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# Random duality

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**Abstract** The purpose of this paper is to provide a random duality theory for the further development of the theory of random conjugate spaces for random normed modules. First, the complicated stratification structure of a module over the algebra $L(\mu, K)$ frequently makes our investigations into random duality theory considerably different from the corresponding ones into classical duality theory, thus in this paper we have to first begin in overcoming several substantial obstacles to the study of stratification structure on random locally convex modules. Then, we give the representation theorem of weakly continuous canonical module homomorphisms, the theorem of existence of random Mackey structure, and the random bipolar theorem with respect to a regular random duality pair together with some important random compatible invariants.

**Keywords: random duality, weakly continuous canonical module homomorphism, random compatible structure, random bipolar theorem**

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## 1 Introduction

In functional analysis, classical duality theory makes a locally convex space and its conjugate (or dual) space enjoy a perfect symmetry (see [1, Chapter 13]). A random locally convex module (also termed as a random seminormed module in [2–4]), as a random generalization of a locally convex space, is playing the same role in random metric theory as a locally convex space has played in functional analysis (see [5]), in particular the theory of random conjugate spaces has played an important role in the development of random locally convex modules just as the theory of classical conjugate spaces has done in the development of locally convex spaces. Thus, only speaking logically, there should be a successful random duality theory to serve for a random locally convex module and its random conjugate space. In the sequel of this section, we will briefly elucidate why random duality theory can play a possibly important role in some interplaying disciplines of functional analysis and probability theory.

Since the book [6] by Schweizer and Sklar was published in 1983, the subject on probabilistic metric spaces has obtained great advances in all the directions (see [7]). Among the directions, random metric theory has grown to a rapidly developed and deep whole. The original notions of random metric spaces and random normed spaces were presented in order to provide a

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measure-theoretic approach to the development of probabilistic metric and normed spaces (see [6, Chapters 9 and 15]); about 1995, the development of random metric and normed spaces in the direction of functional analysis led Guo directly to the respective new versions of random metric and normed spaces (see [8] for details). These new versions have made us successfully introduce the crucial concepts of random normed modules and random inner product modules together with the definitive concept of a random conjugate space of a random normed space[8,9]. Since the theory of random conjugate spaces on random normed modules behaves extremely well as the theory of classical conjugate spaces on normed spaces, it has obtained a profound development in the past ten years[10–14] together with its applications to some topics in functional analysis (see [11–13, 15, 16]) and random analysis (see [11, 16]). In particular the random weak topology for a random normed module and the random weak star topology for a random conjugate space of a random normed space were deeply studied by Guo in [14]. There is no doubt that the further development of these topics strongly suggests that people should deeply study a random locally convex module and its random conjugate space.

Let $(\Omega, \mathcal{A}, \mu)$ be a $\sigma$-finite measure space, $K$ the scalar field of real numbers or complex numbers and $S$ a random locally convex module over $K$ with base $(\Omega, \mathscr{A}, \mu)$ such that $S$ has full support (see the second section of this paper). Then we showed in [17] that $S$ admits a nontrivial continuous linear functional iff there exists a $\mu$-atom in $\mathcal{A}$, and that $S$ admits enough nontrivial continuous linear functionals iff $\mathcal{A}$ is essentially generated by at most countably many disjoint $\mu$-atoms, that is, $\mathcal{A}$ is essentially purely $\mu$-atomic (see [14, 17] for the related terminologies). In fact, the method of proof in [17] also implied that $S$ is locally convex (or has a proper convex open set) iff $\mathcal{A}$ is essentially purely $\mu$-atomic (correspondingly, there exists a $\mu$-atom in $\mathcal{A}$). So the theory of traditional conjugate spaces can not universally apply to the development of random locally convex modules. Fortunately, the Hahn-Banach theorem of random linear functionals guarantees that there always exist enough continuous canonical module homomorphisms on a nontrivial random locally convex module (see [3], also the second section of this paper), which leads to the theory of a random conjugate space for a random locally convex module. The theory of a random conjugate space will, without doubt, occupy a crucial place in the further development of a random locally convex module, which in turn motivates us to develop a successful random duality theory.

The remainder of this paper is organized as follows: Section 2 will briefly list some known basic facts about random locally convex modules together with some new lemmas, which are all necessary preliminaries for the proofs of main results of this paper. Among them, important are Proposition 2.7, Corollary 2.1, Lemma 2.2 and Proposition 2.8; Section 3 will present and prove our main results, the most important of which are Theorem 3.1, Theorem 3.2 and Theorem 3.4.

## 2 Preliminaries

Throughout the sequel of this paper, $(\Omega, \mathscr{A}, \mu)$ always denotes a given $\sigma$-finite space and $K$ the scalar field $R$ of real numbers or $C$ of complex numbers unless stated otherwise.

Let us recall: if $(B, \|\cdot\|)$ is a normed space over $K$, then a mapping from $\Omega$ to $B$ is called a $B$-valued $\mu$-measurable function on $(\Omega, \mathscr{A}, \mu)$ if it is the $\mu$-almost everywhere limit of a sequence of $B$-valued simple $\mathscr{A}$-measurable functions on $(\Omega, \mathscr{A}, \mu)$. The definition of a $\mu$-measurable function coincides with that given in [18] since $\mu$ is $\sigma$-finite. Denote by $L(\mu, B)$ the linear space of $\mu$-equivalence classes of $B$-valued $\mu$-measurable functions on $(\Omega, \mathscr{A}, \mu)$. Clearly when $B = K, L(\mu, K)$ is an algebra over $K$ under the ordinary addition, multiplication and scalar multiplication operations on equivalence classes, the null and unit elements are still denoted by 0 and 1, respectively.

Denote by $\tilde{L}(\mu, R)$ the set of all $\mu$-equivalence classes of extended real-valued $\mu$-measurable functions on $(\Omega, \mathscr{A}, \mu)$. Then $\tilde{L}(\mu, R)$ is a lattice by the ordering $\leqslant$: $\xi \leqslant \eta$ if $\xi^0(\omega) \leqslant \eta^0(\omega)$, $\mu$-a.e. (namely for $\mu$-almost all $\omega$ in $\Omega$), where $\xi^0$ and $\eta^0$ are arbitrarily chosen representatives of $\xi$ and $\eta$, respectively. In particular every set $A$ in $\tilde{L}(\mu, R)$ has a supremum $\bigvee A$ and an infimum $\bigwedge A$. Further, as a sublattice of $\tilde{L}(\mu, R)$, $L(\mu, R)$ is also a complete lattice, namely every set having an upper bound (a lower bound) in it possesses a supremum (accordingly, an infimum). The pleasant properties of $\tilde{L}(\mu, R)$ are summarized as follows:

**Proposition 2.1**[18]**.** *For every subset $A$ of $\tilde{L}(\mu, R)$ there exist countable subsets $\{a_n : n \in \mathbb{N}\}$ and $\{b_n : n \in \mathbb{N}\}$ of $A$ such that $\bigvee A = \bigvee_{n \geqslant 1} a_n$ and $\bigwedge A = \bigwedge_{n \geqslant 1} b_n$, where $\mathbb{N}$ stands for the set of positive integers. Further, if $A$ is directed (dually directed) with respect to $\leqslant$, then the above $\{a_n : n \in \mathbb{N}\}$ (accordingly, $\{b_n : n \in \mathbb{N}\}$) can be chosen as nondecreasing (correspondingly, nonincreasing) with respect to $\leqslant$.*

The lattice $L(\mu, R)$ has the similar properties as above (see [13, Proposition 2.1]). For the sake of convenience and brevity, we introduce the following:

**Definition 2.1.** (1) *Let $\xi$ be an element of $L(\mu, K)$. For an arbitrarily chosen representative $\xi_0$ of $\xi$, define the two $\mu$-measurable functions $\xi_0^{-1}$ and $|\xi_0|$ by $\xi_0^{-1}(\omega) = 1/\xi_0(\omega)$ if $\xi_0(\omega) \neq 0$, and $\xi_0^{-1}(\omega) = 0$ otherwise, and by $|\xi_0|(\omega) = |\xi_0(\omega)|, \forall\ \omega \in \Omega$. Then the $\mu$-equivalence class $Q(\xi)$ of $\xi_0^{-1}$ is called the generalized inverse of $\xi$, clearly $Q(\xi) \cdot \xi = \xi \cdot Q(\xi) = \tilde{I}_A$, where $A = \{\omega \in \Omega : \xi_0(\omega) \neq 0\}$ and $\tilde{I}_A$ is the $\mu$-equivalence class of the indicator function $I_A$ of $A$ (namely, $I_A(\omega) = 1$ if $\omega \in A$, and 0 otherwise); the $\mu$-equivalence class $|\xi|$ of $|\xi_0|$ is called the absolute value of $\xi$.* (2) *If $\xi$ is an element of $L(\mu, C)$, set $\xi = u + iv$, where $u, v \in L(\mu, R)$, then $\bar{\xi} = u - iv$ is called the complex conjugate of $\xi$, and $\operatorname{sgn}(\xi) = (Q(|\xi|)) \cdot \xi$ is called the sign of $\xi$, clearly $\xi \cdot \operatorname{sgn}(\bar{\xi}) = |\xi|$.* (3) *If $\xi$ and $\eta \in L(\mu, R)$, let $\xi_0$ and $\eta_0$ be an arbitrarily chosen representatives of $\xi$ and $\eta$, respectively, and let $A = \{\omega \in \Omega : \xi_0(\omega) > \eta_0(\omega)\}$, then we always use $[\xi > \eta]$ for the $\mu$-equivalence class of $A$, namely $[\xi > \eta] = \{B \in \hat{\mathcal{A}}^\mu : \mu(B \bigtriangleup A) = 0\}$, where $\bigtriangleup$ denotes the symmetric difference operation and $\hat{\mathcal{A}}^\mu$ the Lebesgue completion of $\mathcal{A}$ with respect to $\mu$, that is, the $\sigma$-algebra of $\mu$-measurable sets of $\Omega$. In this paper, we also often write $I_{[\xi > \eta]}$ for $\tilde{I}_A$, one can also understand the implication of such notations as $I_{[\xi \leqslant \eta]}$, $I_{[\xi \neq \eta]}$ and $I_{[\xi = \eta]}$.*

Finally, we specially denote by $\tilde{L}^+(\mu)$ the set $\{\xi \in \tilde{L}(\mu, R) : \xi \geqslant 0\}$ and by $L^+(\mu)$ the set $\{\xi \in L(\mu, R) : \xi \geqslant 0\}$.

