SCIENCE CHINA
Mathematics

• ARTICLES •

May 2010 Vol. 53 No. 5: 1253–1264
doi: 10.1007/s11425-010-0013-y

An extension of Mok's theorem on the generalized Frankel conjecture

GU HuiLing* & ZHANG ZhuHong

Department of Mathematics, Sun Yat-Sen University, Guangzhou 510275*, China*
*Email: ghl*1026*@tom.com, juhoncheung@hotmail.com*

Received October 2, 2007; accepted June 9, 2009; published online December 24, 2009

Abstract In this paper, we will give an extension of Mok's theorem on the generalized Frankel conjecture under the condition of the orthogonal holomorphic bisectional curvature.

Keywords Kähler Ricci flow, orthogonal holomorphic bisectional curvature, first Chern class

MSC(2000): 53C55

Citation: Gu H L, Zhang Z H. An extension of Mok's theorem on the generalized Frankel conjecture. Sci China Math, 2010, 53(5): 1253–1264, doi: 10.1007/s11425-010-0013-y

1 Introduction

Let M^n be a complex n-dimensional compact Kähler manifold. One of the interesting problems is to give the classification of the manifolds under certain curvature conditions. Corresponding to the sectional curvature condition in Riemannian geometry, one usually considers the holomorphic bisectional curvature in complex differential geometry. In 1979 Mori [10] and in 1980 Siu-Yau [12] independently proved the famous Frankel conjecture by using different methods. They proved that *any compact Kähler manifold with positive holomorphic bisectional curvature must be biholomorphic to the complex projective space.* After the work of Mori and Siu-Yau, in 1988, Mok [9] generalized the Frankel conjecture to the nonnegative case, usually we call it the generalized Frankel conjecture which states that *any compact irreducible Kähler manifold with nonnegative bisectional curvature must be either a Hermitian symmetric manifold or biholomorphic to the complex projective space.* Recently, based on the work of Brendle-Schoen [2], the first author [5] gave a simple and completely transcendental proof to Mok's theorem on the generalized Frankel conjecture. In the late 1980's, Cao and Hamilton [3] introduced the concept of orthogonal holomorphic bisectional curvature and observed that the nonnegativity of the orthogonal holomorphic bisectional curvature is preserved under the Kähler-Ricci flow. (For the definition of the orthogonal holomorphic bisectional curvature we will give in the following.) In 2006, Chen [4] generalized the Frankel conjecture in another aspect with the orthogonal holomorphic bisectional curvature but under some additional condition. He proved that *any compact irreducible Kähler manifold with positive orthogonal holomorphic bisectional curvature and* $c_1 > 0$ *must be biholomorphic to the complex projective space.*

*Corresponding author

Definition 1.1. *A complex n-dimensional ($n \geqslant 2$) Kähler manifold (M^n, h) is said to have nonnegative orthogonal holomorphic bisectional curvature if for any orthonormal basis $\{e_\alpha\}$, the following holds $R(e_\alpha, \overline{e_\alpha}, e_\beta, \overline{e_\beta}) = R_{\alpha\bar{\alpha}\beta\bar{\beta}} \geqslant 0$, for any $\alpha \neq \beta$.*

If we consider the Kähler manifold as a Riemannian manifold, then we define the manifold has nonnegative orthogonal holomorphic bisectional curvature by $R(u_i, Ju_i, Ju_j, u_j) \geqslant 0$, for any $\langle u_i, u_j\rangle = \langle u_i, Ju_j\rangle = 0$, where J is the complex structure of M. The above Definition 1.1 is equivalent to that in the Riemannian case. Indeed, we can choose an orthonormal basis $\{u_1, u_2, \ldots, u_{2n}\}$ such that $Ju_i = u_{n+i}$ for $i = 1, 2, \ldots, n$. Set $e_i = \frac{1}{\sqrt{2}}(u_i - \sqrt{-1}Ju_i)$, then $\{e_i\}$ is an orthonormal basis. It follows that

$$R(e_i, \overline{e_i}, e_j, \overline{e_j}) = R_{i\bar{i}j\bar{j}} = R(u_i, Ju_i, Ju_j, u_j),$$

for any $i \neq j$. This implies the two definitions are equivalent.

Recently, Seshadri [11] gave the classification of manifolds with nonnegative isotropic curvature. He proved that *any compact irreducible Kähler manifold with nonnegative isotropic curvature must be either a Hermitian symmetric manifold or biholomorphic to the complex projective space.* From the computation in Lemma 2.1 in [11], we can see that nonnegative isotropic curvature implies the nonnegative orthogonal holomorphic bisectional curvature. However, the converse is not true. Following we give an example and other examples can be given in a similar way:

Example 1.2. Let $(M, h) = (\Sigma, g) \times (CP^n, g_0)$, where Σ is a Riemann surface with the Gauss curvature $\kappa(\Sigma) \geqslant -4$ and $\min(\kappa(\Sigma)) = -4$ and g_0 is the standard Fubini-Study metric such that the sectional curvature of CP^n satisfies $1 \leqslant K(p) \leqslant 4$. In the following, we want to show that M has nonnegative orthogonal holomorphic bisectional curvature but the isotropic curvature is not nonnegative.

Indeed, suppose that $\widehat{e_0}$ and $\{\widehat{e_i}\}(1 \leqslant i \leqslant n)$, are the orthonormal basis of $T_p^{1,0}(\Sigma)$ and $T_q^{1,0}(CP^n)$ respectively. Then we can naturally extend them to the orthonomal basis $\{e_i\}(0 \leqslant i \leqslant n)$, of $T_x^{1,0}(M)$ at the point $x = (p, q) \in M$, such that $pr_1(e_0) = \widehat{e_0}$ and $pr_2(e_i) = \widehat{e_i}$, where pr_1, pr_2 denote the canonical projection onto Σ and CP^n respectively.

Now for any two orthogonal vectors X, Y on M, we assume that $X = \sum_{i=0}^n a_i e_i$ and $Y = \sum_{i=0}^n b_i e_i$, where a_i, b_i are complex numbers satisfying $\sum_{i=0}^n a_i \overline{b_i} = 0$.

Then by a direct computation we can get

$$\begin{aligned} R(X, \overline{X}, Y, \overline{Y}) &= |a_0|^2 |b_0|^2 R_{0\bar{0}0\bar{0}} + 4\sum_{i=1}^n (|a_i|^2|b_i|^2) + 2\sum_{i=1}^n \sum_{j\neq i} (|a_i|^2|b_j|^2 + a_i\overline{a_j}b_j\overline{b_i}) \\ &\geqslant -4\left|\sum_{i=1}^n a_i\overline{b_i}\right|^2 + 4\sum_{i=1}^n (|a_i|^2|b_i|^2) + 2\sum_{i=1}^n \sum_{j\neq i} (|a_i|^2|b_j|^2 + a_i\overline{a_j}b_j\overline{b_i}) \\ &= 2\sum_{1\leqslant i<j\leqslant n} |a_i b_j - a_j b_i|^2 \geqslant 0. \end{aligned}$$

This implies that the orthogonal holomorphic bisectional curvature of M is nonnegative. On the other hand, by a direct computation or the result of [8], it is easy to see that the isotropic curvature is not nonnegative.

