Science in China Series A: Mathematics
Nov., 2009, Vol. 52, No. 11, 2447–2458
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Estimates for the zeros of differences of meromorphic functions

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Abstract Let f be a transcendental meromorphic function and $g(z) = f(z + c_1) + f(z + c_2) - 2f(z)$ and $g_2(z) = f(z + c_1) \cdot f(z + c_2) - f^2(z)$. The exponents of convergence of zeros of differences $g(z)$, $g_2(z)$, $g(z)/f(z)$, and $g_2(z)/f^2(z)$ are estimated accurately.

Keywords: complex difference, zero, exponents of convergence

MSC(2000): 30D35, 39B12

1 Introduction and results

We use the basic notions of Nevanlinna's theory in this work (see e.g., [1–3]). In addition, we use notations $\sigma(f)$ to denote the order of growth of meromorphic function $f(z); \lambda(f)$ and $\overline{\lambda}(f)$ to denote the exponents of the convergence of the zero-sequence and the sequence of distinct zeros of $f(z)$, respectively.

Recently, a number of papers (including [4–11]) have focused on complex difference equations and differences analogues of Nevanlinna's theory. In [5], Bergweiler and Langley first investigated the existence of zeros of $\Delta f(z)$ and $\Delta f(z)/f(z)$, and obtained many profound and significative results. The results may be viewed as discrete analogues of the following existing theorem on the zeros of f'.

Theorem A[12–14]**.** *Let f be transcendental and meromorphic in the plane with*

$$\lim_{r\to\infty} \inf \frac{T(r,f)}{r} = 0, \tag{1.1}$$

then f' has infinitely many zeros.

Theorem A is sharp, as shown by e^z, $\tan z$ and examples of arbitrary order greater than 1 constructed in [15]. For f as in the hypotheses of Theorem A it follows from Hurwitz' theorem that if z_1 is a zero of f' then the difference $f(z+c) - f(z)$ has a zero near z_1 for all sufficiently small $c \in \mathbb{C}\backslash\{0\}$. This makes it natural to ask whether the difference $f(z + c) - f(z)$, for such functions f, must always have infinitely many zeros or not. In [5], Bergeiler and Langley answered this problem, and obtained the following Theorem B, Theorem C and Lemma A.

Received November 19, 2008; accepted April 1, 2009; published online August 6, 2009
DOI: 10.1007/s11425-009-0159-7
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This work was supported by National Natural Science Foundation of China (Grant No. 10871076) and Brain Pool Program of Korean Federation of Science and Technology Societies (Grant No. 072-1-3-0164)

Citation: Chen Z X, Shon K H. Estimates for the zeros of differences of meromorphic functions. Sci China Ser A, 2009, 52(11): 2447–2458, DOI: 10.1007/s11425-009-0159-7

Theorem B[5]**.** *There exists* $\delta_0 \in (0, \frac{1}{2})$ *with the following property. Let* f *be a transcendental entire function with order* $\sigma(f) \leqslant \sigma < \frac{1}{2} + \delta_0 < 1$. *Then*

$$H(z) = \frac{f(z+1) - f(z)}{f(z)}$$

has infinitely many zeros.

Theorem C[5]**.** *Let* f *be a function transcendental and meromorphic of lower order* $\mu(f) < 1$ *in the plane. Let* $c \in \mathbb{C}\backslash\{0\}$ *be such that at most finitely many poles* z_j, z_k *of* f *satisfy* $z_j - z_k = c$. *Then* $h(z) = f(z+c) - f(z)$ *has infinitely many zeros.*

Lemma A[5]**.** *Let* f *be a function transcendental and meromorphic in the plane which satisfies* (1.1), *and let* $h = f(z+1) - f(z)$ *and* $H = h/f$. *Then* h *and* H *are both transcendental.*

For an entire function $f(z)$, evidently $h(z) = f(z+1) - f(z)$ has infinitely many zeros by Lemma A.

The above results show the existence of zeros of differences and divided differences in the complex plane well.

Recently, in a number of papers, differences of the forms $f(z_j + c_1) + f(z_j + c_2)$, $f(z_j + c_1) \cdot f(z_j + c_2)$ often appear (see [4, 6, 9, 11]).

Thus, two natural questions are:

(i) What are the exponents of convergence of zeros of differences and divided differences? In other words, how do we estimate the amount of zeros of differences and divided differences more precisely?

(ii) What can be said about zeros of differences $g(z) = f(z+c_1) + f(z+c_2) - 2f(z)$ and $g_2(z) = f(z+c_1) \cdot f(z+c_2) - f^2(z)$?

In this study, we consider the aforementioned questions. We let

$$g(z) = f(z+c_1) + f(z+c_2) - 2f(z) \quad \text{and} \quad g_2(z) = f(z+c_1) \cdot f(z+c_2) - f^2(z).$$

If $c_2 = 0$, then $g(z)$ and $g_2(z)$ become the problem on $f(z+1) - f(z)$. Hence, discussions on $g(z)$ and $g_2(z)$ are more general.

We will prove the following theorems.

Theorem 1.1. *Let* f *be a transcendental entire function of order of growth* $\sigma(f) = \sigma < 1$. *Let* $c_1, c_2 \in \mathbb{C}\backslash\{0\}$ *be such that* $c_1 + c_2 \neq 0$. *Then* $g(z)$ *has infinitely many zeros and satisfies* $\lambda(g) = \sigma(g) = \sigma$.

