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Error estimate of the homogenization solution for elliptic problems with small periodic coefficients on $L^\infty(\Omega)$

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Abstract In this paper, we discuss the multi-scale homogenization theory for the second order elliptic problems with small periodic coefficients of the form $\frac{\partial}{\partial x_i}(a^{ij}(\frac{x}{\varepsilon})\frac{\partial u^\varepsilon(x)}{\partial x_j}) = f(x)$. Assuming $n = 2$ and $u^0 \in W^{1,\infty}(\Omega)$, we present an error estimate between the homogenization solution $u^0(x)$ and the exact solution $u^\varepsilon(x)$ on the Sobolev space $L^\infty(\Omega)$.

Keywords multi-scale homogenization theory, homogenization solution, second order elliptic equations with small periodic coefficients

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1 Introduction

Composite materials have been widely used in high technology engineering as well as ordinary industrial products since they have many elegant qualities, such as high strength, high stiffness, high temperature resistance, corrosion resistance and fatigue resistance. Most of the composite materials have small periodic configurations. Thus, the static analysis of the structures of composite materials usually leads to the boundary value problems of elliptic partial differential equations with small periodic coefficients. Solving these problems by classical finite element methods is very difficult because it usually requires very fine meshes and this leads to tremendous amount of computer memory and CPU time.

In order to solve this kind of problem, the multi-scale methods, which is thoroughly described in numerous sources [1–25], is introduced. It couples macroscopic scale and microscopic scale together, and it reflects not only global mechanical and physical properties, but also the effect of micro-configuration of composite material.

Consider the following multi-scale elliptic model problem:

$$\begin{cases} L_\varepsilon u^\varepsilon(x) \equiv \dfrac{\partial}{\partial x_i}\left(a^{ij}\left(\dfrac{x}{\varepsilon}\right)\dfrac{\partial u^\varepsilon(x)}{\partial x_j}\right) = f(x), & \text{in} \quad \Omega, \\ u^\varepsilon(x) = g(x), & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

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where Ω is a smooth bounded domain contained in $\mathbb{R}^n$ ($n \geqslant 1$) and the matrix of coefficients $a^{ij}(\frac{x}{\varepsilon}): \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is symmetric and satisfies the following conditions: $\exists\, \Gamma_0 > 0, \Gamma_1 > 0,$ such that $\forall\, \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$,

$$(A_1) \quad a_{ij}(\xi) \text{ is 1-periodic in } \xi,$$

$$(A_2) \quad \Gamma_0 \sum_{i=1}^n \xi_i^2 \leqslant \sum_{i=1}^n \sum_{j=1}^n a^{ij}(\xi)\xi_i\xi_j \leqslant \Gamma_1 \sum_{i=1}^n \xi_i^2. \tag{2}$$

Oleinik et al. [18, 22] presented the following multi-scale homogenization method for (1):

First let $N^k(\xi), 1 \leqslant k \leqslant n$, be the weak solution of

$$\begin{cases} \dfrac{\partial}{\partial \xi_i}\left(a^{ij}(\xi)\dfrac{\partial N^k(\xi)}{\partial \xi_j}\right) = -\dfrac{\partial}{\partial \xi_i} a^{ik}(\xi), \quad \text{in} \quad \mathbb{R}^n, \\ N^k(\xi) \text{ is a 1-periodic function, } \displaystyle\int_Q N^k(\xi)d\xi = 0, \end{cases} \tag{3}$$

where $Q = \{\xi | 0 < \xi_i < 1, 1 \leqslant i \leqslant n\}$. Then, define the matrix of coefficients $\hat{a} = (\hat{a}^{ij})$ by

$$\hat{a}^{ij} = \int_Q \left(a^{ij}(\xi) + a^{ik}(\xi)\frac{\partial N^j(\xi)}{\partial \xi_k}\right) d\xi. \tag{4}$$

Finally, we have an approximation to the weak solution for (1) by

$$\tilde{u}(x) = u^0(x) + \varepsilon N^k\left(\frac{x}{\varepsilon}\right)\frac{\partial u^0(x)}{\partial x_k}, \tag{5}$$

where $u^0(x)$ satisfies the following homogenization problem:

$$\begin{cases} L_0 u^0(x) \equiv \dfrac{\partial}{\partial x_i}\left(\hat{a}^{ij}\dfrac{\partial u^0(x)}{\partial x_j}\right) = f(x), \quad \text{in} \quad \Omega, \\ u^0(x) = g(x), \qquad\qquad\qquad\qquad \text{on} \quad \partial\Omega. \end{cases} \tag{6}$$

Remark 1.1. $\tilde{u}(x)$ is named as the 1-order approximation of $u^\varepsilon(x)$.

The following lemma, which was obtained by Jikov et al. [18, p. 28], gives an error estimate between the solutions $\tilde{u}(x)$ and $u^\varepsilon(x)$.

Lemma 1.1. *Under the assumption that $u^0 \in H^2(\Omega)$, there exists a constant c such that*

$$\|u^\varepsilon - \tilde{u} - \theta_\varepsilon\|_{H^1(\Omega)} \leqslant c\varepsilon \|u^0\|_{H^2(\Omega)}, \tag{7}$$

where θ_ε satisfies

$$\begin{cases} L_\varepsilon \theta_\varepsilon(x) = 0, & \text{in} \quad \Omega, \\ \theta_\varepsilon(x) = -\varepsilon N^k\left(\dfrac{x}{\varepsilon}\right)\dfrac{\partial u^0(x)}{\partial x_k}, & \text{on} \quad \partial\Omega. \end{cases} \tag{8}$$

From (7) we have

$$\|u^\varepsilon - u^0\|_{L^2(\Omega)} \leqslant c\varepsilon \|u^0\|_{H^2(\Omega)}. \tag{9}$$

There are many works (see [6–12, 14, 16, 17, 19, 21, 24, 25] etc.) to discuss the numerical methods of the multi-scale homogenization problem and Lemma 1.1 is the theory basis of them when they discuss the problem (1). We observe that there exists a problem whether we need $u^0 \in W^{2,p}(\Omega)$ when we only discuss the accuracy of displacement of u^0. In this paper, using the homogenization theory of Green function for (1), we propose a new approach to analyze the accuracy of displacement of u^0 and have the following result.

Theorem 1.1. *Assume that $n = 2, \partial\Omega$ is smooth enough and u^0 belongs to the Sobolev space $W^{1,\infty}(\Omega)$. Then there exists a constant c such that*

$$\|u^\varepsilon - u^0\|_{L^\infty(\Omega)} \leqslant c\varepsilon|\ln\varepsilon|^2\|u^0\|_{W^{1,\infty}(\Omega)}. \tag{10}$$

Remark 1.2. Theorem 1.1 shows that $u^0 \in W^{2,p}(\Omega)$ is not necessary when we only discuss the accuracy of displacement of u^0.

The rest of this paper is organized as follows: Assume that $G^\varepsilon_{x_0}(x)$ is the Green function of (1) in point x_0 and $\tilde{G}_{x_0}(x)$ is the 1-order approximation of $G^\varepsilon_{x_0}(x)$, $\theta^\varepsilon_G(x)$ is the boundary corrector of $G^\varepsilon_{x_0}(x)$ and is defined by

$$\begin{cases} L_\varepsilon\theta^\varepsilon_G(x) = 0, & \text{in} \quad \Omega, \\ \theta^\varepsilon_G(x) = -\varepsilon N^k\left(\dfrac{x}{\varepsilon}\right)\dfrac{\partial G^0_{x_0}(x)}{\partial x_k}, & \text{on} \quad \partial\Omega. \end{cases} \tag{11}$$

Assume that $B(x_0, d) = \{x \in \Omega|\mathrm{dist}(x, x_0) \leqslant d\}$. We present an error estimate between the solutions $\tilde{G}_{x_0}(x) + \theta^\varepsilon_G(x)$ and $G^\varepsilon_{x_0}(x)$ on $W^{1,1}(\Omega - B(x_0, \varepsilon))$ in Section 2. Section 3 gives a proof of Theorem 1.1 on the basis of Section 2.

Remark 1.3. In this paper, we establish some notations and conventions. In the following, the Einstein summation convention is used: summation is taken over repeated indices. Throughout this paper, we let Ω be a bounded smooth domain in the n-dimensional space, $Q = \{\xi|0 < \xi_i < 1, i = 1, \ldots, n\}$, $x_0 \in \Omega, B(x_0, d) = \{x \in \Omega|\mathrm{dist}(x, x_0) \leqslant d\}$ and $\mathrm{dist}(x, \partial\Omega)$ denotes the distance between x and $\partial\Omega$. We also assume that c (with or without subscripts) denotes constants not necessarily the same at each occurrence, but always independent of ε and d.

2 Estimate for $\|\tilde{G}_{x_0} + \theta^\varepsilon_G - G^\varepsilon_{x_0}\|_{W^{1,1}(\Omega - B(x_0,\varepsilon))}$

In this section, our main result is as follows.

Theorem 2.1. *There exists a constant c such that*

$$\|G^\varepsilon_{x_0} - \tilde{G}_{x_0} - \theta^\varepsilon_G\|_{W^{1,1}(\Omega - B(x_0,\varepsilon))} \leqslant c\varepsilon|\ln\varepsilon|^2. \tag{12}$$

For proving Theorem 2.1, we need to introduce some lemmas. The following result is presented in [6].

Lemma 2.1. *There exists a constant c such that*

$$|G^\varepsilon_{x_0}(x)| \leqslant \begin{cases} c|\ln|x - x_0||, & n = 2, \\ c|x - x_0|^{2-n}, & n = 3. \end{cases} \tag{13}$$

Using Lemma 2.1, we present a local estimate for $\nabla G^\varepsilon_{x_0}(x)$ as follows.

Lemma 2.2. *Assume that $d > 0$. Then there exists a constant c such that*

$$\|\nabla G^\varepsilon_{x_0}\|_{L^2(\Omega - B(x_0,d))} \leqslant cd^{\frac{2-n}{2}}|\ln d|, \tag{14}$$

and

$$\|\nabla G^\varepsilon_{x_0}\|_{L^1(B(x_0,d))} \leqslant cd|\ln d|. \tag{15}$$

Proof. For simplicity, assume that $\|\nabla G^\varepsilon_{x_0}\|_{L^1(\partial B(x_0,d))}$ satisfies

$$\|\nabla G^\varepsilon_{x_0}\|_{L^1(\partial B(x_0,d))} \leqslant cd^{-1}\|\nabla G^\varepsilon_{x_0}\|_{L^1(B(x_0,2d) - B(x_0,d))},$$

which implies

$$\|\nabla G^\varepsilon_{x_0}\|_{L^1(\partial B(x_0,d))} \leqslant cd^{-1}d^{\frac{n}{2}}\|\nabla G^\varepsilon_{x_0}\|_{L^2(B(x_0,2d) - B(x_0,d))} \leqslant cd^{\frac{n-2}{2}}\|\nabla G^\varepsilon_{x_0}\|_{L^2(\Omega - B(x_0,d))}. \tag{16}$$

Assume that n_i means the normal derivative in the x_i direction. Note that

$$\int_{\Omega-B(x_0,d)} a^{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial G^{\varepsilon}_{x_0}(x)}{\partial x_i}\frac{\partial G^{\varepsilon}_{x_0}(x)}{\partial x_j}dx=\left|\int_{\partial B(x_0,d)} a^{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial G^{\varepsilon}_{x_0}(x)}{\partial x_j}G^{\varepsilon}_{x_0}(x)n_i ds\right|. \tag{17}$$

The combination of (13), (16) and (17) shows

$$\begin{aligned}(\|\nabla G^{\varepsilon}_{x_0}\|_{L^2(\Omega-B(x_0,d))})^2 &\leqslant c\int_{\Omega-B(x_0,d)} a^{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial G^{\varepsilon}_{x_0}(x)}{\partial x_i}\frac{\partial G^{\varepsilon}_{x_0}(x)}{\partial x_j}dx\\ &= c\left|\int_{\partial B(x_0,d)} a^{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial G^{\varepsilon}_{x_0}(x)}{\partial x_j}G^{\varepsilon}_{x_0}(x)n_i ds\right| \leqslant cd^{2-n}|\ln d|\|\nabla G^{\varepsilon}_{x_0}\|_{L^1(\partial B(x_0,d))}\\ &\leqslant cd^{2-n}|\ln d|\cdot d^{\frac{n-2}{2}}\|\nabla G^{\varepsilon}_{x_0}\|_{L^2(\Omega-B(x_0,d))}\\ &\leqslant cd^{\frac{2-n}{2}}|\ln d|\|\nabla G^{\varepsilon}_{x_0}\|_{L^2(\Omega-B(x_0,d))}.\end{aligned} \tag{18}$$

This relation implies the desired result (14).

Now we consider (15). Assume that $d_0=d$ and $d_k=\frac{d}{2^k}$ $(k\geqslant 1, k\in N)$. (14) gives

$$\begin{aligned}\|\nabla G^{\varepsilon}_{x_0}\|_{L^1(B(x_0,d))} &= \sum_{k=0}^{\infty}\|\nabla G^{\varepsilon}_{x_0}\|_{L^1(B(x_0,d_k)-B(x_0,d_{k+1}))} \leqslant c\sum_{k=0}^{\infty} d_k^{\frac{n}{2}}\|\nabla G^{\varepsilon}_{x_0}\|_{L^2(B(x_0,d_k)-B(x_0,d_{k+1}))}\\ &\leqslant c\sum_{k=0}^{\infty} d_k^{\frac{n}{2}} d_k^{\frac{2-n}{2}}|\ln d_k| \leqslant c\sum_{k=0}^{\infty} d_k|\ln d_k| \leqslant cd|\ln d|.\end{aligned}$$

This ends the proof. □

Now we discuss a local estimate for elliptic problems with constant coefficients.