Clearly nonnegative holomorphic bisectional curvature also implies the nonnegative orthogonal holomorphic bisectional curvature, naturally we want to know the relations between holomorphic bisectional curvature and isotropic curvature. By the work of Ivey [7], we know that in the complex 2-dimensional case, nonnegative holomorphic bisectional curvature implies nonnegative isotropic curvature. So the result of Seshadri [11] can be viewed as a generalization of Mok's theorem on the generalized Frankel conjecture in complex 2-dimension. But in higher dimensional case, we do not know whether this is also true, since the nonnegative holomorphic bisectional curvature means the bisectional curvature is nonnegative on any holomorphic complex plane, while nonnegative isotropic curvature requires on any 2-dimensional isotropic plane. Even though, we know that both holomorphic bisectional curvature and isotropic curvature imply orthogonal holomorphic bisectional curvature. So orthogonal holomorphic bisectional curvature is the weakest one among the three curvature conditions. In [4], Chen asked a question: whether a compact

Kähler manifold with positive orthogonal holomorphic bisectional curvature necessary has $c_1 > 0$. In this paper, we give an affirmative answer to this question and hence solve the Question/Conjecture 1.6 in [4]. Moreover, we will also give a complete classification of manifolds with nonnegative orthogonal holomorphic bisectional curvature. This can be considered as an extension of the generalized Frankel conjecture. Our main result is the following

Theorem 1.3. *Suppose (M^n, h) is an n-dimensional $(n \geqslant 2)$ compact Kähler manifold of nonnegative orthogonal holomorphic bisectional curvature. Let $(\tilde{M}^n, \tilde{h})$ be its universal covering space. Then $(\tilde{M}^n, \tilde{h})$ is isometrically biholomorphic to one of the following two cases*

(1) $(C^k, h_0) \times (M_1, h_1) \times \cdots \times (M_l, h_l) \times (CP^{n_1}, \theta_1) \times \cdots \times (CP^{n_r}, \theta_r)$, *where h_0 denotes the Euclidean metric on $C^k, h_i\,(1 \leqslant i \leqslant l)$ are canonical metrics on the irreducible compact Hermitian symmetric spaces M_i of* rank $\geqslant 2$, *and $\theta_j\,(1 \leqslant j \leqslant r)$ is a Kähler metric on CP^{n_j} carrying nonnegative orthogonal holomorphic bisectional curvature*;

(2) $(Y, g_0) \times (M_1, h_1) \times \cdots \times (M_l, h_l) \times (CP^{n_1}, \theta_1) \times \cdots \times (CP^{n_r}, \theta_r)$, *where Y is a simply connected Riemann surface with Gauss curvature negative somewhere or a simply connected noncompact Kähler manifold with* $\dim(Y) \geqslant 2$ *and has nonnegative orthogonal holomorphic bisectional curvature and the minimum of the holomorphic sectional curvature < 0 somewhere, $M_i, CP^{n_j}\,(1 \leqslant i \leqslant l, 1 \leqslant j \leqslant r)$ are the same as in Case* (1). *Moreover, we have the holomorphic sectional curvatures of M_i and CP^{n_j} are not less than* $-\min\{$*holomorphic sectional curvature of* $Y\} > 0$.

This paper contains four sections and the organization is as follows. In Section 2, we will get the nonnegativity and the equation of the orthogonal bisectional curvature under the Ricci flow. In Section 3, we will prove the positivity of the first Chern class under the positive orthogonal holomorphic bisectional curvature condition and give some results on the irreducible manifolds which will be used in the proof of our main theorem. In Section 4, we will complete the proof of Theorem 1.3.

2 Nonnegativity of the orthogonal bisectional curvature under the Ricci flow

In this section, we first give a proof of the preserving nonnegativity of the orthogonal bisectional curvature under the Ricci flow, which is announced by Cao and Hamilton in an unpublished work [3]. And then we give the equation of the orthogonal bisectional curvature under the Ricci flow.

Suppose that (M^n, h) is a compact Kähler manifold with nonnegative orthogonal holomorphic bisectional curvature and $g_{i\bar{j}}(t), 0 \leqslant t \leqslant \delta$, is the solution to the Kähler Ricci flow with the initial data h. Let P be the bundle with the fixed metric h and the fibre over $p \in M$ consists of all orthogonal 2-vectors $\{X, Y\} \subset T_p^{1,0}(M)$. We define a function on $P \times (0, \delta)$ by $u(\{X, Y\}, t) = R(X, \overline{X}, Y, \overline{Y})$, where R denotes the pull-back of the curvature tensor of $g_{i\bar{j}}(t)$.

Proposition 2.1. *The above function u is nonnegative as long as the solution exists.*

Proof. According to Hamilton [6], under the evolving orthonormal frame $\{e_\alpha\}$, we have

$$\begin{aligned}
\frac{\partial}{\partial t} R_{\alpha\bar{\alpha}\beta\bar{\beta}} &= \triangle R_{\alpha\bar{\alpha}\beta\bar{\beta}} + \sum_{\mu,\nu} (R_{\alpha\bar{\alpha}\mu\bar{\nu}} R_{\nu\bar{\mu}\beta\bar{\beta}} - |R_{\alpha\bar{\mu}\beta\bar{\nu}}|^2 + |R_{\alpha\bar{\beta}\mu\bar{\nu}}|^2) \\
&= \triangle R_{\alpha\bar{\alpha}\beta\bar{\beta}} + \sum_{\mu,\nu=\alpha,\beta} (R_{\alpha\bar{\alpha}\mu\bar{\nu}} R_{\nu\bar{\mu}\beta\bar{\beta}} - |R_{\alpha\bar{\mu}\beta\bar{\nu}}|^2 + |R_{\alpha\bar{\beta}\mu\bar{\nu}}|^2) \\
&\quad + \Bigg(\sum_{\substack{\mu=\alpha,\beta \\ \nu\neq\alpha,\beta}} + \sum_{\substack{\nu=\alpha,\beta \\ \mu\neq\alpha,\beta}} \Bigg) (R_{\alpha\bar{\alpha}\mu\bar{\nu}} R_{\nu\bar{\mu}\beta\bar{\beta}} - |R_{\alpha\bar{\mu}\beta\bar{\nu}}|^2 + |R_{\alpha\bar{\beta}\mu\bar{\nu}}|^2) \\
&\quad + \sum_{\mu,\nu\neq\alpha,\beta} (R_{\alpha\bar{\alpha}\mu\bar{\nu}} R_{\nu\bar{\mu}\beta\bar{\beta}} - |R_{\alpha\bar{\mu}\beta\bar{\nu}}|^2 + |R_{\alpha\bar{\beta}\mu\bar{\nu}}|^2) \\
&\triangleq \triangle R_{\alpha\bar{\alpha}\beta\bar{\beta}} + (I) + (II) + (III).
\end{aligned} \tag{2.1}$$

During the following proof, we assume that c denotes the various positive constants which depend on the bound of the curvature and its derivatives.

Suppose at some point $(\{e_\alpha, e_\beta\}, t)$ such that $u = 0$. We want to prove that $I \geqslant 0, II \geqslant 0, III \geqslant 0$, then by the maximal principle, we know that u is nonnegative.

Claim 1. $I \geqslant 0$.

Indeed, by the definition and a direct computation, we have

$$\begin{aligned} I &= \sum_{\mu,\nu=\alpha,\beta} (R_{\alpha\bar{\alpha}\mu\bar{\nu}} R_{\nu\bar{\mu}\beta\bar{\beta}} - |R_{\alpha\bar{\mu}\beta\bar{\nu}}|^2 + |R_{\alpha\bar{\beta}\mu\bar{\nu}}|^2) \\ &= R_{\alpha\bar{\alpha}\beta\bar{\beta}}(R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} + R_{\beta\bar{\beta}\beta\bar{\beta}} - R_{\alpha\bar{\alpha}\beta\bar{\beta}}) + |R_{\alpha\bar{\beta}\alpha\bar{\beta}}|^2 + 2\mathrm{Re}(R_{\alpha\bar{\alpha}\alpha\bar{\beta}} \overline{R_{\alpha\bar{\beta}\beta\bar{\beta}}}). \end{aligned} \tag{2.2}$$

Obviously, the first term equals zero, and the second term is nonnegative. So we need only to deal with the third term.