**Definition 2.2**[5]**.** (1) *Let $S$ be a linear space over $K$, then a mapping $f : S \to L(\mu, K)$ is called a random linear functional on $S$ if $f$ is linear; if $S$ is a linear space over $R$, then a mapping $f : S \to L(\mu, R)$ is called a random sublinear functional on $S$ if $f(\alpha p) = \alpha \cdot f(p), \forall\ \alpha > 0$ and $p \in S$, and $f(p + q) \leqslant f(p) + f(q), \forall\ p, q \in S$.* (2) *Let $S$ be a linear space over $K$, then a mapping $f : S \to L^+(\mu)$ is called a random seminorm on $S$ if $f(\alpha p) = |\alpha| \cdot f(p), \forall\ \alpha \in K$ and $p \in S$, and $f(p + q) \leqslant f(p) + f(q), \forall\ p, q \in S$.* (3) *Let $S$ be a left module over the* *algebra $L(\mu, K)$, then a mapping $f : S \to L^+(\mu)$ is called a module-absolutely homogeneous random seminorm (briefly, an $M$-random seminorm) on $S$ if $f$ is a random seminorm on $S$ and $f(\xi \cdot p) = |\xi| \cdot f(p), \forall\, \xi \in L(\mu, K)$ and $p \in S$.*

**Remark 2.1.** One can easily see that a random sublinear functional $f$ must satisfy $f(\theta) = 0$ ($\theta$ is the null element in the domain of $f$). Let $S$ be a left module over the algebra $L(\mu, K)$. For each $\alpha \in K$, let $\hat{\alpha}$ be the $\mu$-equivalence class of the $\mu$-measurable function with the constant value $\alpha$, that is, $\hat{\alpha} = \alpha \cdot 1$ (note 1 is the unit element in $L(\mu, K)$), then $\hat{\alpha} \cdot p = (\alpha \cdot 1) \cdot p = \alpha \cdot (1 \cdot p) = \alpha \cdot p, \forall p \in S$, since $S$ is a left module over the algebra $L(\mu, K)$. Thus the module multiplication operation $\cdot : L(\mu, K) \times S \to S$ can be naturally regarded as an extension of the scalar multiplication operation: $K \times S \to S$, and hence it would not produce any confusion if we use the same notation $\cdot$ for both the module multiplication and the scalar multiplication.

The following analytic forms of the Hahn-Banach theorems for random linear functionals are due to Guo[8], who also gave the first rigorous proofs of them in [5].

**Proposition 2.2[5,8].** *Let $S$ be a real linear space, $M$ a linear subspace of $S$, $f : M \to L(\mu, R)$ a random linear functional and $\mathcal{X} : S \to L(\mu, R)$ a random sublinear functional such that $f(p) \leqslant X_p, \forall p \in M$, where $X_p$ denotes $\mathcal{X}(p)$. Then there exists a random linear functional $F : S \to L(\mu, R)$ such that $F$ extends $f$ and $F(p) \leqslant X_p, \forall p \in S$.*

**Proposition 2.3[5,8].** *Let $S$ be a linear space over $K$, $M$ a linear subspace of $S$, $f : M \to L(\mu, K)$ a random linear functional and $\mathcal{X} : S \to L^+(\mu)$ a random seminorm such that $|f(p)| \leqslant X_p, \forall\, p \in M$. Then there exists a random linear functional $F : S \to L(\mu, K)$ such that $F$ extends $f$ and $|F(p)| \leqslant X_p, \forall p \in S$.*

**Definition 2.3.** *An ordered pair $(S, \{\mathcal{X}^d\}_{d \in D})$ is called a random locally convex space (also termed a random seminormed space in [2–5]) over $K$ with base $(\Omega, \mathcal{A}, \mu)$ if the following three conditions are satisfied:*

*(1) $S$ is a linear space over $K$;*

*(2) $D$ is an indexing set and for each $d \in D, \mathcal{X}^d : S \to L^+(\mu)$ is a random seminorm on $S$;*

*(3) $\bigvee\{X_p^d : d \in D\} = 0$ if $p = \theta$ (the null in $S$).*

*Further more, if there exists another mapping $* : L(\mu, K) \times S \to S$ such that the following two conditions are also satisfied:*

*(4) $(S, *)$ is a left module over the algebra $L(\mu, K)$;*

*(5) For each $d \in D, \mathcal{X}^d$ is an $M$-random seminorm on $(S, *)$ (see Definition 2.2 for an $M$-random seminorm). Then the ordered triple $(S, \{\mathcal{X}^d\}_{d \in D}, *)$ is a random locally convex module over $K$ with base $(\Omega, \mathcal{A}, \mu)$.*

**Remark 2.2.** In Definition 2.3, since $*$ is a natural extension of the scalar multiplication on $S$ (see Remark 2.1), from now on we will always briefly write $(S, \{\mathcal{X}^d\}_{d \in D})$ for $(S, \{\mathcal{X}^d\}_{d \in D}, *)$, and $\xi \cdot p$ for $\xi * p$ for any $\xi \in L(\mu, K)$ and $p \in S$, once $*$ is understood from the context. Finally, let $(S, \{\mathcal{X}^d\}_{d \in D})$ be a random locally convex space over $K$ with base $(\Omega, \mathcal{A}, \mu)$, set $\xi = \bigvee\{X_p^d : p \in S, \text{and } d \in D\}$. Since $S$ is a linear space, $\mu\{\omega \in \Omega : 0 < \xi(\omega) < +\infty\} = 0$, and hence $[\xi > 0] = [\xi = +\infty]$. An arbitrarily chosen representative of $[\xi > 0]$ is called a support of $S$. Further, if $\Omega$ is a support of $S$, then $S$ is called having full support.

**Example 2.1.** Let $(\Omega, \mathcal{A}, \mu)$ be a trivial probability space, that is, $\mathcal{A} = \{\Omega, \emptyset\}$ and $\mu(\Omega) = 1$, then a random locally convex module over $K$ with base $(\Omega,\mathcal{A},\mu)$ is exactly an ordinary locally convex space (see [19] for details).

**Example 2.2.** Let $(S,\{\mathcal{X}^d\}_{d\in D})$ be a random locally convex module over $K$ with base $(\Omega,\mathcal{A},\mu)$, when $D$ is a singleton, $\{\mathcal{X}^d\}_{d\in D}$ exactly degenerates to a single $M$-random norm $\mathcal{X}$ on $S$, namely $(S,\mathcal{X})$ becomes a random normed module (see [13, Definition 2.1]). Both $L(\mu,B)$ and $L(\mu,K)$ are typical random normed modules over $K$ with base $(\Omega,\mathcal{A},\mu)$, and hence also random locally convex modules (see [13, Example 2.1] for details).

The following lemma is obvious.

**Lemma 2.1.** *Let $(\Omega,\mathcal{A},\mu)$ be a $\sigma$-finite measure space, and $\mu$ not finite. There must be a countable $\mathcal{A}$-measurable partition $\{A_n : n\in\mathbb{N}\}$ of $\Omega$ such that $0<\mu(A_n)<+\infty$, for each $n\in\mathbb{N}$. Define $\tilde{\mu}:\mathcal{A}\to[0,1]$ by: $\tilde{\mu}(A)=\sum_{n\geqslant 1}\frac{1}{2^n}\cdot\frac{\mu(A\bigcap A_n)}{\mu(A_n)},\forall A\in\mathcal{A}$. Then $\tilde{\mu}$ is a probability measure and equivalent with $\mu$. Further, one can easily see that a net $\{p_\alpha:\alpha\in\Gamma\}$ in $L(\mu,K)$ converges locally in measure $\mu$ to $p$ in $L(\mu,K)$, namely $\{p_\alpha:\alpha\in\Gamma\}$ converges in measure $\mu$ to $p$ on each $\mathcal{A}$-measurable set with finite and positive measure, iff $\{p_\alpha:\alpha\in\Gamma\}$ converges in probability $\tilde{\mu}$ to $p$.*