Lemma 2.3. *Let $x_0\in\Omega$ and $f^{\varepsilon}(x)$ satisfy $f^{\varepsilon}(x)=0$ if $x\in B(x_0,d)$. Assume that $\omega_{\varepsilon}(x)$ satisfies the following problem*

$$\begin{cases}\dfrac{\partial}{\partial x_i}\left(\hat{a}^{ij}\dfrac{\partial \omega_{\varepsilon}(x)}{\partial x_j}\right)=f^{\varepsilon}(x), & x\in\Omega,\\ \omega_{\varepsilon}(x)=0, & x\in\partial\Omega.\end{cases} \tag{19}$$

Then there exists a constant c such that

$$\|D^2\omega_{\varepsilon}\|_{L^{\infty}(B(x_0,\frac{d}{2}))} \leqslant cd^{\frac{-n}{2}}\|f^{\varepsilon}\|_{L^2(\Omega-B(x_0,d))}. \tag{20}$$

Proof. Assume that $y\in B(x_0,\frac{d}{2})$ and $G^0_y(x)$ is the Green function of (19) in point y. Note that

$$\omega^0_{\varepsilon}(y)=\int_{\Omega} G^0_y(x)f^{\varepsilon}(x)dx, \quad \omega^0_{\varepsilon}(y+\Delta y)=\int_{\Omega} G^0_{y+\Delta y}(x)f^{\varepsilon}(x)dx. \tag{21}$$

One finds that

$$\frac{\partial\omega^0_{\varepsilon}(y)}{\partial y_k}=\int_{\Omega}\frac{\partial G^0_y(x)}{\partial y_k}f^{\varepsilon}(x)dx. \tag{22}$$

Similarly,

$$\frac{\partial\omega^0_{\varepsilon}(y+\Delta y)}{\partial y_k}=\int_{\Omega}\frac{\partial G^0_{y+\Delta y}(x)}{\partial y_k}f^{\varepsilon}(x)dx. \tag{23}$$

The combination of (22) and (23) yields

$$\frac{\partial^2\omega^0_{\varepsilon}(y)}{\partial y_k\partial y_l}=\int_{\Omega}\frac{\partial^2 G^0_y(x)}{\partial y_k\partial y_l}f^{\varepsilon}(x)dx=\int_{\Omega-B(x_0,d)}\frac{\partial^2 G^0_y(x)}{\partial y_k\partial y_l}f^{\varepsilon}(x)dx. \tag{24}$$

Consequently,

$$|D^2\omega_\varepsilon^0(y)| \leqslant \int_{\Omega-B(y,\frac{d}{2})}\left|\frac{\partial^2 G_y^0(x)}{\partial y_k\partial y_l}f^\varepsilon(x)\right|dx \leqslant \left\|\frac{\partial^2 G_y^0}{\partial y_k\partial y_l}\right\|_{L^2(\Omega-B(y,\frac{d}{2}))}\|f^\varepsilon\|_{L^2(\Omega-B(y,\frac{d}{2}))}$$
$$\leqslant cd^{\frac{-n}{2}}\|f^\varepsilon\|_{L^2(\Omega-B(y,\frac{d}{2}))} = cd^{\frac{-n}{2}}\|f^\varepsilon\|_{L^2(\Omega-B(x_0,d))}. \tag{25}$$

This ends the proof. □

Assume that $d>0$ and $E_d=\{x\in\Omega|\mathrm{dist}(x,\partial\Omega)\leqslant d\}-B(x_0,d)$. Now we are going to estimate $\|G_{x_0}^\varepsilon-\tilde{G}_{x_0}-\theta_G^\varepsilon\|_{L^2(\Omega-B(x_0,d)-E_d)}$.

Lemma 2.4. *There exists a constant c such that*

$$\|G_{x_0}^\varepsilon-\tilde{G}_{x_0}-\theta_G^\varepsilon\|_{L^2(\Omega-B(x_0,d)-E_d)} \leqslant c\varepsilon d^{\frac{2-n}{2}}|\ln d|. \tag{26}$$

Proof. Assume that $k\geqslant 1$ ($k\in N$) and $\Omega_{kd}=\Omega-B(x_0,kd)-E_{kd}$. Define $\phi(x)$ by

$$\begin{aligned}
&0\leqslant\phi(x)\leqslant 1, \quad x\in\Omega,\\
&\phi(x)=1, \quad x\in\Omega_{2d},\\
&\phi(x)=0, \quad x\in\Omega-\Omega_d,\\
&\phi\in C^\infty(\Omega), \quad |D^l\phi(x)|\leqslant c_l d^{-l}, \quad x\in\Omega.
\end{aligned}$$

We introduce $v^\varepsilon(x), v_\varepsilon^0(x)$ and $\tilde{v}_\varepsilon(x)$ by the following (27)–(29),

$$\begin{cases} L_\varepsilon v^\varepsilon(x)\equiv\dfrac{\partial}{\partial x_i}\left(a^{ij}\left(\dfrac{x}{\varepsilon}\right)\dfrac{\partial v^\varepsilon(x)}{\partial x_j}\right)=\phi(x)[G_{x_0}^\varepsilon(x)-\tilde{G}_{x_0}(x)-\theta_G^\varepsilon(x)], & \text{in} \quad \Omega,\\ v^\varepsilon(x)=0, & \text{on} \quad \partial\Omega, \end{cases} \tag{27}$$

$$\begin{cases} L_0 v_\varepsilon^0(x)\equiv\dfrac{\partial}{\partial x_i}\left(\hat{a}^{ij}\dfrac{\partial v_\varepsilon^0(x)}{\partial x_j}\right)=\phi(x)[G_{x_0}^\varepsilon(x)-\tilde{G}_{x_0}(x)-\theta_G^\varepsilon(x)], & \text{in} \quad \Omega,\\ v_\varepsilon^0(x)=0, & \text{on} \quad \partial\Omega, \end{cases} \tag{28}$$

$$\tilde{v}_\varepsilon(x)=v_\varepsilon^0(x)+\varepsilon N^k\left(\frac{x}{\varepsilon}\right)\frac{\partial v_\varepsilon^0(x)}{\partial x_k}. \tag{29}$$

Note that

$$\left.\frac{\partial v_\varepsilon^0(y)}{\partial y_k}\right|_{y=x_0}=\int_\Omega\left.\frac{\partial G_y^0(x)}{\partial y_k}\right|_{y=x_0}\phi(x)[G_{x_0}^\varepsilon(x)-\tilde{G}_{x_0}(x)-\theta_G^\varepsilon(x)]dx.$$

One observes that $\int_\Omega[G_{x_0}^\varepsilon-\tilde{G}_{x_0}-\theta_G^\varepsilon](x)\phi(x)[G_{x_0}^\varepsilon-\tilde{G}_{x_0}-\theta_G^\varepsilon](x)dx$ can be split into

$$\begin{aligned}
&\int_\Omega[G_{x_0}^\varepsilon-\tilde{G}_{x_0}-\theta_G^\varepsilon](x)\phi(x)[G_{x_0}^\varepsilon-\tilde{G}_{x_0}-\theta_G^\varepsilon](x)dx\\
&\quad=\int_\Omega G_{x_0}^\varepsilon(x)\phi(x)[G_{x_0}^\varepsilon-\tilde{G}_{x_0}-\theta_G^\varepsilon](x)dx-\int_\Omega G_{x_0}^0(x)\phi(x)[G_{x_0}^\varepsilon-\tilde{G}_{x_0}-\theta_G^\varepsilon](x)dx\\
&\qquad-\int_\Omega\varepsilon N^k\left(\frac{x}{\varepsilon}\right)\frac{\partial G_{x_0}^0(x)}{\partial x_k}\phi(x)[G_{x_0}^\varepsilon-\tilde{G}_{x_0}-\theta_G^\varepsilon](x)dx-\int_\Omega\theta_G^\varepsilon(x)\phi(x)[G_{x_0}^\varepsilon-\tilde{G}_{x_0}-\theta_G^\varepsilon](x)dx\\
&\quad=(v^\varepsilon-\tilde{v}_\varepsilon)(x_0)-\int_\Omega\theta_G^\varepsilon(x)\phi(x)[G_{x_0}^\varepsilon-\tilde{G}_{x_0}-\theta_G^\varepsilon](x)dx\\
&\qquad+\int_\Omega\varepsilon\left[N^k\left(\frac{x_0}{\varepsilon}\right)\left.\frac{\partial G_y^0(x)}{\partial y_k}\right|_{y=x_0}-N^k\left(\frac{x}{\varepsilon}\right)\frac{\partial G_{x_0}^0(x)}{\partial x_k}\right]\phi(x)[G_{x_0}^\varepsilon-\tilde{G}_{x_0}-\theta_G^\varepsilon](x)dx\\
&\quad=I+II+III.
\end{aligned} \tag{30}$$

First we estimate I. Denote $\xi=\frac{x}{\varepsilon}$ and $e_v^\varepsilon(x)=v^\varepsilon(x)-\tilde{v}_\epsilon(x)$, one finds that I can be split into

$$I=I_1+I_2, \tag{31}$$

where

$$I_1 = \int_\Omega \frac{\partial (a_{ij}(\frac{x}{\varepsilon})\frac{\partial e_v^\varepsilon(x)}{\partial x_j})}{\partial x_i} G_{x_0}^\varepsilon(x)dx, \quad I_2 = -\int_{\partial\Omega} \varepsilon N^k\left(\frac{x}{\varepsilon}\right)\frac{\partial v_\varepsilon^0(x)}{\partial x_k}\frac{\partial G_{x_0}^\varepsilon(x)}{\partial n_x}ds_x. \tag{32}$$

From (27) and (28) we have

$$\int_\Omega a^{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial v^\varepsilon(x)}{\partial x_j}\frac{\partial G_{x_0}^\varepsilon(x)}{\partial x_i}dx = \int_\Omega \hat{a}^{ij}\frac{\partial v_\varepsilon^0(x)}{\partial x_j}\frac{\partial G_{x_0}^\varepsilon(x)}{\partial x_i}dx. \tag{33}$$

Consequently,

$$\begin{aligned}
I_1 &= -\int_\Omega a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial e_v^\varepsilon(x)}{\partial x_j}\frac{\partial G_{x_0}^\varepsilon(x)}{\partial x_i}dx \\
&= -\int_\Omega \left[\hat{a}_{ij}\frac{\partial v_\varepsilon^0(x)}{\partial x_j} - a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial (v_\varepsilon^0(x)+\varepsilon N^k(\frac{x}{\varepsilon})\frac{\partial v_\varepsilon^0(x)}{\partial x_k})}{\partial x_j}\right]\frac{\partial G_{x_0}^\varepsilon(x)}{\partial x_i}dx \\
&= \int_\Omega \left[\left(a_{ij}\left(\frac{x}{\varepsilon}\right) + a_{ik}\left(\frac{x}{\varepsilon}\right)\frac{\partial N^j(\xi)}{\partial \xi_k} - \hat{a}_{ij}\right)\frac{\partial v_\varepsilon^0(x)}{\partial x_j} + \varepsilon a_{ij}\left(\frac{x}{\varepsilon}\right)N^k\left(\frac{x}{\varepsilon}\right)\frac{\partial^2 v_\varepsilon^0(x)}{\partial x_j\partial x_k}\right]\frac{\partial G_{x_0}^\varepsilon(x)}{\partial x_i}dx.
\end{aligned} \tag{34}$$

Set

$$G_i^j(\xi) = a_{ij}(\xi) + a_{ik}(\xi)\frac{\partial N^j(\xi)}{\partial \xi_k} - \hat{a}_{ij}. \tag{35}$$

From the definitions of $\hat{a}_{ij}$ and $N^j(\xi)$, it follows that

$$\int_Q G_i^j(\xi)d\xi = 0, \tag{36}$$

$$\frac{\partial G_i^j(\xi)}{\partial \xi_i} = 0. \tag{37}$$

Then there exist skew-symmetric matrices [18, p. 6] $\alpha^j(\xi) = (\alpha_{ki}^j(\xi))$ satisfying

$$G_i^j(\xi) = \frac{\partial}{\partial \xi_k}(\alpha_{ki}^j(\xi)), \tag{38}$$

$$\int_Q \alpha_{ki}^j(\xi)d\xi = 0. \tag{39}$$

Note that the definition of $\alpha_{ki}^j(\xi)$ implies

$$\int_\Omega \frac{\partial^2(\alpha_{ki}^j(\frac{x}{\varepsilon})\frac{\partial v_\varepsilon^0(x)}{\partial x_j})}{\partial x_i\partial x_k}G_{x_0}^\varepsilon(x)dx = 0.$$

With these notations, one can rewrite I_1 into

$$\begin{aligned}
I_1 &= \int_\Omega \left[\frac{\partial}{\partial \xi_k}(\alpha_{ki}^j(\xi))\frac{\partial v_\varepsilon^0(x)}{\partial x_j} + \varepsilon a_{ij}\left(\frac{x}{\varepsilon}\right)N^k\left(\frac{x}{\varepsilon}\right)\frac{\partial^2 v_\varepsilon^0(x)}{\partial x_j\partial x_k}\right]\frac{\partial G_{x_0}^\varepsilon(x)}{\partial x_i}dx \\
&= -\int_\Omega \varepsilon\frac{\partial^2(\alpha_{ki}^j(\frac{x}{\varepsilon})\frac{\partial v_\varepsilon^0(x)}{\partial x_j})}{\partial x_i\partial x_k}G_{x_0}^\varepsilon(x)dx - \varepsilon\int_\Omega \frac{\partial}{\partial x_i}\left[\left(-\alpha_{ki}^j\left(\frac{x}{\varepsilon}\right) + a_{ij}\left(\frac{x}{\varepsilon}\right)N^k\left(\frac{x}{\varepsilon}\right)\right)\frac{\partial^2 v_\varepsilon^0(x)}{\partial x_j\partial x_k}\right]G_{x_0}^\varepsilon(x)dx \\
&= \varepsilon\int_\Omega \left[-\alpha_{ki}^j\left(\frac{x}{\varepsilon}\right) + a_{ij}\left(\frac{x}{\varepsilon}\right)N^k\left(\frac{x}{\varepsilon}\right)\right]\frac{\partial^2 v_\varepsilon^0(x)}{\partial x_j\partial x_k}\frac{\partial G_{x_0}^\varepsilon(x)}{\partial x_i}dx = I_{1,1} + I_{1,2},
\end{aligned} \tag{40}$$

where

$$I_{1,1} = \varepsilon\int_{B(x_0,\frac{d}{2})}\left[-\alpha_{ki}^j\left(\frac{x}{\varepsilon}\right) + a_{ij}\left(\frac{x}{\varepsilon}\right)N^k\left(\frac{x}{\varepsilon}\right)\right]\frac{\partial^2 v_\varepsilon^0(x)}{\partial x_j\partial x_k}\frac{\partial G_{x_0}^\varepsilon(x)}{\partial x_i}dx, \tag{41}$$

$$I_{1,2} = \varepsilon \int_{\Omega - B(x_0,\frac{d}{2})} \left[-\alpha_{ki}^j\left(\frac{x}{\varepsilon}\right) + a_{ij}\left(\frac{x}{\varepsilon}\right) N^k\left(\frac{x}{\varepsilon}\right)\right] \frac{\partial^2 v_\varepsilon^0(x)}{\partial x_j \partial x_k} \frac{\partial G_{x_0}^\varepsilon(x)}{\partial x_i} dx. \tag{42}$$

Note that the combination of Lemmas 2.2, 2.3, (28) and (41) gives

$$\begin{aligned}|I_{1,1}| &\leqslant c\varepsilon \|D^2 v_\varepsilon^0\|_{L^\infty(B(x_0,\frac{d}{2}))} \int_{B(x_0,\frac{d}{2})} |DG_{x_0}^\varepsilon(x)| dx \leqslant c\varepsilon d^{\frac{-n}{2}} d |\ln d| \|\phi(G_{x_0}^\varepsilon - \tilde{G}_{x_0} - \theta_G^\varepsilon)\|_{L^2(\Omega)} \\ &\leqslant c\varepsilon d^{\frac{2-n}{2}} |\ln d| \|\phi(G_{x_0}^\varepsilon - \tilde{G}_{x_0} - \theta_G^\varepsilon)\|_{L^2(\Omega)}, \end{aligned} \tag{43}$$

and the combination of Lemma 2.2, (28) and (42) implies

$$|I_{1,2}| \leqslant c\varepsilon \|D^2 v_\varepsilon^0\|_{L^2(\Omega - B(x_0,\frac{d}{2}))} \|DG_{x_0}^\varepsilon\|_{L^2(\Omega - B(x_0,\frac{d}{2}))} \leqslant c\varepsilon d^{\frac{2-n}{2}} |\ln d| \|\phi(G_{x_0}^\varepsilon - \tilde{G}_{x_0} - \theta_G^\varepsilon)\|_{L^2(\Omega)}. \tag{44}$$