Now we consider the orthogonal 2-frames $\{\cos\theta e_\alpha + \sin\theta e_\beta, -\sin\theta e_\alpha + \cos\theta e_\beta\}$, we have

$$\begin{aligned} &u(\{\cos\theta e_\alpha + \sin\theta e_\beta, -\sin\theta e_\alpha + \cos\theta e_\beta\}, t) \\ &\quad = R(\cos\theta e_\alpha + \sin\theta e_\beta, \cos\theta \overline{e_\alpha} + \sin\theta \overline{e_\beta}, -\sin\theta e_\alpha + \cos\theta e_\beta, -\sin\theta \overline{e_\alpha} + \cos\theta \overline{e_\beta}). \end{aligned}$$

Then

$$\left.\frac{du}{d\theta}\right|_{\theta=0} = 2\mathrm{Re}(R_{\alpha\bar{\beta}\beta\bar{\beta}} - R_{\alpha\bar{\alpha}\alpha\bar{\beta}}).$$

So

$$|\mathrm{Re}(R_{\alpha\bar{\beta}\beta\bar{\beta}} - R_{\alpha\bar{\alpha}\alpha\bar{\beta}})| \leqslant c \sup_{\xi \in V, |\xi|=1} Du(\{e_\alpha, e_\beta\}, t)(\xi), \tag{2.3}$$

for some constant $c > 0$.

Similarly, if we change e_β by $\sqrt{-1}e_\beta$, and consider the orthogonal 2-frames $\{\cos\theta e_\alpha + \sin\theta\sqrt{-1}e_\beta, -\sin\theta e_\alpha + \cos\theta\sqrt{-1}e_\beta\}$, noting that

$$\begin{aligned} &R(\cos\theta e_\alpha + \sin\theta\sqrt{-1}e_\beta, \cos\theta\overline{e_\alpha} - \sin\theta\sqrt{-1}\overline{e_\beta}, -\sin\theta e_\alpha + \cos\theta\sqrt{-1}e_\beta, -\sin\theta\overline{e_\alpha} - \cos\theta\sqrt{-1}\overline{e_\beta}) \\ &\quad = R(\cos\theta e_\alpha + \sin\theta\sqrt{-1}e_\beta, \cos\theta\overline{e_\alpha} - \sin\theta\sqrt{-1}\overline{e_\beta}, \sqrt{-1}\sin\theta e_\alpha + \cos\theta e_\beta, -\sqrt{-1}\sin\theta\overline{e_\alpha} + \cos\theta\overline{e_\beta}), \end{aligned}$$

we can obtain that

$$|\mathrm{Im}(R_{\alpha\bar{\beta}\beta\bar{\beta}} - R_{\alpha\bar{\alpha}\alpha\bar{\beta}})| \leqslant c \sup_{\xi \in V, |\xi|=1} Du(\{e_\alpha, e_\beta\}, t)(\xi). \tag{2.4}$$

By (2.3) and (2.4) we get

$$|R_{\alpha\bar{\beta}\beta\bar{\beta}} - R_{\alpha\bar{\alpha}\alpha\bar{\beta}}|^2 \leqslant c \sup_{\xi \in V, |\xi|=1} Du(\{e_\alpha, e_\beta\}, t)(\xi)$$

i.e.,

$$|R_{\alpha\bar{\beta}\beta\bar{\beta}}|^2 + |R_{\alpha\bar{\alpha}\alpha\bar{\beta}}|^2 - 2\mathrm{Re}(R_{\alpha\bar{\alpha}\alpha\bar{\beta}}\overline{R_{\alpha\bar{\beta}\beta\bar{\beta}}}) \leqslant c \sup_{\xi \in V, |\xi|=1} Du(\{e_\alpha, e_\beta\}, t)(\xi).$$

So we have

$$2\mathrm{Re}(R_{\alpha\bar{\alpha}\alpha\bar{\beta}}\overline{R_{\alpha\bar{\beta}\beta\bar{\beta}}}) \geqslant -c \sup_{\xi \in V, |\xi|=1} Du(\{e_\alpha, e_\beta\}, t)(\xi) \tag{2.5}$$

for some constant $c > 0$.

Since the function u attains its minimum at the point $(\{e_\alpha, e_\beta\}, t)$, we obtain that $Du(\{e_\alpha, e_\beta\}, t) = 0$. And hence by (2.5), we know that $2\mathrm{Re}(R_{\alpha\bar{\alpha}\alpha\bar{\beta}}\overline{R_{\alpha\bar{\beta}\beta\bar{\beta}}}) \geqslant 0$, that is, the third term in (2.1) is nonnegative. So we have proved Claim 1.

Claim 2. $II \geqslant 0$.

Indeed, by the definition and a direct computation, we have

$$II = \Bigg(\sum_{\substack{\mu\neq\alpha,\beta \\ \nu=\alpha,\beta}} + \sum_{\substack{\nu\neq\alpha,\beta \\ \mu=\alpha,\beta}} \Bigg) (R_{\alpha\bar{\alpha}\mu\bar{\nu}} R_{\nu\bar{\mu}\beta\bar{\beta}} - |R_{\alpha\bar{\mu}\beta\bar{\nu}}|^2 + |R_{\alpha\bar{\beta}\mu\bar{\nu}}|^2)$$

$$= \sum_{\mu\neq\alpha,\beta} 2\mathrm{Re}(R_{\alpha\bar{\alpha}\alpha\bar{\mu}}R_{\mu\bar{\alpha}\beta\bar{\beta}} + R_{\alpha\bar{\alpha}\beta\bar{\mu}}R_{\mu\bar{\beta}\beta\bar{\beta}}) + \sum_{\mu\neq\alpha,\beta} (|R_{\alpha\bar{\beta}\mu\bar{\beta}}|^2 - |R_{\alpha\bar{\mu}\beta\bar{\beta}}|^2 + |R_{\alpha\bar{\beta}\alpha\bar{\mu}}|^2 - |R_{\alpha\bar{\alpha}\beta\bar{\mu}}|^2). \tag{2.6}$$

Now for $\mu \neq \alpha, \beta$, we consider the orthogonal 2-vectors $\{e_\alpha + se_\mu, e_\beta\}$, and we have

$$u(\{e_\alpha + se_\mu, e_\beta\}, t) = R(e_\alpha + se_\mu, \overline{e_\alpha} + s\overline{e_\mu}, e_\beta, \overline{e_\beta}).$$

Then

$$\frac{du}{ds}\Big|_{s=0} = R_{\mu\bar{\alpha}\beta\bar{\beta}} + R_{\alpha\bar{\mu}\beta\bar{\beta}} = 2\mathrm{Re}(R_{\alpha\bar{\mu}\beta\bar{\beta}}).$$

So we have

$$|\mathrm{Re}(R_{\alpha\bar{\mu}\beta\bar{\beta}})| \leqslant c \sup_{\xi\in V,|\xi|=1} Du(\{e_\alpha, e_\beta\}, t)(\xi), \tag{2.7}$$

for some constant $c > 0$.