**Proposition 2.4**[5]**.** *Let $(S,\{\mathcal{X}^d\}_{d\in D})$ be a random locally convex space over $K$ with base $(\Omega,\mathcal{A},\mu)$. Take $\tilde{\mu}$ to be the same as one in Lemma 2.1 if $\mu$ is not finite, and $\tilde{\mu}(A)=\mu(A)/\mu(\Omega)$ if $\mu(\Omega)<+\infty,\forall A\in\mathcal{A}$. For the null element $\theta$ in $S$ and for any positive $\varepsilon$ and $\lambda$ with $0<\lambda<1$, set $\mathcal{N}_\theta(d,\varepsilon,\lambda)=\{p\in S:\tilde{\mu}\{\omega\in\Omega:X_p^d(\omega)<\varepsilon\}>1-\lambda\}$. Then (1) $\mathcal{N}_\theta=\{\bigcap_{i=1}^n\mathcal{N}_\theta(d_i,\varepsilon_i,\lambda_i):n\in N,d_i\in D,\varepsilon_i>0,0<\lambda_i<1$ for any $i$ such that $1\leqslant i\leqslant n\}$ forms a local base at $\theta$ of some Hausdorff linear topology, called the $(\varepsilon,\lambda)$-linear topology generated by $\{\mathcal{X}^d\}_{d\in D}$, denoted by the $\{\mathcal{X}^d\}_{d\in D}$-topology. (2) A net $\{p_\alpha:\alpha\in\Gamma\}$ in $S$ converges in the $\{\mathcal{X}^d\}_{d\in D}$-topology to $p$ in $S$ iff for each $d\in D,\{X^d_{p_\alpha-p}:\alpha\in\Gamma\}$ converges locally in measure $\mu$ to 0. (3) As special random locally convex spaces, the $(\varepsilon,\lambda)$-linear topologies for $L(\mu,B)$ and $L(\mu,K)$ are both the ordinary topologies of convergence locally in measure $\mu$, namely the ones of convergence in probability $\tilde{\mu}$. (4) Furthermore, if $(S,\{\mathcal{X}^d\}_{d\in D})$ is a random locally convex module, then it is also a Hausdorff topological module over the topological algebra $L(\mu,K)$ when $S$ and $L(\mu,K)$ are both endowed with their $(\varepsilon,\lambda)$-linear topologies, namely the multiplication $\cdot: L(\mu,K)\times S\to S$ are jointly continuous. Clearly $L(\mu,K)$ is a topological algebra endowed with its $(\varepsilon,\lambda)$-topology, namely the algebraic multiplication $\cdot: L(\mu,K)\times L(\mu,K)\to L(\mu,K)$ is jointly continuous.*

**Remark 2.3.** From now on, we always denote by $\mathcal{F}(H)$ the family of all nonempty finite subsets of a given set $H$. Let $(S,\{\mathcal{X}^d\}_{d\in D})$ be the same as in Proposition 2.4, for each $F\in\mathcal{F}(D)$, define $\mathcal{X}^F:S\to L^+(\mu)$ by $X_p^F=\sum_{d\in F}X_p^d,\forall p\in S$, then $\{\mathcal{X}^F\}_{F\in\mathcal{F}(D)}$ is a family of random seminorms, and each $\mathcal{X}^F$ is also an $M$-random seminorm if each $\mathcal{X}^d$ is an $M$-random seminorm, and it is easy to see that $\{\mathcal{X}^F\}_{F\in\mathcal{F}(D)}$-topology coincides with $\{\mathcal{X}^d\}_{d\in D}$-topology. An advantage of $\{\mathcal{X}^F\}_{F\in\mathcal{F}(D)}$ over $\{\mathcal{X}^d\}_{d\in D}$ is that $\{\mathcal{X}^F\}_{F\in\mathcal{F}(D)}$ is directed: for any $F_1,F_2\in\mathcal{F}(D)$, set $F_3=F_1\cup F_2$, then $X_p^{F_3}\geqslant X_p^{F_1}\vee X_p^{F_2},\forall p\in S$. So we often suppose $D$ is a directed set and $X_p^{d_1}\geqslant X_p^{d_2},\forall p\in S$, when $d_1\geqslant d_2$, for example, in the papers [2, 3] we just adopted such a way of definition. From now on, when we speak of a topology for a random locally convex space, this topology always means its $(\varepsilon,\lambda)$-topology.

**Definition 2.4**[5]**.** *Let $(S,\{\mathcal{X}^d\}_{d\in D})$ be a random locally convex space over $K$ with base*

*$(\Omega, \mathcal{A}, \mu)$. A random linear functional $f : S \to L(\mu, K)$ is called $\mu$-a.e. bounded if there exist a countable $\mathcal{A}$-measurable partition $\{A_n : n \in \mathbb{N}\}$ of $\Omega$ (namely $A_n \in \mathcal{A}, A_n \cap A_m = \emptyset$ $(n \neq m)$ and $\Omega = \sum_{n\geqslant 1} A_n$), a sequence $\{\xi_n : n \in \mathbb{N}\}$ in $L^+(\mu)$ and a countable subfamily $\{F_n : n \in \mathbb{N}\}$ of $\mathcal{F}(D)$ such that $|f(p)| \leqslant \sum_{n\geqslant 1} \tilde{I}_{A_n} \cdot \xi_n \cdot X_p^{F_n}, \forall p \in S$.*

**Remark 2.4.** In [2], a random linear functional $f : S \to L(\mu, K)$ was called $\mu$-a.e. bounded if there exist $\xi \in L^+(\mu)$ and $F \in \mathcal{F}(D)$ such that $|f(p)| \leqslant \xi \cdot X_p^F, \forall p \in S$. Thus [2] only considered a special case, which in turn caused that the theory of random duality given in [20] was seriously limited. The following Proposition 2.5 shows that Definition 2.4 is proper. Usually, we called the linear space of $\mu$-a.e. bounded random linear functionals on $(S, \{\mathcal{X}^d\}_{d\in D})$, denoted by $S^*$, the random conjugate space of $S$. Clearly $S^*$ is also left module over the algebra $L(\mu, K)$ endowed with the module multiplication $\cdot : L(\mu, K) \times S^* \to S^*$ by $(\xi \cdot f)(p) = \xi \cdot (f(p)), \forall (\xi, f) \in L(\mu, K) \times S^*$ and $p \in S$. When $(S, \{\mathcal{X}^d\}_{d\in D})$ is an RN-module $(S, \mathcal{X})$, $S^*$ can also be endowed with a random norm $\mathcal{X}^*$ so that $(S^*, \mathcal{X}^*)$ becomes an RN-modules (see [13, Definition 2.2] for details).

**Proposition 2.5**[4,5]**.** *Let $(S, \{\mathcal{X}^d\}_{d\in D})$ be a random locally convex module over $K$ with base $(\Omega, \mathcal{A}, \mu)$. Then a random linear functional $f : S \to L(\mu, K)$ is $\mu$-a.e. bounded if $f$ is a continuous module homomorphism. When $f : S \to L^+(\mu)$ is an $M$-random seminorm on $S$, then the process of proof in* [4] *also implies that $f$ is $\mu$-a.e. bounded (the definition of $f$ being $\mu$-a.e. bounded is completely similar to Definition* 2.4) *iff $f$ is continuous.*

The second part of Proposition 2.5 motivates the following Definition 2.5 and Proposition 2.6, which deepen the understanding of the $(\varepsilon, \lambda)$-topologies.

**Definition 2.5.** *Let $S$ be a left module over the algebra $L(\mu, K)$. A family $\{\mathcal{X}^d\}_{d\in D}$ of $M$-random seminorms on $S$ is called saturated if the family also includes any $M$-random seminorm $\mathcal{X}$ on $S$ with the following property $(B)$: there exist a countable $\mathcal{A}$-measurable partition $\{A_n : n \in \mathbb{N}\}$ of $\Omega$, a sequence $\{\xi_n : n \in \mathbb{N}\}$ in $L^+(\mu)$ and a countable subfamily $\{F_n : n \in \mathbb{N}\}$ of $\mathcal{F}(D)$ such that $X_p \leqslant \sum_{n\geqslant 1} \tilde{I}_{A_n} \cdot \xi_n \cdot X_p^{F_n}, \forall p \in S$. For any family $\{\mathcal{X}^d\}_{d\in D}$ of $M$-random seminorms on $S$, denote by $s(\{\mathcal{X}^d\}_{d\in D})$ the family of all the $M$-random seminorms having the above property $(B)$. Then $s(\{\mathcal{X}^d\}_{d\in D})$ is called the saturation of $\{\mathcal{X}^d\}_{d\in D}$, which is the smallest of all the saturated families containing $\{\mathcal{X}^d\}_{d\in D}$.*

**Proposition 2.6.** *Let $S$ be a left module over the algebra $L(\mu, K)$, and $\{\mathcal{X}^d\}_{d\in D}$ and $\{\mathcal{X}^e\}_{e\in\Gamma}$ the two families of $M$-random seminorms on $S$. Then $\{\mathcal{X}^d\}_{d\in D}$ and $\{\mathcal{X}^e\}_{e\in\Gamma}$ generate the same $(\varepsilon, \lambda)$-topology iff $s(\{\mathcal{X}^d\}_{d\in D}) = s(\{\mathcal{X}^e\}_{e\in\Gamma})$.*

*Proof.* It immediately follows from the second part of Proposition 2.5.

It is well known from Proposition 2.3 that for a nontrivial random locally convex space $(S, \{\mathcal{X}^d\}_{d\in D})$ over $K$ with base $(\Omega, \mathcal{A}, \mu)$ there always exist enough $\mu$-a.e. bounded random linear functionals on $S$, further, if $(S, \{\mathcal{X}^d\}_{d\in D})$ is a random locally convex module, then $S$ admits enough continuous module homomorphisms from $S$ to $L(\mu, K)$. As another interesting application of Proposition 2.3, we obtain the following Proposition 2.7 and Corollary 2.1, which are crucial in this paper.

**Proposition 2.7.** *Let $S$ be a linear space over $K$, $\{\mathcal{X}^{(n)}\}_{n\in\mathbb{N}}$ a sequence of random seminorms on $S$ such that the series $\sum_{n\geqslant 1} X_p^{(n)}$ converges locally in measure $\mu$ for each $p$ in $S$, and*

*$f : S \to L(\mu, K)$ a random linear functional such that $|f(p)| \leqslant \sum_{n\geqslant 1} X_p^{(n)}, \forall p \in S$. Then there exists a sequence $\{f_n : n \in \mathbb{N}\}$ of random linear functionals on $S$ such that the following two conditions are satisfied:*

(1) $|f_n(p)| \leqslant X_p^{(n)}, \forall\ p \in S$;

(2) $f(p) = \sum_{n\geqslant 1} f_n(p), \forall\ p \in S$.

