$

Step 5. Consider every state in $P_5 = \{a_{102}, a_{001}, a_{012}\}$. The results are

$$r^5(a_{102}) = (19, 31, 23)^{\mathrm{T}}, \quad s_2^*(a_{102}) = a_{101}, \quad r^5(a_{001}) = (20, 28, 22)^{\mathrm{T}}, \quad s_3^*(a_{001}) = a_{101}.$$

For $a_{012} \in P_5$, since $a_{012} \in A_2$, and $h_2(a_{012}, a_{022}, a_{122}, a_{121}, a_{120}, b_{12}) = 24 = h_2(a_{012}, a_{112}, a_{122}, a_{121}, a_{120}, b_{12})$, Player 2 can arbitrarily choose a_{112} or a_{022}. If Player 2 chooses state a_{022}, denote $r^5(a_{012}) = (21, 24, 21)^{\mathrm{T}}$. Then $s_2^*(a_{012}) = a_{022}$.

Step 6. Now consider the unique state of rank 6, $a_{002} \in P_6$. We have

$$\gamma'\langle a_{002}\rangle = Z_0\langle a_{002}\rangle \cup Z_1\langle a_{002}\rangle \cup Z_2\langle a_{002}\rangle \cup Z_3\langle a_{002}\rangle \cup Z_4\langle a_{002}\rangle \cup Z_5\langle a_{002}\rangle,$$

where $Z_0\langle a_{002}\rangle = \{b_1, b_2, b_3\}, Z_1\langle a_{002}\rangle = Z_2\langle a_{002}\rangle = Z_3\langle a_{002}\rangle = Z_4\langle a_{002}\rangle = \emptyset, Z_5\langle a_{002}\rangle = \{a_{102}, a_{001}, a_{012}\}$. For $a_{002} \in A_1$, noting $r^6(a_{002}) = (23, 28, 26)^{\mathrm{T}}$, we get $s_1^*(a_{002}) = a_{012}$.

Finally we get the absolute equilibrium of the game, which is noted by bold black line in Figure 1.

6 Conclusion

The result of this paper is valid for the random finite connected graph, no matter it is two dimension or three dimension (where the dimension is usually in the sense of Euclid space). Of course the result includes the game tree we have known. We choose the three-dimensional mesh-like graph in this paper only because the initial inspiration comes from the exception for carrying out game's research on three-dimensional space. Moreover, on usual dynamic games, usually the absolute equilibrium is perfect equilibrium. But from a different aspect, actually the absolute equilibrium is stronger than the perfect equilibrium.

The inductive method of the rank of state on the graph used in this paper will show its power on the research of game theory on complex graph. For simple structure graph, by listing all likely appeared play, we can always transform the game graph to the game tree. Then we can solve it by common method

[4]. However, considering games on the three-dimensional connected graph, the complex computation led by the large amounts of play can hinder and conceal some important research for the nature of game [2]. We have reasons to believe that the arithmetic of the absolute equilibrium about the game for limited connected graph established by this paper can be popularized to partial cooperation game of finite three-dimensional graph with variable coalitional structure and so on.

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