Changing e_μ by $\sqrt{-1}e_\mu$, we can obtain

$$|\mathrm{Im}(R_{\alpha\bar{\mu}\beta\bar{\beta}})| \leqslant c \sup_{\xi\in V,|\xi|=1} Du(\{e_\alpha, e_\beta\}, t)(\xi). \tag{2.8}$$

By (2.7) and (2.8) we get

$$|R_{\alpha\bar{\mu}\beta\bar{\beta}}| \leqslant c \sup_{\xi\in V,|\xi|=1} Du(\{e_\alpha, e_\beta\}, t)(\xi). \tag{2.9}$$

Similarly, we can obtain

$$|R_{\alpha\bar{\alpha}\beta\bar{\mu}}| \leqslant c \sup_{\xi\in V,|\xi|=1} Du(\{e_\alpha, e_\beta\}, t)(\xi). \tag{2.10}$$

By (2.6), (2.9) and (2.10) we know that

$$II \geqslant -c \sup_{\xi\in V,|\xi|=1} Du(\{e_\alpha, e_\beta\}, t)(\xi),$$

for some constant $c > 0$. Similarly as above, we obtain that $II \geqslant 0$. Hence we proved Claim 2.

Claim 3. $III \geqslant 0$.

Indeed, in the following we will prove that

$$\sum_{\mu,\nu\neq\alpha,\beta} (R_{\alpha\bar{\alpha}\mu\bar{\nu}}R_{\nu\bar{\mu}\beta\bar{\beta}} - |R_{\alpha\bar{\mu}\beta\bar{\nu}}|^2) \geqslant 0.$$

For any vectors $\omega_\alpha, \omega_\beta$ orthogonal to e_α, e_β, we define an orthogonal 2-vectors $\{v_\alpha(s), v_\beta(s)\}$ by

$$v_\alpha(s) = e_\alpha + s\omega_\alpha - \frac{1}{2}s^2 \sum_{j=\alpha,\beta} \langle\omega_\alpha, \omega_j\rangle e_j + O(s^3),$$
$$v_\beta(s) = e_\beta + s\omega_\beta - \frac{1}{2}s^2 \sum_{j=\alpha,\beta} \langle\omega_\beta, \omega_j\rangle e_j + O(s^3).$$

Then consider $u(\{v_\alpha(s), v_\beta(s)\}, t) = R(v_\alpha, \overline{v_\alpha}, v_\beta, \overline{v_\beta})$. By a direct computation, we have

$$\begin{aligned}\frac{1}{2}\frac{d^2u(s)}{ds^2}\Big|_{s=0} &= R(\omega_\alpha, \overline{\omega_\alpha}, e_\beta, \overline{e_\beta}) + R(e_\alpha, \overline{e_\alpha}, \omega_\beta, \overline{\omega_\beta}) \\ &\quad + 2\mathrm{Re}(R(\omega_\alpha, \overline{e_\alpha}, e_\beta, \overline{\omega_\beta})) + 2\mathrm{Re}(R(e_\alpha, \overline{\omega_\alpha}, e_\beta, \overline{\omega_\beta})) \\ &\quad - (\langle\omega_\alpha, \omega_\alpha\rangle + \langle\omega_\beta, \omega_\beta\rangle)R_{\alpha\bar{\alpha}\beta\bar{\beta}} - \mathrm{Re}(\langle\omega_\alpha, \omega_\beta\rangle R_{\beta\bar{\alpha}\beta\bar{\beta}} + \langle\omega_\beta, \omega_\alpha\rangle R_{\alpha\bar{\alpha}\alpha\bar{\beta}}).\end{aligned}$$

So we have

$$(\omega_\alpha, \overline{\omega_\alpha}, e_\beta, \overline{e_\beta}) + R(e_\alpha, \overline{e_\alpha}, \omega_\beta, \overline{\omega_\beta}) + 2\mathrm{Re}(R(\omega_\alpha, \overline{e_\alpha}, e_\beta, \overline{\omega_\beta})) + 2\mathrm{Re}(R(e_\alpha, \overline{\omega_\alpha}, e_\beta, \overline{\omega_\beta}))$$

$$
\begin{aligned}
&-(\langle\omega_\alpha,\omega_\alpha\rangle+\langle\omega_\beta,\omega_\beta\rangle)R_{\alpha\bar\alpha\beta\bar\beta}-\mathrm{Re}(\langle\omega_\alpha,\omega_\beta\rangle R_{\beta\bar\alpha\beta\bar\beta}+\langle\omega_\beta,\omega_\alpha\rangle R_{\alpha\bar\alpha\alpha\bar\beta})\\
&\geqslant c\inf_{\xi\in V,|\xi|=1}D^2u(\{e_\alpha,e_\beta\},t)(\xi,\xi),
\end{aligned}\tag{2.11}
$$

for some constant $c>0$. If we change e_α by $-\sqrt{-1}e_\alpha$ and e_β by $\sqrt{-1}e_\beta$, we can obtain that

$$
\begin{aligned}
&R(\omega_\alpha,\overline{\omega_\alpha},e_\beta,\overline{e_\beta})+R(e_\alpha,\overline{e_\alpha},\omega_\beta,\overline{\omega_\beta})-2\mathrm{Re}(R(\omega_\alpha,\overline{e_\alpha},e_\beta,\overline{\omega_\beta}))+2\mathrm{Re}(R(e_\alpha,\overline{\omega_\alpha},e_\beta,\overline{\omega_\beta}))\\
&\quad-(\langle\omega_\alpha,\omega_\alpha\rangle+\langle\omega_\beta,\omega_\beta\rangle)R_{\alpha\bar\alpha\beta\bar\beta}+\mathrm{Re}(\langle\omega_\alpha,\omega_\beta\rangle R_{\beta\bar\alpha\beta\bar\beta}+\langle\omega_\beta,\omega_\alpha\rangle R_{\alpha\bar\alpha\alpha\bar\beta})\\
&\quad\geqslant c\inf_{\xi\in V,|\xi|=1}D^2u(\{e_\alpha,e_\beta\},t)(\xi,\xi),
\end{aligned}\tag{2.12}
$$

By (2.11) and (2.12) we have

$$
R(\omega_\alpha,\overline{\omega_\alpha},e_\beta,\overline{e_\beta})+R(e_\alpha,\overline{e_\alpha},\omega_\beta,\overline{\omega_\beta})+2\mathrm{Re}(R(e_\alpha,\overline{\omega_\alpha},e_\beta,\overline{\omega_\beta}))\geqslant c\inf_{\xi\in V,|\xi|=1}D^2u(\{e_\alpha,e_\beta\},t)(\xi,\xi).\tag{2.13}
$$

If we set

$$
A(X,\overline{Y})=R(X,\overline{Y},e_\beta,\overline{e_\beta}),\quad B(X,Y)=R(\overline{e_\alpha},X,\overline{e_\beta},Y),\quad C(X,\overline{Y})=R(e_\alpha,\overline{e_\alpha},X,\overline{Y}).
$$

Then by (2.13) we know that

$$
\begin{pmatrix} A & \overline{B}\\ B^T & \overline{C}\end{pmatrix}\geqslant c\inf_{\xi\in V,|\xi|=1}D^2u(\{e_\alpha,e_\beta\},t)(\xi,\xi).
$$

Hence we have

$$
\mathrm{tr}(AC)-\mathrm{tr}(B\overline{B})\geqslant c\inf_{\xi\in V,|\xi|=1}D^2u(\{e_\alpha,e_\beta\},t)(\xi,\xi),
$$

where $c>0$ is a constant depending on the bound of the curvature and its derivatives. i.e.,

$$
\sum_{\mu,\nu\neq\alpha,\beta}(R_{\alpha\bar\alpha\mu\bar\nu}R_{\nu\bar\mu\beta\bar\beta}-|R_{\alpha\bar\mu\beta\bar\nu}|^2)\geqslant c\inf_{\xi\in V,|\xi|=1}D^2u(\{e_\alpha,e_\beta\},t)(\xi,\xi),\tag{2.14}
$$

for some constant $c>0$.