One finds from (40), (43) and (44) that

$$|I_1| \leqslant c\varepsilon d^{\frac{2-n}{2}} |\ln d| \|\phi(G_{x_0}^\varepsilon - \tilde{G}_{x_0} - \theta_G^\varepsilon)\|_{L^2(\Omega)}. \tag{45}$$

Now we estimate I_2. Similarly to Lemma 2.3, we deduce from the definition of $\phi(x)$ and (28) that

$$|Dv_\varepsilon^0|_{L^\infty(\partial\Omega)} \leqslant cd^{\frac{2-n}{2}} |\ln d| \|\phi(G_{x_0}^\varepsilon - \tilde{G}_{x_0} - \theta_G^\varepsilon)\|_{L^2(\Omega_d)} = cd^{\frac{2-n}{2}} |\ln d| \|\phi(G_{x_0}^\varepsilon - \tilde{G}_{x_0} - \theta_G^\varepsilon)\|_{L^2(\Omega)}. \tag{46}$$

Consquently,

$$\begin{aligned}|I_2| &= \left| \int_{\partial\Omega} \varepsilon N^k\left(\frac{x}{\varepsilon}\right) \frac{\partial v_\varepsilon^0(x)}{\partial x_i} \frac{\partial G_{x_0}^\varepsilon(x)}{\partial n_x} ds_x \right| \\ &\leqslant c\varepsilon d^{\frac{2-n}{2}} |\ln d| \|\phi(G_{x_0}^\varepsilon - \tilde{G}_{x_0} - \theta_G^\varepsilon)\|_{L^2(\Omega)} \int_{\partial\Omega} \left| \frac{\partial G_{x_0}^\varepsilon(x)}{\partial n_x} \right| ds_x \\ &\leqslant c\varepsilon d^{\frac{2-n}{2}} |\ln d| \|\phi(G_{x_0}^\varepsilon - \tilde{G}_{x_0} - \theta_G^\varepsilon)\|_{L^2(\Omega)}. \end{aligned} \tag{47}$$

The combination of (31), (45) and (47) shows that

$$|I| \leqslant c\varepsilon d^{\frac{2-n}{2}} |\ln d| \|\phi(G_{x_0}^\varepsilon - \tilde{G}_{x_0} - \theta_G^\varepsilon)\|_{L^2(\Omega)}. \tag{48}$$

Now we estimate II and III. Using the definitions of $\theta_G^\varepsilon(x)$ and $\phi(x)$, one finds that

$$\begin{aligned}|II| &= \left| \int_{\Omega - B(x_0,d) - E_d} \theta_G^\varepsilon(x)\phi(x)(G_{x_0}^\varepsilon - \tilde{G}_{x_0} - \theta_G^\varepsilon)(x) dx \right| \\ &\leqslant c\varepsilon \|G_{x_0}^0\|_{H^1(\Omega_d)} \|\phi(G_{x_0}^\varepsilon - \tilde{G}_{x_0} - \theta_G^\varepsilon)\|_{L^2(\Omega_d)} \leqslant c\varepsilon d^{\frac{2-n}{2}} |\ln d| \|\phi(G_{x_0}^\varepsilon - \tilde{G}_{x_0} - \theta_G^\varepsilon)\|_{L^2(\Omega_d)} \\ &\leqslant c\varepsilon d^{\frac{2-n}{2}} |\ln d| \|\phi(G_{x_0}^\varepsilon - \tilde{G}_{x_0} - \theta_G^\varepsilon)\|_{L^2(\Omega)}. \end{aligned} \tag{49}$$

Similarly,

$$|III| \leqslant c\varepsilon d^{\frac{2-n}{2}} |\ln d| \|\phi[G_{x_0}^\varepsilon - \tilde{G}_{x_0} - \theta_G^\varepsilon]\|_{L^2(\Omega)}. \tag{50}$$

Hence by the definition of $\phi(x)$, (30) and (48)–(50),

$$\|\phi^{\frac{1}{2}}(G_{x_0}^\varepsilon - \tilde{G}_{x_0} - \theta_G^\varepsilon)\|_{L^2(\Omega)} \leqslant c\varepsilon d^{\frac{2-n}{2}} |\ln d|. \tag{51}$$

Note that d is an arbitrary positive number. This relation implies the desired result (26). □

Now we estimate $\|G_{x_0}^\varepsilon - \tilde{G}_{x_0} - \theta_G^\varepsilon\|_{H^1(\Omega - B(x_0,d))}$.

Lemma 2.5. *Assume that $d \geqslant \varepsilon$ and $B(x_0,d)$ is defined as above. Then*

$$\|G_{x_0}^\varepsilon - \tilde{G}_{x_0} - \theta_G^\varepsilon\|_{H^1(\Omega - B(x_0,d))} \leqslant c\varepsilon d^{\frac{-n}{2}} |\ln \varepsilon|. \tag{52}$$

Proof. Assume that $k \geqslant 1$ ($k \in N$) and E_d is defined as above. Set

$$\Omega_{kd} = \Omega - B(x_0, kd) - E_{kd}, \qquad e_G^\varepsilon(x) = (\tilde{G}_{x_0} + \theta_G^\varepsilon - G_{x_0}^\varepsilon)(x). \tag{53}$$

One finds that $e^{\varepsilon}_G(x)$ can be decomposed into

$$e^{\varepsilon}_G(x) = e^{\varepsilon}_{G,1}(x) + e^{\varepsilon}_{G,2}(x), \tag{54}$$

where $e^{\varepsilon}_{G,1}(x)$ satisfies

$$\begin{cases} L_\varepsilon e^{\varepsilon}_{G,1}(x) \equiv \dfrac{\partial}{\partial x_i}\left(a^{ij}\left(\dfrac{x}{\varepsilon}\right)\dfrac{\partial e^{\varepsilon}_{G,1}(x)}{\partial x_j}\right) = \dfrac{\partial}{\partial x_i}\left(a^{ij}\left(\dfrac{x}{\varepsilon}\right)\dfrac{\partial e^{\varepsilon}_{G}(x)}{\partial x_j}\right), & \text{in} \quad \Omega_{kd}, \\ e^{\varepsilon}_{G,1}(x) = 0, & \text{on} \quad \partial\Omega_{kd}, \end{cases} \tag{55}$$

and $e^{\varepsilon}_{G,2}(x)$ satisfies

$$\begin{cases} L_\varepsilon e^{\varepsilon}_{G,2}(x) \equiv \dfrac{\partial}{\partial x_i}\left(a^{ij}\left(\dfrac{x}{\varepsilon}\right)\dfrac{\partial e^{\varepsilon}_{G,2}(x)}{\partial x_j}\right) = 0, & \text{in} \quad \Omega_{kd}, \\ e^{\varepsilon}_{G,2}(x) = e^{\varepsilon}_{G}(x), & \text{on} \quad \partial\Omega_{kd}. \end{cases} \tag{56}$$

First we estimate $\|e^{\varepsilon}_{G,1}\|_{H^1(\Omega_{kd})}$. Assume that $\alpha^j(\xi) = (\alpha^j_{ki}(\xi))$ are defined as Lemma 2.4. Similarly to Lemma 2.4, one observes that, for any $v \in H^1_0(\Omega_{kd})$,

$$\begin{aligned} &\int_{\Omega_{kd}} a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial e^{\varepsilon}_{G,1}(x)}{\partial x_j}\frac{\partial v(x)}{\partial x_i}dx \\ &= \int_{\Omega_{kd}}\left[\varepsilon\frac{\partial}{\partial x_k}\left(\alpha^j_{ki}\left(\frac{x}{\varepsilon}\right)\frac{\partial G^0_{x_0}(x)}{\partial x_j}\right) + \varepsilon\left(-\alpha^j_{ki}\left(\frac{x}{\varepsilon}\right) + a_{ij}\left(\frac{x}{\varepsilon}\right)N^k\left(\frac{x}{\varepsilon}\right)\right)\frac{\partial^2 G^0_{x_0}(x)}{\partial x_j \partial x_k}\right]\frac{\partial v(x)}{\partial x_i}dx. \end{aligned} \tag{57}$$

Notice that the definition of α^j_{ki} implies, for any $v \in H^1_0(\Omega)$,

$$\int_{\Omega_{kd}}\frac{\partial}{\partial x_k}\left(\alpha^j_{ki}\left(\frac{x}{\varepsilon}\right)\frac{\partial G^0_{x_0}(x)}{\partial x_j}\right)\frac{\partial v(x)}{\partial x_i}dx = -\int_{\Omega_{kd}}\frac{\partial^2}{\partial x_k \partial x_i}\left(\alpha^j_{ki}\left(\frac{x}{\varepsilon}\right)\frac{\partial G^0_{x_0}(x)}{\partial x_j}\right)v dx = 0.$$

Hence by (57),

$$\begin{aligned} (\|e^{\varepsilon}_{G,1}\|_{H^1(\Omega_{kd})})^2 &\leqslant c\left|\int_{\Omega_{kd}}\varepsilon\left[-\alpha^j_{ki}\left(\frac{x}{\varepsilon}\right) + a_{ij}\left(\frac{x}{\varepsilon}\right)N^k\left(\frac{x}{\varepsilon}\right)\right]\frac{\partial^2 G^0_{x_0}(x)}{\partial x_j \partial x_k}\frac{\partial e^{\varepsilon}_{G,1}(x)}{\partial x_i}dx\right| \\ &\leqslant c\varepsilon\|G^0_{x_0}\|_{H^2(\Omega_{kd})}\|e^{\varepsilon}_{G,1}\|_{H^1(\Omega_{kd})} \leqslant c\varepsilon d^{\frac{-n}{2}}\|e^{\varepsilon}_{G,1}\|_{H^1(\Omega_{kd})}, \end{aligned}$$

which gives

$$\|e^{\varepsilon}_{G,1}\|_{H^1(\Omega_{kd})} \leqslant c\varepsilon d^{\frac{-n}{2}}. \tag{58}$$

We now turn to the estimation of $\|e^{\varepsilon}_{G,2}\|_{H^1(\Omega_{kd})}$. Define the function $\phi(x)$ by

$$\begin{aligned} &0 \leqslant \phi(x) \leqslant 1, \quad x \in \Omega_{kd}, \\ &\phi(x) = 1, \quad x \in \partial\Omega_{kd}, \\ &\phi(x) = 0, \quad x \in \Omega_{(k+1)d}, \\ &\phi \in C^\infty(\Omega_{kd}), \quad |D^l\phi(x)| \leqslant \frac{c_l}{d^l}. \end{aligned}$$

One observes that $e^{\varepsilon}_{G,2}(x)$ can be split into two parts:

$$e^{\varepsilon}_{G,2}(x) = e^{\varepsilon}_{G,2,1}(x) + e^{\varepsilon}_{G,2,2}(x), \tag{59}$$

where

$$e^{\varepsilon}_{G,2,1}(x) = e^{\varepsilon}_{G}(x)\phi(x), \tag{60}$$

and $e^\varepsilon_{G,2,2}(x)$ satisfies the problem

$$\begin{cases} L_\varepsilon e^\varepsilon_{G,2,2}(x) = -\dfrac{\partial}{\partial x_i}\left(a^{ij}\left(\dfrac{x}{\varepsilon}\right)\dfrac{\partial e^\varepsilon_{G,2,1}(x)}{\partial x_j}\right), & \text{in} \quad \Omega_{kd}, \\ e^\varepsilon_{G,2,2}(x) = 0, & \text{on} \quad \partial\Omega_{kd}. \end{cases} \tag{61}$$

Note that (60) leads to

$$\begin{aligned} \|e^\varepsilon_{G,2,1}\|_{H^1(\Omega_{kd})} &= \|e^\varepsilon_{G,2,1}\|_{H^1(\Omega_{kd}-\Omega_{(k+1)d})} \\ &\leqslant \|\phi\|_{L^\infty(\Omega_{kd}-\Omega_{(k+1)d})}\|e^\varepsilon_G\|_{H^1(\Omega_{kd}-\Omega_{(k+1)d})} + \|\phi\|_{W^{1,\infty}(\Omega_{kd}-\Omega_{(k+1)d})}\|e^\varepsilon_G\|_{L^2(\Omega_{kd}-\Omega_{(k+1)d})} \\ &\leqslant \|e^\varepsilon_G\|_{H^1(\Omega_{kd}-\Omega_{(k+1)d})} + c\varepsilon d^{\frac{-n}{2}}|\ln d|, \end{aligned} \tag{62}$$

and

$$\|e^\varepsilon_{G,2,2}\|_{H^1(\Omega_{kd})} \leqslant c\|e^\varepsilon_{G,2,1}\|_{H^1(\Omega_{kd})} \leqslant c_1\|e^\varepsilon_G\|_{H^1(\Omega_{kd}-\Omega_{(k+1)d})} + c_2\varepsilon d^{\frac{-n}{2}}|\ln d|. \tag{63}$$

The combination of (62) and (63) implies

$$\|e^\varepsilon_{G,2}\|_{H^1(\Omega_{kd})} \leqslant c_1\|e^\varepsilon_G\|_{H^1(\Omega_{kd}-\Omega_{(k+1)d})} + c_2\varepsilon d^{\frac{-n}{2}}|\ln d|. \tag{64}$$

Note that $d \geqslant \varepsilon$. One finds from (58) and (64) that

$$\|e^\varepsilon_G\|_{H^1(\Omega_{kd})} \leqslant c_1\,\|e^\varepsilon_G\|_{H^1(\Omega_{kd}-\Omega_{(k+1)d})} + c_2\varepsilon d^{\frac{-n}{2}}|\ln \varepsilon|. \tag{65}$$

Now we use (65) to estimate $\|e^\varepsilon_G\|_{H^1(\Omega_d)}$. The inequality (65) gives

$$\|e^\varepsilon_G\|^2_{H^1(\Omega_{kd})} \leqslant c_1[\|e^\varepsilon_G\|^2_{H^1(\Omega_{kd})} - \|e^\varepsilon_G\|^2_{H^1(\Omega_{(k+1)d})}] + c_2\varepsilon^2 d^{-n}|\ln\varepsilon|^2, \tag{66}$$

which leads to

$$\begin{aligned} c_1\,\|e^\varepsilon_G\|^2_{H^1(\Omega_{(k+1)d})} &\leqslant (c_1-1)\,\|e^\varepsilon_G\|^2_{H^1(\Omega_{kd})} + c_2\varepsilon^2 d^{-n}|\ln\varepsilon|^2 \\ &= (c_1-1)[\|e^\varepsilon_G\|^2_{H^1(\Omega_{kd})} - c_2\varepsilon^2 d^{-n}|\ln\varepsilon|^2] + c_2c_1\varepsilon^2 d^{-n}|\ln\varepsilon|^2. \end{aligned}$$

