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Eigenvalue approximation from below using non-conforming finite elements

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Abstract This is a survey article about using non-conforming finite elements in solving eigenvalue problems of elliptic operators, with emphasis on obtaining lower bounds. In addition, this article also contains some new materials for eigenvalue approximations of the Laplace operator, which include: 1) the proof of the fact that the non-conforming Crouzeix-Raviart element approximates eigenvalues associated with smooth eigenfunctions from below; 2) the proof of the fact that the non-conforming $EQ_1^{\rm rot}$ element approximates eigenvalues from below on polygonal domains that can be decomposed into rectangular elements; 3) the explanation of the phenomena that numerical eigenvalues $\lambda_{1,h}$ and $\lambda_{3,h}$ of the non-conforming $Q_1^{\rm rot}$ element approximate the true eigenvalues from below for the L-shaped domain. Finally, we list several unsolved problems.

Keywords non-conforming element, eigenvalue, lower bound

MSC(2000): 65N25, 65N30, 35P15, 65N15

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1 Introduction

Let V and H be two Hilbert spaces, where $V \subset H$ with a compact embedding, let $a(\cdot,\cdot)$ be a bilinear form on $V \times V$ which is symmetric, V-elliptic, and continuous, and let $b(\cdot,\cdot)$ be a bilinear form on $H \times H$ which is continuous and symmetric positive definite. Furthermore, we define an inner product and norm on H by $b(\cdot,\cdot)$ and $\|\cdot\|_b = \sqrt{b(\cdot,\cdot)}$, respectively.

Consider the weak form of eigenvalue problems of the self-adjoint elliptic differential operator:

Find $\lambda \in \mathbb{R}, u \in V, \|u\|_b = 1$, such that

$$a(u,v) = \lambda b(u,v), \quad \forall\, v \in V. \tag{1.1}$$

Let $S^h \subset V$ be a conforming element space. The conforming finite element approximation of (1.1) is: Find $\lambda_h \in \mathbb{R}, u_h \in S^h, \|u_h\|_b = 1$, such that

$$a(u_h,v) = \lambda_h b(u_h,v), \quad \forall\, v \in S^h. \tag{1.2}$$

As Strang and Fix [12] pointed out, the following Rayleigh quotient and minimum-maximum principle discovered by Rayleigh, Poincare, Courant and Fischer etc., play a fundamental role in finite element eigenvalue approximation.

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Rayleigh quotient. $R(v) = \frac{a(v,v)}{b(v,v)}$ $(v \neq 0)$ is called as Rayleigh quotient of v.

Minimum-maximum principle. Let λ_j be the j-th eigenvalue of (1.1), and $\lambda_{j,h}$ be the j-th eigenvalue of (1.2), respectively. We arrange eigenvalues by the increasing order, with each eigenvalue counted repeatedly according to its algebraic multiplicity. Then the minimum-maximum principle states,

$$\lambda_j = \min_{V_j \subset V, \dim V_j = j} \max_{v \in V_j} R(v), \quad \lambda_{j,h} = \min_{V_j \subset S^h, \dim V_j = j} \max_{v \in V_j} R(v).$$

By the minimum-maximum principle and $S^h \subset V$, one can deduce

$$\lambda_j \leqslant \lambda_{j,h}, \quad j = 1, 2, \ldots, \dim(S^h). \tag{1.3}$$

The minimum-maximum principle insures that numerical eigenvalues by conforming elements approximate exact eigenvalues from above.

The minimum-maximum principle is still valid for the non-conforming element. However, $\lambda_j \leqslant \lambda_{j,h}$ is no longer true since $S^h \not\subset V$. Then how about the opposite, i.e., $\lambda_j \geqslant \lambda_{j,h}$ for non-conforming elements? This problems have attracted many attentions from mathematics and engineering community.

Weinstein and Qian [13] obtained lower bounds of eigenvalues for the plate vibration problem using relaxed boundary conditions (breaking conforming properties) and provided numerical examples in 1943. Zienkiewicz et al. [18] discovered that the non-conforming Morley element approximates eigenvalues of the plate from below in 1967. Rannacher [9] provided some numerical results in 1979 for plate vibration problems, which indicate that the non-conforming Morley and Adini elements can be used to obtain lower bounds of eigenvalues. Rannacher also pointed out that this property is in general true for non-conforming elements. However, there are exceptions, e.g., the Adini element approximates from above under mixed boundary conditions ($w = \partial_n w = 0$ on one side and free boundary conditions on the other three sides) on a square domain.

In 2005, Liu and Yan [8] provided some numerical results for non-conforming Wilson and $EQ_1^{\rm rot}$ elements, which approximates eigenvalues of the Laplace operator from below. Their numerical results also indicated that $Q_1^{\rm rot}$ element gives lower bounds for some eigenvalues and upper bounds for others. In the same year, Chen and Yang [4] gave numerical examples of the three-dimensional Wilson's brick, which approximates eigenvalues of the Laplace operator from below.

Theoretical analysis seems a little behind. In 2000, Yang [14] proved that for the plate vibration problem on rectangular domain, the Adini element approximates exact eigenvalues from below. As for the Laplace operator eigenvalue problems, here is a list of some theoretical results:

Armentano and Duran [1] proved that piecewise linear non-conforming Crouzeix-Raviart element approximates exact eigenvalues associated with singular eigenfunctions from below in 2004.

Lin and Lin [6] proved that the non-conforming $EQ_1^{\rm rot}$ rectangular element approximates exact eigenvalues associated with smooth eigenfunctions from below in 2006.

Zhang et al. [16] proved that the non-conforming Wilson element approximates exact eigenvalues associated with smooth eigenfunctions from below in 2006.

In this article, we discuss some popular non-conforming elements in approximating eigenvalues of plate vibration problems and the Laplace operator, with emphasizing on obtaining lower bounds. In addition, for the Laplace operator, we obtained some new results which are listed here:

1) The non-conforming Crouzeix-Raviart element approximates exact eigenvalues from below on general polygonal domains. In the literature, the result obtained by Armentano and Duran [1] was restricted to eigenvalues associated with singular eigenfunctions. Our results include smooth eigenfunctions as well.

2) The non-conforming $EQ_1^{\rm rot}$ rectangular element approximates exact eigenvalues from below. In the literature, the result obtained by Lin and Lin [6] was restricted to eigenvalues associated with smooth eigenfunctions. Our results include singular eigenfunctions as well.

3) For non-conforming rectangular $Q_1^{\rm rot}$ element, we explain why numerical eigenvalues $\lambda_{1,h}$ and $\lambda_{3,h}$ approximate the exact eigenvalues from below for the L-shaped domain.

At the end, we list three related open problems.