In particular, if f *has at most finitely many zeros* z_j *satisfying* $f(z_j + c_1) + f(z_j + c_2) = 0$, *then* $G(z) = g(z)/f(z)$ *satisfies* $\lambda(G) = \sigma(G) = \sigma$.

Theorem 1.2. *Let* f, c_1, c_2 *satisfy the conditions of Theorem* 1.1 *Then* $g_2(z)$ *has infinitely many zeros and satisfies* $\lambda(g_2) = \sigma(g_2) = \sigma$.

In particular, if f *has at most finitely many zeros* z_j, z_s *satisfying* $z_j - z_s = c_1$ *or* c_2, *then* $G_2(z) = g_2(z)/f^2(z)$ *has infinitely many zeros and satisfies* $\lambda(G_2) = \sigma(G_2) = \sigma$.

Theorem 1.3. *Let* f *be a transcendental and meromorphic function of the order of growth* $\sigma(f) = \sigma < 1$. *Let* $c_1, c_2 \in \mathbb{C}\backslash\{0\}$ *be such that* $c_1 + c_2 \neq 0$. *If* f *has at most finitely many poles* b_j, b_s *satisfying*

$$b_j - b_s = k_1 c_1 + k_2 c_2 \quad (k_d = 0, \pm 1,\ d = 1, 2),$$

then $g(z)$ has infinitely many zeros and satisfies $\lambda(g) = \sigma(g) = \sigma$.

In particular, if f has at most finitely many zeros a_j satisfying $f(a_j + c_1) + f(a_j + c_2) = 0$, then $G(z) = g(z)/f(z)$ has infinitely many zeros and satisfies $\lambda(G) = \sigma(G) = \sigma$.

Theorem 1.4. *Let f, c_1, c_2 satisfy the conditions of Theorem 1.3. If f has at most finitely many poles b_j satisfying*

$$f(b_j + k_1c_1 + k_2c_2) = 0, \infty \quad (k_d = 0, \pm 1,\ d = 1, 2),$$

then $g_2(z)$ has infinitely many zeros and satisfies $\lambda(g_2) = \sigma(g_2) = \sigma$.

In particular, if f has at most finitely many zeros $a_j,\ a_s$ satisfying $a_j - a_s = c_1$ or c_2, then $G_2(z) = g_2(z)/f^2(z)$ has infinitely many zeros and satisfies $\lambda(G_2) = \sigma(G_2) = \sigma$.

In the above theorems, if we suppose $c_2 = 0$, then the following theorems are more easily obtained. So, we omit their proofs.

Theorem 1.5. *Let f be a transcendental entire function of the order of growth $\sigma(f) = \sigma < 1$. Let $c \in \mathbb{C}\backslash\{0\}$. Then $h(z) = f(z + c) - f(z)$ has infinitely many zeros and satisfies $\lambda(h) = \sigma(h) = \sigma$.*

In particular, if f has at most finitely many zeros $z_j,\ z_s$ satisfying $z_j - z_s = c$, then $H(z) = h(z)/f(z)$ has infinitely many zeros and satisfies $\lambda(H) = \sigma(H) = \sigma$.

Theorem 1.6. *Let f be a transcendental and meromorphic function of order of growth $\sigma(f) = \sigma < 1$ and $\overline{\lambda}(\frac{1}{f}) < \sigma$. Let $c \in \mathbb{C}\backslash\{0\}$. If f has at most finitely many poles $b_j,\ b_s$ satisfying $b_j - b_s = c$ then $h(z) = f(z+c) - f(z)$ has infinitely many zeros and satisfies $\lambda(h) = \sigma(h) = \sigma$.*

In particular, if f has at most finitely many zeros $a_j,\ a_s$ satisfying $a_j - a_s = c$, then $H(z) = h(z)/f(z)$ has infinitely many zeros and satisfies $\lambda(H) = \sigma(H) = \sigma$.

Remark 1.1. Suppose that f is a transcendental and meromorphic function of the order of growth $\sigma(f) = \sigma < 1$. Set $h_j(z) = f(z + c_j) - f(z), c_j \in \mathbb{C}\backslash\{0\}, j = 1, 2$. Then $g(z) = h_1(z) + h_2(z), g_2(z) = h_1(z)h_2(z) + f(z)[h_1(z) + h_2(z)]$. From Theorems 1.3, 1.4, 1.6 and latter their proofs, we know that $g(z),\ g_2(z)$ are greatly different from $h_1(z)$. First, although $h_1(z)$ is simpler than $g(z)$ and $g_2(z)$, we need to add a condition $\overline{\lambda}(1/f) < \sigma$ to prove $\lambda(h_1) = \sigma$ in the proof of Theorem 1.6. But the condition is not required in the proof of Theorems 1.3 and 1.4. Second, the proofs that $g(z)$ and $g_2(z)$ are transcendental are more difficult than the case of $h_1(z)$ (see the proofs of Lemmas 2.2, 3.1, 3.2 and Lemma A[5]).

2 Proof of Theorem 1.1

For proofs of Theorems 1.1–1.4, we need to deal with the term ε-set.