Consquently,

$$\|e^\varepsilon_G\|^2_{H^1(\Omega_{(k+1)d})} - c_2\varepsilon^2 d^{-n}|\ln\varepsilon|^2 \leqslant \frac{c_1-1}{c_1}(\|e^\varepsilon_G\|^2_{H^1(\Omega_{kd})} - c_2\varepsilon^2 d^{-n}|\ln\varepsilon|^2). \tag{67}$$

Assume that c_1 and c_2 satisfy (67). Let $m_1 = c_1$ and $m_2 = c_2$. Note that the combination of (13) and (14) gives

$$\|e^\varepsilon_G\|^2_{H^1(\Omega_d)} \leqslant \|G^\varepsilon_{x_0}\|^2_{H^1(\Omega_d)} + \|\tilde{G}_{x_0}\|^2_{H^1(\Omega_d)} \leqslant m_3\varepsilon^2 d^{-n}|\ln\varepsilon|^2, \quad \text{if} \quad d \in \left[\frac{\varepsilon}{2}, \varepsilon\right]. \tag{68}$$

Define a_s $(s \in N)$ by

$$a_s = \frac{(2s)^n}{1 - (\frac{m_1-1}{m_1})^s 2^n}. \tag{69}$$

Let $s_0 \in N$ satisfy $a_{s_0} > 0$ and $a_{s_0} \leqslant a_s$ if $a_s > 0$. Define a constant m by

$$m = \max\{m_2 a_{s_0}, m_3\}. \tag{70}$$

Observing that m is a constant, independent of d and ε, now we use (67) to prove

$$\|e^\varepsilon_G\|^2_{H^1(\Omega_d)} \leqslant m\varepsilon^2 d^{-n}|\ln\varepsilon|^2, \quad \text{if} \quad d \geqslant \frac{\varepsilon}{2}. \tag{71}$$

(68) shows that (71) holds if $d \in [\frac{\varepsilon}{2}, \varepsilon]$. Assume that $d_0 \geqslant \frac{\varepsilon}{2}$ such that $\|e_G^\varepsilon\|^2_{H^1(\Omega_{d_0})} \leqslant m\varepsilon^2 d_0^{-n}|\ln\varepsilon|^2$. Let $d_1 = 2d_0$. Now we prove $\|e_G^\varepsilon\|^2_{H^1(\Omega_{d_1})} \leqslant m\varepsilon^2 d_1^{-n}|\ln\varepsilon|^2$. Set $d = \frac{d_1}{2s_0}$. Note that (67) implies

$$\begin{aligned}&\|e_G^\varepsilon\|^2_{H^1(\Omega_{d_1})} - m_2\varepsilon^2 d^{-n}|\ln\varepsilon|^2\\ &\quad= \|e_G^\varepsilon\|^2_{H^1(\Omega_{2s_0 d})} - m_2\varepsilon^2 d^{-n}|\ln\varepsilon|^2 \leqslant \frac{m_1-1}{m_1}[\|e_G^\varepsilon\|^2_{H^1(\Omega_{(2s_0-1)d})} - m_2\varepsilon^2 d^{-n}|\ln\varepsilon|^2]\\ &\quad\leqslant\cdots\leqslant\left(\frac{m_1-1}{m_1}\right)^{s_0}[\|e_G^\varepsilon\|^2_{H^1(\Omega_{s_0 d})} - m_2\varepsilon^2 d^{-n}|\ln\varepsilon|^2].\end{aligned}\tag{72}$$

One observes from (72) that

$$\begin{aligned}\|e_G^\varepsilon\|^2_{H^1(\Omega_{d_1})} - m_2\varepsilon^2(2s_0)^n d_1^{-n}|\ln\varepsilon|^2 &\leqslant \left(\frac{m_1-1}{m_1}\right)^{s_0}\|e_G^\varepsilon\|^2_{H^1(\Omega_{d_0})}\\ \leqslant \left(\frac{m_1-1}{m_1}\right)^{s_0} m\varepsilon^2 d_0^{-n}|\ln\varepsilon|^2 &= \left(\frac{m_1-1}{m_1}\right)^{s_0} 2^n m\varepsilon^2 d_1^{-n}|\ln\varepsilon|^2.\end{aligned}\tag{73}$$

Hence by (69), (70) and (73),

$$\|e_G^\varepsilon\|^2_{H^1(\Omega_{d_1})} \leqslant \left[(2s_0)^n m_2 + \left(\frac{m_1-1}{m_1}\right)^{s_0} 2^n m\right]\varepsilon^2 d_1^{-n}|\ln\varepsilon|^2 = m\varepsilon^2 d_1^{-n}|\ln\varepsilon|^2.\tag{74}$$

The combination of (69)–(74) implies

$$\|e_G^\varepsilon\|_{H^1(\Omega - B(x_0,d) - E_d)} = \|e_G^\varepsilon\|_{H^1(\Omega_d)} \leqslant c\varepsilon d^{\frac{-n}{2}}|\ln\varepsilon|, \quad \text{if} \quad d \geqslant \varepsilon.\tag{75}$$

Assume that $E = \{x \in \Omega | \operatorname{dist}(x, \partial\Omega) \leqslant 2d\} - B(x_0, d)$. One sees that $E_d \subset E$. Now we estimate $\|e_G^\varepsilon\|_{H^1(E)}$. Similarly to (54), one observes that $e_G^\varepsilon(x)$ can be decomposed into

$$e_G^\varepsilon(x) = e_{G,1}^\varepsilon(x) + e_{G,2}^\varepsilon(x), \quad x \in E,\tag{76}$$

where $e_{G,1}^\varepsilon(x)$ satisfies the problem

$$\begin{cases} L_\varepsilon e_{G,1}^\varepsilon(x) \equiv \dfrac{\partial}{\partial x_i}\left(a^{ij}\left(\dfrac{x}{\varepsilon}\right)\dfrac{\partial e_{G,1}^\varepsilon(x)}{\partial x_j}\right) = \dfrac{\partial}{\partial x_i}\left(a^{ij}\left(\dfrac{x}{\varepsilon}\right)\dfrac{\partial e_G^\varepsilon(x)}{\partial x_j}\right), & \text{in} \quad E,\\ e_{G,1}^\varepsilon(x) = 0, & \text{on} \quad \partial E,\end{cases}\tag{77}$$

and $e_{G,2}^\varepsilon(x)$ satisfies the problem

$$\begin{cases} L_\varepsilon e_{G,2}^\varepsilon(x) \equiv \dfrac{\partial}{\partial x_i}\left(a^{ij}\left(\dfrac{x}{\varepsilon}\right)\dfrac{\partial e_{G,2}^\varepsilon(x)}{\partial x_j}\right) = 0, & \text{in} \quad E,\\ e_{G,2}^\varepsilon(x) = e_G^\varepsilon(x), & \text{on} \quad \partial E.\end{cases}\tag{78}$$

Similarly to (58), one finds that

$$\|e_{G,1}^\varepsilon\|_{H^1(E)} \leqslant c\varepsilon d^{\frac{-n}{2}}.\tag{79}$$

Similarly to (64), one observes from (26), (75) and (78) that

$$\begin{aligned}\|e_{G,2}^\varepsilon\|_{H^1(E)} &\leqslant c\|e_G^\varepsilon\|_{H^1(E-E_d)} + cd^{-1}\|e_G^\varepsilon\|_{L^2(E-E_d)} \leqslant c\|e_G^\varepsilon\|_{H^1(\Omega_d)} + cd^{-1}\|e_G^\varepsilon\|_{L^2(\Omega_d)}\\ &\leqslant c\varepsilon d^{\frac{-n}{2}}|\ln\varepsilon| + c\varepsilon d^{\frac{-n}{2}}|\ln\varepsilon| \leqslant c\varepsilon d^{\frac{-n}{2}}|\ln\varepsilon|.\end{aligned}\tag{80}$$

Hence by (79)–(80),

$$\|e_G^\varepsilon\|_{H^1(E)} \leqslant c\varepsilon d^{\frac{-n}{2}}|\ln\varepsilon|.\tag{81}$$

The combination of (75) and (81) gives the desired result (52). □

Finally, we are now in a position to give a proof of Theorem 2.1.

Assume that $l \in N$ such that $2^{l-1}\varepsilon < \sup\{|x-x_0\| x \in \Omega\} \leqslant 2^l\varepsilon$. Let $d_0 = \varepsilon, d_1 = 2\varepsilon, \ldots, d_l = 2^l\varepsilon$, it follows from Lemma 2.5 that

$$\begin{aligned}\|G^\varepsilon_{x_0} - \tilde{G}_{x_0} - \theta^\varepsilon_G\|_{W^{1,1}(\Omega - B(x_0,\varepsilon))} &= \sum_{i=1}^{l} \|G^\varepsilon_{x_0} - \tilde{G}_{x_0} - \theta^\varepsilon_G\|_{W^{1,1}(\Omega\cap(B(x_0,d_i)-B(x_0,d_{i-1}))} \\ &\leqslant \sum_{i=1}^{l} c d_i^{\frac{n}{2}} \|G^\varepsilon_{x_0} - \tilde{G}_{x_0} - \theta^\varepsilon_G\|_{H^1(\Omega\cap(B(x_0,d_i)-B(x_0,d_{i-1}))} \\ &\leqslant \sum_{i=1}^{l} c d_i^{\frac{n}{2}} \|G^\varepsilon_{x_0} - \tilde{G}_{x_0} - \theta^\varepsilon_G\|_{H^1(\Omega-B(x_0,d_{i-1}))} \\ &\leqslant \sum_{i=1}^{l} c d_i^{\frac{n}{2}} \varepsilon d_i^{\frac{-n}{2}} |\ln\varepsilon| \leqslant cl\varepsilon|\ln\varepsilon| \leqslant c\varepsilon|\ln\varepsilon|^2. \end{aligned} \tag{82}$$

This ends the proof. □

3 Proof of Theorem 1.1

We observe that Theorem 1.1 can be deduced from the following result, directly.

Theorem 3.1. *Assume that $n = 2, x_0 \in \Omega$, $u^\varepsilon(x)$ and $\tilde{u}(x)$ are the solutions of* (1) *and its* 1*-order approximation, respectively. Under the assumption that $\partial\Omega$ is smooth enough and $u^0 \in W^{1,\infty}(\Omega), N^k \in W^{1,\infty}(Q)$, there exists a constant c, independent of x_0, such that*

$$|(u^\varepsilon - \tilde{u})(x_0)| \leqslant c\varepsilon|\ln\varepsilon|^2 \|u^0\|_{W^{1,\infty}(\Omega)}. \tag{83}$$

Assume that $r = 2\varepsilon$. For proving Theorem 3.1, we shall construct an auxiliary function $u^\varepsilon_r(x)$ depending on x_0 and estimate $|(u^\varepsilon - u^\varepsilon_r)(x_0)|$, $|(u^\varepsilon_r - \tilde{u}_r)(x_0)|$ and $\|\tilde{u}_r - \tilde{u}\|_{L^\infty(\Omega)}$, respectively where $\tilde{u}_r(x)$ is the 1-order approximation of $u^\varepsilon_r(x)$.

The organization of this section is as follows. Assume that $r = 2\varepsilon$ and x_0 is defined as above, we construct auxiliary functions $u^0_r(x) \in C^\infty(\Omega)$ and $u^\varepsilon_r(x)$ in Subsection 3.1. Subsection 3.2 presents an error estimate between the solutions $\tilde{u}(x_0)$ and $u^\varepsilon(x_0)$.

3.1 The construction of auxiliary functions $u^0_r(x)$ and $u^\varepsilon_r(x)$

Let $N_0 = 2$ and $r = N_0\varepsilon$. We will introduce two functions $u^0_{r,1}(x)$ and $u^0_{r,2}(x)$ to construct $u^0_r(x)$.

Assume that $\hat{u}^0$ satisfies $\|\hat{u}^0\|_{W^{1,\infty}(\mathbb{R}^2)} \leqslant c\|u^0\|_{W^{1,\infty}(\Omega)}$, $\hat{u}^0(x) = u^0(x)$ if $x \in \Omega$. We define $u^0_{r,1}(x)$ by

$$u^0_{r,1}(x) = \frac{\int_{\mathbb{R}^2} \hat{u}^0(y) e^{\frac{-|x-y|}{r}} dy}{\int_{\mathbb{R}^2} e^{\frac{-|x-y|}{r}} dy}, \quad x \in \Omega. \tag{84}$$

Now we construct $u^0_{r,2}(x)$. Assume that $x = (x_1, x_2), x_0 = (x_{0,1}, x_{0,2}), z \in \mathbb{Z}^2$. We define Ω_2 by

$$E_2 = \{r(Q+z) \subset \Omega | \rho(\varepsilon(Q+z), x_0) \geqslant r, \rho(r(Q+z), \partial\Omega) \geqslant r\}, \quad \Omega_2 = \bigsqcup_{r(Q+z)\in E_2} r(Q+z). \tag{85}$$

And the function space V is given by

$$V = \{\psi(\xi) \in C^\infty(\mathbb{R}^2) | \psi(\xi) \text{ is an } N_0\text{-periodic function, } |\nabla^k\psi(\xi)| \leqslant N_0^{-k}\ (k \geqslant 0, k \in N)\}, \tag{86}$$

and integer l $(1 \leqslant l \leqslant 2)$ satisfies

$$\sup_{\psi\in V} \int_{N_0Q} \hat{a}_{ij} \frac{\partial\psi(\xi)}{\partial\xi_j} \frac{\partial N^l(\xi)}{\partial\xi_i} d\xi \geqslant \sup_{\psi\in V} \int_{N_0Q} \hat{a}_{ij} \frac{\partial\psi(\xi)}{\partial\xi_j} \frac{\partial N^{l'}(\xi)}{\partial\xi_i} d\xi, \quad \text{where} \quad l' = 3-l. \tag{87}$$

We introduce $\hat{\psi}(\xi) \in V$ and $u^0_{r,2}(x)$ by

$$\int_{N_0Q} \hat{a}_{ij} \frac{\partial\hat{\psi}(\xi)}{\partial\xi_j}\frac{\partial N^l(\xi)}{\partial\xi_i} d\xi \geqslant \frac{1}{2}\sup_{\psi\in V}\int_{N_0Q} \hat{a}_{ij} \frac{\partial\psi(\xi)}{\partial\xi_j}\frac{\partial N^l(\xi)}{\partial\xi_i} d\xi, \tag{88}$$

and

$$u^0_{r,2}(x) = M\hat{\psi}\left(\frac{x}{\varepsilon}\right)(x_l - x_{0,l}), \tag{89}$$

where M is defined by

$$M = \frac{N_0^2\varepsilon\int_{\Omega_2}\hat{a}_{ij}\frac{\partial(u^0-u^0_{r,1})(x)}{\partial x_j}\frac{\partial N^k(\xi)}{\partial\xi_i}\frac{\partial G^0_{x_0}(x)}{\partial x_k}dx}{\int_{\Omega_2}\frac{\partial G^0_{x_0}(x)}{\partial x_l}(x_l - x_{0,l})dx_1dx_2 \times \int_{N_0Q}\hat{a}_{ij}\frac{\partial\hat{\psi}(\xi)}{\partial\xi_j}\frac{\partial N^l(\xi)}{\partial\xi_i}d\xi}. \tag{90}$$

Finally, we define $u^0_r(x), g_r(x)$ and $u^\varepsilon_r(x)$ by

$$u^0_r(x) = u^0_{r,1}(x) + u^0_{r,2}(x), \quad g_r(x) = u^0_r(x), \tag{91}$$

and

$$\begin{cases} L_\varepsilon u^\varepsilon_r(x) = \dfrac{\partial}{\partial x_i}\left(\hat{a}^{ij}\dfrac{\partial u^0_r(x)}{\partial x_j}\right), & \text{in} \quad \Omega, \\ u^\varepsilon_r(x) = g_r(x), & \text{on} \quad \partial\Omega. \end{cases} \tag{92}$$

One observes that $u^0_r(x)$ is the homogenization solution for (92). In the next subsection, we shall estimate $(u^0 - u^0_r)(x_0), (u^\varepsilon_r - u^\varepsilon)(x_0)$ and $(\tilde{u}_r - u^\varepsilon_r)(x_0)$, respectively.