*Proof.* Denote by $S^\infty$ the self-product space of countably many copies of $S$, by $M$ the linear space of such elements $q$ in $S^\infty$ that $\sum_{n\geqslant 1} X_{q_n}^{(n)}$ converges locally in measure $\mu$, where $q_n$ is the $n$-th coordinate of $q$ for each $n \geqslant 1$, by $\triangle$ the linear space of diagonal elements in $S^\infty$ and by $\bigoplus_{n\geqslant 1} S$ the linear space of such elements $q$ in $S^\infty$ that all but finitely many coordinates of $q$ are the null $\theta$ in $S$. It is clear that $M \supset \triangle \cup (\bigoplus_{n\geqslant 1} S)$.

Define $g : \triangle \to L(\mu, K)$ by $g(q(p)) = f(p), \forall p \in S$, where $q(p)$ is the element in $\triangle$ such that each coordinate of $q(p)$ is $p$. Then $g$ is a random linear functional on $\triangle$. Again define $\mathcal{X} : M \to L^+(\mu)$ by $X_q = \sum_{n\geqslant 1} X_{q_n}^{(n)}, \forall q = (q_n, n \geqslant 1) \in M$, then one can easily see that $|g(q(p))| = |f(p)| \leqslant \sum_{n\geqslant 1} X_p^{(n)} = X_{q(p)}, \forall p \in S$. Thus by Proposition 2.3 there exists a random linear functional $G : M \to L(\mu, K)$ such that $G$ extends $g$ and $|G(q)| \leqslant X_q, \forall q \in M$.

Since $M \supset \bigoplus_{n\geqslant 1} S$, and for each $p \in S$ and each $n \geqslant 1$ it is also clear that $p^{(n)}$ belongs to $\bigoplus_{n\geqslant 1} S$, where $p^{(n)}$ is the element whose all but the $n$-th coordinate are the null in $S$ and whose $n$-th coordinate is $p$. Define a sequence $\{f_n : n \geqslant 1\}$ of random linear functionals on $S$ as follows: $f_n(p) = G(p^{(n)}), \forall n \geqslant 1$ and $p \in S$. Then it is very easy to verify that $\{f_n : n \geqslant 1\}$ are just desired: in fact, first, it immediately follows from $\sum_{n\geqslant 1} |G(p^n)| \leqslant \sum_{n\geqslant 1} X_p^{(n)}, \forall p \in S$ that the series $\sum_{n\geqslant 1} G(p^n)$ is absolutely convergent locally in measure $\mu$; second, $|f(p) - \sum_{k=1}^n f_k(p)| = |G(q(p)) - \sum_{k=1}^n G(p^{(k)})| = |G(q(p) - \sum_{k=1}^n p^{(k)})| = |G(R_n(p))| \leqslant \sum_{k\geqslant n+1} X_p^{(k)}$ implies that $\{f(p) - \sum_{k=1}^n f_k(p) : n \in \mathbb{N}\}$ converges to 0 locally in measure $\mu$,where the $i$-th coordinate of $R_n(p)$ is $\theta$ when $1 \leqslant i \leqslant n$, and $p$ when $i \geqslant n + 1$, so $f(p) = \sum_{n\geqslant 1} f_n(p), \forall p \in S$.

**Corollary 2.1.** *Let $S$ be a linear space over $K$, $f : S \to L(\mu, K)$ a random linear functional, $\{A_n : n \in \mathbb{N}\}$ a countable $\mathcal{A}$-measurable partition of $\Omega$, and each $\mathcal{X}^{(n)} : S \to L^+(\mu)$ a random seminorm on $S$ for each $n \geqslant 1$ such that $|f(p)| \leqslant \sum_{n\geqslant 1} \tilde{I}_{A_n} \cdot X_p^{(n)}, \forall p \in S$. Then there exists a sequence $\{f_n : n \geqslant 1\}$ of random linear functionals on $S$ such that the following two conditions are satisfied:*

(1) $|f_n(p)| \leqslant X_p^{(n)}, \forall\ p \in S$;

(2) $f(p) = \sum_{n\geqslant 1} \tilde{I}_{A_n} \cdot f_n(p), \forall\ p \in S$.

*Proof.* For each $n \geqslant 1$, define the random seminorm $\tilde{\mathcal{X}}^{(n)} : S \to L^+(\mu)$ by $\tilde{X}_p^{(n)} = \tilde{I}_{A_n} \cdot X_p^{(n)}, \forall p \in S$. Notice such series as $\sum_{n\geqslant 1} \tilde{X}_p^{(n)}$ always converge locally in measure $\mu$ for each $p$ in $S$, one can obtain by Proposition 2.7 a sequence $\{f_n : n \geqslant 1\}$ of random linear functionals on $S$ such that the following are satisfied:

(1)′ $|f_n(p)| \leqslant \tilde{X}_p^{(n)}, \forall\ p \in S$ and $n \geqslant 1$;

(2)′ $f(p) = \sum_{n\geqslant 1} f_n(p), \forall\ p \in S$.

(1)′ also implies that $|f_n(p)| \leqslant \tilde{I}_{A_n} \cdot X_p^{(n)} \leqslant X_p^{(n)}, \forall p \in S$, and that $\tilde{I}_{A_n^c} \cdot f_n(p) = 0$, where $A_n^c = \Omega \setminus A_n$. So $f_n(p) = \tilde{I}_{A_n} \cdot f_n(p) + \tilde{I}_{A_n^c} \cdot f_n(p) = \tilde{I}_{A_n} \cdot f_n(p), \forall p \in S$ and $n \geqslant 1$. To sum up, $\{f_n : n \geqslant 1\}$ are just desired.

The following Proposition 2.8 plays a crucial role in the proof of main results in this paper, whereas the following Lemma 2.2 provides a key step in the proof of Proposition 2.8.

**Lemma 2.2.** *Let $S$ be a left module over the algebra $L(\mu, K)$, $f$ and $g : S \to L(\mu, K)$ two module homomorphisms such that $N(f) \subset N(g)$, where $N(f)$ and $N(g)$ are the null spaces of $f$ and $g$, respectively, namely $N(f) = \{p \in S : f(p) = 0\}$ and $N(g) = \{p \in S : g(p) = 0\}$. Then there exists $\xi \in L(\mu, K)$ such that $g = \xi \cdot f$. That is, $g(p) = \xi \cdot (f(p)), \forall\, p \in S$.*

*Proof.* We can, without loss of generality, suppose $\mu(\Omega) = 1$ (or we employ $\tilde{\mu}$ in place of $\mu$). Since $S$ is an $L(\mu, K)$-module, then $\{|f(p)| : p \in S\}$ is directed with respect to $\leqslant$: in fact, $\forall p_1, p_2 \in S$, let $A = [|f(p_1)| \geqslant |f(p_2)|]$ and $p_3 = I_A \cdot p_1 + (1 - I_A) \cdot p_2$, then it is easy to see that $|f(p_3)| = I_A \cdot |f(p_1)| + (1 - I_A) \cdot |f(p_2)| = |f(p_1)| \vee |f(p_2)|$, which, of course, implies $\{|f(p)| : p \in S\}$ is directed. So, by Proposition 2.1 there exists a sequence $\{p_n\}$ in $S$ such that $\{|f(p_n)| : n \in \mathbb{N}\} \nearrow \xi := \vee\{|f(p)| : p \in S\}$.

Denote $[\xi > 0]$ by $B$, and $[|f(p_n)| > 0]$ by $B_n$ for each $n \geqslant 1$. We can, without loss of generality, choose a representative $C$ of $B$, and a representative $C_n$ of $B_n$ for each $n \geqslant 1$ such that $C = \bigcup_{n=1}^{\infty} C_n$ and $C_n \subset C_{n+1}$ for each $n \geqslant 1$. Putting $C_0 = \emptyset, A_n = C_n \backslash C_{n-1}, n \geqslant 1$, then one can have $C = \sum_{n \geqslant 1} A_n, A_n \cap A_m = \emptyset$ $(n \neq m)$. Again letting $q_i = \tilde{I}_{A_i} \cdot Q(f(p_i)) \cdot p_i, \forall i \geqslant 1$, yields $f(q_i) = \tilde{I}_{A_i} \cdot Q(f(p_i)) \cdot f(p_i) = \tilde{I}_{A_i} \cdot I_{B_i} = \tilde{I}_{A_i} \cdot \tilde{I}_{C_i} = \tilde{I}_{A_i}$ (note $A_i \subset C_i$), $\forall\, i \geqslant 1$.

Clearly, $f(\tilde{I}_{A_i} \cdot p - f(p) \cdot q_i) = \tilde{I}_{A_i} \cdot f(p) - \tilde{I}_{A_i} \cdot f(p) = 0, \forall p \in S$, and hence also $g(\tilde{I}_{A_i} \cdot p - f(p) \cdot q_i) = 0, \forall\, p \in S$ by $N(f) \subset N(g)$, namely,

$$\tilde{I}_{A_i} \cdot g(p) = g(q_i) \cdot f(p), \quad \forall\, i \geqslant 1 \text{ and } p \in S. \tag{1}$$

By the definition of $B$, one can easily observe that $f((1 - I_B) \cdot p) = (1 - I_B) f(p) = 0, \forall p \in S$, again by $N(f) \subset N(g)$ one can have $g((1 - I_B) \cdot p) = 0, \forall p \in S$, namely,

$$g(p) = I_B \cdot g(p), \quad \forall\, p \in S. \tag{2}$$

By putting $\xi_n = \sum_{i=1}^{n} g(q_i), \forall n \geqslant 1$, and observing $g(q_i) = \tilde{I}_{A_i} \cdot Q(f(p_i)) \cdot g(p_i), \forall i \geqslant 1$, one can have that $\{\xi_n : n \in \mathbb{N}\}$ is a Cauchy sequence in $L(\mu, K)$ in convergence in measure $\mu$, since $\sum_{i=1}^{\infty} \mu(A_i) = \mu(C) < +\infty$. Completeness of $L(\mu, K)$ produces a $\xi$ in $L(\mu, K)$ such that $\{\xi_n : n \in \mathbb{N}\}$ converges in measure $\mu$ to $\xi$.