2 Fundamental properties of non-conforming element methods in eigenvalues problems

Let $S^h \subset H$, $S^h \not\subset V$ be a non-conforming finite dimensional subspace. The non-conforming finite element approximation of (1.1) states:

Find $\lambda_h \in \mathbb{R}$, $u_h \in S^h$, $\|u_h\|_b = 1$, such that

$$a_h(u_h, v) = \lambda_h b(u_h, v), \quad \forall v \in S^h, \tag{2.1}$$

where $a_h(u, v) = \sum_{\kappa\in\pi_h} a_\kappa(u, v)$ and a_κ denotes restriction of a on $\kappa \in \pi_h$, where π_h is a partition. Yang [14] proved the following fundamental identity:

Lemma 2.1. *Let* (λ_h, u_h) *be an eigenpair of* (2.1) *with* $\|u_h\|_b = 1$, *and* (λ, u) *be an eigenpair of* (1.1) *with* $\|u\|_b = 1$, *then the following identity holds*:

$$\lambda_h - \lambda = a_h(u - u_h, u - u_h) - \lambda b(u - u_h, u - u_h) + 2D_h, \tag{2.2}$$

where $D_h = a_h(u, u_h) - \lambda b(u, u_h)$.

Lemma 2.1 is a generalization of fundamental identities in [12] and [2] for conforming element eigenvalue approximation. The difference is the non-conforming term $2D_h$. We shall see from the following sections, the sign and order of D_h are crucial for determining whether the numerical eigenvalues are below or above the true ones.

The following lemma developed by Zhang et al. [16], which is a generalization of an identity in [1], is an effective tool in studying the non-conforming finite element eigenvalue approximations.

Lemma 2.2. *Assume that* $(\lambda, u) \in \mathbb{R} \times V$ *is an eigenpair of* (1.1), $(\lambda_h, u_h) \in \mathbb{R} \times S^h$ *is an eigenpair of* (2.1), $I_h u \in S^h$ *is the non-conforming element interpolation of* u. *Then the following identity is valid*:

$$\lambda - \lambda_h = \|u - u_h\|_h^2 - \lambda_h\|I_h u - u_h\|_b^2 + \lambda_h(\|I_h u\|_b^2 - \|u\|_b^2) + 2a_h(u - I_h u, u_h). \tag{2.3}$$

By the Lax-Milgram Lemma, we know that the source problem of (1.1) has a unique solution. Therefore we can define an operator $T : H \to V$,

$$a(Tf, v) = b(f, v), \quad \forall f \in H, \quad v \in V. \tag{2.4}$$

Babuška and Osborn [2] proved that (1.1) is equivalent to the following operator form: $Tu = \lambda^{-1}u$, where $T : H \to H$ is a self-adjoint completely continuous operator.

Let the bilinear form $a_h(\cdot, \cdot)$ be S^h-elliptic, continuous and symmetric over the space S^h, and $\|v\|_h = \sqrt{a_h(v, v)}$ be a norm over the non-conforming element space S^h. Then the source problem of (2.1) has a unique solution, and we can define an operator $T_h : H \to S^h$ by

$$a_h(T_h f, v) = b(f, v), \quad \forall f \in H, \quad v \in S^h. \tag{2.5}$$

Therefore, (2.1) is equivalent to the following operator form: $T_h u_h = \lambda_h^{-1} u_h$. One can prove that $T_h : H \to H$ is a self-adjoint finite rank operator.

Let (λ_j, u_j) be the j-th eigenpair of (1.1) with $\|u_j\|_b = 1$, and let $M(\lambda_j)$ denote the space of all eigenfunctions of λ_j.

Lemma 2.3. *Let $T : H \to H$ be a self-adjoint completely continuous operator with $\|T_h - T\|_b \to 0$, and let $(\lambda_{j,h}, u_{j,h})$ be the jth eigenpair of (2.1) with $\|u_{j,h}\|_b = 1$. Then $\lambda_{j,h} \to \lambda_j$, and there exists $u_j \in M(\lambda_j)$ with $\|u_j\|_b = 1$, such that*

$$\lambda_{j,h} - \lambda_j = \frac{\lambda_j \lambda_{j,h}}{b(u_j, u_{j,h})} b((T - T_h)u_j, u_j) + R_1, \tag{2.6}$$

$$\|u_{j,h} - u_j\|_b \leqslant C\lambda_j^2 \|(T - T_h)u_j\|_b, \tag{2.7}$$

$$\|u_{j,h} - u_j\|_h = \lambda_j \|Tu_j - T_h u_j\|_h + R_2, \tag{2.8}$$

where $|R_1| \leqslant C\|(T - T_h)u_j\|_b^2$, $|R_2| \leqslant C\|(T - T_h)u_j\|_b$.

Proof. Combining Theorem 1 in [15] with Lemma 1 in [14], we have $\|u_{j,h} - u_j\|_b \leqslant C\lambda_j^2 \|(T - T_h)u_{j,h}\|_b$. By simple calculation shows that

$$\|u_{j,h} - u_j\|_b \leqslant C\lambda_j^2 \|(T - T_h)(u_{j,h} - u_j + u_j)\|_b \leqslant C\lambda_j^2 (\|T - T_h)u_j\|_b + \|T - T_h\|_b \|u_{j,h} - u_j\|_b),$$

which, together with $\|T - T_h\|_b \to 0$ $(h \to 0)$, yields (2.7). Since

$$\begin{aligned} b(Tu_{j,h} - T_h u_{j,h}, u_j) &= b(Tu_{j,h}, u_j) - b(\lambda_{j,h}^{-1} u_{j,h}, u_j) \\ &= b(u_{j,h}, Tu_j) - b(\lambda_{j,h}^{-1} u_{j,h}, u_j) = (\lambda_j^{-1} - \lambda_{j,h}^{-1}) b(u_{j,h}, u_j), \end{aligned}$$

we have

$$\begin{aligned} \lambda_{j,h} - \lambda_j &= \frac{\lambda_j \lambda_{j,h}}{b(u_j, u_{j,h})} b((T - T_h)u_{j,h}, u_j) \\ &= \frac{\lambda_j \lambda_{j,h}}{b(u_j, u_{j,h})} (b((T - T_h)u_j, u_j) + b((T - T_h)(u_{j,h} - u_j), u_j)) \\ &= \frac{\lambda_j \lambda_{j,h}}{b(u_j, u_{j,h})} b((T - T_h)u_j, u_j) + R_1. \end{aligned}$$

Moreover, using the facts that T and T_h are symmetric, $\lambda_{j,h} \to \lambda_j$, and (2.7), we can infer that

$$\begin{aligned} |R_1| &= \left| \frac{\lambda_j \lambda_{j,h}}{b(u_j, u_{j,h})} b((T - T_h)(u_{j,h} - u_j), u_j) \right| \leqslant C|b(u_{j,h} - u_j, (T - T_h)u_j)| \\ &\leqslant \|u_{j,h} - u_j\|_b \|(T - T_h)u_j\|_b \leqslant C\|(T - T_h)u_j\|_b^2. \end{aligned}$$

Therefore, we get (2.6) immediately.