Since the function u attains its minimum at the point $(\{e_\alpha,e_\beta\},t)$, we obtain that $D^2u(\{e_\alpha,e_\beta\},t)\geqslant 0$. By the definition of III, we get $III\geqslant 0$. Therefore we have proved Claim 3.

By (2.1) and Claims 1–3, we can get that the nonnegative orthogonal bisectional curvature is preserved under the Ricci flow. This completes the proof of Proposition 2.1. □

In the following, we can obtain the equation of the orthogonal bisectional curvature under the Ricci flow.

Proposition 2.2. *There exists $c>0$ such that*

$$
\frac{\partial u}{\partial t}\geqslant Lu+c\inf_{\xi\in V,|\xi|=1}D^2u(\xi,\xi)-c\sup_{\xi\in V,|\xi|=1}Du(\xi)-cu,
$$

where L is the horizontal Laplacian on P and V denotes the vertical subspace of the bundle.

Proof. During the above proof of the preserving nonnegativity of the orthogonal bisectional curvature, we actually obtain that

Claim 1. There exist constants $c_1>0, c_2>0$ such that

$$
I\geqslant -c_1\cdot u(\{e_\alpha,e_\beta\},t)-c_2\sup_{\xi\in V,|\xi|=1}Du(\{e_\alpha,e_\beta\},t)(\xi).
$$

Claim 2. There exists constant $c_3>0$, such that

$$
II\geqslant -c_3\sup_{\xi\in V,|\xi|=1}Du(\{e_\alpha,e_\beta\},t)(\xi).
$$

Claim 3. There exists constant $c_4 > 0$, such that

$$III \geqslant c_4 \inf_{\xi \in V, |\xi|=1} D^2 u(\{e_\alpha, e_\beta\}, t)(\xi, \xi).$$

By (2.1) and Claims 1–3, we can get

$$\frac{\partial u}{\partial t} \geqslant Lu + c \inf_{\xi \in V, |\xi|=1} D^2 u(\xi, \xi) - c \sup_{\xi \in V, |\xi|=1} Du(\xi) - cu,$$

for some constant $c > 0$, where L is the horizontal Laplacian on P and V denotes the vertical subspace of the bundle.

This completes the proof of Proposition 2.2. □

3 Some results on irreducible manifolds

In the following we first give a similar result to [8] in terms of the orthogonal holomorphic bisectional curvature in the Kähler manifolds. We will show that the curvature term in the Weitzenböck formula on $(1,1)$-forms involves only the orthogonal holomorphic bisectional curvature. This also gives the answer to the positivity of the first Chern class under the positive orthogonal holomorphic bisectional curvature condition. In this section we always assume that the complex dimension n of the Kähler manifold M^n satisfies $n \geqslant 2$.

Theorem 3.1. *Let (M^n, h) be a compact Kähler manifold with nonnegative orthogonal holomorphic bisectional curvature. Then all real harmonic $(1,1)$-forms are parallel. Furthermore, we have*

(i) *If $b_{1,1}(M) = \dim H^{1,1}(M) = 1$, then $c_1(M) > 0$;*

(ii) *If in addition M is locally irreducible, then we have $b_{1,1}(M) = \dim H^{1,1}(M) = 1$ and hence by* (i) *we have $c_1(M) > 0$.*

Proof. Suppose (M^n, h) is a compact Kähler manifold with nonnegative orthogonal holomorphic bisectional curvature and J is the complex structure. Let η be a nontrivial harmonic $(1,1)$-form on M.

In the following, we want to show that η is parallel. Indeed, the parallelity of η was already obtained by [13] under the condition of nonnegative holomorphic bisectional curvature. For the completeness of our paper, we will adapt the argument in [13, 8] to show that η is parallel under the condition of nonnegative orthogonal holomorphic bisectional curvature.

Now we can choose an orthonormal basis $\{e_\beta\}_{\beta=1}^n$ such that under this basis

$$\eta = \frac{\sqrt{-1}}{2} \sum_{\beta=1}^{n} 2a_\beta \cdot e_\beta \wedge \overline{e_\beta}.$$

Set $e_\beta = \frac{1}{\sqrt{2}}(u_\beta - \sqrt{-1} J u_\beta)$, $1 \leqslant \beta \leqslant n$, where $\{u_1, Ju_1, \ldots, u_n, Ju_n\}$ is an orthonormal basis of M in the sense of considering M as a Riemannian manifold. So in the basis $\{u_1, Ju_1, \ldots, u_n, Ju_n\}$, η becomes $\eta = -\sum_{i=1}^n a_i \cdot u_i \wedge Ju_i$. By the Bochner formula we have $\triangle \eta = \nabla^* \nabla \eta + \pounds(\eta)$, where $\pounds(\eta) = -\frac{1}{4}\sum_{i,j}(R(\eta_i), \eta_j)[\eta_i, [\eta_j, \eta]]$ and $(\cdot, \cdot)$ denotes the corresponding Riemannian metric. Then by the same argument as in [8], we know that

$$(\pounds(\eta), \eta) = \frac{1}{2} \sum_{\alpha > 0} (R(X_\alpha), X_{-\alpha})(-\alpha(\eta) X_{-\alpha}, \alpha(\eta) X_\alpha) = \frac{1}{2} \sum_{\alpha > 0} -\alpha(\eta)^2 (R(X_\alpha), X_{-\alpha}),$$

where $\alpha(\eta)$ satisfies $[X_\alpha, \eta] = -[\eta, X_\alpha] = -\alpha(\eta) X_\alpha$ and $-\alpha(\eta)^2$ is nonnegative and the symbols are the same as in [4]. Now for the positive roots $x_i + x_j$ $(1 \leqslant i < j \leqslant n)$, we have

$$X_\alpha = \frac{1}{2}(u_i + \sqrt{-1} J u_i) \wedge (u_j + \sqrt{-1} J u_j) = \overline{e_i} \wedge \overline{e_j},$$
$$X_{-\alpha} = \frac{1}{2}(u_i - \sqrt{-1} J u_i) \wedge (u_j - \sqrt{-1} J u_j) = e_i \wedge e_j.$$

For the positive roots $x_i - x_j$ $(1 \leqslant i < j \leqslant n)$, we have

$$X_\alpha = \frac{1}{2}(u_i + \sqrt{-1}Ju_i) \wedge (u_j - \sqrt{-1}Ju_j) = \overline{e_i} \wedge e_j,$$
$$X_{-\alpha} = \frac{1}{2}(u_i - \sqrt{-1}Ju_i) \wedge (u_j + \sqrt{-1}Ju_j) = e_i \wedge \overline{e_j}.$$

So $(R(X_\alpha), X_{-\alpha}) = 0$ the previous case and for the other case, we have

$$(R(X_\alpha), X_{-\alpha}) = R(\overline{e_i} \wedge e_j, e_i \wedge \overline{e_j}) = R(\overline{e_i}, e_j, \overline{e_j}, e_i) = R(e_i, \overline{e_i}, e_j, \overline{e_j}) \geqslant 0,$$

since the orthogonal holomorphic bisectional curvature is nonnegative. Then by the standard Bochner argument we can obtain that all real harmonic $(1,1)$-forms are parallel.