Furthermore, Combining (1) and (2) above yields

$$\begin{aligned} g(p) = I_B \cdot g(p) = \tilde{I}_C \cdot g(p) &= \left( \sum_{i \geqslant 1} \tilde{I}_{A_i} \right) \cdot g(p) \\ &= \left( \mu - \lim_{n \to \infty} \left( \sum_{i=1}^{n} \tilde{I}_{A_i} \right) \right) \cdot g(p) = \mu - \lim_{n \to \infty} \left( \sum_{i=1}^{n} \tilde{I}_{A_i} \cdot g(p) \right) \\ &= \mu - \lim_{n \to \infty} \left( \sum_{i=1}^{n} g(q_i) \right) \cdot f(p) = \xi \cdot f(p), \quad \forall p \in S. \end{aligned}$$

**Proposition 2.8.** *Let $S$ be a left module over the $L(\mu, K)$, and $f_1, f_2, \ldots, f_n$ and $g$ module homomorphisms from $S$ to $L(\mu, K)$ such that $\cap_{i=1}^{n} N(f_i) \subset N(g)$. Then there exist $\xi_1, \xi_2, \ldots, \xi_n$ in $L(\mu, K)$ such that $g = \sum_{i=1}^{n} \xi_i \cdot f_i$, i.e., $g(p) = \sum_{i=1}^{n} \xi_i \cdot (f_i(p)), \forall p \in S$.*

*Proof.* We proceed by induction as follows.

When $n=1$, this proposition is exactly Lemma 2.2.

When $n=k$, this proposition is assumed to be valid, we will prove it also valid for $n=k+1$. Assume $\bigcap_{i=1}^{k+1} N(f_i) \subset N(g)$, and denote by $\hat{g}$ the restriction of $g$ to $N(f_{k+1})$, and by $\hat{f}_i$ the restriction of $f_i$ to $N(f_{k+1})$ for each $1 \leqslant i \leqslant k$. Since $N(\hat{f}_i) = N(f_{k+1}) \cap N(f_i), \forall\ 1 \leqslant i \leqslant k$, and $N(\hat{g}) = N(f_{k+1}) \cap N(g)$, clearly $\bigcap_{i=1}^{k+1} N(f_i) \subset N(g)$ implies $\bigcap_{i=1}^{k} N(\hat{f}_i) = \bigcap_{i=1}^{k+1} N(f_i) \subset N(f_{k+1}) \cap N(g) = N(\hat{g})$, then there exist $\xi_1, \xi_2, \ldots, \xi_k$ in $L(\mu, K)$ such that $\hat{g} = \sum_{i=1}^{k} \xi_i \cdot \hat{f}_i$. Namely,

$$g(p) = \hat{g}(p) = \sum_{i=1}^{k} \xi_i \cdot \hat{f}_i(p) = \sum_{i=1}^{k} \xi_i \cdot f_i(p), \quad \forall p \in N(f_{k+1}),$$

which is equivalent to saying $N(f_{k+1}) \subset N\big(g - \sum_{i=1}^{k} \xi_i \cdot f_i\big)$. So by Lemma 2.2 there exists $\xi_{k+1}$ in $L(\mu, K)$ such that $g - \sum_{i=1}^{k} \xi_i \cdot f_i = \xi_{k+1} \cdot f_{k+1}$, that is, $g(p) = \sum_{i=1}^{k+1} \xi_i \cdot (f_i(p)), \forall p \in S$.

**Remark 2.5.** When $(\Omega, \mathcal{A}, \mu)$ is a trivial probability space, Proposition 2.8 is exactly a known and quite simple fact: Let $S$ be a linear space over $K$, $f_1, f_2, \ldots, f_n$ and $g$ linear functionals on $S$ such that $\bigcap_{i=1}^{n} N(f_i) \subset N(g)$, then there exist $\alpha_1, \alpha_2, \ldots, \alpha_n$ in $K$ such that $g = \sum_{i=1}^{n} \alpha_i \cdot f_i$ (see [1, Chapter 2]). Readers will find that the substantive difficulty in generalizing this result to the random setting of Proposition 2.8 occurs in Lemma 2.2 where we are forced to construct a Cauchy sequence to produce our desired $\xi$! this idea is just motivated by the work [10] where we arrived at our aim by constructing a Cauchy sequence and making full use of the completeness of a complete random inner product module, such complications in the study of random setting are involved completely because of the arbitrariness of the $\sigma$-algebra $\mathcal{A}$ of $(\Omega, \mathcal{A}, \mu)$, which has been further elucidated in detail in [14] from the point of view on the stratification structure of modules over the algebra $L(\mu, K)$. Let $S$ be a left module over the algebra $L(\mu, K)$, as interpreted in [14], $p_A = \tilde{I}_A \cdot p$ is called the $A$-stratification of $p$ for each $\mu$-measurable set $A$ of $\Omega$ and $p$ in $S$. Clearly, $p_A = \theta$ when $\mu(A) = 0$, and $p_A = p$ when $\mu(\Omega \backslash A) = 0$, so $\theta$ and $p$ are just the two trivial stratifications of $p$. When $(\Omega, \mathcal{A}, \mu)$ is a trivial probability space, every element in $S$ possesses merely the above two trivial stratifications since $\mathcal{A} = \{\Omega, \emptyset\}$, but when $(\Omega, \mathcal{A}, \mu)$ is arbitrary, every element in $S$ may have many nontrivial intermediate stratifications. The essence of Proposition 2.8 together with much of our work in this paper is just to overcome the difficulty with the kind of complicated stratification structure. Lemma 2.2 and Proposition 2.8 were mentioned with a sketch proof of them in [5, 20]. The proofs of them given in this paper are an improvement of those in [5, 20].

## 3 Random duality together with some basic theorems on it

**Definition 3.1.** *The left modules $S_1$ and $S_2$ over the algebra $L(\mu, K)$ are called a pair in random duality over $K$ with base $(\Omega, \mathcal{A}, \mu)$ with respect to the bi-module homomorphism $\langle \cdot, \cdot \rangle : S_1 \times S_2 \to L(\mu, K)$ (namely both $\langle \cdot, q \rangle : S_1 \to L(\mu, K)$ and $\langle p, \cdot \rangle : S_2 \to L(\mu, K)$ are module homomorphisms for each given $p \in S_1$ and $q \in S_2$) if the following two conditions are satisfied:*

(1) $\langle p, q \rangle = 0, \forall p \in S_1$ *implies* $q = \theta$ *(the null in* $S_2$*)*;

(2) $\langle p, q \rangle = 0, \forall q \in S_2$ *implies* $p = \theta$ *(the null in* $S_1$*)*.

From now on, for the sake of brevity we only say $\langle S_1, S_2 \rangle$ is a random duality pair over $K$ with base $(\Omega, \mathcal{A}, \mu)$ if $S_1, S_2$ and $\langle \cdot, \cdot \rangle$ satisfy Definition 3.1.

**Definition 3.2.** *Let $\langle S_1, S_2\rangle$ be a random duality pair over $K$ with base $(\Omega, \mathcal{A}, \mu)$. $S_2$ is called regular (standard in terms of [5]) with respect to $\langle\cdot,\cdot\rangle$ and $S_1$, at this time $\langle S_1, S_2\rangle$ is also called right regular, if there always exists $q$ in $S_2$ for any given countable set $\{q_n : n \in \mathbb{N}\}$ in $S_2$ and any given countable $\mathcal{A}$-measurable partition $\{A_n : n \in \mathbb{N}\}$ of $\Omega$ such that $\langle p, q\rangle = \sum_{n\geqslant 1} \tilde{I}_{A_n} \cdot \langle p, q_n\rangle, \forall p \in S_1$(notice: the series on the right side always converges both $\mu$-almost everywhere and locally in measure $\mu$ in $L(\mu, K)$). In a complete symmetry one can speak of the regularity of $S_1$ with respect to $S_2$ and $\langle\cdot,\cdot\rangle$. Further if $S_1$ and $S_2$ are both regular with respect to each other and $\langle\cdot,\cdot\rangle$, then we say $\langle S_1, S_2\rangle$ is regular random duality pair.*

**Definition 3.3.** *Let $\langle S_1, S_2\rangle$ be a random duality pair over $K$ with base $(\Omega, \mathcal{A}, \mu)$. For each given $q$ in $S_2$, $\hat{q}$ denotes the module homomorphism $\langle\cdot, q\rangle$ from $S_1$ to $L(\mu, K)$ and $\mathcal{X}^q : S_1 \to L^+(\mu)$ the random seminorm defined by $X^q_p = |\langle p, q\rangle|, \forall p \in S_1$. Denote by $\sigma(S_1, S_2)$ the family $\{\mathcal{X}^q : q \in S_2\}$, then the $\sigma(S_1, S_2)$-topology, namely the $(\varepsilon, \lambda)$-topology generated by $\sigma(S_1, S_2)$ is called the random weak topology for $S_1$ with respect to $S_2$ and $\langle\cdot,\cdot\rangle$. One can easily understand the notation $\sigma(S_2, S_1)$ and the terminology "the $\sigma(S_2, S_1)$-topology".*

**Remark 3.1.** It follows immediately from Proposition 2.4 that $\sigma(S_1, S_2)$ makes $S_1$ a Hausdorff topological module over the Hausdorff topological algebra $L(\mu, K)$, and it is easy to see that $\sigma(S_1, S_2)$-topology is the smallest linear topology for $S_1$ such that $\hat{q}$ is a continuous module homomorphism for each $q$ in $S_2$.