Remark 2.1 (On the ε-set). Following Hayman[16, pp. 75–76], we define an ε-set E to be a countable union of open discs, not containing the origin and subtending angles at the origin, whose sum is finite. If E is an ε-set then the set of $r \geqslant 1$ for which the circle $S(0, r)$ meets E has finite logarithmic measure, and for almost all real θ the intersection of E with the ray $\arg z = \theta$ is bounded.

We need the following lemma to prove Theorem 1.1, Lemmas 2.2 and 3.2.

Lemma 2.1[5]**.** *Let f be transcendental and meromorphic of order less than 1 in the plane. Let $h > 0$. Then, there exists an ε-set E such that*

$$f(z + c) - f(z) = cf'(z)(1 + o(1)) \quad \textit{as } z \to \infty \textit{ in } \mathbb{C}\backslash E,$$

uniformly in c *for* $|c| \leqslant h$.

We need also the following lemma to prove Theorems 1.1 and 1.3. The proof of Lemma 2.2 is greatly different from that of Lemma A[5].

Lemma 2.2. *Let* f *be a transcendental and meromorphic function of order less than* 1. *Let* $c_1, c_2 \in \mathbb{C}\backslash\{0\}$ *be such that* $c_1+c_2 \neq 0$. *Then* $g(z)$ *and* $G(z) = g(z)/f(z)$ *are both transcendental.*

Proof. First, we prove that $g(z)$ is transcendental. Without loss of generality, it may be assumed that $c_1 = 1$, thus, $1 + c_2 \neq 0$ and

$$g(z) = f(z+1) + f(z+c_2) - 2f(z).$$

Assume that $g(z)$ is a rational function. Then

$$f(z+1) + f(z+c_2) = R(z) + 2f(z), \tag{2.1}$$

where $R(z)$ is a rational function. Suppose that a set $A = \{x_j + iy_j \mid j = 1, \ldots, s\}$ consists of all poles of $R(z)$, and set $M = \max\{|x_j| + |y_j| + 1 + |c_2| : 1 \leqslant j \leqslant s\}$. Then there is no pole of $R(z)$ in regions $D_1 = \{z : \text{Re } z > M\}$, $D_2 = \{z : \text{Re } z < -M\}$, $D_3 = \{z : \text{Im } z > M\}$ and $D_4 = \{z : \text{Im } z < -M\}$.

Now we prove that $f(z)$ has at most finitely many poles. Suppose to the contrary. Then there is some D_j, say D_1, in which there are infinitely many poles. So, there is a pole z_0 of $f(z)$ in D_1. If Re $c_2 \geqslant 0$, then for any $n, m \in \mathbb{N}^0$ ($\mathbb{N}^0 = \{0, 1, 2, \ldots\}$), $z_{n,m} = z_0 + n + mc_2 \in D_1$, i.e., $z_{n,m}$ is not a pole of $R(z)$. By (2.1), we can see that $f(z)$ has a sequence of poles which is of the form

$$\{z_{n,m} = z_0 + n + mc_2,\ n, m \in \mathbb{N}^0,\ \text{at least one of } n, m \text{ gets all over } 1, 2, \ldots\}.$$

So that $\lambda(1/f) = 1$. This is a contradiction.

If Re $c_2 < 0$, then we can write (2.1) to

$$f(z+1-c_2) - 2f(z-c_2) = R(z-c_2) - f(z). \tag{2.2}$$

For any $n, m \in \mathbb{N}^0$, $z'_{n,m} = z_0 + n - mc_2 \in D_1$, and $z'_{n,m}$ is not a pole of $R(z)$. By (2.2), we can see that $f(z)$ has a sequence of poles which is of the form

$$\{z'_{n,m} = z_0 + n - mc_2,\ n, m \in \mathbb{N}^0,\ \text{at least one of } n, m \text{ gets all over } 1, 2, \ldots\}.$$

So that $\lambda(1/f) = 1$. This is also a contradiction.

If in D_2 (or D_3, or D_4), f has infinitely many poles, we use the similar method to obtain a contradiction. Hence f has at most finitely many poles.

Thus, there exists a rational function $R_1(z)$ such that $h(z) = f(z) - R_1(z)$ is a transcendental entire function. By (2.1), we obtain that

$$h(z+1) + h(z+c_2) = 2h(z) + P(z), \tag{2.3}$$

where

$$P(z) = R(z) + 2R_1(z) - R_1(z+1) - R_1(z+c_2).$$

Since $h(z+1)$, $h(z+c_2)$ and $h(z)$ are entire functions, we see $P(z)$ is a polynomial. By Lemma 2.1, we see that there is an ε-set E such that as $z \to \infty$ in $\mathbb{C}\backslash E$,

$$h(z+1) - h(z) = h'(z)(1+o(1)), \quad h(z+c_2) - h(z) = c_2 h'(z)(1+o(1)). \tag{2.4}$$

If $P(z) \equiv 0$, by (2.3) and (2.4), we get as $z \to \infty$ in $\mathbb{C}\backslash E$,

$$h'(z)(1+o(1)) = -c_2 h'(z)(1+o(1)).$$

Since $h'(z) \neq 0$ (as $z \notin E$), we have $1 + c_2 = 0$. This contradicts our assumption. Hence $P(z) \not\equiv 0$. Set $\deg P = d \geqslant 0$. Then $P(z) = az^d(1+o(1))$ where $a\,(\neq 0)$ is a constant. By (2.3) and (2.4), we obtain that

$$(1+c_2)h'(z)(1+o(1)) = az^d(1+o(1)), \quad \text{as } \ z \to \infty \text{ in } \mathbb{C}\backslash E.$$

This is a contradiction since $h'(z)$ is transcendental.