From (2.1) and the definition of T_h, we get

$$\begin{aligned} \|u_{j,h} - \lambda_j T_h u_j\|_h^2 &= a_h(u_{j,h} - \lambda_j T_h u_j, u_{j,h} - \lambda_j T_h u_j) = b(\lambda_{j,h} u_{j,h} - \lambda_j u_j, u_{j,h} - \lambda_j T_h u_j) \\ &\leqslant \|\lambda_{j,h} u_{j,h} - \lambda_j u_j\|_b \|u_{j,h} - \lambda_j T_h u_j\|_b \leqslant (\|\lambda_{j,h} u_{j,h} - \lambda_j u_j\|_b + \|u_{j,h} - \lambda_j T_h u_j\|_b)^2, \end{aligned}$$

which, together with (2.6) and (2.7), yields $\|u_{j,h} - \lambda_j T_h u_j\|_h \leqslant C\|(T - T_h)u_j\|_b$. Denote $\|u_{j,h} - u_j\|_h = \lambda_j \|Tu_j - T_h u_j\|_h + R_2$. Using the triangle inequality, we deduce that

$$\begin{aligned} |R_2| &= |\|u_{j,h} - u_j\|_h - \lambda_j \|Tu_j - T_h u_j\|_h| = |\|u_{j,h} - u_j\|_h - \|u_j - \lambda_j T_h u_j\|_h| \\ &\leqslant \|u_{j,h} - \lambda_j T_h u_j\|_h \leqslant C\|(T - T_h)u_j\|_b. \end{aligned}$$

Therefore, we obtain (2.8).

According to Lemma 2.3, the error estimates of non-conforming eigenvalue approximations attribute to the error estimates of the corresponding non-conforming element approximation of their source problems. Therefore the error estimates of Wilson, Crouzeix-Raviart, Adini, Morley, Q_1^{rot}, and EQ_1^{rot} elements in eigenvalue approximations can be derived from error estimates of corresponding source problems.

3 Adini element and Morley element

Consider the plate vibration problem (1.1) with $V = H_0^2(\Omega)$, $H = L_2(\Omega)$,

$$a(u,v) = \int_\Omega (\sigma \Delta u \Delta v + (1-\sigma)(2\partial_{12}u\partial_{12}v + \partial_{11}u\partial_{11}v + \partial_{22}u\partial_{22}v))dx,$$
$$b(u,v) = \int_\Omega uvdx, \quad \|u\|_b = \|u\|_0,$$

where $\Omega \subset \mathbb{R}^2$ is a polygonal domain, $\sigma \in [0, 0.5)$ is the Poisson ratio. Clearly, the bilinear form $a(u,v)$ is symmetric; and according to [5], it is also continuous and $H_0^2(\Omega)$-elliptic.

We illustrate the Adini element by Figure 1.

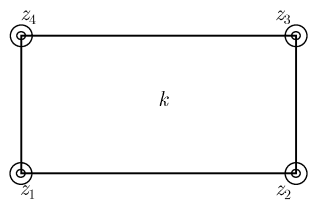

Figure 1

The degrees of freedom (interpolation conditions) are: function values and gradients $(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$ at the four vertices (nodes).

The Adini finite element space is:

$S^h = \{v \in C^0(\Omega) : v|_\kappa \in P_3(\kappa) \oplus \text{span}\{x_1^3x_2, x_1x_2^3\}, \kappa \in \pi_h, v, \partial_1 v, \partial_2 v$ are continuous at element vertices and zeros on boundary nodes $\}$,

where $P_3(\kappa)$ is the complete polynomial space of degree$\leqslant 3$ on κ, π_h is a rectangular mesh.

We see that $S^h \subset C^0$. However, $S^h \not\subset H^2(\Omega)$. In this case,

$$a_h(u_h, v) = \sum_{\kappa \in \pi_h} \int_\kappa (\sigma \Delta u_h \Delta v + (1-\sigma)(2\partial_{12}u_h\partial_{12}v + \partial_{11}u_h\partial_{11}v + \partial_{22}u_h\partial_{22}v))dx.$$

Denote $\partial_i = \frac{\partial}{\partial x_i}$, $\partial_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}, \ldots$ Inspired with [3, 17], Yang established the following results in [14].

Lemma 3.1. *Let eigenfunction $u \in H^4(\Omega)$, and let π_h be a uniform rectangular partition, then we have*

$$D_h = -\frac{1}{3}h_1^2 \int_\Omega [(\partial_{221}u)^2 + \sigma(\partial_{112}u)^2] - \frac{1}{3}h_2^2 \int_\Omega [(\partial_{112}u)^2 + \sigma(\partial_{221}u)^2] + O(h^3), \tag{3.1}$$

where h_1 and h_2 are horizontal and vertical side lengths of κ, respectively.

Theorem 3.1. *Let λ_j be the j-th eigenvalue of the plate vibration problem with $M(\lambda_j) \subset H^4(\Omega)$, let π_h be a uniform rectangular partition of Ω, and $(\lambda_{j,h}, u_{j,h})$ be the j-th Adini element eigenpair of the plate vibration problem with $\|u_{j,h}\|_0 = 1$. Then there exists $u_j \in M(\lambda_j)$ with $\|u_j\|_0 = 1$, such that*

$$\lambda_{j,h} - \lambda_j = -\frac{2}{3}h_1^2 \int_\Omega [(\partial_{221}u_j)^2 + \sigma(\partial_{112}u_j)^2]dx - \frac{2}{3}h_2^2 \int_\Omega [(\partial_{112}u_j)^2 + \sigma(\partial_{221}u_j)^2]dx + O(h^3). \tag{3.2}$$

Proof. According to Ciarlet [5], we immediately obtain

$$\|(T - T_h)f\|_0 \leqslant Ch^2, \tag{3.3}$$
$$\|(T - T_h)f\|_h \leqslant Ch^2. \tag{3.4}$$

Substituting (3.3) and (3.4) into (2.7) and (2.8) respectively, we have

$$\|u_{j,h} - u_j\|_0 \leqslant Ch^2, \tag{3.5}$$

$$\|u_{j,h} - u_j\|_h \leqslant Ch^2. \tag{3.6}$$

Substituting (3.1), (3.5) and (3.6) into (2.2) yields (3.2).