3.2 Estimate for $|(u^0 - u^\varepsilon)(x_0)|$

First we estimate $\|u^0_r - u^0\|_{L^\infty(\Omega)}$ and $\|u^0_r\|_{W^{p,\infty}(\Omega)}$.

Lemma 3.1. *There exists a constant c, independent of p, such that*

$$\|u^0_r - u^0\|_{L^\infty(\Omega)} \leqslant c\varepsilon\|u^0\|_{W^{1,\infty}(\Omega)}, \tag{93}$$
$$\|u^0_r\|_{W^{p,\infty}(\Omega)} \leqslant cr^{1-p}\|u^0\|_{W^{1,\infty}(\Omega)}. \tag{94}$$

Proof. From the definition of $u^0_{r,1}(x)$ one gets

$$\|u^0 - u^0_{r,1}\|_{L^\infty(\Omega)} \leqslant cr\|u^0\|_{W^{1,\infty}(\Omega)} \leqslant c\varepsilon\|u^0\|_{W^{1,\infty}(\Omega)}. \tag{95}$$

Now we estimate $\|u^0_{r,2}\|_{L^\infty(\Omega)}$. Assume that M is defined by (90). Note that

$$\left|\int_{\Omega_2}\frac{\partial G^0_{x_0}(x)}{\partial x_l}(x_l - x_{0,l})dx\right| \geqslant c, \quad \int_{N_0Q}\hat{a}_{ij}\frac{\partial\hat{\psi}(\xi)}{\partial\xi_j}\frac{\partial N^l(\xi)}{\partial\xi_i}d\xi \geqslant c. \tag{96}$$

We have

$$\begin{aligned} |M| &\leqslant c\varepsilon\|u^0 - u^0_{r,1}\|_{W^{1,\infty}(\Omega_2)}\|N^l\|_{W^{1,\infty}(Q)}\|G^0_{x_0}\|_{W^{1,1}(\Omega_2)} \\ &\leqslant c\varepsilon\|u^0 - u^0_{r,1}\|_{W^{1,\infty}(\Omega_2)}\|N^l\|_{W^{1,\infty}(Q)} \leqslant c\varepsilon\|u^0\|_{W^{1,\infty}(\Omega_2)}. \end{aligned} \tag{97}$$

Hence by (86),

$$\|u^0_{r,2}\|_{L^\infty(\Omega)} \leqslant c\varepsilon\|u^0\|_{W^{1,\infty}(\Omega)}, \tag{98}$$

which, together with (95), gives the desired result (93).

Similarly, one concludes from the definitions of $u^0_{r,1}(x)$ and $u^0_{r,2}(x)$ that

$$|u^0_{r,1}\|_{W^{p,\infty}(\Omega)} \leqslant cr^{1-p}\|u^0\|_{W^{1,\infty}(\Omega)}, \quad |u^0_{r,2}\|_{W^{p,\infty}(\Omega)} \leqslant cr^{1-p}\|u^0\|_{W^{1,\infty}(\Omega)}. \tag{99}$$

These relations imply the desired result (94). □

Now we use Theorem 2.1, Lemmas 2.2 and 3.1 to estimate $(u^\varepsilon - u^\varepsilon_r)(x_0)$.

Lemma 3.2. *Under the assumptions of Theorem* 3.1, *there exists a constant c such that*

$$|(u^\varepsilon - u^\varepsilon_r)(x_0)| \leqslant c\varepsilon|\ln\varepsilon|^2\|u^0\|_{W^{1,\infty}(\Omega)}. \tag{100}$$

Proof. Assume that r, E_2 and Ω_2 are defined as Subsection 3.1, $E_1 = \{r(Q+z) \subset \Omega | \rho(\varepsilon(Q+z), x_0) < r\}$. Note that $\Omega - \Omega_2$ can be split into

$$\Omega - \Omega_2 = \Omega_1 \cup \Omega_3, \tag{101}$$

where

$$\Omega_1 = \bigsqcup_{r(Q+z)\in E_1} r(Q+z), \quad \Omega_3 = \Omega - \Omega_1 - \Omega_2. \tag{102}$$

One finds that $(u^\varepsilon - u^\varepsilon_r)(x_0)$ can be decomposed into

$$\begin{aligned}(u^\varepsilon - u^\varepsilon_r)(x_0) &= -\int_\Omega \hat{a}^{ij}\frac{\partial(u^0-u^0_r)(x)}{\partial x_j}\frac{\partial G^\varepsilon_{x_0}(x)}{\partial x_i}dx \\ &= \int_{\Omega_1} -\hat{a}^{ij}\frac{\partial(u^0-u^0_r)(x)}{\partial x_j}\frac{\partial G^\varepsilon_{x_0}(x)}{\partial x_i}dx + \int_{\Omega-\Omega_1} -\hat{a}^{ij}\frac{\partial(u^0-u^0_r)(x)}{\partial x_j}\frac{\partial G^\varepsilon_{x_0}(x)}{\partial x_i}dx \\ &= A + B. \end{aligned} \tag{103}$$

From (15) it follows that

$$|A| \leqslant c\|G^\varepsilon_{x_0}\|_{W^{1,1}(\Omega_1)}\|u^0\|_{W^{1,\infty}(\Omega)} \leqslant c\varepsilon|\ln\varepsilon|\|u^0\|_{W^{1,\infty}(\Omega)}. \tag{104}$$

Now we estimate B. Observe that B can be split into

$$\begin{aligned} B &= \int_{\Omega-\Omega_1} -\hat{a}^{ij}\frac{\partial(u^0-u^0_r)(x)}{\partial x_j}\frac{\partial G^0_{x_0}(x)}{\partial x_i}dx + \int_{\Omega-\Omega_1} -\hat{a}^{ij}\frac{\partial(u^0-u^0_r)(x)}{\partial x_j}\frac{\partial(\varepsilon N^k(\frac{x}{\varepsilon})\frac{\partial G^0_{x_0}(x)}{\partial x_k})}{\partial x_i}dx \\ &\quad + \int_{\Omega-\Omega_1} -\hat{a}^{ij}\frac{\partial(u^0-u^0_r)(x)}{\partial x_j}\frac{\partial\theta^\varepsilon_G(x)}{\partial x_i}dx + \int_{\Omega-\Omega_1} -\hat{a}^{ij}\frac{\partial(u^0-u^0_r)(x)}{\partial x_j}\frac{\partial(G^\varepsilon_{x_0} - \tilde{G}_{x_0} - \theta^\varepsilon_G)(x)}{\partial x_i}dx \\ &= B_1 + B_2 + B_3 + B_4. \end{aligned} \tag{105}$$

Notice that

$$(u^0 - u^0_r)(x_0) = \int_{\Omega_1} -\hat{a}^{ij}\frac{\partial(u^0-u^0_r)(x)}{\partial x_j}\frac{\partial G^0_{x_0}(x)}{\partial x_i}dx + \int_{\Omega-\Omega_1} -\hat{a}^{ij}\frac{\partial(u^0-u^0_r)(x)}{\partial x_j}\frac{\partial G^0_{x_0}(x)}{\partial x_i}dx. \tag{106}$$

One finds from Lemma 3.1 and (106) that

$$\begin{aligned} |B_1| &\leqslant \|u^0 - u^0_r\|_{L^\infty(\Omega)} + \left|\int_{\Omega_1} -\hat{a}^{ij}\frac{\partial(u^0-u^0_r)(x)}{\partial x_j}\frac{\partial G^0_{x_0}(x)}{\partial x_i}dx\right| \\ &\leqslant c\varepsilon\|u^0\|_{W^{1,\infty}(\Omega)} + c\varepsilon\|u^0\|_{W^{1,\infty}(\Omega)} \leqslant c\varepsilon\|u^0\|_{W^{1,\infty}(\Omega)}. \end{aligned} \tag{107}$$

We now turn to the estimation of B_2. Assume that Ω_2, Ω_1 and Ω_3 are defined as (85) and (102), respectively. With these notations, we split B_2 into

$$B_2 = B_{2,1} + B_{2,2} + B_{2,3}, \tag{108}$$

where

$$B_{2,1} = -\varepsilon \int_{\Omega-\Omega_1} \hat{a}^{ij} \frac{\partial (u^0 - u_r^0)(x)}{\partial x_j} N^k\left(\frac{x}{\varepsilon}\right) \frac{\partial^2 G_{x_0}^0(x)}{\partial x_i \partial x_k} dx,$$

$$B_{2,2} = -\int_{\Omega_2} \hat{a}^{ij} \frac{\partial (u^0 - u_r^0)(x)}{\partial x_j} \frac{\partial N^k(\xi)}{\partial \xi_i} \frac{\partial G_{x_0}^0(x)}{\partial x_k} dx,$$

$$B_{2,3} = -\int_{\Omega_3} \hat{a}^{ij} \frac{\partial (u^0 - u_r^0)(x)}{\partial x_j} \frac{\partial N^k(\xi)}{\partial \xi_i} \frac{\partial G_{x_0}^0(x)}{\partial x_k} dx.$$

One finds that

$$|B_{2,1}| \leqslant c\varepsilon \left| \int_{\Omega-\Omega_1} \frac{\partial^2 G_{x_0}^0(x)}{\partial x_i \partial x_k} dx \right| \|u^0\|_{W^{1,\infty}(\Omega)} \leqslant c\varepsilon |\ln \varepsilon| \|u^0\|_{W^{1,\infty}(\Omega)}. \tag{109}$$

Now we estimate $B_{2,2}$. Assume that E_2 and l are defined as (85) and (87), respectively. Let $\omega(x) = \frac{\partial G_{x_0}^0(x)}{\partial x_l}(x_l - x_{0,l})$. For any $r(Q+z) \in E_2$, let $x_z \in r(Q+z)$ satisfy $\omega(x_z) = r^{-2}\int_{r(Q+z)} \omega(x)dx$. Note that $\int_{\Omega_2} \hat{a}^{ij} \frac{\partial u_{r,2}^0(x)}{\partial x_j} \frac{\partial N^k(\xi)}{\partial \xi_i} \frac{\partial G_{x_0}^0(x)}{\partial x_k} dx$ can be rewritten into

$$\int_{\Omega_2} \hat{a}^{ij} \frac{\partial u_{r,2}^0(x)}{\partial x_j} \frac{\partial N^k(\xi)}{\partial \xi_i} \frac{\partial G_{x_0}^0(x)}{\partial x_k} dx = \int_{\Omega_2} \hat{a}^{ij} \frac{\partial [M\hat{\psi}(\xi)(x_l - x_{0,l})]}{\partial x_j} \frac{\partial N^k(\xi)}{\partial \xi_i} \frac{\partial G_{x_0}^0(x)}{\partial x_k} dx$$
$$= \sum_{r(Q+z)\in E_2} \int_{r(Q+z)} \hat{a}^{ij} \frac{\partial [M\hat{\psi}(\xi)(x_l - x_{0,l})]}{\partial x_j} \frac{\partial N^k(\xi)}{\partial \xi_i} \frac{\partial G_{x_0}^0(x)}{\partial x_k} dx = I_1 + I_2 + I_3 + I_4, \tag{110}$$

where

$$I_1 = \frac{M}{\varepsilon} \sum_{r(Q+z)\in E_2} \int_{r(Q+z)} \hat{a}^{ij} \frac{\partial \hat{\psi}(\xi)}{\partial \xi_j} \frac{\partial N^l(\xi)}{\partial \xi_i} \omega(x_z) dx, \tag{111}$$

$$I_2 = \frac{M}{\varepsilon} \sum_{r(Q+z)\in E_2} \int_{r(Q+z)} \hat{a}^{ij} \frac{\partial \hat{\psi}(\xi)}{\partial \xi_j} \frac{\partial N^l(\xi)}{\partial \xi_i} (\omega(x) - \omega(x_z)) dx, \tag{112}$$

$$I_3 = \frac{M}{\varepsilon} \int_{\Omega_2} \hat{a}^{ij} \frac{\partial \hat{\psi}(\xi)}{\partial \xi_j} \frac{\partial N^{l'}(\xi)}{\partial \xi_i} \frac{\partial G_{x_0}^0(x)}{\partial x_{l'}} (x_l - x_{0,l}) dx, \tag{113}$$

$$I_4 = M \int_{\Omega_2} \hat{a}^{il} \hat{\psi}(\xi) \frac{\partial N^k(\xi)}{\partial \xi_i} \frac{\partial G_{x_0}^0(x)}{\partial x_k} dx. \tag{114}$$

and the definition of M (see (90)) implies

$$I_1 = \frac{M}{\varepsilon} \sum_{r(Q+z)\in E_2} \left[\omega(x_z) \times \varepsilon^2 \int_{N_0 Q} \hat{a}^{ij} \frac{\partial \hat{\psi}(\xi)}{\partial \xi_j} \frac{\partial N^l(\xi)}{\partial \xi_i} d\xi \right]$$
$$= \frac{M}{\varepsilon} \int_{N_0 Q} \hat{a}^{ij} \frac{\partial \hat{\psi}(\xi)}{\partial \xi_j} \frac{\partial N^l(\xi)}{\partial \xi_i} d\xi \sum_{r(Q+z)\in B_2} \omega(x_z)\varepsilon^2 = \frac{M}{\varepsilon N_0^2} \int_{N_0 Q} \hat{a}^{ij} \frac{\partial \hat{\psi}(\xi)}{\partial \xi_j} \frac{\partial N^l(\xi)}{\partial \xi_i} d\xi \int_{\Omega_2} \omega(x) dx$$
$$= \int_{\Omega_2} \hat{a}^{ij} \frac{\partial (u^0 - u_{r,1}^0)(x)}{\partial x_j} \frac{\partial N^k(\xi)}{\partial \xi_i} \frac{\partial G_{x_0}^0(x)}{\partial x_k} dx. \tag{115}$$