Second, we prove that $G(z)$ is transcendental. Suppose that $G(z)$ is a rational function. Then

$$\frac{f(z+1) + f(z+c_2) - 2f(z)}{f(z)} = R_0(z),$$

where $R_0(z)$ is a rational function. By Lemma 2.1, we see that there is an ε-set E_1 such that as $z \to \infty$ in $\mathbb{C}\backslash E_1$,

$$\frac{(1+c_2)f'(z)(1+o(1))}{f(z)} = R_0(z). \tag{2.5}$$

Since $f(z)$ is transcendental and has either infinitely many zeros or infinitely many poles, we see that $f'(z)/f(z)$ must be transcendental. Thus (2.5) is a contradiction.

Proof of Theorem 1.1. By Lemma 2.2, we see that $g(z)$ is a transcendental function. By Lemma 2.1, we see that there is an ε-set E such that as $z \to \infty$ in $\mathbb{C}\backslash E$,

$$g(z) = (c_1 + c_2)f'(z)(1+o(1)). \tag{2.6}$$

Set

$$H = \{|z| : z \in E, \text{ or } g(z) = 0, \text{ or } f'(z) = 0\}.$$

Then the linear measure of H is finite. On $|z| = r \notin H$, entire functions $g(z)$ and $f'(z)$ have no zero. By (2.6),

$$|g(z) - (c_1 + c_2)f'(z)| = |o(f'(z))| < |g(z)| + |(c_1 + c_2)f'(z)|.$$

Thus $g(z)$ and $-(c_1+c_2)f'(z)$ satisfy the conditions of Rouché's theorem. By Rouché's theorem, we obtain that for $|z| = r \notin H$,

$$n\left(r, \frac{1}{g}\right) - n(r, g) = n\left(r, \frac{1}{f'}\right) - n(r, f').$$

Since f is a transcendental entire function and $\sigma(f) < 1$, we obtain that

$$\lambda(g) = \lambda(f') = \sigma(f') = \sigma(f) = \sigma. \tag{2.7}$$

Now we prove that $\lambda(G)=\sigma(G)=\sigma(f)=\sigma$. Suppose that z_j is a zero of $g(z)$, then there are two cases: (1) $f(z_j)=0$; (2) $f(z_j)\neq 0$. If $f(z_j)=0$, then $f(z_j+c_1)+f(z_j+c_2)=0$. By the hypotheses of the theorem, at most there exist finitely many such points. If $f(z_j)\neq 0$, then z_j must be a zero of $G(z)$. Hence

$$n\left(r,\frac{1}{G}\right)=n\left(r,\frac{1}{g}\right)+O(1). \tag{2.8}$$

Thus, (2.7) and (2.8) show $\lambda(G)=\sigma(G)=\sigma(f)=\sigma$. Theorem 1.1 is thus proved.

3 Proof of Theorem 1.2

To prove Theorems 1.2 and 1.4, we need the following Lemmas 3.1 and 3.2. The proofs of Lemmas 3.1 and 3.2 are greatly different from Lemmas A and 2.2.

Lemma 3.1. *Let f be a transcendental and meromorphic function of order less than 1. Let $c_1,c_2\in\mathbb{C}\backslash\{0\}$ be such that $c_1+c_2\neq 0$. If $g_2(z)$ is a rational function, then $f(z)$ has at most finitely many poles.*

Proof. Without loss of generality, it may be assumed that $c_1=1$, thus, $1+c_2\neq 0$ and

$$f(z+1)\cdot f(z+c_2)=R(z)+f^2(z), \tag{3.1}$$

where $R(z)$ is a rational function. Set

$$B=\{b_j: R(b_j)=\infty,\ j=1,2,\dots,s\},\quad M=2\max\{|b_j|: j=1,2,\dots,s\}+1,$$

and

$$D_1=\{z:\mathrm{Re}\ z>M\},\quad D_2=\{z:\mathrm{Re}\ z<-M\},$$
$$D_3=\{z:\mathrm{Im}\ z>M\},\quad D_4=\{z:\mathrm{Im}\ z<-M\}.$$

Suppose that $f(z)$ has infinitely many poles. Then there exists at least one of $D_j (j=1,\dots,4)$, say D_1, such that $f(z)$ has infinitely many poles in D_1. Assume a set $A=\{z_j\}$ consists of all poles z_j of $f(z)$ in D_1, and the poles z_j satisfy $M\leqslant |z_1|\leqslant |z_2|\leqslant\cdots$. Then there exist only the following two cases.

Case 1. There exists pole $z_d\in A$ such that for any $b_j\in B$, it is not of the form z_d+n+mc_2 (for any $n,m\in\mathbb{N}^0$). So that $R(z_d+n+mc_2)\neq\infty$. Thus, by (3.1), we see that there is one infinite sequence $A_1=\{z_d+n+mc_2\}$ such that each $z_d+n+mc_2\in A_1$ is a pole of $f(z)$, and there exists at least one of n,m, we say m, such that m gets all over $1,2,\dots$ Hence $\lambda(\frac{1}{f})=1$. This contradicts our assumption.