**Example 3.1.** Let $\langle S_1, S_2\rangle$ be a duality pair over $K$ in the sense of classical duality theory (see [1, Chapter 13]). Then $\langle S_1, S_2\rangle$ is a regular random duality pair over $K$ with base a trivial probability space.

**Example 3.2.** Let $(S, \{\mathcal{X}^d\}_{d\in D})$ be a random locally convex module over $K$ with base $(\Omega, \mathcal{A}, \mu)$ and $S^*$ the left module over the algebra $L(\mu, K)$ of continuous module homomorphisms from $S$ to $L(\mu, K)$ (namely $S^*$ is the random conjugate space of $(S, \{\mathcal{X}^d\}_{d\in D})$). Define $\langle\cdot,\cdot\rangle : S \times S^* \to L(\mu, K)$ by $\langle p, f\rangle = f(p), \forall p \in S$ and $f \in S^*$, then $\langle S, S^*\rangle$ is a random duality pair over $K$ with base $(\Omega, \mathcal{A}, \mu)$, called a canonical random duality pair. In fact, $S^*$ is also regular with respect to $S$ and the canonical random duality relation $\langle\cdot,\cdot\rangle$, which can be seen from the following observation: for a given countable set $\{f_n : n \in \mathbb{N}\}$ in $S^*$ and a given countable $\mathcal{A}$-measurable partition $\{A_n : n \in \mathbb{N}\}$ of $\Omega$, define $f : S \to L(\mu, K)$ by $f(p) = \sum_{n\geqslant 1} \tilde{I}_{A_n} \cdot f_n(p) = \tilde{\mu}$-limit of $\{\sum_{n=1}^{k} \tilde{I}_{A_n} \cdot f_n(p) : k \in \mathbb{N}\}$ (the limit of convergence in probability $\tilde{\mu}$ must exist by $\sum_{n=1}^{\infty} \tilde{\mu}(A_n) = 1 < +\infty$), where $\tilde{\mu}$ is defined as in Lemma 2.1. $f$ is also continuous according to $\sum_{n=1}^{\infty} \tilde{\mu}(A_n) = 1 < +\infty$, so $f$ is in $S^*$ such that $f(p) = \sum_{n=1}^{\infty} \tilde{I}_{A_n} \cdot f_n(p), \forall p \in S$.

**Example 3.3.** Let $\langle S_1, S_2\rangle$ be a random duality pair over $K$ with base $(\Omega, \mathcal{A}, \mu)$. Denote by $\hat{S}_2$ the set of all such module homomorphisms as $\hat{q}$, $q \in S_2$, where $\hat{q}$ is the same as in Definition 3.3, again denote by $S_2^r$ the set of all such module homomorphisms $f$ from $S_1$ to $L(\mu, K)$ that $f(p) = \sum_{n=1}^{\infty} \tilde{I}_{A_n} \cdot \hat{q}_n(p) = \sum_{n=1}^{\infty} \tilde{I}_{A_n} \cdot \langle p, q_n\rangle, \forall p \in S_1$, for some given countable set $\{q_n : n \in \mathbb{N}\}$ of $S_2$ and some given countable $\mathcal{A}$-measurable partition $\{A_n : n \in \mathbb{N}\}$ of $\Omega$. Similarly, one can understand such notations as $\hat{S}_1$ and $S_1^r$. Clearly, when $S_1$ and $S_2$ are identified with $\hat{S}_1$ and $\hat{S}_2$, respectively, they can be regarded as the $L(\mu, K)$-submodules of $S_1^r$ and $S_2^r$, respectively. Since for any given $f \in S_1^r$ and $g \in S_2^r$ there always exist countable sets $\{p_n : n \in \mathbb{N}\}$ in $S_1$, $\{q_n : n \in \mathbb{N}\}$ in $S_2$ and some common countable $\mathcal{A}$-measurable partition $\{A_n : n \in \mathbb{N}\}$ of $\Omega$

such that $f(q) = \sum_{n\geqslant 1} \tilde{I}_{A_n} \cdot \langle p_n, q\rangle, \forall q \in S_2$, while $g(p) = \sum_{n\geqslant 1} \tilde{I}_{A_n} \cdot \langle p, q_n\rangle, \forall p \in S_1$, now define $\langle\langle f, g\rangle\rangle = \sum_{n\geqslant 1} \tilde{I}_{A_n} \cdot \langle p_n, q_n\rangle$. Then $\langle\langle \cdot, \cdot\rangle\rangle : S_1^r \times S_2^r \to L(\mu, K)$ is a natural extension of $\langle \cdot, \cdot\rangle$ from $S_1 \times S_2$ to $S_1^r \times S_2^r$, and $\langle\langle S_1^r, S_2^r\rangle\rangle$ is a regular random duality pair, called *the regularization* of $\langle S_1, S_2\rangle$, still denoted by $\langle S_1^r, S_2^r\rangle$ for brevity. By the way, $\langle\langle S_1, S_2^r\rangle\rangle$ is right regular and $\langle\langle S_1^r, S_2\rangle\rangle$ is left regular when $\langle\langle \cdot, \cdot\rangle\rangle$ is restricted to $S_1 \times S_2^r$ or $S_1^r \times S_2$.

**Example 3.4.** In Example 3.2, if $(S, \{\mathcal{X}^d\}_{d\in D})$ is sequentially complete, $\langle S, S^*\rangle$ is a regular random duality pair. Finally when $S$ is this form of $L(\mu, B)$ for some normed space $B$ over $K$, it is easy to see that $\langle S, S^*\rangle$ is always regular (see [13] for the $RN$-module $L(\mu, B)$). This form of canonical random duality pair $\langle S, S^*\rangle$ for the case when $S$ is an $RN$-module was deeply studied in [14].

**Theorem 3.1.** *Let $\langle S_1, S_2\rangle$ be a random duality pair over $K$ with base $(\Omega, \mathcal{A}, \mu)$. Then any $f \in (S_1, \sigma(S_1, S_2))^*$ can be represented in the following way: there exist a countable set $\{q_n : n \in \mathbb{N}\}$ in $S_2$ and a countable $\mathcal{A}$-measurable partition $\{A_n : n \in \mathbb{N}\}$ of $\Omega$ such that $f(p) = \sum_{n\geqslant 1} \tilde{I}_{A_n} \cdot \langle p, q_n\rangle, \forall p \in S_1$.*

*Proof.* According to Definition 2.4, there exist a countable $\mathcal{A}$-measurable partition $\{A_n : n \in \mathbb{N}\}$ of $\Omega$, a sequence $\{\xi_n : n \in \mathbb{N}\}$ in $L^+(\mu)$ and a countable subfamily $\{F_n : n \in \mathbb{N}\}$ of $\mathcal{F}(S_2)$ such that $|f(p)| \leqslant \sum_{n=1}^{\infty} \tilde{I}_{A_n} \cdot \xi_n \cdot (\sum_{q\in F_n} |\langle p, q\rangle|), \forall p \in S_1$. Let $F_n' = \{\xi_n \cdot q : q \in F_n\}, \forall n \geqslant 1$, then $F_n'$ still belongs to $\mathcal{F}(S_2)$, so we have that $|f(p)| \leqslant \sum_{n=1}^{\infty} \tilde{I}_{A_n} \cdot \big(\sum_{q\in F_n'} |\langle p, q\rangle|\big), \forall p \in S_1$.

By Corollary 2.1 there exists a sequence $\{f_n : n \in \mathbb{N}\}$ of random linear functionals on $S_1$ such that the following two conditions are satisfied:

(1) $|f_n(p)| \leqslant \sum_{q\in F_n'} |\langle p, q\rangle|, \forall\, p \in S_1$;

(2) $f(p) = \sum_{n\geqslant 1} \tilde{I}_{A_n} \cdot f_n(p),\ \forall\, p \in S_1$.

(1) clearly implies $f_n \in (S_1, \sigma(S_1, S_2))^*$ and $\bigcap_{q\in F_n'} N(\hat{q}) \subset N(f_n)$, so by Proposition 2.8 there exist $\xi_1^{(n)}, \xi_2^{(n)}, \ldots, \xi_{k_n}^{(n)}$ such that $f_n(p) = \sum_{i=1}^{k_n} \xi_i^{(n)} \cdot \langle p, q_i^{(n)}\rangle = \langle p, \sum_{i=1}^{k_n} \xi_i^{(n)} \cdot q_i^{(n)}\rangle$, $\forall p \in S_1$, where $F_n' = \{q_1^{(n)}, q_2^{(n)}, \ldots, q_{k_n}^{(n)}\}$ for each $n \geqslant 1$.

Finally, taking $q_n = \sum_{i=1}^{k_n} \xi_i^{(n)} \cdot q_i^{(n)}$ and considering (2) end the proof of Theorem 3.1.

**Remark 3.2.** Theorem 3.1 generalizes the main result of [20] where $f$ was required as follows: $|f(p)| \leqslant \xi \cdot (\sum_{q\in F} |\langle p, q\rangle|)$ for some $\xi$ in $L^+(\mu)$ and some $F \in \mathcal{F}(S_2)$. Theorem 3.1 was given in [5] without a complete proof except a very short sketch proof.