Since the dominant sign of the asymptotic expansion is negative, therefore, the numerical eigenvalue $\lambda_{j,h}$ obtained by the Adini element approximates exact eigenvalue λ_j from below. In other words, when h is sufficiently small, $\lambda_{j,h} \leqslant \lambda_j$.

Note. When $\sigma = 0$, our model reduces to the biharmonic operator. In this case, Theorem 3.1 is still valid. Indeed, Lin and Lin [6] proved that the Adini element approximates exact eigenvalues from below for the biharmonic operator, which is a special case here. Numerical evidence for this special case was provided by Rannacher [9] as early as 1979.

Consider the following eigenvalue problem:$\triangle^2 u = \lambda u$, in Ω; $u = \partial_n u = 0$, on $\partial\Omega$.

Let Ω be a unite square. Under uniform square partition, set the diameter of elements $h = \frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{6}, \ldots, \frac{\sqrt{2}}{18}$, numerical results of $\lambda^A_{1,h}$ using the Adini element are listed in the third column of Table 1. Under triangular partition (see Figure 2), set the diameter of elements $h = \frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{6}, \ldots, \frac{\sqrt{2}}{18}$, numerical results of $\lambda^M_{1,h}$ by the Morley element are listed in the second column of Table 1. The smallest eigenvalue of this problem is approximately $\lambda_1 = 1,295$.

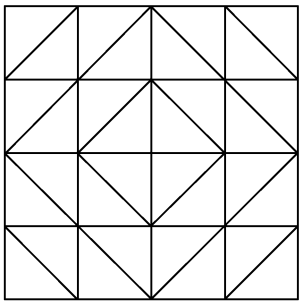

Figure 2

Remark 1. The above numerical example was provided by [9] when σ is zero. Since the symmetric property of the eigenvalue problem, the author used the symmetric grids intentionally for the test on the Morley element.

Table 1 Numerical eigenvalues of the unit square using Morley and Adini elements

h	$\lambda^M_{1,h}$	$\lambda^A_{1,h}$
$\frac{\sqrt{2}}{4}$	546	1177
$\frac{\sqrt{2}}{6}$	785	1231
$\frac{\sqrt{2}}{8}$	941	1256
$\frac{\sqrt{2}}{10}$	1040	1269
$\frac{\sqrt{2}}{12}$	1105	1277
$\frac{\sqrt{2}}{14}$	1149	1281
$\frac{\sqrt{2}}{16}$	1180	1284
$\frac{\sqrt{2}}{18}$	–	1286

4 Crouzeix-Raviart element

Consider the Laplace operator eigenvalue problem:

$$-\Delta u = \lambda u, \text{ in } \Omega; \quad u = 0, \text{ on } \partial\Omega, \tag{4.1}$$

where $\Omega \subset \mathbb{R}^2$ is a polygonal domain with the maximum interior angle ω.

If $\omega > \pi$, and the eigenfunction is singular, we denote $r_0 = \frac{\pi}{\omega}$, $r < r_0$ which is sufficiently close to r_0, and $q = 2/(2-r)$; and if $\omega < \pi$, or the eigenfunction is smooth, we denote $r_0 = r = 1$ and $q = 2$.

The weak form of (4.1) is (1.1) with $V = H_0^1(\Omega)$, $H = L_2(\Omega)$, and

$$a(u,v) = \int_\Omega \nabla u \nabla v dx, \quad b(u,v) = \int_\Omega uv dx, \quad \|u\|_b = \|u\|_0.$$

We illustrate the Crouzeix-Raviart triangular element by Figure 3. The degrees of freedom (interpolation conditions) are the function values at the midpoints z_1,z_2 and z_3 of the three sides.

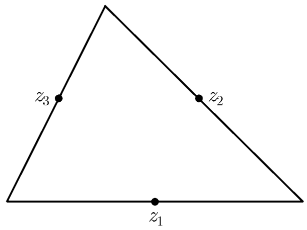

Figure 3

Let π_h be a regular partition (see [5, p. 131]). The non-conforming Crouzeix-Raviart element space is:

$S^h = \{v \in L_2(\Omega) : v|_\kappa \in P_1(\kappa), \kappa \in \pi_h,$ v is continuous at all midpoints of internal element edges and is zero at all midpoints of boundary element edges$\}$.

And $\varphi_i(x)$ is the basic function, satisfying $\varphi_i \in S^h, \varphi_i(z_j) = \delta_{i,j}$. Note that $S^h \not\subset H^1(\Omega)$ and in this case

$$a_h(u_h, v) = \sum_{\kappa \in \pi_h} \int_\kappa \nabla u_h \nabla v dx.$$

The side average interpolation operator $I_h : H_0^1(\Omega) \to S^h$ is defined by:

$$\int_l I_h u ds = \int_l u ds, \quad \forall\, l, u \in H_0^1(\Omega),$$

where l is an arbitrary element side of π_h.

In this case, we estimate the four terms of the right-hand side of (2.3) in order to prove λ_h approximating λ from below.

In the following, Lemma 4.1 is due to Armentano and Duran [1].

Lemma 4.1. *Let $u_j \in W_{2,q}(\Omega)$, then we have*:

$$a_h(u_j - I_h u_j, v) = 0, \quad \forall v \in S^h. \tag{4.2}$$

By the above Lemma 2.3 and (2.13) in [1], the following Lemma 4.2 can be obtained.

Lemma 4.2. *Let $u_j \in W_{2,q}(\Omega)$, then the following estimates are valid*:

$$\|u_j - I_h u_j\|_0 \leqslant C\lambda_j h^{1+r} \|u_j\|_{0,q}, \tag{4.3}$$

$$\|u_j - u_{j,h}\|_h \leqslant C\lambda_j h^r \|u_j\|_{0,q}, \tag{4.4}$$

$$\|u_j - u_{j,h}\|_0 \leqslant C\lambda_j^2 h^{2r} \|u_j\|_{0,q}. \tag{4.5}$$

Lemma 4.3. *Let the eigenfunctions of* (4.1) *be smooth, then we have*:

$$\lambda_{j,h}(\|I_h u_j\|_0^2 - \|u_j\|_0^2) = R(j) + O(h^3), \tag{4.6}$$

$$|R(j)| = \left| \frac{1}{36} \lambda_j \sum_{\kappa \in \pi_h} \sum_{i=1}^{3} \text{meas}(l_i)^2 \int_\kappa \frac{\partial^2 u_j(x)}{\partial l_i^2} u_j dx \right| \leqslant C(j) \lambda_j^2 h^2, \tag{4.7}$$

where $C(j) \leqslant \frac{1}{12}$ $(j = 1, 2, 3, \ldots)$.