Case 2. For all $z_j\in A$, there exists some $b_{t_j}\in B$ satisfying $b_{t_j}=z_j+n_j+m_jc_2$ (for some $n_j,m_j\in\mathbb{N}^0$). Since Re $z_j>M$, and Re $b_{t_j}<M$, we see Re $c_2<0$. Since A is an infinite set and B is a finite set, we see that there is a $b_j\in B$, say b_1, satisfying

$$b_1=z_1+n_1+m_1c_2=z_2+n_2+m_2c_2=\cdots.$$

We may rearrange $z_j+n_j+m_jc_2$ according to $n_1\leqslant n_2\leqslant\cdots, m_1\leqslant m_2\leqslant\cdots$, and still use original notations. Thus, $z_2=z_1+(n_1-n_2)+(m_1-m_2)c_2, z_3=z_1+(n_1-n_3)+(m_1-m_3)c_2,\dots$ and z_j $(j=2,3,\dots)$ satisfy

$$0\geqslant n_1-n_2\geqslant n_1-n_3\geqslant\cdots,\quad 0\geqslant m_1-m_2\geqslant m_1-m_3\geqslant\cdots.$$

Now set

$$z_{3ij} = z_1 + (n_1 - n_3 + i) + (m_1 - m_3 + j)c_2,$$

where

$$i = 0, 1, \dots, (n_1 - n_2) - (n_1 - n_3) - 1,$$
$$j = 0, 1, \dots, (m_1 - m_2) - (m_1 - m_3) - 1.$$

Since Re z_{3ij} (Im z_{3ij}) is between Re z_2 (Im z_2) and Re z_3 (Im z_3) respectively, we see that $z_{3ij} \in D_1$, i.e., $R(z_{3ij}) \neq \infty$. Since $f(z_3) = \infty$ and $R(z_{3ij}) \neq \infty$, by (3.1), we see that either z_{310} or z_{301} is a pole of $f(z)$. If z_{310} is the pole of $f(z)$, then z_{320} or z_{311} is a pole of $f(z)$; if z_{301} is the pole of $f(z)$, then z_{311} or z_{302} is a pole of $f(z); \cdots$.

Thus from these poles, we see that at least one of i, j, we say j, gets all over $0, 1, \dots, (m_1 - m_2) - (m_1 - m_3) - 1$.

We may repeat the above processes to $z_4 \to z_3;\ z_5 \to z_4; \cdots$. We can see that $f(z)$ has infinitely many poles being of the form

$$z_1 + (n_1 - n_2 + i) + (m_1 - m_2 + j)c_2$$

and at least one of i, j, we say j, gets all over $0, 1, \dots$ Thus, $\lambda(\frac{1}{f}) = 1$ which contradicts our assumption.

Lemma 3.2. *Let f be a transcendental and meromorphic function of order less than* 1*. Let $c_1, c_2 \in \mathbb{C}\backslash\{0\}$ be such that $c_1 + c_2 \neq 0$. Then $g_2(z)$ is transcendental.*

Proof. Assume

$$f(z + c_1) \cdot f(z + c_2) - f^2(z) = R(z), \tag{3.2}$$

where $R(z)$ is a rational function. By Lemma 3.1, we see that f has at most finitely many poles. By Lemma 2.1, we know that there exists an ε-set E, such that as $z \to \infty$ in $\mathbb{C}\backslash E$,

$$f(z + c_1) - f(z) = c_1 f'(z)(1 + o(1)), \quad f(z + c_2) - f(z) = c_2 f'(z)(1 + o(1)). \tag{3.3}$$

By (3.2) and (3.3), we obtain that

$$f'(z)\left[c_1 c_2 f'(z)(1 + o(1)) + (c_1 + c_2)f(z)(1 + o(1))\right] = R(z). \tag{3.4}$$

Set $f(z) = f_0(z)/d(z)$, where $d(z)$ is a polynomial formed by all poles of $f(z)$, $f_0(z)$ is a transcendental entire function with $\sigma(f_0) = \sigma(f) = \sigma < 1$. From the Wiman-Valiron theory (see [2, 17]), we see that there exists a subset $F \subset (1, \infty)$ of finite logarithmic measure such that for large $r \notin F$, for all z satisfying $|z| = r$ and $|f_0(z)| = M(r, f_0)$,

$$\frac{f_0'(z)}{f_0(z)} = \frac{\nu(r)}{z}(1 + o(1)), \tag{3.5}$$

where $\nu(r)$ is the central index of $f_0(z)$. By (3.5) and $f(z) = f_0(z)/d(z)$, we use the same method as in [18], and can see that for all z satisfying $|z| = r$ and $|f_0(z)| = M(r, f_0)$,

$$\frac{f'(z)}{f(z)} = \frac{f_0'(z)}{f_0(z)} - \frac{d'(z)}{d(z)} = \frac{\nu(r)}{z}(1 + o(1)). \tag{3.6}$$