In order to prove the left conclusions (i) and (ii), we evolve the metric by the Kähler Ricci flow

$$\begin{cases} \dfrac{\partial}{\partial t} g_{i\bar{j}}(x,t) = -R_{i\bar{j}}(x,t), \\ g_{i\bar{j}}(x,0) = h_{i\bar{j}}(x). \end{cases}$$

Then by Shi's short-time existence theorem, we know that there is a $T > 0$ such that the Ricci flow has a smooth bounded curvature solution $(M, g_{i\bar{j}}(t))$ for $t \in [0,T)$. By Proposition 2.1 that the solution $g_{i\bar{j}}(t)$ still has the nonnegative orthogonal holomorphic bisectional curvature. Suppose $\{e_\alpha\}$ is an orthonormal basis, then for any $\alpha \neq \beta$, we have

$$R(e_\alpha - e_\beta, \overline{e_\alpha} - \overline{e_\beta}, e_\alpha + e_\beta, \overline{e_\alpha} + \overline{e_\beta}) = R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} + R_{\beta\bar{\beta}\beta\bar{\beta}} - R_{\alpha\bar{\beta}\alpha\bar{\beta}} - R_{\beta\bar{\alpha}\beta\bar{\alpha}} \geqslant 0, \tag{3.1}$$

where we have used the assumption and the result that the nonnegativity of the orthogonal holomorphic bisectional curvature is preserved under the Ricci flow. Similarly changing e_β by $\sqrt{-1}e_\beta$, we have

$$R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} + R_{\beta\bar{\beta}\beta\bar{\beta}} + R_{\alpha\bar{\beta}\alpha\bar{\beta}} + R_{\beta\bar{\alpha}\beta\bar{\alpha}} \geqslant 0. \tag{3.2}$$

By (3.1) and (3.2) we obtain that

$$R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} + R_{\beta\bar{\beta}\beta\bar{\beta}} \geqslant 0 \tag{3.3}$$

for any orthonormal 2-frames $\{e_\alpha, e_\beta\}$. So by the assumption and (3.3), we have

$$R = \sum_{\alpha,\beta} R_{\alpha\bar{\alpha}\beta\bar{\beta}} = \sum_\alpha \sum_{\beta\neq\alpha} R_{\alpha\bar{\alpha}\beta\bar{\beta}} + \sum_\alpha R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} \geqslant 0. \tag{3.4}$$

If $b_{1,1}(M) = \dim H^{1,1}(M) = 1$, let ρ and ω denote the Ricci form and Kähler form respectively, then by the Hodge theory, we have $\rho = \lambda\omega + \eta$, where λ is a real number and $\int_M \langle\omega, \eta\rangle = 0$. On the other hand, we have

$$\int_M \langle\rho, \omega\rangle = \frac{1}{4}\int_M R = \lambda\|\omega\|^2 \geqslant 0,$$

since the scalar curvature $R \geqslant 0$ by (3.4). Hence we have $c_1(M) \geqslant 0$. Moreover if the scalar curvature R at some point is positive, then $c_1(M) > 0$. So now we can assume that the scalar curvature $R(t) \equiv 0$ for all sufficiently small t. Then by the evolution equation of the scalar curvature $\frac{\partial R}{\partial t} = \triangle R + |\mathrm{Ric}|^2$ we know that for all sufficiently small t,

$$\mathrm{Ric}(t) \equiv 0. \tag{3.5}$$

We claim that for all α, the holomorphic sectional curvature $R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} = 0$.

Indeed, by (3.3)–(3.5), we know that for any $\alpha \neq \beta$ $R_{\alpha\bar{\alpha}\beta\bar{\beta}} = 0$ and $R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} + R_{\beta\bar{\beta}\beta\bar{\beta}} = 0$. Suppose there exists $1 \leqslant \alpha \leqslant n$ such that $R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} \neq 0$, then $R_{\alpha\bar{\alpha}} = \sum_{\beta\neq\alpha} R_{\alpha\bar{\alpha}\beta\bar{\beta}} + R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} \neq 0$. And this contradicts with (3.5). So we have proved the claim and hence the curvature operator is equal to zero. Therefore (M^n, h) is flat. However, note that $n \geqslant 2$, we know that there exists no compact and flat Kähler manifold satisfying $b_{1,1}(M) = \dim H^{1,1}(M) = 1$. Thus the scalar curvature must be positive at some point. Hence $c_1(M) > 0$. This completes the proof of (i).

In the following we will give the proof of (ii). We argue by contradiction. Suppose $b_{1,1}(M) = \dim H^{1,1}(M) > 1$, then by the same argument as in [8] in the proof of Theorem 2.1 (b) and note that M is locally irreducible, we know that h is hyper-Kähler and hence Ricci flat. So by the argument above, we know that M is flat. And this is a contradiction with the local irreducibility of M. So $b_{1,1}(M) = \dim H^{1,1}(M) = 1$. Then by (i) we know that $c_1(M) > 0$. This completes the proof of (ii).

Therefore we complete the proof of Theorem 3.1. □

From Theorem 3.1 and the result of [4], we immediately obtain

Corollary 3.2. *Let (M^n, h) be a compact Kähler manifold with positive orthogonal holomorphic bisectional curvature. Then the first Chern class $c_1(M) > 0$. Moreover, the underlying manifold is biholomorphic to CP^n.*

Proof. Since (M^n, h) has positive orthogonal holomorphic bisectional curvature, we get that M is locally irreducible. Then by Theorem 3.1 (ii) we know that $c_1(M) > 0$. Combining the result of [4], we obtain that M is biholomorphic to the complex projective space CP^n. □

In the following we will give a result on the irreducible compact Kähler manifold with nonnegative orthogonal holomorphic bisectional curvature.

Proposition 3.3. *Let (M^n, h) be a compact irreducible Kähler manifold with nonnegative orthogonal holomorphic bisectional curvature. Then either M is biholomorphic to the complex projective space or (M, h) is isometrically biholomorphic to an irreducible compact Hermitian symmetric manifold of rank $\geqslant 2$.*

Proof. Suppose (M^n, h) is a compact irreducible Kähler manifold with nonnegative orthogonal holomorphic bisectional curvature, then by Theorem 3.1 (ii), we know that $c_1(M) > 0$.

First we evolve the metric by the Kähler Ricci flow

$$\begin{cases} \dfrac{\partial}{\partial t} g_{i\bar{j}}(x,t) = -R_{i\bar{j}}(x,t), \\ g_{i\bar{j}}(x,0) = h_{i\bar{j}}(x). \end{cases}$$

According to Bando [1], we know that the evolved metric $g_{i\bar{j}}(t), t \in (0,T)$, remains Kähler. Then by Proposition 2.1, we know that for $t \in (0,T)$, $g_{i\bar{j}}(t)$ has nonnegative orthogonal holomorphic bisectional curvature. Moreover, according to Hamilton [6], under the evolving orthonormal frame $\{e_\alpha\}$, we have

$$\frac{\partial}{\partial t} R_{\alpha\bar{\alpha}\beta\bar{\beta}} = \triangle R_{\alpha\bar{\alpha}\beta\bar{\beta}} + \sum_{\mu,\nu} (R_{\alpha\bar{\alpha}\mu\bar{\nu}} R_{\nu\bar{\mu}\beta\bar{\beta}} - |R_{\alpha\bar{\mu}\beta\bar{\nu}}|^2 + |R_{\alpha\bar{\beta}\mu\bar{\nu}}|^2).$$

Suppose (M, h) is not locally symmetric. In the following, we want to show that M is biholomorphic to the complex projective space CP^n.