Proof. Let I_0 be a piecewise constant interpolation operator, I_h^p be a piecewise linear interpolation operator defined by $I_h^p : C^0 \to S^h$, $I_h^p v(z_i) = v(z_i)$. Note that the three-midpoint Gauss quadrature rule on a triangle has second-order algebraic accuracy (see [12, Table 4.1]), and we have

$$\left|\int_\Omega (u_j - I_h^p u_j) I_0(u_j + I_h u_j)\right| = \left|I_0(u_j + I_h u_j)\int_\Omega (u_j - I_h^p u_j)\right| \leqslant Ch^3. \tag{4.8}$$

By computing, we obtain

$$\left|\frac{\partial^2 u(x)}{\partial l_i^2}\right| \leqslant \left(\left(\frac{\partial^2 u(x)}{\partial x_1^2}\right)^2 + \left(\frac{\partial^2 u(x)}{\partial x_1 \partial x_2}\right)^2 + \left(\frac{\partial^2 u(x)}{\partial x_2 \partial x_1}\right)^2 + \left(\frac{\partial^2 u(x)}{\partial x_2^2}\right)^2\right)^{\frac{1}{2}}.$$

Note that the error estimate of midpoint rectangular quadrature rule, we can conclude that

$$\begin{aligned}
&\int_\kappa (I_h^p u_j - I_h u_j) I_0(u_j + I_h u_j) dx \\
&\quad = I_0(u_j + I_h u_j) \int_\kappa \sum_{i=1}^3 \frac{1}{\text{meas}(l_i)} \int_{l_i} (I_h^p u_j - I_h u_j) ds \varphi_i(x) dx \\
&\quad = I_0(u_j + I_h u_j) \int_\kappa \sum_{i=1}^3 \frac{1}{\text{meas}(l_i)} \int_{l_i} (I_h^p u_j - u_j) ds \varphi_i(x) dx \\
&\quad = \frac{1}{72} I_0(u_j + I_h u_j) \sum_{i=1}^3 \text{meas}(l_i)^2 \frac{\partial^2 u_j(\eta_i)}{\partial l_i^2} \text{meas}(\kappa) \\
&\quad = \frac{1}{72} I_0(u_j + I_h u_j) \sum_{i=1}^3 \text{meas}(l_i)^2 \int_\kappa \left(\frac{\partial^2 u_j(\eta_i)}{\partial l_i^2} - \frac{\partial^2 u_j(x)}{\partial l_i^2} + \frac{\partial^2 u_j(x)}{\partial l_i^2}\right) dx \\
&\quad = \frac{1}{72} \sum_{i=1}^3 \text{meas}(l_i)^2 \int_\kappa \frac{\partial^2 u_j(x)}{\partial l_i^2} I_0(u_j + I_h u_j) dx + O(h^5) \\
&\quad = R_\kappa(j) + O(h^5),
\end{aligned} \tag{4.9}$$

where

$$|R_\kappa(j)| = \left|\frac{1}{72} \sum_{i=1}^3 \text{meas}(l_i)^2 \int_\kappa \frac{\partial^2 u_j(x)}{\partial l_i^2} \times 2u_j dx\right| \leqslant \frac{1}{12} h^2 |u_j|_{2,\kappa} \|u_j\|_{0,\kappa}.$$

Let $R(j) = \lambda_j \sum_\kappa R_\kappa(j)$. Then we get

$$|R(j)| = \frac{1}{36}\lambda_j \left|\sum_\kappa \sum_{i=1}^3 \text{meas}(l_i)^2 \int_\kappa \frac{\partial^2 u_j(x)}{\partial l_i^2} u_j dx\right| \leqslant \lambda_j \frac{1}{12} h^2 |u_j|_2 \|u_j\|_0 \leqslant \frac{1}{12}\lambda_j^2 h^2.$$

By (4.9), we have

$$\lambda_j \int_\Omega (I_h^p u_j - I_h u_j) I_0(u_j + I_h u_j) dx = R(j) + O(h^3). \tag{4.10}$$

Applying (4.8) and (4.10), we have

$$\begin{aligned}
&\lambda_{j,h} \int_\Omega (u_j - I_h u_j) I_0(u_j + I_h u_j) dx \\
&\quad = \lambda_{j,h} \int_\Omega (u_j - I_h^p u_j + I_h^p u_j - I_h u_j) I_0(u_j + I_h u_j) dx \\
&\quad = O(h^3) + \lambda_{j,h} \int_\Omega (I_h^p u_j - I_h u_j) I_0(u_j + I_h u_j) dx \\
&\quad = R(j) + O(h^3).
\end{aligned} \tag{4.11}$$

Using piecewise constant interpolation error estimate and (4.11), we conclude that

$$\begin{aligned}\lambda_{j,h}(\|I_h u_j\|_0^2 - \|u_j\|_0^2) &= \lambda_{j,h}\int_\Omega (u_j - I_h u_j)(u_j + I_h u_j)dx \\ &= \lambda_{j,h}\int_\Omega (u_j - I_h u_j)[(I - I_0)(u_j + I_h u_j) + I_0(u_j + I_h u_j)]dx \\ &= R(j) + O(h^3).\end{aligned}$$

Namely (4.6) and (4.7) can be obtained.

Theorem 4.1. *Let λ_j and $\lambda_{j,h}$ be the j-th eigenvalue of* (4.1) *and the j-th eigenvalue of* (2.1) *by the non-conforming Crouzeix-Raviart element, respectively. Let h be sufficiently small. Then if $\omega > \pi$ and the eigenfunctions of* (4.1) *are singular and $\|u_j - u_{j,h}\|_h \geqslant Ch^{r_0}$, or if the eigenfunctions of* (4.1) *are smooth and $\|u_j - u_{j,h}\|_h \geqslant C_j\lambda_j h$ with $C_j > \sqrt{C(j)}$, we have*

$$\lambda_{j,h} \leqslant \lambda_j. \tag{4.12}$$

Proof. We consider the terms on the right-hand side of (2.3). The first and fourth terms on the right-hand side of (2.3) are estimated by (4.4) and (4.2), respectively. The second and third terms on the right hand side of (2.3) can be estimated as the following. Applying (4.3), (4.5), and the triangular inequality, we derive

$$\|I_h u_j - u_{j,h}\|_0 \leqslant C\lambda_j^2 h^{2r}\|u_j\|_{0,q}. \tag{4.13}$$

Using interpolation error estimate, we conclude

$$|\|I_h u_j\|_0^2 - \|u_j\|_0^2| = \left|\int_\Omega (u_j - I_h u_j)(u_j + I_h u_j)dx\right| \leqslant C\lambda_j h^2\|u_j\|_{0,q}. \tag{4.14}$$

If $\omega > \pi$ and the eigenfunctions of (4.1) are singular, using (4.4), (4.13), (4.14), and (4.2) to estimate each term on the right hand side of (2.3), note that $r < 1$, we have (4.12) immediately.