Set $F_1 = \{|z| : z \in E\}$. Since E is an ε-set, F_1 is of finite logarithmic measure. By (3.4) and (3.6), we obtain that for all z satisfying $|z| = r \notin [0,1] \cup F \cup F_1$ and $|f_0(z)| = M(r, f_0)$,

$$\frac{\nu(r)}{z}(1+o(1))\left[c_1c_2\frac{\nu(r)}{z}(1+o(1)) + (c_1+c_2)(1+o(1))\right] = \frac{R(z)d^2(z)}{[M(r,f_0)]^2}, \tag{3.7}$$

i.e.,

$$c_1c_2\frac{\nu(r)}{z}(1+o(1)) + (c_1+c_2)(1+o(1)) = \frac{R(z)d^2(z)z}{[M(r,f_0)]^2}\frac{1}{\nu(r)}(1+o(1)). \tag{3.8}$$

Since $\sigma(f_0) = \sigma < 1$ and $f_0(z)$ is transcendental, we see

$$\nu(r) \to \infty, \quad \frac{\nu(r)}{z} \to 0 \quad (z \to \infty) \tag{3.9}$$

and

$$\frac{R(z)d^2(z)z}{[M(r,f_0)]^2}\frac{1}{\nu(r)}(1+o(1)) \to 0 \quad (z \to \infty). \tag{3.10}$$

By (3.9), (3.10) and $c_1+c_2 \neq 0$, we see that (3.8) is a contradiction. Hence $g(z)$ is transcendental.

Proof of Theorem 1.2. First, we prove that $\lambda(g_2) = \sigma(g_2) = \sigma(f) = \sigma$.

By Lemma 3.2 and the fact that f is transcendental, we see that $g_2(z)$ is a transcendental entire function. Thus, $\sigma(g_2) \leqslant \sigma(f)$. If $\sigma(g_2) < \sigma(f)$, then there exist real numbers δ, α satisfying

$$\sigma(g_2) < \delta < \alpha < \sigma(f) = \sigma. \tag{3.11}$$

Using the same method as in the proof of Lemma 3.2, we can obtain

$$\frac{\nu(r)}{z}(1+o(1))\left[c_1c_2\frac{\nu(r)}{z}(1+o(1)) + (c_1+c_2)(1+o(1))\right] = \frac{g_2(z)}{[M(r,f)]^2}, \tag{3.12}$$

where $\nu(r)$ is the central index of $f(z)$, z satisfies $|z| = r \notin [0,1] \cup F \cup F_1$ and $|f(z)| = M(r,f)$, F, F_1 and E are defined as in the proof of Lemma 3.2. By (3.11), there exists a sequence $\{r_n\}$ $(r_n \to \infty)$ such that for any given ε $(0 < \varepsilon < 1-\sigma)$,

$$M(r_n, f) > e^{\alpha r_n}, \quad \nu(r_n) < r_n^{\sigma+\varepsilon}, \quad |g_2(z_n)| < e^{\delta r_n} \quad (|z_n| = r_n). \tag{3.13}$$

By (3.13) and $c_1 + c_2 \neq 0$, we see that (3.12) is a contradiction. Hence $\sigma(g_2) = \sigma(f)$, so, $\lambda(g_2) = \sigma(g_2) = \sigma(f)$.

Second, we prove that $\lambda(G_2) = \sigma(G_2) = \sigma(f) = \sigma$.

By $G_2(z) = g_2(z)/f^2(z)$ and the fact that f is an entire function, we see that if z_0 is a zero of $G_2(z)$, then z_0 is also a zero of $g_2(z)$. If z_0 is a zero of $g_2(z)$ and is not a zero of $G_2(z)$, then z_0 must be a zero of $f(z)$. Thus $f(z_0+c_1) = 0$ or $f(z_0+c_2) = 0$, by the condition of the theorem, $f(z)$ has only finitely many such zeros z_0, so that

$$n\left(r, \frac{1}{G_2}\right) = n\left(r, \frac{1}{g_2}\right) + O(1).$$

Thus Theorem 1.2 is proved.

4 Proof of Theorem 1.3

We need the following lemma. We may use the number c instead of 1 in Lemma 3.6 of [5].

Lemma 4.1[5]. *Let f be a function transcendental and meromorphic in the plane of the lower order $\mu(f) < \mu < 1$. Then, there exists arbitrarily large R with the following properties. First, $T(32R, f') < R^{\mu}$. Second, there exists a set $J_R \subseteq [R/2, R]$ of linear measure $(1 - o(1))R/2$ such that, for $r \in J_R$,*

$$f(z + c) - f(z) \sim cf'(z) \quad \text{on } |z| = r.$$

Proof of Theorem 1.3. Let f and c_1, c_2 be as in the hypotheses. By Lemma 2.2, we see $g(z)$ is transcendental. By Lemma 4.1, there exist arbitrarily large R and σ_1 ($\sigma < \sigma_1 < 1$) satisfying

$$T(32R, f') < R^{\sigma_1}, \tag{4.1}$$

and there exists a set $J_R \subset [R/2, R]$ of linear measure $(1 - o(1))R/2$ such that for $r \in J_R$,

$$f(z + c_1) + f(z + c_2) - 2f(z) = (c_1 + c_2)f'(z)(1 + o(1)) \quad \text{on } |z| = r. \tag{4.2}$$

Let ε-set E contain all zeros and poles of $g(z), f(z), f(z + c_1), f(z + c_2), f'(z)$, and the set for $R \in (1, \infty)$,

$$E_R = \{r : z \in E, |z| = r < R\}, \quad E_\infty = \{r : z \in E, |z| = r < \infty\}.$$

Then by the property of ε-set and $\sigma_1 < 1$, we see that E_∞ has finite linear measure and the linear measure of E_R satisfies $m(E_R) = o(1)R/2$ for sufficiently large R.