Since the smooth limit of locally symmetric space is also locally symmetric, we can obtain that there exists $\delta \in (0,T)$ such that $(M, g_{i\bar{j}}(t))$ is not locally symmetric for $t \in (0,\delta)$. Combining the Kählerity of $g_{i\bar{j}}(t)$ and Berger's holonomy theorem and note that $c_1(M) > 0$, we know that the holonomy group $\mathrm{Hol}(g(t)) = U(n)$.

As above, let P be the fiber bundle with the fixed metric h and the fiber over $p \in M$ consist of all orthogonal 2-vectors $\{X, Y\} \subset T^{1,0}_p(M)$. We define a function u on $P \times (0,\delta)$ by $u(\{X,Y\},t) = R(X, \overline{X}, Y, \overline{Y})$, where R denotes the pull-back of the curvature tensor of $g_{i\bar{j}}(t)$. Clearly we have $u \geqslant 0$ by Proposition 2.1. Denote $F = \{(\{X,Y\},t) | u(\{X,Y\},t) = 0, X \neq 0, Y \neq 0\} \subset P \times (0,\delta)$ consists of all pairs $(\{X,Y\},t)$ such that $\{X,Y\}$ has zero orthogonal holomorphic bisectional curvature with respect to $g_{i\bar{j}}(t)$. By Proposition 2.2, we know that

$$\frac{\partial u}{\partial t} \geqslant Lu + c \inf_{\xi \in V, |\xi|=1} D^2 u(\xi,\xi) - c \sup_{\xi \in V, |\xi|=1} Du(\xi) - cu,$$

for some constant $c > 0$, where L is the horizontal Laplacian on P and V denotes the vertical subspace of the bundle.

In the following, we use Proposition 5 in [2] to obtain that the set

$$F = \{(\{X,Y\},t)|u(\{X,Y\},t) = 0, X \neq 0, Y \neq 0\} \subset P \times (0,\delta)$$

is invariant under parallel transport.

Indeed, we know that the vector field $\widetilde{Y}$, defined in Proposition 5 in [2], is the horizontal lift of the vector field $\frac{\partial}{\partial t}$ on $M \times (0,\delta)$, and the vector fields $\widetilde{X}_1, \dots, \widetilde{X}_n$ are tangential to P. Hence $\widetilde{Y}(u) = \frac{\partial u}{\partial t}$, and $\sum_{j=1}^n \widetilde{X}_j(\widetilde{X}_j(u)) = Lu$, then by Proposition 5 in [2] we get that the set

$$F = \{(\{X,Y\},t)|u(\{X,Y\},t) = 0, X \neq 0, Y \neq 0\} \subset P \times (0,\delta)$$

is invariant under parallel transport.

Next, by adapting the argument in [5], we claim that $R_{\alpha\bar{\alpha}\beta\bar{\beta}} > 0$ for all $t \in (0,\delta)$ and all $\alpha \neq \beta$.

Indeed, suppose not. Then $R_{\alpha\bar{\alpha}\beta\bar{\beta}} = 0$ for some $t \in (0,\delta)$ and some $\alpha \neq \beta$. Therefore $(\{e_\alpha, e_\beta\},t) \in F$. Combining $R_{\alpha\bar{\alpha}\beta\bar{\beta}} = 0$ and the computation for (2.2), (2.6) and (2.14) in Proposition 2.1, it is easy to obtain that

$$\begin{cases} \sum\limits_{\mu,\nu} (R_{\alpha\bar{\alpha}\mu\bar{\nu}} R_{\nu\bar{\mu}\beta\bar{\beta}} - |R_{\alpha\bar{\mu}\beta\bar{\nu}}|^2) = 0, \\ R_{\alpha\bar{\beta}\mu\bar{\nu}} = 0, \quad \forall \mu,\nu, \\ R_{\alpha\bar{\alpha}\mu\bar{\beta}} = R_{\beta\bar{\beta}\mu\bar{\alpha}} = 0, \quad \forall \mu. \end{cases} \tag{3.6}$$

We define an orthonormal 2-frames $\{\widetilde{e_\alpha}, \widetilde{e_\beta}\} \subset T_p^{1,0}(M)$ by

$$\widetilde{e_\alpha} = \sin\theta \cdot e_\alpha - \cos\theta \cdot e_\beta, \quad \widetilde{e_\beta} = \cos\theta \cdot e_\alpha + \sin\theta \cdot e_\beta.$$

Then

$$\overline{\widetilde{e_\alpha}} = \sin\theta \cdot \overline{e_\alpha} - \cos\theta \cdot \overline{e_\beta}, \quad \overline{\widetilde{e_\beta}} = \cos\theta \cdot \overline{e_\alpha} + \sin\theta \cdot \overline{e_\beta}.$$

Since F is invariant under parallel transport and $(M, g_{i\bar{j}}(t))$ has the holonomy group $U(n)$, we obtain that $(\{\widetilde{e_\alpha}, \widetilde{e_\beta}\},t) \in F$, that is, $R(\widetilde{e_\alpha}, \overline{\widetilde{e_\alpha}}, \widetilde{e_\beta}, \overline{\widetilde{e_\beta}}) = 0$. On the other hand, by a direct computation and (3.6),

$$R(\widetilde{e_\alpha}, \overline{\widetilde{e_\alpha}}, \widetilde{e_\beta}, \overline{\widetilde{e_\beta}}) = \cos^2\theta \sin^2\theta (R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} + R_{\beta\bar{\beta}\beta\bar{\beta}}).$$

So we have $R_{\beta\bar{\beta}\beta\bar{\beta}} + R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} = 0$, if we choose θ such that $\cos^2\theta \sin^2\theta \neq 0$.

Clearly we can find an element of $U(n)$ such that it changes e_α to e_μ and fixed e_β. Then we can see that $(\{e_\mu, e_\beta\},t) \in F$. By the same argument as above, we get $R_{\beta\bar{\beta}\mu\bar{\mu}} = R_{\beta\bar{\beta}\beta\bar{\beta}} + R_{\mu\bar{\mu}\mu\bar{\mu}} = 0$. Similarly we can obtain that for any e_μ and e_ν with $\mu \neq \nu$, the following holds

$$R_{\mu\bar{\mu}\nu\bar{\nu}} = R_{\nu\bar{\nu}\nu\bar{\nu}} + R_{\mu\bar{\mu}\mu\bar{\mu}} = 0. \tag{3.7}$$

So we have the scalar curvature

$$R = \sum_\alpha R_{\alpha\bar{\alpha}} = \sum_\alpha \sum_{\beta\neq\alpha} R_{\alpha\bar{\alpha}\beta\bar{\beta}} + \sum_\alpha R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} = 0. \tag{3.8}$$

Then by the same argument as in Theorem 3.1, we can obtain that the manifold is flat, and this contradicts with the irreducibility of M. Hence we prove that $R_{\alpha\bar{\alpha}\beta\bar{\beta}} > 0$, for all $t \in (0,\delta)$ and all $\alpha \neq \beta$.

Therefore note that $c_1(M) > 0$ and then using the result of [4], we can get M is biholomorphic to the complex projective space CP^n.