One concludes that

$$B_{2,2} = -\int_{\Omega_2} \hat{a}^{ij} \frac{\partial (u^0 - u_{r,1}^0)(x)}{\partial x_j} \frac{\partial N^k(\xi)}{\partial \xi_i} \frac{\partial G_{x_0}^0(x)}{\partial x_k} dx + \int_{\Omega_2} \hat{a}^{ij} \frac{\partial u_{r,2}^0(x)}{\partial x_j} \frac{\partial N^k(\xi)}{\partial \xi_i} \frac{\partial G_{x_0}^0(x)}{\partial x_k} dx$$
$$= I_2 + I_3 + I_4. \tag{116}$$

(90) gives

$$|I_2| \leqslant \frac{M}{\varepsilon} \sum_{r(Q+z)\in B_2} \int_{r(Q+z)} \left| \hat{a}^{ij} \frac{\partial \hat{\psi}(\xi)}{\partial \xi_j} \frac{\partial N^l(\xi)}{\partial \xi_i} (\omega(x) - \omega(x_z)) \right| dx$$

$$\leqslant c\varepsilon\|\omega\|_{W^{1,1}(\Omega_2)} \leqslant c\varepsilon\|u^0\|_{W^{1,\infty}(\Omega)}\|G_{x_0}\|_{W^{2,1}(\Omega_2)} \leqslant c\varepsilon|\ln\varepsilon|\|u^0\|_{W^{1,\infty}(\Omega)}, \tag{117}$$

$$|I_4| \leqslant c\varepsilon|\ln\varepsilon|\|u^0\|_{W^{1,\infty}(\Omega)}. \tag{118}$$

Now we estimate I_3. Assume that l, l' are defined in (87) and $\hat{\omega}(x) = \frac{\partial G^0_{x_0}(x)}{\partial x'_l}(x_l - x_{0,l})$. For any $r(Q+z) \in E_2$, let $\hat{x}_z \in r(Q+z)$ satisfy $\hat{\omega}(\hat{x}_z) = r^{-2}\int_{r(Q+z)}\hat{\omega}(x)dx$. We split I_3 into

$$\begin{aligned} I_3 = &\frac{M}{\varepsilon}\sum_{r(Q+z)\in E_2}\int_{r(Q+z)}\hat{a}^{ij}\frac{\partial\hat{\psi}(\xi)}{\partial\xi_j}\frac{\partial N^l(\xi)}{\partial\xi_i}\hat{\omega}(x_z)dx \\ &+\frac{M}{\varepsilon}\sum_{r(Q+z)\in E_2}\int_{r(Q+z)}\hat{a}^{ij}\frac{\partial\hat{\psi}(\xi)}{\partial\xi_j}\frac{\partial N^l(\xi)}{\partial\xi_i}(\hat{\omega}(x)-\hat{\omega}(x_z))dx = I_{3,1}+I_{3,2}. \end{aligned} \tag{119}$$

Note that $l \neq l'$. Similarly to (115) and (117), one finds that

$$\begin{aligned} |I_{3,1}| &= \left|\frac{M}{\varepsilon N_0^2}\int_{N_0Q}\hat{a}^{ij}\frac{\partial\hat{\psi}(\xi)}{\partial\xi_j}\frac{\partial N^l(\xi)}{\partial\xi_i}d\xi\int_{\Omega_2}\hat{\omega}(x)dx\right| \\ &\leqslant c\|u^0\|_{W^{1,\infty}(\Omega)}\left|\int_{\Omega_2}\frac{\partial G^0_{x_0}(x)}{\partial x'_l}(x_l - x_{0,l})dx\right| \leqslant c\varepsilon|\ln\varepsilon|\|u^0\|_{W^{1,\infty}(\Omega)}, \end{aligned} \tag{120}$$

$$|I_{3,2}| \leqslant c\varepsilon|\ln\varepsilon|\|u^0\|_{W^{1,\infty}(\Omega)}. \tag{121}$$

The combination of (119)–(121) gives

$$|I_3| \leqslant c\varepsilon|\ln\varepsilon|\|u^0\|_{W^{1,\infty}(\Omega)}. \tag{122}$$

Hence by (116)–(118) and (122),

$$|B_{2,2}| \leqslant c\varepsilon|\ln\varepsilon|\|u^0\|_{W^{1,\infty}(\Omega)}. \tag{123}$$

Note that

$$|B_{2,3}| \leqslant c\|u^0\|_{W^{1,\infty}(\Omega)}\|G^0_{x_0}\|_{W^{1,1}(\Omega_3)} \leqslant c\varepsilon|\ln\varepsilon|\|u^0\|_{W^{1,\infty}(\Omega)}. \tag{124}$$

The combination of (108), (109), (123) and (124) shows that

$$|B_2| \leqslant c\varepsilon|\ln\varepsilon|\|u^0\|_{W^{1,\infty}(\Omega)}. \tag{125}$$

Note that

$$|B_3| \leqslant c\|u^0\|_{W^{1,\infty}(\Omega)}\|\theta^\varepsilon_G\|_{W^{1,1}(\Omega-\Omega_1)} \leqslant c\varepsilon|\ln\varepsilon|\|u^0\|_{W^{1,\infty}(\Omega)}, \tag{126}$$

and Theorem 2.1 gives

$$|B_4| \leqslant c\|u^0\|_{W^{1,\infty}(\Omega)}\|G^\varepsilon_{x_0} - \tilde{G}_{x_0} - \theta^\varepsilon_G\|_{W^{1,1}(\Omega-\Omega_1)} \leqslant c\varepsilon|\ln\varepsilon|^2\|u^0\|_{W^{1,\infty}(\Omega)}. \tag{127}$$

Hence by (105), (107) and (125)–(127),

$$|B| \leqslant c\varepsilon|\ln\varepsilon|^2\|u^0\|_{W^{1,\infty}(\Omega)}. \tag{128}$$

The combination of (103), (104) and (128) gives the desired result (100). □

For estimating $|(\tilde{u}_r - u^\varepsilon_r)(x_0)|$, we need the following Taylor expansion.

Lemma 3.3. *Let $\xi = \frac{x}{\varepsilon}$ and $r = 2\varepsilon$. Assume that $|\beta|_{L^\infty(Q)} \leqslant c$ and $f \in C^\infty(\mathbb{R}^2), |D^l f(x)| \leqslant cr^{-l}$ where c is independent of x and r. Then, for any $M \in N \cup \{0\}$, there exist positive constants $q < 2$, $\hat{\beta}_{l,\alpha}$ $(0 \leqslant \alpha \leqslant l \leqslant M)$ and 1-periodic functions $\beta_{M,l,\alpha}(\xi)$ $(l \geqslant M+1, 0 \leqslant \alpha \leqslant l)$ such that*

$$\int_{\varepsilon Q}\beta(\xi)f(x)dx = \sum_{l=0}^{M}\frac{\varepsilon^l}{2^l}\sum_{\alpha=0}^{l}C^\alpha_l\hat{\beta}_{l,\alpha}\int_{\varepsilon Q}\frac{\partial^l f(x)}{\partial x_1^\alpha\partial x_2^{l-\alpha}}dx$$

$$+\sum_{l=M+1}^{\infty}\frac{\varepsilon^l}{2^l}\sum_{\alpha=0}^{l}C_l^{\alpha}\int_{\varepsilon Q}\beta_{M,l,\alpha}(\xi)\frac{\partial^l f(x)}{\partial x_1^{\alpha}\partial x_2^{l-\alpha}}dx, \tag{129}$$

where $\hat{\beta}_{l,\alpha}\,(0\leqslant\alpha\leqslant l\leqslant M)$ *and* $\beta_{M,l,\alpha}(\xi)\,(l\geqslant M+1, 0\leqslant\alpha\leqslant l)$ *satisfy*

$$|\hat{\beta}_{l,\alpha}|\leqslant q^l\|\beta\|_{L^\infty(Q)},\quad \left|\int_Q\beta_{M,M+1,\alpha}(\xi)d\xi\right|\leqslant q^{M+1}\|\beta\|_{L^\infty(Q)}, \tag{130}$$

$$\left|\int_Q\beta_{M,l,\alpha}(\xi)d\xi\right|\leqslant\frac{q^{M+1}\|\beta\|_{L^\infty(Q)}}{(\frac{3}{2})^{l-M-1}},\quad l\geqslant M+2. \tag{131}$$

Proof. First we obtain (129) by induction. Assume that $M=0$. Let $x_0=(0,0)$. Using the Taylor expansion, one finds that

$$f(x)=f(x_0)-\sum_{l=1}^{\infty}\frac{1}{l!}\sum_{\alpha=0}^{l}(-1)^lC_l^{\alpha}x_1^{\alpha}x_2^{l-\alpha}\frac{\partial^l f(x)}{\partial x_1^{\alpha}\partial x_2^{l-\alpha}}=f(x_0)-\sum_{l=1}^{\infty}\frac{\varepsilon^l}{l!}\sum_{\alpha=0}^{l}(-1)^lC_l^{\alpha}\xi_1^{\alpha}\xi_2^{l-\alpha}\frac{\partial^l f(x)}{\partial x_1^{\alpha}\partial x_2^{l-\alpha}},$$

which implies

$$\int_{\varepsilon Q}\beta(\xi)f(x)dx=\int_Q\beta(\xi)d\xi\int_{\varepsilon Q}f(x_0)dx-\sum_{l=1}^{\infty}\frac{(-\varepsilon)^l}{l!}\sum_{\alpha=0}^{l}\int_{\varepsilon Q}C_l^{\alpha}\beta(\xi)\xi_1^{\alpha}\xi_2^{l-\alpha}\frac{\partial^l f(x)}{\partial x_1^{\alpha}\partial x_2^{l-\alpha}}dx. \tag{132}$$

Similarly, from

$$f(x_0)=f(x)+\sum_{l=1}^{\infty}\frac{1}{l!}\sum_{\alpha=0}^{l}(-1)^lC_l^{\alpha}x_1^{\alpha}x_2^{l-\alpha}\frac{\partial^l f(x)}{\partial x_1^{\alpha}\partial x_2^{l-\alpha}},$$

we obtain

$$\int_{\varepsilon Q}f(x_0)dx=\int_{\varepsilon Q}f(x)dx+\sum_{l=1}^{\infty}\frac{\varepsilon^l}{l!}\sum_{\alpha=0}^{l}(-1)^l\int_{\varepsilon Q}C_l^{\alpha}\xi_1^{\alpha}\xi_2^{l-\alpha}\frac{\partial^l f(x)}{\partial x_1^{\alpha}\partial x_2^{l-\alpha}}dx. \tag{133}$$

Set $\hat{\beta}_{0,0}=\int_Q\beta(\xi)d\xi$. Using (132) together with (133), one finds that

$$\int_{\varepsilon Q}\beta(\xi)f(x)dx=\hat{\beta}_{0,0}\int_{\varepsilon Q}f(x)dx+\sum_{l=1}^{\infty}\frac{\varepsilon^l}{l!}\sum_{\alpha=0}^{l}\int_{\varepsilon Q}(-1)^lC_l^{\alpha}[\hat{\beta}_{0,0}-\beta(\xi)]\xi_1^{\alpha}\xi_2^{l-\alpha}\frac{\partial^l f(x)}{\partial x_1^{\alpha}\partial x_2^{l-\alpha}}dx. \tag{134}$$

For $l\geqslant 1, 0\leqslant\alpha\leqslant l$, set

$$\beta_{0,l,\alpha}(\xi)=\frac{2^l(-1)^l}{l!}[\hat{\beta}_{0,0}-\beta(\xi)]\xi_1^{\alpha}\xi_2^{l-\alpha}. \tag{135}$$

With this notation, we can rewrite (134) as

$$\int_{\varepsilon Q}\beta(\xi)f(x)dx=\hat{\beta}_{0,0}\int_{\varepsilon Q}f(x)dx+\sum_{l=1}^{\infty}\frac{\varepsilon^l}{2^l}\sum_{\alpha=0}^{l}\int_{\varepsilon Q}C_l^{\alpha}\beta_{0,l,\alpha}(\xi)\frac{\partial^l f(x)}{\partial x_1^{\alpha}\partial x_2^{l-\alpha}}dx. \tag{136}$$

Then we have obtained (129) for $M=0$. Assume that (129) holds for $M=m$. Now we consider the case that $M=m+1$. Set

$$\hat{\beta}_{m+1,\alpha}=\int_Q\beta_{m,m+1,\alpha}(\xi)d\xi. \tag{137}$$

Similarly to (134), one gets

$$\int_{\varepsilon Q}\beta_{m,m+1,\alpha}(\xi)\frac{\partial^{m+1}f(x)}{\partial x_1^{\alpha}\partial x_2^{m+1-\alpha}}dx=\hat{\beta}_{m+1,\alpha}\int_{\varepsilon Q}\frac{\partial^{m+1}f(x)}{\partial x_1^{\alpha}\partial x_2^{m+1-\alpha}}dx$$

$$+\sum_{s=1}^{\infty}\frac{\varepsilon^s}{s!}\sum_{\gamma=0}^{s}\int_{\varepsilon Q}(-1)^s(\hat{\beta}_{m+1,\alpha}-\beta_{m,m+1,\alpha}(\xi))C_s^{\gamma}\xi_1^{\gamma}\xi_2^{s-\gamma}\frac{\partial^s}{\partial x_1^{\gamma}\partial x_2^{s-\gamma}}\left(\frac{\partial^{m+1}f(x)}{\partial x_1^{\alpha}\partial x_2^{m+1-\alpha}}\right)dx. \tag{138}$$

Assume that $l\geqslant m+2$. Let $\alpha_0=\min\{\alpha,l-m-1\}$. Define $\beta_{m+1,l,\alpha}(\xi)$ by

$$\begin{aligned}\beta_{m+1,l,\alpha}(\xi)&=\beta_{m,l,\alpha}(\xi)\\&\quad+\frac{2^{l-m-1}(-1)^{l-m-1}}{C_l^{\alpha}(l-m-1)!}\sum_{\gamma=0}^{\alpha_0}C_{l-m-1}^{\gamma}C_{m+1}^{\alpha-\gamma}\xi_1^{\gamma}\xi_2^{l-m-1-\gamma}(\hat{\beta}_{m+1,\alpha-\gamma}-\beta_{m,m+1,\alpha-\gamma}(\xi)).\end{aligned} \tag{139}$$

So that (129) holds for $M=m+1$.

