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Symmetric jump processes and their heat kernel estimates

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Abstract We survey the recent development of the DeGiorgi-Nash-Moser-Aronson type theory for a class of symmetric jump processes (or equivalently, a class of symmetric integro-differential operators). We focus on the sharp two-sided estimates for the transition density functions (or heat kernels) of the processes, a priori Hölder estimate and parabolic Harnack inequalities for their parabolic functions. In contrast to the second order elliptic differential operator case, the methods to establish these properties for symmetric integro-differential operators are mainly probabilistic.

Keywords: symmetric jump process, diffusion with jumps, pseudo-differential operator, Dirichlet form, a prior Hölder estimates, parabolic Harnack inequality, global and Dirichlet heat kernel estimates, Lévy system

MSC(2000): 60J35, 47G30, 60J45, 31C05, 31C25, 60J75

1 Introduction

Second order elliptic differential operators and diffusion processes take up, respectively, an central place in the theory of partial differential equations (PDE) and the theory of probability. There are close relationships between these two subjects. For a large class of second order elliptic differential operators $\mathcal{L}$ on $\mathbb{R}^n$, there is a diffusion process X on $\mathbb{R}^n$ associated with it so that $\mathcal{L}$ is the infinitesimal generator of X, and vice versa. The connection between $\mathcal{L}$ and X can also be seen as follows. The fundamental solution (also called heat kernel) for $\mathcal{L}$ is the transition density function of X. For example, when $\mathcal{L} = \frac{1}{2}\sum_{i,j=1}^n \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j})$, where $(a_{ij}(x))_{1\leqslant i,j\leqslant n}$ is a measurable $n \times n$ matrix-valued function on $\mathbb{R}^n$ that is uniformly elliptic and bounded, there is a symmetric diffusion X having $\mathcal{L}$ as its L^2-infinitesimal generator. The celebrated DeGiorgi-Nash-Moser-Aronson theory tells us that every bounded parabolic function of $\mathcal{L}$ (or equivalently, of X) is locally Hölder continuous and the parabolic Harnack inequality holds for non-negative parabolic functions of $\mathcal{L}$. Moreover, $\mathcal{L}$ has a jointly continuous heat kernel $p(t, x, y)$ with respect to the Lebesgue measure on $\mathbb{R}^n$ that enjoys the following Aronson's estimate: there are constants $c_k > 0$, $k = 1, \ldots, 4$, so that

$$c_1\, p^c(t, c_2|x-y|) \leqslant p(t, x, y) \leqslant c_3\, p^c(t, c_4|x-y|) \qquad \text{for } t > 0 \text{ and } x, y \in \mathbb{R}^n. \tag{1.1}$$

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Here

$$p^c(t,r) := t^{-n/2}\exp(-r^2/t). \tag{1.2}$$

See [1] for some history and a survey on this subject, where a mixture of analytic and probabilistic methods is presented.

Recently there has been intense interest in studying discontinuous Markov processes, due to their importance both in theory and in applications. Many physical and economic systems should be and in fact have been successfully modeled by non-Gaussian jump processes; see for example, [2–5] and the references therein. The infinitesimal generator of a discontinuous Markov process in $\mathbb{R}^n$ is no longer a differential operator but rather a non-local (or, integro-differential) operator. For instance, the infinitesimal generator of an isotropically symmetric α-stable process in $\mathbb{R}^n$ with $\alpha \in (0,2)$ is a fractional Laplacian operator $c\,\Delta^{\alpha/2} := -c\,(-\Delta)^{\alpha/2}$. During the past several years there is also much interest from the theory of PDE (such as singular obstacle problems) to study non-local operators; see, for example, [6, 7] and the references therein.

In this paper, we survey recent development of the DeGiorgi-Nash-Moser-Aronson type theory for the following type of non-local (integro-differential) operators $\mathcal{L}$ on $\mathbb{R}^n$:

$$\mathcal{L}u(x) = \frac{1}{2}\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u(x)}{\partial x_j}\right) + \lim_{\varepsilon\downarrow 0}\int_{\{y\in\mathbb{R}^n:\,|y-x|>\varepsilon\}} (u(y)-u(x))J(x,y)dy, \tag{1.3}$$

where either $(a_{ij}(x))_{1\leqslant i,j\leqslant n}$ is identically zero or $(a_{ij}(x))_{1\leqslant i,j\leqslant n}$ is a measurable $n\times n$ matrix-valued measurable function on $\mathbb{R}^n$ that is uniformly elliptic and bounded, and J is a measurable non-negative symmetric kernel satisfying certain conditions. Associated with such a non-local operator $\mathcal{L}$ is an $\mathbb{R}^n$-valued symmetric jump process X with jumping kernel $J(x,y)$ and with possible diffusive components when $(a_{ij}(x))_{1\leqslant i,j\leqslant n}$ is non-degenerate. Note that the jumping kernel J determines a Lévy system for X, which describes the jumps of the process X: for any non-negative measurable function f on $\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}^n$, $t\geqslant 0$, $x\in\mathbb{R}^n$ and stopping time T (with respect to the minimal admissible filtration of X),

$$\mathbb{E}_x\left[\sum_{s\leqslant T} f(s, X_{s-}, X_s)\right] = \mathbb{E}_x\left[\int_0^T\left(\int_{\mathbb{R}^n} f(s,X_s,y)J(X_s,y)dy\right)ds\right]. \tag{1.4}$$