Let

$$F_R = \left\{r : r \in \left[\frac{R}{2}, R\right], n(r, f) = n(r - (|c_1| + |c_2|), f)\right\}. \tag{4.3}$$

We note that there are at most $o(R)$ points $q_k \in [R/3, R]$ at which $n(t, f)$ is discontinuous by (4.1), and if $r \in [R/2, R]$ is such that $n(t, f) > n(t-(|c_1|+|c_2|), f)$, then $r \in [q_k, q_k+|c_1|+|c_2|]$ for some k. So, F_R has linear measure

$$m(F_R) \geqslant (1 - o(1))\frac{R}{2}. \tag{4.4}$$

By (4.2)–(4.4), we see that there exists $r \in (F_R \cap J_R)\backslash E_R$ such that $g(z), f(z), f(z+c_1), f(z+c_2)$ and $f'(z)$ have no zero and pole on $|z| = r$.

Without loss of generality, we may assume that $b_j + k_1c_1 + k_2c_2$ ($k_d = 0, \pm 1, d = 1, 2$) are not poles for all poles b_j of $f(z)$.

Now by the hypotheses, there exists $r_0 > 0$, independent of R and r, such that if $f(z)$ has a pole of multiplicity m at z_0 and $r_0 \leqslant |z_0| \leqslant r - (|c_1| + |c_2|)$, then by the conditions of the theorem and the expressions of $g(z),\ g(z-c_1),\ g(z-c_2)$, we see that $g(z)$ has poles at $z_0,\ z_0-c_1$ and $z_0 - c_2$ of multiplicity m respectively. So,

$$n(r, g) \geqslant 3n(r - (|c_1| + |c_2|), f) = 3n(r, f) \quad \text{as } r \in (F_R \cap J_R)\backslash E_R. \tag{4.5}$$

By (4.2) and the fact that $g(z)$ and $f'(z)$ have no zero and pole on $|z| = r \in (F_R \cap J_R)\backslash E_R$, we have

$$|g(z) - (c_1 + c_2)f'(z)| = |o(1)(c_1 + c_2)f'(z)| < |g(z)| + |(c_1 + c_2)f'(z)|. \tag{4.6}$$

Thus, $g(z)$ and $-(c_1+c_2)f'(z)$ satisfy the conditions of Rouché's theorem. By (4.2), (4.3), (4.5) and Rouché's theorem, we deduce that

$$\begin{aligned} n\left(r,\frac{1}{g}\right) &= n\left(r,\frac{1}{f'}\right) - n(r,f') + n(r,g) \\ &\geqslant n\left(r,\frac{1}{f'}\right) - n(r,f') + 3\ n(r,f) + O(1) \\ &\geqslant n\left(r,\frac{1}{f'}\right) + n(r,f) + O(1) \end{aligned} \tag{4.7}$$

for $|z| = r \in (F_R \cap J_R)\backslash E_R$. If $\lambda(1/f) < \sigma(f)$, then $\lambda(1/f') < \sigma(f)$, so that $\lambda(f') = \sigma(f)$. Hence $\lambda(g) = \sigma(g) = \sigma(f)$. If $\lambda(f') < \sigma(f') = \sigma(f)$, then $\lambda(1/f') = \sigma(f') = \sigma(f)$, so that $\lambda(1/f) = \sigma(f)$. Hence $\lambda(g) = \sigma(g) = \sigma(f)$.

Finally, using the same method as in the proof of Theorem 1.2, we can prove that if $f(z)$ has at most finitely many zeros a_j satisfying $f(a_j + c_1) + f(a_j + c_2) = 0$, then $G(z)$ has infinitely many zeros and $\lambda(G) = \sigma(G) = \sigma(f)$. Thus Theorem 1.3 is proved.

5 Proof of Theorem 1.4

The proof of Theorem 1.4 is different from that of Theorem 1.3. To estimate the exponent of convergence of zeros of $g_2(z)$ more precisely, we need the following lemma.

Lemma 5.1. *Let f be a transcendental and meromorphic function of the order of growth $\sigma(f) = \sigma < 1$. Let $a, b \in \mathbb{C}\backslash\{0\}$. If $\overline{\lambda}(1/f) = \lambda(1/f)$, then $\max\{\lambda(f'),\ \lambda(af'+bf)\} = \sigma(f) = \sigma$.*

Proof. If we suppose that $\lambda(f') < \sigma$, then $\lambda(\frac{1}{f'}) = \sigma$. So, by the condition of the lemma, we have

$$\lambda\left(\frac{1}{f'}\right) = \lambda\left(\frac{1}{f}\right) = \overline{\lambda}\left(\frac{1}{f}\right) = \sigma(f) = \sigma. \tag{5.1}$$