If the eigenfunctions of (4.1) are smooth, clearly, we have (4.2), (4.4), (4.6) and (4.13) with $q = 2, r = 1$. Thus, the second term is an infinitesimal of higher order than the order of the first term. According to (4.6) and (4.7), as a general rule, the third term is an infinitesimal of the same order as that of the first term, therefore the sign of $\lambda_j - \lambda_{j,h}$ is determined by the first term coefficient C_j and the third term coefficient $C(j)$. Note the theorem condition that $C_j > \sqrt{C(j)}$. So, we can obtain the lower bounds of the eigenvalues.

Remark 2. When Ω is unit square, π_h is uniform isoceles right triangle partition, by computing we can obtain that $C(1)$=0.000 351 809 665 6, $C(2)$=$C(3)$=0.000 140 723 866 2, $C(4)$=0.000 087 952 416 38, and $C_j > \sqrt{C(j)}$ (j=1,2,3,4). Namely the condition in Theorem 4.1 is satisfied.

Armentano and Duran proved that the Crouzeix-Raviart element results in a lower bound in the singular eigenfunction case, and provided numerical results including the L-shaped domain in [1]. Here we discussed that the Crouzeix-Raviart element results in a lower bound not only in the singular eigenfunction case, but also in the smooth eigenfunction case. In the following, we carry out some numerical experiments on both L-shaped domain and the unit square.

Let Ω be the L-shaped domain ($[0,2]\times[0,1]\cup[0,1]\times[1,2]$). The exact eigenvalues are $\lambda_1 = 9.639\,7\cdots$, $\lambda_2 = 15.197\,2\cdots$, $\lambda_3 = 19.739\,2\cdots$, $\lambda_4 = 29.521\,4\cdots$. We use the uniform isoceles right triangle partition. Numerical eigenvalues $\lambda_{j,h}$, j=1, 2, 3, 4, by the Crouzeix-Raviart element are listed in Table 2.

Let Ω be the unit square. The exact eigenvalues are $\lambda_1 = 2\pi^2 = 19.739\,20\cdots$, $\lambda_2 = \lambda_3 = 5\pi^2 = 49.348\,02\cdots$, $\lambda_4 = 8\pi^2 = 78.956\,83\cdots$. Again, we use the uniform triangulation of the regular pattern. Numerical eigenvalues $\lambda_{j,h}, j = 1, 2, 3, 4$, are listed in the Table 3.

Table 2 Numerical eigenvalues of the L-shaped domain using Crouzeix-Raviart element

h	$\lambda_{1,h}$	$\lambda_{2,h}$	$\lambda_{3,h}$	$\lambda_{4,h}$
$\frac{\sqrt{2}}{4}$	9.133400	14.86528	19.39847	28.13676
$\frac{\sqrt{2}}{8}$	9.461196	15.10970	19.65450	29.18283
$\frac{\sqrt{2}}{16}$	9.574822	15.17460	19.71806	29.43694
$\frac{\sqrt{2}}{32}$	9.615485	15.19146	19.73392	29.50032
$\frac{\sqrt{2}}{64}$	9.630487	15.19578	19.73789	29.51618
$\frac{\sqrt{2}}{128}$	9.636154	15.19688	19.73888	29.52016
trend	$\nearrow$	$\nearrow$	$\nearrow$	$\nearrow$

Table 3 Numerical eigenvalues of the unit square using Crouzeix-Raviart element

h	$\lambda_{1,h}$	$\lambda_{2,h}, \lambda_{3,h}$	$\lambda_{4,h}$
$\frac{\sqrt{2}}{4}$	19.39846	44.86754	73.33747
$\frac{\sqrt{2}}{8}$	19.65450	48.24394	77.59386
$\frac{\sqrt{2}}{16}$	19.71806	49.07292	78.61802
$\frac{\sqrt{2}}{32}$	19.73392	49.27930	78.87224
$\frac{\sqrt{2}}{64}$	19.73789	49.33085	78.93569
$\frac{\sqrt{2}}{128}$	19.73888	49.34373	78.95155
$\frac{\sqrt{2}}{256}$	19.73913	49.34695	78.95551
$\frac{\sqrt{2}}{512}$	19.73919	49.34775	78.95651
trend	$\nearrow$	$\nearrow$	$\nearrow$

5 Wilson's element

Zhang et al. [16] proved that the Wilson element provides lower bounds for eigenvalues of the Laplace operator under a rectangular domain, and thereby settled a long standing conjecture in the finite element method.

Consider the Laplace operator eigenvalue problem (4.1), where $\Omega \subset \mathbb{R}^2$ is a rectangular domain.

We illustrate the Wilson element by Figure 4. The degrees of freedom (interpolation condition) are: function values at the four nodes, and two second order partial derivatives $\frac{\partial^2}{\partial x_1^2}$ and $\frac{\partial^2}{\partial x_2^2}$ at the central point of each element (we can also use average values $\int_\kappa \frac{\partial^2}{\partial x_i^2}$ as degrees of freedom).

The Wilson element space is:

$S^h = \{v \in L_\infty(\Omega) : v|_\kappa \in P_2(\kappa), \kappa \in \pi_h, v$ is continuous at four vertices of each element and is set to be zero at boundary nodes$\}$.

Note that $S^h \not\subset H^1(\Omega)$. Again, in this case, $a_h(u_h, v) = \sum_{\kappa \in \pi_h} \int_\kappa \nabla u_h \nabla v dx$.

We cite the following lemma of Lin and Lin (see [6, Lemma 3.8]).

Lemma 5.1. *Let $u \in H^3(\Omega)$, π_h be a rectangular partition, and $u_h \in S^h$. Then we have*

$$a_h(u - I_h u, u_h) = \frac{1}{3} \sum_{\kappa \in \pi_h} (h_1^2 + h_2^2) \int_\kappa \partial_{11} u \partial_{22} u dx + O(h^3), \tag{5.1}$$

where h_1 and h_2 are horizontal and vertical side lengths of κ, respectively.