**Corollary 3.1.** *Let $\langle S_1, S_2\rangle$ be the same as in Theorem* 3.1. *Then we have the following statements*:

(1) *If $\langle S_1, S_2\rangle$ is right regular, then $(S_1, \sigma(S_1, S_2))^* = S_2$, where $S_2$ is identified with $\hat{S}_2$(see Example* 3.3 *for $\hat{S}_2$*);

(2) *If $\langle S_1, S_2\rangle$ is left regular, then $(S_2, \sigma(S_2, S_1))^* = S_1$;*

(3) *If $\langle S_1, S_2\rangle$ is regular, then the conclusions of both* (1) *and* (2) *hold.*

**Definition 3.4.** *Let $\langle S_1, S_2\rangle$ be a right regular random duality pair over $K$ with base $(\Omega, \mathcal{A}, \mu)$. A family $\{\mathcal{X}^d\}_{d\in D}$ of $M$-random seminorms on $S_1$ is called a random compatible structure with $S_2$* (*such a random compatible structure is called a right random compatible structure of $\langle S_1, S_2\rangle$*) *if $(S_1, \{\mathcal{X}^d\}_{d\in D})^* = S_2$. A property of $S_1$ is called a right random compatible invariant* (*or called a right random duality invariant*) *if it also holds for any other right random compatible structure on $S_1$ whenever it does for any given right random compatible structure. One can also define* *a left random compatible structure and a left random compatible invariant in a similar way if* $\langle S_1, S_2\rangle$ *is left regular.*

**Remark 3.3.** In functional analysis, the device "Minkowski functionals" makes any locally convex topology for a linear space be generated by some family of seminorms on the space, and any linear topology generated by a family of seminorms is always locally convex (see [19, Section 37] for details). But at the present time there exist no random counterparts to Minkowski functionals and locally convex topologies except that the random seminorms are a random correspondence to ordinary seminorms. Thus we are forced to work with only random seminorms.

**Theorem 3.2.** *There is a greatest one in all right random compatible structures for a given right regular random duality pair. Dually, a statement about a left random compatible structure is also valid.*

*Proof.* We only need to give the proof for a right random compatible structure.

Let $\langle S_1, S_2\rangle$ be a right regular random duality pair over $K$ with base $(\Omega, \mathcal{A}, \mu)$ and $\{\{\mathcal{X}^d\}_{d\in D_\alpha} : \alpha \in \Lambda\}$ all the family of right random compatible structures on $S_1$. Denote $D = \cup\{D_\alpha : \alpha \in \Lambda\}$, then we will verify that $\{\mathcal{X}^d\}_{d\in D}$ is a right random compatible structure on $S_1$ as follows.

Clearly, $S_2 = (S_1, \{\mathcal{X}^d\}_{d\in D_\alpha})^* \subset (S_1, \{\mathcal{X}^d\}_{d\in D})^*$ for any given $\alpha \in \Lambda$, we will prove that $(S_1, \{\mathcal{X}^d\}_{d\in D})^* \subset S_2$ also holds. Let $f$ be in $(S_1, \{\mathcal{X}^d\}_{d\in D})^*$. Then according to Definition 2.4 there exist a countable $\mathcal{A}$-measurable partition $\{A_n : n \in \mathbb{N}\}$ of $\Omega$, a countable set $\{\xi_n : n \in \mathbb{N}\}$ in $L^+(\mu)$ and a countable subfamily $\{F_n : n \in \mathbb{N}\}$ of $\mathcal{F}(D)$ such that $|f(p)| \leqslant \sum_{n\geqslant 1} \tilde{I}_{A_n} \cdot \xi_n \cdot X_p^{F_n}, \forall p \in S_1$.

Again by Corollary 2.1 there exists a sequence $\{f_n : n \in \mathbb{N}\}$ of random linear functionals on $S_1$ such that the following two conditions are satisfied:

(1) $|f_n(p)| \leqslant \xi_n \cdot X_p^{F_n}, \forall\, p \in S_1$;

(2) $f(p) = \sum_{n\geqslant 1} \tilde{I}_{A_n} \cdot f_n(p), \forall\, p \in S_1$.

We can easily see from (1) that each $f_n \in (S_1, \{\mathcal{X}^d\}_{d\in D})^*$, we will prove that there exists $q_n \in S_2$ such that $f_n(p) = \langle p, q_n\rangle, \forall\, p \in S_1$ and $n \geqslant 1$.

Now, let us fix $n$. Since $F_n \in \mathcal{F}(D)$ and $D = \cup\{D_\alpha : \alpha \in \Lambda\}$, then there must exist finitely many indexes $\alpha_1, \alpha_2, \ldots, \alpha_k$ in $\Lambda$ such that $F_n = \cup_{i=1}^k F_i'$, where $F_i' \in \mathcal{F}(D_{\alpha_i}), \forall 1 \leqslant i \leqslant k$. Since $X_p^{F_n} = \sum_{d\in F_n} X_p^d = \sum_{i=1}^k (\sum_{d\in F_i'} X_p^d), \forall\, p \in S_1$ (we can, without any, assume that $\{F_i' : i = 1, \ldots, k\}$ are pairwise disjoint). Then again by Proposition 2.7 there exist random linear functionals $f_1', f_2', \ldots, f_k'$ on $S_1$ such that $|f_i'(p)| \leqslant \xi_n \cdot (\sum_{d\in F_i'} X_p^d), \forall\, p \in S_1$ and $1 \leqslant i \leqslant k$, and such that $f_n(p) = \sum_{i=1}^k f_i'(p), \forall\, p \in S_1$. Clearly, each $f_i' \in (S_1, \{\mathcal{X}^d\}_{d\in D_{\alpha_i}})^* = S_2$ means that there exists a $q_i'$ in $S_2$ such that $f_i'(p) = \langle p, q_i'\rangle, \forall\, p \in S_1$ and $1 \leqslant i \leqslant k$. To sum up, $f_n(p) = \sum_{i=1}^k f_i'(p) = \sum_{i=1}^k \langle p, q_i'\rangle = \langle p, \sum_{i=1}^k q_i'\rangle = \langle p, q_n\rangle$, for any $p$ in $S_1$, where $q_n = \sum_{i=1}^k q_i'$.

Finally, the above (2) shows that $f(p) = \sum_{n\geqslant 1} \tilde{I}_{A_n} \cdot \langle p, q_n\rangle, \forall\, p \in S_1$. Thus the regularity of $S_2$ implies the existence of a point $q$ in $S_2$ satisfying $f(p) = \langle p, q\rangle, \forall\, p \in S_1$. Since $f$ is arbitrary, one can have $(S_1, \{\mathcal{X}^d\}_{d\in D})^* \subset S_2$.

**Remark 3.4.** $\{\mathcal{X}^d\}_{d\in D}$ obtained in the proof of Theorem 3.2 is called the random Mackey structure on $S_1$ with respect to $S_2$, which must be saturated. This family also generates exactly the ordinary Mackey topology on $S_1$ when $(\Omega, \mathcal{A}, \mu)$ is a trivial probability space.

Besides the usual geometric notions such as convex sets, balanced sets and absorbent sets in linear spaces, the following strengthened forms or similar variants of them in a left module over the algebra $L(\mu, K)$ will also be used in random metric theory and play a crucial role at some time.

**Definition 3.5.** *Let $S$ be a left module over the algebra $L(\mu, K)$. A set $A$ of $S$ is called $M$-convex* (*namely convex in the sense of module*) *if $\xi \cdot p + \eta \cdot q$ still belongs to $A$ whenever $p$ and $q$ are in $A$ and whenever $\xi$ and $\eta$ are in $L^+(\mu)$ such that $\xi + \eta = 1$; $A$ is called $M$-balanced if $\xi \cdot p$ is still in $A$ whenever $\xi$ is in $L(\mu, K)$ such that $|\xi| \leqslant 1$ and $p$ in $A$; $A$ is $M$-absorbed by a subset $B$ of $S$ if there exists a $\xi \in L^+(\mu)$ such that $\xi(\omega) > 0$ $\mu$-a.e. and $\eta \cdot p \in B$ whenever $p$ is in $A$ and $\eta \in L(\mu, K)$ such that $|\eta| \leqslant \xi$; further $A$ is called $M$-absorbent if it $M$-absorbs any point of $S$.*

**Remark 3.5.** Since the module multiplication naturally extends the scalar multiplication, then one can easily see that if a set is $M$-convex ($M$-balanced) then it is also convex (accordingly, balanced), Obviously the converse does not generally hold. Furthermore, neither of the two concepts "$M$-absorbent" and "absorbent" implies each other, they will play their respective roles in random metric theory.

**Definition 3.6.** *Let $(S, \{\mathcal{X}^d\}_{d\in D})$ be a random locally convex space over $K$ with base* $(\Omega, \mathcal{A}, \mu)$. *A set $A$ of $S$ is called bounded if it is bounded with respect to the $\{\mathcal{X}^d\}_{d\in D}$-linear topology* (*if* $(\Omega, \mathcal{A}, \mu)$ *is a probability space, such a terminology is equivalent with the one "stochastically bounded" in probability theory in Banach spaces, and also with the one "probabilistically bounded" in the theory of probabilistic metric spaces*); *A subset $A$ is called $\mu$-a.e. bounded if $\bigvee\{X_p^d : p \in A\}$ is in $L^+(\mu)$ for each $d \in D$.*

**Remark 3.6.** It is obvious that $A$ is $\mu$-a.e. bounded iff $\bigvee\{X_p : p \in A\}$ is in $L^+(\mu)$ for each $\mathcal{X}$ in $s(\{\mathcal{X}^d\}_{d\in D})$, the saturation of $\{\mathcal{X}^d\}_{d\in D}$. By Proposition 2.1 one can further see that an $M$-convex set in a random locally convex module is $\mu$-a.e. bounded iff it is bounded. Generally, a $\mu$-a.e. bounded set must be bounded, but not conversely.