Now we prove (130) and (131) by induction. Let $q=\frac{5}{3}$. Assume that $M=0$ and $\beta_{0,l,\alpha}(\xi)$ is defined as (135). One gets

$$|\hat{\beta}_{0,0}|=\left|\int_Q\beta(\xi)d\xi\right|\leqslant\|\beta\|_{L^\infty(Q)}, \tag{140}$$

$$\begin{aligned}\left|\int_Q\beta_{0,1,\alpha}(\xi)d\xi\right|&=2\left|\int_Q[\beta_0-\beta(\xi)]\xi_1^{\alpha}\xi_2^{1-\alpha}d\xi\right|=2\left|\int_Q[\beta_0-\beta(\xi)]\left(\xi_1^{\alpha}\xi_2^{1-\alpha}-\frac{1}{2}\right)d\xi\right|\\&=2\left|\int_Q\beta(\xi)\left(\xi_1^{\alpha}\xi_2^{1-\alpha}-\frac{1}{2}\right)d\xi\right|\leqslant2\|\beta\|_{L^\infty(Q)}\int_Q\left|\left(\xi_1^{\alpha}\xi_2^{1-\alpha}-\frac{1}{2}\right)\right|d\xi\\&=\frac{1}{2}\|\beta\|_{L^\infty(Q)}.\end{aligned} \tag{141}$$

Note that $\int_Q\xi_i^2d\xi=\frac{1}{3},\int_Q\xi_1\xi_2d\xi=\frac{1}{4}$. Let $E=\{\xi\in Q|\xi_1\xi_2\geqslant\frac{1}{4}\}$. We have

$$\int_Q\left|\xi_1\xi_2-\frac{1}{4}\right|d\xi=\int_E\left(\xi_1\xi_2-\frac{1}{4}\right)d\xi+\int_{Q-E}\left(-\xi_1\xi_2+\frac{1}{4}\right)d\xi\leqslant\frac{4+4\ln2-1}{64}+\frac{3-2\ln2}{4\times4}\leqslant\frac{1}{4},$$

$$\left|\int_Q\beta(\xi)\xi_i^2d\xi\right|\leqslant\|\beta\|_{L^\infty(Q)}\left[\int_0^{\frac{\sqrt{3}}{3}}\left(\frac{1}{3}-\xi_i^2\right)d\xi+\int_{\frac{\sqrt{3}}{3}}^{1}\left(\xi_i^2-\frac{1}{3}\right)d\xi\right]=\frac{4\sqrt{3}}{27}\|\beta\|_{L^\infty(Q)}.$$

Consquently,

$$\left|\int_Q\beta_{0,2,\alpha}(\xi)d\xi\right|\leqslant\frac{2^2}{2!}\times\max\left\{\frac{4\sqrt{3}}{27}\|\beta\|_{L^\infty(Q)},\frac{1}{4}\|\beta\|_{L^\infty(Q)}\right\}\leqslant\frac{8\sqrt{3}}{27}\|\beta\|_{L^\infty(Q)}. \tag{142}$$

Similarly, we find

$$\left|\int_Q\beta_{0,3,\alpha}(\xi)d\xi\right|\leqslant\frac{2^3}{3!}\times\frac{1}{3}\|\beta\|_{L^\infty(Q)}=\frac{4}{9}\|\beta\|_{L^\infty(Q)}. \tag{143}$$

For $l\geqslant4$, one gets

$$\begin{aligned}\left|\int_Q\beta_{0,l,\alpha}(\xi)d\xi\right|&=\frac{2^l}{l!}\left|\int_Q[\beta_0-\beta(\xi)]\xi_1^{\alpha}\xi_2^{l-\alpha}d\xi\right|=\frac{2^l}{l!}\left|\int_Q[\beta_0-\beta(\xi)]\left[\xi_1^{\alpha}\xi_2^{l-\alpha}-\int_Q\xi_1^{\alpha}\xi_2^{l-\alpha}d\xi\right]d\xi\right|\\&=\frac{2^l}{l!}\left|\int_Q\beta(\xi)\left[\xi_1^{\alpha}\xi_2^{l-\alpha}-\int_Q\xi_1^{\alpha}\xi_2^{l-\alpha}d\xi\right]d\xi\right|\leqslant\frac{2^l}{l!}\|\beta\|_{L^\infty(Q)}\times2\int_Q\xi_1^{\alpha}\xi_2^{l-\alpha}d\xi\\&\leqslant\frac{2^l}{l!}\|\beta\|_{L^\infty(Q)}\times\frac{2}{l+1}=\frac{2^{l+1}}{(l+1)!}\|\beta\|_{L^\infty(Q)}.\end{aligned} \tag{144}$$

The combination of (140)–(144) leads to (130) and (131) for $M=0$.

Assume that (130) and (131) hold when $M=m$. Now we consider the case that $M=m+1$. It follows from (137) that

$$|\hat{\beta}_{m+1,\alpha}|=\left|\int_Q\beta_{m,m+1,\alpha}(\xi)d\xi\right|\leqslant q^{m+1}\|\beta\|_{L^\infty(Q)}. \tag{145}$$

Now we estimate $|\int_Q(\beta_{m+1,l,\alpha}(\xi)d\xi|$ $(l \geqslant m+2, 0 \leqslant \alpha \leqslant l)$. Note that the definition of α_0 implies $C_l^\alpha = \sum_{\gamma=0}^{\alpha_0} C_{l-m-1}^{\gamma} C_{m+1}^{\alpha-\gamma}$. One observes from (139) that

$$\left|\int_Q \beta_{m+1,l,\alpha}(\xi)d\xi\right| \leqslant \frac{2^{l-m-1}}{(l-m-1)!}\max_{0\leqslant\gamma\leqslant l-m-1}\left|\int_Q(\hat{\beta}_{m+1,\alpha-\gamma} - \beta_{m,m+1,\alpha-\gamma}(\xi))\xi_1^{\gamma}\xi_2^{l-m-1-\gamma}d\xi\right| + \left|\int_Q \beta_{m,l,\alpha}(\xi)d\xi\right| = I + II.$$

Similarly to (141)–(144), one finds that

$$I \leqslant \begin{cases} \dfrac{1}{2}q^{m+1}\|\beta\|_{L^\infty(Q)}, & \text{if} \quad l = m+2, \\ \dfrac{8\sqrt{3}}{27}q^{m+1}\|\beta\|_{L^\infty(Q)}, & \text{if} \quad l = m+3, \\ \dfrac{4}{9}q^{m+1}\|\beta\|_{L^\infty(Q)}, & \text{if} \quad l = m+4, \\ \dfrac{2^{l-m}}{(l-m)!}q^{m+1}\|\beta\|_{L^\infty(Q)}, & \text{if} \quad l \geqslant m+5. \end{cases} \tag{146}$$

Note that

$$II \leqslant \left(\frac{2}{3}\right)^{l-m-1} q^{m+1}\|\beta\|_{L^\infty(Q)}. \tag{147}$$

Hence by (146) and (147),

$$\left|\int_Q \beta_{m+1,l,\alpha}(\xi)d\xi\right| \leqslant \begin{cases} \dfrac{7}{6}q^{m+1}\|\beta\|_{L^\infty(Q)} \leqslant q^{m+2}\|\beta\|_{L^\infty(Q)}, & \text{if} \quad l = m+2, \\ \dfrac{12+8\sqrt{3}}{27}q^{m+1}\|\beta\|_{L^\infty(Q)} \leqslant \dfrac{2}{3}q^{m+2}\|\beta\|_{L^\infty(Q)}, & \text{if} \quad l = m+3, \\ \dfrac{20}{27}q^{m+1}\|\beta\|_{L^\infty(Q)} = \left(\dfrac{2}{3}\right)^2 q^{m+2}\|\beta\|_{L^\infty(Q)}, & \text{if} \quad l = m+4, \\ \left[\left(\dfrac{2}{3}\right)^{l-m-1} + \dfrac{2^{l-m}}{(l-m)!}\right]q^{m+1}\|\beta\|_{L^\infty(Q)} & \\ \qquad \leqslant \left(\dfrac{2}{3}\right)^{l-m-2} q^{m+2}\|\beta\|_{L^\infty(Q)}, & \text{if} \quad l \geqslant m+5. \end{cases} \tag{148}$$

The combination of (145) and (148) shows that (130) and (131) hold when $M = m+1$. The proof is completed. □

The following result can be concluded from Lemma 3.3, directly.

Lemma 3.4. *Let $\xi = \frac{x}{\varepsilon}$ and $r = 2\varepsilon$. Assume that $|\beta|_{L^\infty(Q)} \leqslant c$ and $f \in C^\infty(\mathbb{R}^2), |D^l f(x)| \leqslant cr^{-l}$ where c is independent of x and r. Then there exist constants $\hat{\beta}_{l,\alpha}$ such that*

$$\int_{\varepsilon Q} \beta(\xi)f(x)dx = \sum_{l=0}^{\infty}\sum_{\alpha=0}^{l} \frac{\varepsilon^l C_l^\alpha}{2^l}\hat{\beta}_{l,\alpha}\int_{\varepsilon Q}\frac{\partial^l f(x)}{\partial x_1^\alpha \partial x_2^{l-\alpha}}dx,$$

where $\hat{\beta}_{l,\alpha}$ satisfies

$$\sum_{l=0}^{\infty}\sum_{\alpha=0}^{l}|\hat{\beta}_{l,\alpha}|\frac{C_l^\alpha}{2^l}r^{-l}\varepsilon^l \leqslant c|\beta|_{L^\infty(Q)}. \tag{149}$$

Now we use Theorem 2.1, Lemmas 2.2, 3.1 and 3.4 to estimate $|(\tilde{u}_r - u_r^\varepsilon)(x_0)|$.

Lemma 3.5. *Under the assumptions of Theorem* 3.1, *there exists a constant c such that*

$$|(\tilde{u}_r - u_r^\varepsilon)(x_0)| \leqslant c\varepsilon|\ln\varepsilon|^2\|u^0\|_{W^{1,\infty}(\Omega)}. \tag{150}$$

Proof. Set $e_r^\varepsilon(x) = u_r^\varepsilon(x) - \tilde{u}_r(x)$. We split $e_r^\varepsilon(x)$ into

$$e_r^\varepsilon(x) = w_r^\varepsilon(x) + \theta_r^\varepsilon(x), \tag{151}$$

where $w_r^\varepsilon(x)$ satisfies the problem

$$\begin{cases} L_\varepsilon w_r^\varepsilon(x) \equiv \dfrac{\partial}{\partial x_i}\left(a^{ij}\left(\dfrac{x}{\varepsilon}\right)\dfrac{\partial w_r^\varepsilon(x)}{\partial x_j}\right) = \dfrac{\partial}{\partial x_i}\left(a^{ij}\left(\dfrac{x}{\varepsilon}\right)\dfrac{\partial e_r^\varepsilon(x)}{\partial x_j}\right), & \text{in} \quad \Omega, \\ w_r^\varepsilon(x) = 0, & \text{on} \quad \partial\Omega, \end{cases} \tag{152}$$

and $\theta_r^\varepsilon(x)$ satisfies the problem

$$\begin{cases} L_\varepsilon \theta_r^\varepsilon(x) \equiv \dfrac{\partial}{\partial x_i}\left(a^{ij}\left(\dfrac{x}{\varepsilon}\right)\dfrac{\partial \theta_r^\varepsilon(x)}{\partial x_j}\right) = 0, & \text{in} \quad \Omega, \\ \theta_r^\varepsilon(x) = -\varepsilon N^k\left(\dfrac{x}{\varepsilon}\right)\dfrac{\partial u_r^0(x)}{\partial x_k}, & \text{on} \quad \partial\Omega. \end{cases} \tag{153}$$

First we estimate $|w_r^\varepsilon(x_0)|$. Assume that $\alpha^j(\xi) = (\alpha_{ki}^j(\xi))$ is defined as Lemma 2.4. Similarly to Lemma 2.4, one observes that, for any $v \in H_0^1(\Omega)$,

$$\begin{aligned} &\int_\Omega a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial w_r^\varepsilon(x)}{\partial x_j}\frac{\partial v(x)}{\partial x_i}dx \\ &\quad = -\int_\Omega \varepsilon\left[\frac{\partial}{\partial x_k}\left(\alpha_{ki}^j\left(\frac{x}{\varepsilon}\right)\frac{\partial u_r^0(x)}{\partial x_j}\right) + \left(a_{ij}\left(\frac{x}{\varepsilon}\right)N^k\left(\frac{x}{\varepsilon}\right) - \alpha_{ki}^j\left(\frac{x}{\varepsilon}\right)\right)\frac{\partial^2 u_r^0(x)}{\partial x_j \partial x_k}\right]\frac{\partial v(x)}{\partial x_i}dx \\ &\quad = -\int_\Omega \varepsilon\left[\left(a_{ij}\left(\frac{x}{\varepsilon}\right)N^k\left(\frac{x}{\varepsilon}\right) - \alpha_{ki}^j\left(\frac{x}{\varepsilon}\right)\right)\frac{\partial^2 u_r^0(x)}{\partial x_j \partial x_k}\right]\frac{\partial v(x)}{\partial x_i}dx. \end{aligned} \tag{154}$$

Consquently,

$$\begin{aligned} w_r^\varepsilon(x_0) &= -\int_\Omega a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial w_r^\varepsilon(x)}{\partial x_j}\frac{\partial G_{x_0}^\varepsilon(x)}{\partial x_i}dx \\ &= \varepsilon\int_\Omega\left[-\alpha_{ki}^j\left(\frac{x}{\varepsilon}\right) + a_{ij}\left(\frac{x}{\varepsilon}\right)N^k\left(\frac{x}{\varepsilon}\right)\right]\frac{\partial^2 u_r^0(x)}{\partial x_j \partial x_k}\frac{\partial G_{x_0}^\varepsilon(x)}{\partial x_i}dx = \int_\Omega \hat{F}_{r,i}(x)\frac{\partial G_{x_0}^\varepsilon(x)}{\partial x_i}dx, \end{aligned}$$

where

$$\hat{F}_{r,i}(x) = \varepsilon\left[-\alpha_{ki}^j\left(\frac{x}{\varepsilon}\right) + a_{ij}\left(\frac{x}{\varepsilon}\right)N^k\left(\frac{x}{\varepsilon}\right)\right]\frac{\partial^2 u_r^0(x)}{\partial x_j \partial x_k}.$$

Assume that Ω_1 is defined as (102). We split $w_r^\varepsilon(x_0)$ into

$$w_r^\varepsilon(x_0) = \int_{\Omega_1}\hat{F}_{r,i}(x)\frac{\partial G_{x_0}^\varepsilon(x)}{\partial x_i}dx + \int_{\Omega-\Omega_1}\hat{F}_{r,i}(x)\frac{\partial G_{x_0}^\varepsilon(x)}{\partial x_i}dx = A + B. \tag{155}$$

It follows from Lemmas 2.2 and 3.1 that

$$|A| \leqslant c\varepsilon\|u_r^0\|_{W^{2,\infty}(\Omega)}\|G_{x_0}^\varepsilon\|_{W^{1,1}(\Omega_1)} \leqslant c\varepsilon|\ln\varepsilon|\|u^0\|_{W^{1,\infty}(\Omega)}. \tag{156}$$

Now we estimate B. One finds that B can be decomposed into

$$\begin{aligned} B = &\int_{\Omega-\Omega_1}\hat{F}_{r,i}(x)\frac{\partial G_{x_0}^0(x)}{\partial x_i}dx + \int_{\Omega-\Omega_1}\hat{F}_{r,i}(x)\frac{\partial(\varepsilon N^k(\frac{x}{\varepsilon})\frac{\partial G_{x_0}^0(x)}{\partial x_k})}{\partial x_i}dx \\ &+ \int_{\Omega-\Omega_1}\hat{F}_{r,i}(x)\frac{\partial(G_{x_0}^\varepsilon - \tilde{G}_{x_0})(x)}{\partial x_i}dx = B_1 + B_2 + B_3. \end{aligned} \tag{157}$$