Theorem 5.1. *Let λ_j and $\lambda_{j,h}$ be the j-th eigenvalue of (4.1) and the j-th eigenvalue of (2.1) by the Wilson element, respectively. Let $u_j \in H^3(\Omega)$ and λ_j be a simple eigenvalue, and let π_h be a regular rectangular partition. Then we have $\lambda_{j,h} \nearrow \lambda_j$, namely $\lambda_{j,h}$ approximates λ_j from below.*

Proof. We estimate each of the four terms on the right hand side of (2.3). It is well-known (see [5, 11]) that for the Wilson element under the regular rectangular mesh,

$$\|u_j - u_{j,h}\|_h^2 = O(h^2), \quad \|I_h u_j - u_{j,h}\|_0^2 = O(h^4). \tag{5.2}$$

According to the triangular inequality we obtain

$$\begin{aligned} |\, \|I_h u_j\|_0^2 - \|u_j\|_0^2 \,| &= |\, \|u_j\|_0 - \|I_h u_j\|_0 \,| \, (\|u_j\|_0 + \|I_h u_j\|_0) \\ &\leqslant \|u_j - I_h u_j\|_0 (\|u_j\|_0 + \|I_h u_j\|_0). \end{aligned}$$

By the standard polynomial approximation theory, $\|u_j - I_h u_j\|_0 \leqslant Ch^3 |u_j|_3$, and consequently

$$|\, \|I_h u_j\|_0^2 - \|u_j\|_0^2 \,| \leqslant Ch^3. \tag{5.3}$$

When $\Omega = (0, L_1) \times (0, L_2)$ is a rectangular domain, the eigenfunctions are

$$u_j(x_1, x_2) = \sin \frac{m\pi}{L_1} x_1 \sin \frac{n\pi}{L_2} x_2.$$

Therefore, $\partial_{11} u \partial_{22} u > 0$ on Ω. Applying (5.1) we deduce

$$a_h(u_j - I_h u_j, u_{j,h}) > 0, \quad a_h(u_j - I_h u_j, u_{j,h}) = O(h^2). \tag{5.4}$$

Using (5.2)–(5.4) to estimate terms in (2.3), we see that $\lambda_j - \lambda_{j,h} > 0$ for sufficiently small h. In other words, $\lambda_{j,h}$ approximates λ_j from below.

In the following, we provide some numerical experiments on the unit square as well as the L-shaped domain. Note that Theorem 5.1 covers only the case of smooth eigenfunctions. However, it is not the case for singular eigenfunctions as demonstrated by our numerical experiments.

Consider eigenvalue problem (4.1).

Let Ω be a unit square. Under uniform square partition, set the diameter of element $h = \frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{8}, \ldots, \frac{\sqrt{2}}{512}$, the numerical results of $\lambda_{j,h}$, $j = 1, 2, 3, 4$, using the Wilson element are listed in Table 4. Let Ω be the L-shaped domain $([0, 2] \times [0, 1] \cup [0, 1] \times [1, 2])$. Under the uniform square partition, we set the element diameter $h = \frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{8}, \ldots, \frac{\sqrt{2}}{256}$, and apply the Wilson element. The numerical eigenvalues $\lambda_{j,h}$, $j = 1, 2, 3, 4$, are listed in Table 5.

Table 4 Numerical eigenvalues of the unit square using the Wilson element

h	$\lambda_{1,h}$	$\lambda_{2,h}$	$\lambda_{3,h}$	$\lambda_{4,h}$
$\frac{\sqrt{2}}{4}$	17.29601	38.61814	38.61814	53.28405
$\frac{\sqrt{2}}{8}$	19.02322	45.70096	45.70096	69.18404
$\frac{\sqrt{2}}{16}$	19.55192	48.33936	48.33936	76.09289
$\frac{\sqrt{2}}{32}$	19.69183	49.08886	49.08886	78.20768
$\frac{\sqrt{2}}{64}$	19.72734	49.28278	49.28279	78.76733
$\frac{\sqrt{2}}{128}$	19.73620	49.33165	49.33172	78.90934
$\frac{\sqrt{2}}{256}$	19.73905	49.34302	49.34558	78.94539
$\frac{\sqrt{2}}{512}$	19.73902	49.34700	49.34700	78.95386
trend	$\nearrow$	$\nearrow$	$\nearrow$	$\nearrow$

Table 5 Numerical eigenvalues of the L-shaped domain using the Wilson element

h	$\lambda_{1,h}$	$\lambda_{2,h}$	$\lambda_{3,h}$	$\lambda_{4,h}$
$\frac{\sqrt{2}}{4}$	9.222409	13.81224	17.29601	24.91406
$\frac{\sqrt{2}}{8}$	9.549170	14.80791	19.02322	28.09714
$\frac{\sqrt{2}}{16}$	9.626115	15.09681	19.55192	29.14208
$\frac{\sqrt{2}}{32}$	9.640134	15.17195	19.69183	29.42504
$\frac{\sqrt{2}}{64}$	9.641356	15.19092	19.72733	29.49728
$\frac{\sqrt{2}}{128}$	9.640722	15.19556	19.73620	29.51541
$\frac{\sqrt{2}}{256}$	9.640218	15.19686	19.73847	29.51997
trend		$\nearrow$	$\nearrow$	$\nearrow$

6 $EQ_1^{\rm rot}$ element

The $EQ_1^{\rm rot}$ element is a non-conforming rectangular element proposed by Lin et al. [7] in 2005. We illustrate $EQ_1^{\rm rot}$ element by Figure 5.

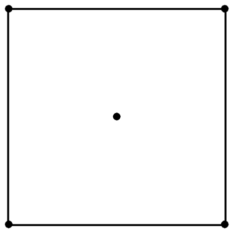

Figure 4

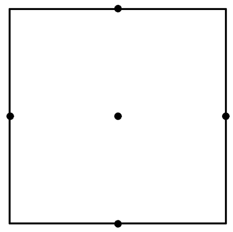

Figure 5

The $EQ_1^{\rm rot}$ finite element space is defined as:

$$V_h = \left\{ v \in L_2(\Omega) : v|_\kappa \in \text{span}\{1, x, y, x^2, y^2\}, \int_l v|_{\kappa_1} ds = \int_l v|_{\kappa_2} ds \text{ if } \kappa_1 \cap \kappa_2 = l \right\},$$
$$S^h = \left\{ v \in V_h, \int_l v|_\kappa ds = 0 \text{ if } \kappa \cap \partial\Omega = l \right\},$$

where $\kappa, \kappa_1, \kappa_2 \in \pi_h$.

Regarding eigenvalue problem (4.1), Lin and Lin [6] proved:

$$\lambda_h - \lambda = -\frac{1}{3}(h_1^2 + h_2^2) \int_\Omega (\partial_{12} u)^2 dx + O(h^4) \tag{6.1}$$

for smooth eigenfunctions under the uniform rectangular partition. Here h_1 and h_2 are horizontal and vertical side lengths of k, respectively.

We see that $EQ_1^{\rm rot}$ element approximates eigenvalues from below. In the following, we show that this is also the case for singular eigenfunctions. This result has not been reported in the literature.