**Lemma 3.1.** *Let $(S, \{\mathcal{X}^d\}_{d\in D})$ be the same as in Definition* 3.6, *and $A$ a subset of $S$. Then we have the following statements:*

(1) *$A$ is bounded iff $f(A)$ is bounded in $L(\mu, K)$ for each $f$ in $S^*$;*

(2) *$A$ is $\mu$-a.e. bounded iff $f(A)$ is $\mu$-a.e. bounded in $L(\mu, K)$ for each $f$ in $S^*$.*

*Proof.* Denote $S_0^* = \{f \in S^*$ : there exist $\xi \in L^+(\mu)$ and $F \in \mathcal{F}(D)$ such that $|f(p)| \leqslant \xi \cdot X_p^F \equiv \xi \cdot (\sum_{d\in F} X_p^d), \forall p \in S\}$. The paper [2] already proved that (1) and (2) are both valid if $S^*$ is replaced by $S_0^*$.

(1) If $A$ is bounded, then $f(A)$ is bounded since each $f$ in $S^*$ is continuous; the converse implies, of course, that $f(A)$ is bounded for each $f$ in $S_0^*$, and hence $A$ also bounded.

(2) Its proof is similar to that of the above (1), and one only needs to make use of Definition 2.4.

**Theorem 3.3.** *Let $\langle S_1, S_2\rangle$ be a right regular random duality pair over $K$ with base $(\Omega, \mathcal{A}, \mu)$, $\{\mathcal{X}^d\}_{d\in D}$ and $\{\mathcal{X}^e\}_{e\in\Gamma}$ the two right random compatible structures on $S_1$, and $A \subset S_1$ a subset of $S_1$. Then we have the following statements:*

(1) *$A$ is bounded in $(S_1, \{\mathcal{X}^d\}_{d\in D})$ iff it is bounded in $(S_1, \{\mathcal{X}^e\}_{e\in\Gamma})$;*

(2) *$A$ is $\mu$-a.e. bounded in $(S_1, \{\mathcal{X}^d\}_{d\in D})$ iff it is $\mu$-a.e. bounded in $(S_1, \{\mathcal{X}^e\}_{e\in\Gamma})$.*

*Proof.* It follows immediately from Lemma 3.1 and the fact that $(S_1, \{\mathcal{X}^d\}_{d\in D})^* = S_2 = (S_1, \{\mathcal{X}^e\}_{e\in\Gamma})^*$.

**Definition 3.7.** *Let $\langle S_1, S_2\rangle$ be a random duality pair over $K$ with base $(\Omega, \mathcal{A}, \mu)$, and $A$ and $B$ subsets of $S_1$ and $S_2$, respectively. The set $A^0 = \{q : q \in S_2$ and $\bigvee\{|\langle p, q\rangle| : p \in A\} \leqslant 1\}$ is called the polar of $A$, and the set $B^0 = \{p : p \in S_1$ and $\bigvee\{|\langle p, q\rangle| : q \in B\} \leqslant 1\}$ is called the polar of $B$.*

**Lemma 3.2**[21]**.** *Let $(S, \{\mathcal{X}^d\}_{d\in D})$ be a random locally convex module over $K$ with base $(\Omega, \mathcal{A}, \mu)$, $p \in S$, $A \subset S$ a closed $M$-convex set, and $p \notin A$. Then there exists an $f$ in $S^*$ such that* $\mathrm{Re}(f(p)) > \bigvee\{\mathrm{Re}(f(q)) : q \in A\}$, *where* Re *stands for "the real part", "$>$" means "$\geqslant$" but "$\neq$". Furthermore, if $A$ is also $M$-balanced, we can require that $|f(p)| > \bigvee\{|f(q)| : q \in A\}$.*

**Theorem 3.4.** *Let $\langle S_1, S_2\rangle$ be a random duality pair over $K$ with base $(\Omega, \mathcal{A}, \mu)$, and $A$ a subset of $S_1$. Then we have the following statements:*

(1) *$A^0$ is $M$-convex, $M$-balanced and $\sigma(S_2, S_1)$-closed;*

(2) *If $A$ is $\mu$-a.e. bounded in $(S_1, \sigma(S_1, S_2))$, then $A^0$ is $M$-absorbent;*

(3) *If $\langle S_1, S_2\rangle$ is right regular, then $A$ has the same closed $M$-convex hull and the same closed $M$-convex and $M$-balanced hull for all the random compatible structures on $S_1$;*

(4) *If $\langle S_1, S_2\rangle$ is right regular, and $B$ is the closed $M$-convex and $M$-balanced hull of $A$ with respect to any given random compatible structure on $S_1$, then $B = A^{00}$.*

*Proof.* (1) It is a straightforward verification.

(2) Since $A$ is $\mu$-a.e. bounded with respect to $\sigma(S_1, S_2)$, then $\bigvee\{|\langle p, q\rangle| : p \in A\} \in L^+(\mu)$ for any given $q$ in $S_2$. Now fix $q$ in $S_2$ and denote $\xi = \bigvee\{|\langle p, q\rangle| : p \in A\}$ and $\xi_q = \frac{1}{\xi+1}$, then $\xi_q(\omega) > 0$ $\mu$-a.e. Let $\eta \in L(\mu, K)$ be such that $|\eta| \leqslant \xi_q$, then $\bigvee\{|\langle p, \eta\cdot q\rangle| : p \in A\} = |\eta|\cdot\xi \leqslant \xi_q\cdot\xi \leqslant 1$, this means $\eta\cdot q \in A^0$, namely $A^0$ $M$-absorbs $q$.

(3) Let $(S, \{\mathcal{X}^d\}_{d\in D})$ be the same as in Lemma 3.2, then Lemma 3.2 implies an $M$-convex set of $S$ is closed in $\{\mathcal{X}^d\}_{d\in D}$-topology if it is closed in $\sigma(S, S^*)$-topology. Now if $\langle S_1, S_2\rangle$ is right regular, take the above $S = S_1$, $\{\mathcal{X}^d\}_{d\in D}$ is a given random compatible structure on $S_1$ with respect to $S_2$, then $S^* = S_2$, so our desired conclusions follow.

(4) Since $A^{00}$ includes $A$ and is $\sigma(S_1, S_2)$-closed, $M$-convex and $M$-balanced, then $A^{00}$ includes $B$ by the above (3). If there exists $p \in A^{00}\setminus B$, then by Lemma 3.2 there exists a $q$ in $S_2$ such that $|\langle p, q\rangle| > \bigvee\{|\langle b, q\rangle| : b \in B\} = \bigvee\{|\langle a, q\rangle| : a \in A\}$, the lastest equality follows from the fact that $B$ is the $\sigma(S_1, S_2)$-closed $M$-convex and $M$-balanced hull of $A$ (one can easily verify it by Proposition 2.1 and by noticing $\{|\langle h, q\rangle| : h \in H\}$ is directed, where $H$ is the $M$-convex hull of A).

Denote $|\langle p, q\rangle|$ by $\xi$, and $\bigvee\{|\langle a, q\rangle| : a \in A\}$ by $\eta$, and let $\xi^0$ and $\eta^0$ be arbitrarily chosen representatives of $\xi$ and $\eta$, respectively, and $E = \{\omega \in \Omega : \xi^0(\omega) > \eta^0(\omega)\}$, then $\mu(E) > 0$ since $\xi > \eta$. We will divide the remainder of our proof into the following two cases.

Case 1. Let $F = \{\omega \in \Omega : \eta^0(\omega) > 0\}$ and $G = E \cap F$. If $\mu(G) > 0$, let $q^* = \tilde{I}_G\cdot Q(\eta)\cdot q$,where $Q(\eta)$ is the generalized inverse of $\eta$ (see Definition 2.1), then $|\langle p, q^*\rangle| = \tilde{I}_G\cdot Q(\eta)\cdot|\langle p, q\rangle| = \tilde{I}_G\cdot Q(\eta)\cdot\xi$, and $\bigvee\{|\langle a, q^*\rangle| : a \in A\} = \tilde{I}_G\cdot Q(\eta)\cdot\eta = \tilde{I}_G\cdot\tilde{I}_F = \tilde{I}_G$. $\tilde{I}_G \leqslant 1$ implies $q^* \in A^0$. But since $\xi^0(\omega)\cdot(\eta^0(\omega))^{-1} > 1$ for all $\omega$ in $G$, and $I_G\cdot(\eta^0)^{-1}\cdot\xi^0$ is a representative of $|\langle p, q^*\rangle|$.

$\mu(G) > 0$ means $|\langle p, q^*\rangle| \nleq 1$, namely $p \notin A^{00}$, which contradicts the assumption on $p$.

Case 2. If $\mu(G) = 0$, namely $\eta^0(\omega) = 0$ $\mu$-a.e. on $E$, so $\tilde{I}_E \cdot \eta = 0$, which implies $\bigvee\{|\langle a, q^*\rangle| : a \in A\} = 0$, where $q^* = \tilde{I}_E \cdot q$. Since $|\langle p, q^*\rangle| = \tilde{I}_E \cdot \xi$ and $\xi^0(\omega) > \eta^0(\omega) = 0$ $\mu$-a.e. on $E$. Choosing a positive integer $n$ so large that $\mu\{\omega \in E : n \cdot \xi^0(\omega) > 1\} > 0$, and putting $\tilde{q} = n \cdot q^*$, then $|\langle p, \tilde{q}\rangle| = \tilde{I}_E \cdot n \cdot \xi \nleq 1$. But $\bigvee\{|\langle a, \tilde{q}\rangle| : a \in A\} = 0$ still holds, this means $\tilde{q} \in A^0$, again $p \notin A^{00}$ contradicts the assumption on $p$.

**Remark 3.7.** Theorem 3.4 (2) shows the concept of the $\mu$-a.e. boundedness is more important than that of the boundedness; Theorem 3.4 (4), namely the random bipolar theorem, shows the concept of polar and that of the $\mu$-a.e. boundedness will help us develop a kind of theory of random uniform convergence structure and random admissible structure with respect to a random duality pair. This will be the subject of the forthcoming paper.

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