First we estimate B_1. Assume that Ω_2 and Ω_3 are defined by (85) and (102), respectively. We split B_1 into

$$B_1 = \int_{\Omega_2} \hat{F}_{r,i}(x)\frac{\partial G^0_{x_0}(x)}{\partial x_i}dx + \int_{\Omega_3} \hat{F}_{r,i}(x)\frac{\partial G^0_{x_0}(x)}{\partial x_i}dx = B_{1,1} + B_{1,2}. \tag{158}$$

Set

$$\beta_{ijk}(\xi) = -\alpha^j_{ki}(\xi) + a_{ij}(\xi)N^k(\xi), \qquad w_{ijk}(x) = \frac{\partial^2 u^0_r(x)}{\partial x_j \partial x_k}\frac{\partial G^0_{x_0}(x)}{\partial x_i}. \tag{159}$$

With these notations, we rewrite $B_{1,1}$ into

$$B_{1,1} = \varepsilon \int_{\Omega_2} \beta_{ijk}\left(\frac{x}{\varepsilon}\right) w_{ijk}(x)dx. \tag{160}$$

The combination of Lemma 3.4 and (160) shows that there exist constants $\hat{\beta}_{l,\alpha,ijk}$ such that

$$\begin{aligned} |B_{1,1}| &\leqslant \varepsilon \sum_{l=0}^{\infty}\sum_{\alpha=0}^{l}\sum_{1\leqslant i,j,k\leqslant 2} \frac{\varepsilon^l C^\alpha_l}{2^l}|\hat{\beta}_{l,\alpha,ijk}|\left|\int_{\Omega_2}\frac{\partial^l w_{ijk}(x)}{\partial x_1^\alpha \partial x_2^{l-\alpha}}dx\right| \\ &\leqslant c\varepsilon \sum_{1\leqslant i,j,k\leqslant 2}\left|\int_{\Omega_2} w_{ijk}(x)dx\right| + \sum_{k=1}^{\infty}\sum_{\alpha=0}^{l}\sum_{1\leqslant i,j,k\leqslant 2}\frac{\varepsilon^{l+1}C^\alpha_l}{2^l}|\hat{\beta}_{l,\alpha,ijk}|\|w_{ijk}\|_{W^{l-1,1}(\partial\Omega_2)} \\ &= B_{1,1,1} + B_{1,1,2}. \end{aligned} \tag{161}$$

Assume that n_k means the normal derivative in the x_k direction. Note that

$$\begin{aligned} |B_{1,1,1}| &\leqslant c\varepsilon \sum_{1\leqslant i,j,k\leqslant 2}\left|\int_{\Omega_2}\frac{\partial^2 u^0_r(x)}{\partial x_j\partial x_k}\frac{\partial G^0_{x_0}(x)}{\partial x_i}dx\right| \\ &\leqslant c\varepsilon \sum_{1\leqslant i,j,k\leqslant 2}\left|\int_{\Omega_2}\frac{\partial u^0_r(x)}{\partial x_j}\frac{\partial^2 G^0_{x_0}(x)}{\partial x_k\partial x_i}dx\right| + c\varepsilon\sum_{1\leqslant i,j,k\leqslant 2}\left|\int_{\partial\Omega_2}\frac{\partial u^0_r(x)}{\partial x_j}\frac{\partial G^0_{x_0}(x)}{\partial x_i}n_k ds\right| \\ &\leqslant c\varepsilon\|u^0\|_{W^{1,\infty}(\Omega)}\|G^0_{x_0}\|_{W^{2,1}(\Omega_2)} + c\varepsilon\|u^0\|_{W^{1,\infty}(\Omega)}\|G^0_{x_0}\|_{W^{1,1}(\partial\Omega_2)} \leqslant c\varepsilon|\ln\varepsilon|\|u^0\|_{W^{1,\infty}(\Omega)}, \end{aligned} \tag{162}$$

and Lemma 3.4 gives

$$\begin{aligned} |B_{1,1,2}| &\leqslant c\sum_{l=1}^{\infty}\sum_{\alpha=0}^{l}\sum_{1\leqslant i,j,k\leqslant 2}\frac{\varepsilon^{l+1}C^\alpha_l}{2^l}|\hat{\beta}_{l,\alpha,ijk}|r^{-l}\|u^0_r\|_{W^{1,\infty}(\Omega_2)}\|G^0_{x_0}\|_{W^{1,1}(\partial\Omega_2)} \\ &\leqslant c\varepsilon|\ln\varepsilon|\|u^0\|_{W^{1,\infty}(\Omega)}. \end{aligned} \tag{163}$$

One concludes from (161)–(163) that

$$|B_{1,1}| \leqslant c\varepsilon|\ln\varepsilon|\|u^0\|_{W^{1,\infty}(\Omega)}. \tag{164}$$

Note that

$$|B_{1,2}| \leqslant c\varepsilon\|u^0_r\|_{W^{2,\infty}(\Omega)}\|G^0_{x_0}\|_{W^{1,1}(\Omega_3)} \leqslant c\varepsilon\varepsilon^{-1}\|u^0\|_{W^{1,\infty}(\Omega)}c\varepsilon|\ln\varepsilon| \leqslant c\varepsilon|\ln\varepsilon|\|u^0\|_{W^{1,\infty}(\Omega)}. \tag{165}$$

Hence by (158), (164) and (165),

$$|B_1| \leqslant c\varepsilon|\ln\varepsilon|\|u^0\|_{W^{1,\infty}(\Omega)}. \tag{166}$$

We now turn to the estimation of B_2 and B_3. Similarly to (164) and (165), we have

$$\left|\int_{\Omega_2}\hat{F}_{r,i}(x)\frac{\partial(\varepsilon N^k(\frac{x}{\varepsilon})\frac{\partial G^0_{x_0}(x)}{\partial x_k})}{\partial x_i}dx\right| \leqslant c\varepsilon|\ln\varepsilon|\|u^0\|_{W^{1,\infty}(\Omega)}, \tag{167}$$

$$\left|\int_{\Omega_3}\hat{F}_{r,i}(x)\frac{\partial(\varepsilon N^k(\frac{x}{\varepsilon})\frac{\partial G^0_{x_0}(x)}{\partial x_k})}{\partial x_i}dx\right| \leqslant c\varepsilon\|u^0_r\|_{W^{2,\infty}(\Omega)}\varepsilon\|G^0_{x_0}\|_{W^{2,1}(\Omega_3)} + c\varepsilon\|u^0_r\|_{W^{2,\infty}(\Omega)}\|G^0_{x_0}\|_{W^{1,1}(\Omega_3)}$$
$$\leqslant c\varepsilon\varepsilon^{-1}\|u^0\|_{W^{1,\infty}(\Omega)}\varepsilon|\ln\varepsilon| + c\varepsilon\varepsilon^{-1}\|u^0\|_{W^{1,\infty}(\Omega)}\varepsilon|\ln\varepsilon| \leqslant c\varepsilon|\ln\varepsilon|\|u^0\|_{W^{1,\infty}(\Omega)}. \quad (168)$$

The combination of (167) and (168) shows

$$|B_2| \leqslant c\varepsilon|\ln\varepsilon|\|u^0\|_{W^{1,\infty}(\Omega)}. \quad (169)$$

It follows from Theorem 2.1 that

$$|B_3| \leqslant \left|\int_{\Omega-\Omega_1}\hat{F}_{r,i}(x)\frac{\partial(G^\varepsilon_{x_0}-\tilde{G}_{x_0}-\theta^\varepsilon_G)(x)}{\partial x_i}dx\right| + \left|\int_{\Omega-\Omega_1}\hat{F}_{r,i}(x)\frac{\partial\theta^\varepsilon_G(x)}{\partial x_i}dx\right|$$
$$\leqslant c\varepsilon\|u^0_r\|_{W^{2,\infty}(\Omega)}\|G^\varepsilon_{x_0}-\tilde{G}_{x_0}-\theta^\varepsilon_G\|_{W^{1,1}(\Omega-\Omega_1)} + c\varepsilon\|u^0_r\|_{W^{2,\infty}(\Omega)}\|\theta^\varepsilon_G\|_{W^{1,1}(\Omega-\Omega_1)}$$
$$\leqslant c\varepsilon\varepsilon^{-1}\|u^0\|_{W^{1,\infty}(\Omega)}\varepsilon|\ln\varepsilon|^2 + c\varepsilon\varepsilon^{-1}\|u^0\|_{W^{1,\infty}(\Omega)}\varepsilon|\ln\varepsilon| \leqslant c\varepsilon|\ln\varepsilon|^2\|u^0\|_{W^{1,\infty}(\Omega)}, \quad (170)$$

which, together with (157), (166) and (169), leads to

$$|B| \leqslant c\varepsilon|\ln\varepsilon|^2\|u^0\|_{W^{1,\infty}(\Omega)}. \quad (171)$$

Hence, by (155), (156) and (171),

$$|w^\varepsilon_r(x_0)| \leqslant c\varepsilon|\ln\varepsilon|^2\|u^0\|_{W^{1,\infty}(\Omega)}. \quad (172)$$

Notice that

$$|\theta^\varepsilon_r(x_0)| \leqslant c\varepsilon|\ln\varepsilon|\|u^0\|_{W^{1,\infty}(\Omega)}. \quad (173)$$

The combination of (172) and (173) implies the desired result (150). □

Now we give an error estimate between the solutions $\tilde{u}_r(x)$ and $\tilde{u}(x)$.

Lemma 3.6. *Under the assumptions of Theorem* 3.1, *there exists a constant c such that*

$$\|\tilde{u}_r-\tilde{u}\|_{L^\infty(\Omega)} \leqslant c\varepsilon\|u^0\|_{W^{1,\infty}(\Omega)}. \quad (174)$$

Proof. Using Lemma 3.2, we have

$$\|\tilde{u}_r-\tilde{u}\|_{L^\infty(\Omega)} \leqslant \|u^0_r-u^0\|_{L^\infty(\Omega)} + c\varepsilon\|u^0_r-u^0\|_{W^{1,\infty}(\Omega)}$$
$$\leqslant c\varepsilon\|u^0\|_{W^{1,\infty}(\Omega)} + c\varepsilon\|u^0\|_{W^{1,\infty}(\Omega)} \leqslant c\varepsilon\|u^0\|_{W^{1,\infty}(\Omega)}. \quad (175)$$

This completes the proof. □

Finally, we are now in a position to give a proof of Theorem 3.1.

Proof of Theorem 3.1. Combining Lemma 3.2 with Lemmas 3.5 and 3.6, we have

$$|(u^\varepsilon-\tilde{u})(x_0)| \leqslant |(u^\varepsilon-u^\varepsilon_r)(x_0)| + |(u^\varepsilon_r-\tilde{u}_r)(x_0)| + \|\tilde{u}_r-\tilde{u}\|_{L^\infty(\Omega)}$$
$$\leqslant c\varepsilon|\ln\varepsilon|^2\|u^0\|_{W^{1,\infty}(\Omega)} + c\varepsilon|\ln\varepsilon|^2\|u^0\|_{W^{1,\infty}(\Omega)} + c\varepsilon\|u^0\|_{W^{1,\infty}(\Omega)}$$
$$\leqslant c\varepsilon|\ln\varepsilon|^2\|u^0\|_{W^{1,\infty}(\Omega)}. \quad (176)$$

This ends the proof. □

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References

1 Allaire G. Homogenization and two-scale convergence. SIAM Math Anal, 1992, 23: 1482–1518
2 Avellaneda M, Lin F H. Theory of homogenization. Comm Pure Appl Math, 1989, 42: 803–847
3 Azzam A. Smoothness properties of bounded solution of dirichlet problem for elliptic equations in regions with corners on the boundaries. Canad Math Bull, 1980, 23: 213–239
4 Bachvalov N S, Panasenko G P. Homogenization of Processes in Periodic Media. Moscow: Nauka, 1984
5 Bensussan A, Lions J L, Papanicolou G. Asymptotic Analysis of Periodic Structures. Amsterdan: North-Holland, 1978
6 Cao L Q. Multi-scale asymptotic expansion and finite element methods for the mixed boundary value problems of second order elliptic equation in perforated domains. Numer Math, 2006, 103: 11–45
7 Cao L Q, Cui J Z. Asymptotic expansions and numerical algorithms of eigenvalues and eigenfunctions of the Dirichlet problems for second order elliptic equations in perforated domains. Numer Math, 2004, 96: 525–581
8 Cao L Q, Luo J L. Multiscale numerical algorithm for the elliptic eigenvalue problem with the mixed boundary in perforated domains. Appl Numer Math, 2007, 57: 1201–1226
9 Chen J R, Cui J Z. Two-scale fem for elliptic mixed boundary value problems with small periodic coefficients. J Comput Math, 2001, 19: 549–560
10 Chen Z M, Deng W B, Ye H. A new upscaling method for the solute transport equations. Discrete Contin Dynam Sys Ser B, 2005, 13: 493–515
11 Chen Z M, Hou T Y. A mixed multi-scale finite element method for elliptic problems with oscillating coefficients. Math Comp, 2002, 72: 541–576
12 Chen Z M, Yue X Y. Numerical homogenization of well singularites in the flow transport through heterogeneous porous media. SIAM Multiscale Model Simul, 2003, 1: 260–303
13 Doina C, Jeannine S, Jean P. Homogenization of Reticulated Structures. Berlin: Springer-Verlag, 1998
14 E W, Ming P B, Zhang P W. Analysis of the heterogeneous multi-scale method for elliptic homogenization problems. J Amer Math Soc, 2005, 18: 121–156
15 He W M, Cui J Z, A pointwise estimate on the 1-order approximation of $G^{\varepsilon}_{x_0}$. IMA J Appl Math, 2005, 70: 241–269
16 He W M, Cui J Z. A finite element method for elliptic problems with rapidly oscillating coefficients. BIT Numer Math, 2007, 47: 77–102
17 Hou T Y. Convergence of a multi-scale finite element method for elliptic problem with oscillation coefficients. Math Comp, 1999, 68: 913–943
18 Jikov V V, Kozlov S M, Oleinik O A. Homogenization of Differential Operators and Integral Functionals. Berlin: Springer-Verlag, 1994
19 Li Y Y, Cui J Z. Two-scale analysis method for predicting heat transfer performance of composite materials with random grain distribution. Sci China Ser A, 2004, 47: 101–110
20 Ladyzhenskaia O A, Uraltseva N N. Linear and Quailinear Elliptic Equations. New York: Academic Press, 1968
21 Ming P B, Yue X Y. Numerical methods for multiscale elliptic problems. J Comput Phys, 2006, 214: 421–445
22 Oleinik O A, Shamaev A S, Yosifian G A. Mathematical Problems in Elasticity and Homogenization. Amsterdam: North-Holland, 1992
23 Sanchez-Palencia E. Nonhomogeneous Media and Vibration Theory. In: Lect Notes in Phys, vol. 127. Berlin: Springer-Verlag, 1980
24 Wang X, Cao L Q. The hole-filling method and the uniform multiscale computation of the elastic equations in perforated domains. Int J Numer Anal Model, 2008, 5: 612–634
25 Yalchin R E, Hou T Y, Wu X H. Convergence of nonconforming multi-scale finite element method. SIAM J Numer Anal, 2000, 37: 888–910