Set

$$f(z) = \frac{p(z)}{q(z)}, \quad f'(z) = \frac{p_1(z)}{q_1(z)},$$

where $p(z)$ ($p_1(z)$) and $q(z)$ ($q_1(z)$) are canonical products (or polynomials) formed by the zeros and poles of $f(z)$ ($f'(z)$) respectively, such that $p(z)$ and $q(z)$ ($p_1(z)$ and $q_1(z)$) are irreducible. By $\lambda(f') < \sigma$ and (5.1), we have

$$\sigma(q) = \sigma(q_1) = \sigma(f), \quad \lambda(p_1) = \sigma(p_1) < \sigma(f).$$

Since a pole z_0 of $f(z)$ with multiplicity m must be the pole of $f'(z)$ with multiplicity $m+1$, we can assume $q_1(z) = q(z)d(z)$, where $d(z)$ is canonical product formed by different poles of $f(z)$. By (5.1), we see that

$$\sigma(d) = \lambda(d) = \overline{\lambda}\left(\frac{1}{f}\right) = \sigma(f) = \sigma. \tag{5.2}$$

Since

$$af'(z) + bf(z) = \frac{ap_1(z) + bp(z)d(z)}{q_1(z)},$$

we see that if z_0 is a pole of $f'(z)$ (i.e., a zero of $q_1(z)$), then $d(z_0)=0$, but $p_1(z_0)\neq 0$. So, z_0 is not a zero of $ap_1(z)+bp(z)d(z)$. Hence $ap_1(z)+bp(z)d(z)$ and $q_1(z)$ are irreducible. By (5.2), we get that

$$\lambda(af'+bf)=\lambda(ap_1+bpd)=\sigma(ap_1+bpd)\geqslant\sigma(d)=\sigma(f).$$

Thus, Lemma 5.1 is proved.

Proof of Theorem 1.4. Set

$$F(z)=f'(z)[c_1c_2f'(z)+(c_1+c_2)f(z)]. \tag{5.3}$$

By Lemma 3.2, we see that $g_2(z)$ is transcendental. Using the similar method to the proof of Theorem 1.3, we can obtain that as $|z|\to\infty$ and $|z|=r\in J_R$, $g_2(z)=F(z)(1+o(1))$ and for $|z|=r\in(F_R\cap J_R)\backslash E_R$,

$$n\left(r,\frac{1}{g_2}\right)=n\left(r,\frac{1}{F}\right)-n(r,F)+n(r,g_2), \tag{5.4}$$

where $F_R,\ J_R,\ E$ and E_R are defined as in the proof of Theorem 1.3, E contains all zeros and poles of $g_2,\ F,\ f,\ f(z+c_1),\ f(z+c_2)$ and f'.

By the hypotheses, there exists $r_0>0$, independent of R and r, such that if $f(z)$ has a pole of multiplicity m at z_0 and $r_0\leqslant|z_0|\leqslant r-(|c_1|+|c_2|)$, then by the conditions of the theorem and the expressions of $g_2(z),\ g_2(z-c_1),\ g_2(z-c_2)$, we see that $g_2(z)$ has poles at $z_0,\ z_0-c_1$ and z_0-c_2 of multiplicity $2m,\ m,\ m$ respectively. So,

$$n(r,g_2)\geqslant 4n(r-(|c_1|+|c_2|),f)+O(1)=4n(r,f)+O(1). \tag{5.5}$$

Since $F(z)$ has a pole at z_0 of multiplicity $2m+2$, we have

$$n(r,F)=2n(r,f)+2\overline{n}(r,f). \tag{5.6}$$

By (5.3), we have

$$n\left(r,\frac{1}{F}\right)=n\left(r,\frac{1}{f'}\right)+n\left(r,\frac{1}{c_1c_2f'+(c_1+c_2)f}\right). \tag{5.7}$$

By (5.4)–(5.7), we deduce that

$$n\left(r,\frac{1}{g_2}\right)\geqslant n\left(r,\frac{1}{f'}\right)+n\left(r,\frac{1}{c_1c_2f'+(c_1+c_2)f}\right)+2n(r,f)-2\overline{n}(r,f)+O(1). \tag{5.8}$$

If $\overline{\lambda}\big(\frac{1}{f}\big)<\lambda\big(\frac{1}{f}\big)$, then

$$\overline{n}(r,f)=o(n(r,f)). \tag{5.9}$$

And (5.8) and (5.9) give that

$$n\left(r,\frac{1}{g_2}\right)\geqslant n\left(r,\frac{1}{f'}\right)+n(r,f)+O(1). \tag{5.10}$$

Using the same method as in the proof of Theorem 1.3, by (5.10), we can get $\lambda(g_2)=\sigma(g_2)=\sigma(f)$.

If $\overline{\lambda}\left(\frac{1}{f}\right) = \lambda\left(\frac{1}{f}\right)$, then by (5.8), we get

$$n\left(r, \frac{1}{g_2}\right) \geqslant n\left(r, \frac{1}{f'}\right) + n\left(r, \frac{1}{c_1c_2f' + (c_1 + c_2)f}\right) + O(1). \tag{5.11}$$

By Lemma 5.1 and (5.11), we derive $\lambda(g_2) = \sigma(g_2) = \sigma(f)$.

Finally, using the same method as in the proof of Theorem 1.3, we can prove $\lambda(G_2) = \sigma(f)$.

Acknowledgements The authors cordially thank referees for their valuable comments which lead to the improvement of this paper.

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