Consider the Laplace operator eigenvalue problem (4.1), where $\Omega \subset \mathbb{R}^2$ is a polygonal domain, $u \in W_{2,q}(\Omega)$ (q see Section 4). The interpolation function $I_h u \in S^h$ of u is defined as follows:

$$\int_l I_h u ds = \int_l u ds, \quad \forall\, l, \tag{6.2}$$
$$\int_\kappa I_h u dx = \int_\kappa u dx, \quad \forall \kappa \in \pi_h, \tag{6.3}$$

where l is a side of an arbitrary element in π_h.

Applying Lemma 2.2 and following the similar argument as in the proof of Theorem 4.1, we have:

Theorem 6.1. *Let λ_j and $\lambda_{j,h}$ be the j-th exact eigenvalue and the j-th numerical eigenvalue by non-conforming $EQ_1^{\rm rot}$ element of* (4.1), *respectively. Let $u_j \in W_{2,q}(\Omega)$, and $\|u_j - u_{j,h}\|_h \geqslant Ch^{r_0}$, and let π_h be a regular rectangular partition. Then we have*

$$\lambda_{j,h} \leqslant \lambda_j, \tag{6.4}$$

when h is sufficiently small.

Proof. We estimate each of the four terms on the right-hand side of (2.3). According to [6] we have

$$a_h(u_j - I_h u_j, v) = 0, \quad \forall\, v \in S^h. \tag{6.5}$$

Applying the following interpolation error estimates and non-conforming error estimates

$$\|u_j - I_h u_j\|_0 \leqslant Ch^{1+r} \|u_j\|_{2,q}, \tag{6.6}$$

$$\|I_h u_j\|_{1,2} \leqslant C\|u_j\|_{2,q}, \tag{6.7}$$

$$\|u_j - u_{j,h}\|_h \leqslant Ch^r\|u\|_{2,q}, \tag{6.8}$$

$$\|u_j - u_{j,h}\|_0 \leqslant Ch^{2r}\|u\|_{2,q} \tag{6.9}$$

and (6.3), we conclude

$$\|I_h u_j - u_{j,h}\|_0 \leqslant Ch^{2r}\|u\|_{2,q}, \tag{6.10}$$

$$\begin{aligned} |\|I_h u_j\|_0^2 - \|u_j\|_0^2| &= \left|\int_\Omega (u_j - I_h u_j)(u_j + I_h u_j)dx\right| \\ &= \left|\int_\Omega (u_j - I_h u_j)((u_j + I_h u_j) - I_0(u_j + I_h u_j))dx\right| \leqslant Ch^{2+r}\|u_j\|_{2,q}, \end{aligned} \tag{6.11}$$

where I_0 is a piecewise constant interpolation operator. Combining (6.5), (6.8), (6.10) and (6.11) to estimate terms in (2.3), we are able to obtain (6.4).

7 Q_1^{rot} element

The Q_1^{rot} element is a non-conforming rectangular element proposed by Rannacher-Turek [10]. The Q_1^{rot} finite element space is:

$$V_h = \left\{v \in L_2(\Omega) : v|_\kappa \in \mathrm{span}\{1, x, y, x^2 - y^2\}, \int_l v|_{\kappa_1} ds = \int_l v|_{\kappa_2} ds, \text{ if } \kappa_1 \cap \kappa_2 = l\right\}, \tag{6.12}$$

$$S^h = \left\{v \in V_h, \int_l v|_\kappa ds = 0, \text{ if } \kappa \cap \partial\Omega = l\right\}, \tag{6.13}$$

where $\kappa, \kappa_1, \kappa_2 \in \pi_h$.

Regarding eigenvalue problem (4.1), Liu and Yan [8] made the following numerical observation for the Q_1^{rot} element.

1) Under a square domain, numerical eigenvalues $\lambda_{1,h}$ and $\lambda_{4,h}$ approximate exact eigenvalues from below, while $\lambda_{2,h}$ and $\lambda_{3,h}$ approximate from above.

2) Under the L-shaped domain, numerical eigenvalues $\lambda_{1,h}$ and $\lambda_{3,h}$ approximate exact eigenvalues from below, while $\lambda_{2,h}$ and $\lambda_{4,h}$ approximate from above.

Liu and Yan [8] explained the phenomenon 1) for the square domain. In the following we shall analyze the phenomenon 2) for the L-shaped domain.

Let $\Omega \subset \mathbb{R}^2$ be a concave polygonal domain, $u \in W_{2,q}(\Omega)$ (q see Section 4). A side average interpolant $I_h u \in S^h$ is defined as the following: $\int_l I_h u ds = \int_l u ds$, $\forall\, l$, where l is an arbitrary side of element in π_h.

We prove Theorem 7.1 using the similar argument as in the proof of Theorem 6.1:

Theorem 7.1. *Assume that $\omega > \pi$ and the eigenfunction is singular. Let λ_j and $\lambda_{j,h}$ be the j-th exact eigenvalue and the j-th numerical eigenvalue of* (4.1) *by the non-conforming Q_1^{rot} element, respectively. Let $u_j \in W_{2,q}(\Omega)$, and $\|u_j - u_{j,h}\|_h \geqslant Ch^{r_0}$. Then when h is sufficiently small, the following inequality holds*:

$$\lambda_{j,h} \leqslant \lambda_j. \tag{7.1}$$

According to Theorem 7.1, we see that under the L-shaped domain the non-conforming Q_1^{rot} element approximates λ_1 from below due to the condition $\|u_j - u_{j,h}\|_h \geqslant Ch^{r_0}$; on the other hand, the corresponding eigenfunction of λ_3 is sufficiently smooth for the L-shaped domain. Therefore, $\lambda_{3,h}$ obtained by the Q_1^{rot} element approximates exact eigenvalue from below for the same reason as for the case of the square domain (see [8]).

Since the numerical eigenvalues $\lambda_{2,h}$ and $\lambda_{4,h}$ approximate exact eigenvalues from above, we believe that $\|u_2 - u_{2,h}\|_h \geqslant Ch^{r_0}$, $\|u_4 - u_{4,h}\|_h \geqslant Ch^{r_0}$ are not valid.

Three open problems:

(1) Numerical results illustrated that the three-dimensional Wilson's brick approximates exact eigenvalues of the Laplace operator from below. However, we have not seen a theoretical proof.

(2) Numerical results demonstrated that the Morley element approximates exact eigenvalues of the plate vibration problems from below. Again, the theoretical justification is lacking. In addition, we do not know if the result is valid for any Poisson ratio σ (not only $\sigma = 0$) ranging from 0 to 0.5 as indicated by a referee.

(3) The higher dimensional generalization of the Crouzeix-Raviart element.

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