This completes the proof of Proposition 3.3. □

4 Proof of the main theorem

Proof of Theorem 1.3. Suppose (M^n, h) is an n-dimensional ($n \geqslant 2$) compact Kähler manifold of non-negative orthogonal holomorphic bisectional curvature. By applying the standard de Rham decomposition theorem, we know that the universal cover $(\tilde{M}, \tilde{h})$ can be isometrically and holomorphically splitted as

$$(C^k, h_0) \times (M_1^{n_1}, h_1) \times \cdots \times (M_l^{n_l}, h_l),$$

where each $(M_i^{n_i}, h_i)\, 1 \leqslant i \leqslant l$, is irreducible and non-flat, h_0 is the standard flat metric on C^k and $k, n_1, \ldots, n_l$ are nonnegative integers.

In the following we divide it into three cases.

Case 1. $k = 0$ and in the de Rham decomposition there exists a complex 1-dimensional irreducible factor $\Sigma = M_1$ with Gauss curvature $\kappa(\Sigma)$ negative somewhere.

In this case, let e_1 be the unit basis of $T_p^{1,0}(\Sigma)$ and $\{e_j^i\}$ $(1 \leqslant j \leqslant n_i, 2 \leqslant i \leqslant l)$, be the orthonormal basis of $T_{q_i}^{1,0}(M_i)$ for arbitrary points $p \in \Sigma, q_i \in M_i$. Naturally we can extend e_1 and $\{e_j^i\}$ to an orthonormal basis of $T_x^{1,0}(\tilde{M})$ for $x = (p, q_2, \ldots, q_l) \in \tilde{M}$, still we denote by e_1 and e_j^i $(1 \leqslant j \leqslant n_i, 2 \leqslant i \leqslant l)$. Since M has nonnegative orthogonal holomorphic bisectional curvature, we obtain

$$R(e_1 - e_j^i, \overline{e_1} - \overline{e_j^i}, e_1 + e_j^i, \overline{e_1} + \overline{e_j^i}) = R^{(1)}_{1\bar{1}1\bar{1}} + R^{(i)}_{j\bar{j}j\bar{j}} = \kappa(p) + R^{(i)}_{j\bar{j}j\bar{j}} \geqslant 0,$$

where $R^{(i)}$ denotes the curvature on M_i. So for each $i \neq 1$ we have $R^{(i)}_{j\bar{j}j\bar{j}} \geqslant -\kappa(p)$. By the arbitrariness of p, q_i, we know that

$$\min\{\text{holomorphic sectional curvature of } M_i\} \geqslant -\min\{\kappa(\Sigma)\} > 0.$$

So we have proved that all M_i $(i \neq 1)$, have nonnegative holomorphic bisectional curvature. If $\dim(M_i) = n_i \geqslant 2$, then we know that it also has nonnegative Ricci curvature. So M_i is compact, otherwise, it will split off a line and we can obtain a contradiction with the irreducibility of M_i. Then by Proposition 3.3 we obtain that either M_i is biholomorphic to the complex projective space CP^{n_i} or M_i is isometrically biholomorphic to an irreducible compact Hermitian symmetric manifold of rank $\geqslant 2$. If $\dim(M_i) = n_i = 1\, (i \neq 1)$, then by the Gauss-Bonnet Theorem, we know that M_i is $S^2\, (= CP^1)$ with a nonnegatively curved metric. Hence this case is contained in (2).

Case 2. $k = 0$ and in the de Rham decomposition there exists no complex 1-dimensional irreducible factor or there exist complex 1-dimensional irreducible factors and all these complex 1-dimensional irreducible factors have nonnegatively curved metric.

In this case, we know that all the complex 1-dimensional irreducible factors, if exists, are compact by the Gauss-Bonnet Theorem and are $S^2\, (= CP^1)$.

If all the irreducible factors M_i with $\dim(M_i) \geqslant 2$ are compact, then by Proposition 3.3 we obtain that either M_i is biholomorphic to the complex projective space CP^{n_i} or M_i is isometrically biholomorphic to an irreducible compact Hermitian symmetric manifold of rank $\geqslant 2$. Hence this is contained in (1).

If there exists an irreducible factor, without loss of generality, denoted by M_1, is noncompact, then we claim that the minimal of the holomorphic sectional curvature of $M_1 < 0$ somewhere. Otherwise, suppose the holomorphic bisectional curvature of $M_1 \geqslant 0$ and hence it has nonnegative Ricci curvature, so it is compact which contradicts to the noncompactness of M_1. So we have proved the claim. Then by the nonnegativity of the orthogonal holomorphic bisectional curvature and the same argument as in Case 1, we know that all the other irreducible factors M_i $(i \neq 1)$, have nonnegative holomorphic bisectional curvature and hence are compact. Therefore as above, by Proposition 3.3 we obtain that either M_i $(i \neq 1)$, is biholomorphic to the complex projective space CP^{n_i} or M_i $(i \neq 1)$, is isometrically biholomorphic to an irreducible compact Hermitian symmetric manifold of rank $\geqslant 2$. This is contained in (2).

Case 3. $k \geqslant 1$.

In this case, by the nonnegativity of the orthogonal holomorphic bisectional curvature of $\tilde{M}$ and the same argument as in Case 1, we know that all the other irreducible factors M_i have nonnegative holomorphic bisectional curvature. Again by the same argument as in Case 1, we can obtain that if $\dim(M_i) = n_i \geqslant 2$, then either M_i is biholomorphic to the complex projective space CP^{n_i} or M_i is isometrically biholomorphic to an irreducible compact Hermitian symmetric manifold of rank $\geqslant 2$. If $\dim(M_i) = n_i = 1$, then by the Gauss-Bonnet Theorem, we know that M_i is $S^2 (= CP^1)$ with a nonnegatively curved metric. This case is contained in (1).

Hence from the above argument, we have proved Theorem 1.3. □

Acknowledgements The first author was partially supported by the Young Faculty Career Start Program (Grant No. 34000-3171917), Natural Science Foundation of Guangdong Province (Grant No. 9451027501002600) and National Natural Science Foundation of China (Grant No. 10901165). We would be indebted to our advisor Professor X. P. Zhu for provoking our interest to this problem. We are grateful to Professor B. L. Chen for many suggestions and discussions. We would also like to thank Professor S. H. Tang for discussions.

References

1 Bando S. On three-dimensional compact Kähler manifolds of nonnegative bisectional curvature. J Diff Geom, 1984, 19: 283–297
2 Brendle S, Schoen R. Classification of manifolds with weakly 1/4-pinched curvatures. Acta Math, 2008, 200: 1–13
3 Cao H D, Hamilton R S. Unpublished work
4 Chen X X. On Kähler manifolds with positive orthogonal bisectional curvature. ArXiv:math.DG/0606229 v1 June 2006
5 Gu H L. A new proof of Mok's generalized Frankel conjecture theorem. Proc Amer Math Soc, 2009, 137: 1063–1068
6 Hamilton R S. Four-manifolds with positive curvature operator. J Diff Geom, 1986, 24: 153–179
7 Ivey T. Ricci solitons on compact Kähler surfaces. Proc Amer Math Soc, 1997, 125: 1203–1208
8 Micallef M, Wang M. Metrics with nonnegative isotropic curvature. Duke Math J, 1993, 72: 649–672
9 Mok N. The uniformization theorem for compact Kähler manifolds of nonnegative bisectional curvature. J Diff Geom, 1988, 27: 179–214
10 Mori S. Projective manifolds with ample tangent bundles. Ann of Math, 1979, 110: 593–606
11 Seshadri H. Manifolds with nonnegative isotropic curvature. ArXiv:math.DG/0707.3894 v1 July 2007
12 Siu Y T, Yau S T. Complex Kähler manifolds of positive bisectional curvature. Invent Math, 1980, 59: 189–204
13 Wu H. On compact Kähler manifolds of nonnegative bisectional curvature II. Acta Math, 1981, 147: 57–70