Our focus will be on sharp two-sided heat kernel estimates for $\mathcal{L}$ (or equivalently, transition density function estimates for X), as well as parabolic Harnack inequality and a priori joint Hölder continuity estimate for parabolic functions of $\mathcal{L}$. When $(a_{ij}(x))_{1\leqslant i,j\leqslant n}\equiv 0$ and $J(x,y) = c|x-y|^{-n-\alpha}$ for some $\alpha\in(0,2)$ in (1.3), $\mathcal{L}$ is a fractional Laplacian $c_1\Delta^{\alpha/2}$ on $\mathbb{R}^n$ and its associated process X is a rotationally symmetric α-stable process on $\mathbb{R}^n$. Unlike the Brownian motion case, the explicit formula for the density function $p(t,x,y)$ of X with respect to the Lebesgue measure is only known for a few special α, such as $\alpha = 1$. However due to the scaling property of X, one has

$$p(t,x,y) = t^{-n/\alpha}p(1, t^{-1/\alpha}x, t^{-1/\alpha}y) = t^{-n/\alpha}f(t^{-1/\alpha}(x-y)) \quad \text{for } t>0 \text{ and } x,y\in\mathbb{R}^n,$$

where $f(z)$ is the density function of the symmetric α-stable random variable $X_1 - X_0$ in $\mathbb{R}^n$. Using Fourier transform, it is not difficult to show (see [8, Theorem 2.1]) that $f(z)$ is a continuous strictly positive function on $\mathbb{R}^n$ depending on z only through $|z|$ and that $f(z) \asymp |z|^{-n-\alpha}$ at infinity. Consequently

$$p(t,x,y) \asymp t^{-n/\alpha}\left(1 \wedge \frac{t^{1/\alpha}}{|x-y|}\right)^{n+\alpha} \qquad \text{on } \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n. \tag{1.5}$$

In this paper, for two non-negative functions f and g, the notation $f \asymp g$ means that there are positive constants c_1 and c_2 so that $c_1 g(x) \leqslant f(x) \leqslant c_2 g(x)$ in the common domain of definition for f and g. For $a, b \in \mathbb{R}$, $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. However such kind of simple argument for (1.5) breaks down for the symmetric α-stable-like processes on $\mathbb{R}^n$ when $(a_{ij}(x))_{1 \leqslant i,j \leqslant n} \equiv 0$ and $J(x,y) = c(x,y)|x-y|^{-n-\alpha}$ for some $\alpha \in (0,2)$ and a symmetric function $c(x,y)$ that is bounded between two positive constants in (1.3), as in this case, X is no longer a Lévy process.

Two-sided heat kernel estimates for jump processes in $\mathbb{R}^n$ have only been studied recently. In [9], Kolokoltsov obtained two-sided heat kernel estimates for certain stable-like processes in $\mathbb{R}^n$, whose infinitesimal generators are a class of pseudo-differential operators having smooth symbols. Bass and Levin[10] used a completely different approach to obtain similar estimates for discrete time Markov chain on $\mathbb{Z}^n$ where the conductance between x and y is comparable to $|x-y|^{-n-\alpha}$ for $\alpha \in (0,2)$. In [11], two-sided heat kernel estimates and a scale-invariant parabolic Harnack inequality (PHI in abbreviation) for symmetric α-stable-like processes on d-sets are obtained. Recently in [12], PHI and two-sided heat kernel estimates are even established for non-local operators of variable order. Finite range stable-like processes on $\mathbb{R}^n$ are studied in [13]. This class of processes is very natural in applications where jumps only up to a certain size are allowed. The heat kernel estimates obtained in [13] shows finite range stable-like processes behave like discontinuous stable-like processes in small scale and behave like Brownian motion in large scale. Processes having such properties may be useful in applications. For example, in mathematical finance, it has been observed that even though discontinuous stable processes provide better representations of financial data than Gaussian processes[14], financial data tend to become more Gaussian over a longer time-scale (see [15] and the references therein). Our heat kernel estimates in [13] show that finite range stable-like processes have this type of property. Moreover, finite range stable-like processes avoid large sizes of jumps which can be considered as impossibly huge changes of financial data in short time. See [16] for some results on parabolic Harnack inequality and heat kernel estimate for more general non-local operators of variable order on $\mathbb{R}^n$, whose jumping kernel is supported on jump size less than or equal to 1. The DeGiorgi-Nash-Moser-Aronson type theory is studied very recently in [17] for diffusions with jumps whose infinitesimal generator is of type (1.3) with uniformly elliptic and bounded diffusion matrix $(a_{ij}(x))_{1 \leqslant i,j \leqslant n}$ and non-degenerate measurable jumping kernel J.

Quite often we need to consider part process X^D of X killed upon exit an open set $D \subset \mathbb{R}^n$. When X is a Brownian motion, the infinitesimal generator of X^D is the Dirichlet Laplacian $\frac{1}{2}\Delta_D$. When X is a rotationally symmetric α-stable process in $\mathbb{R}^n$, the infinitesimal generator of X^D is a Dirichlet fractional Laplacian $c\,\Delta^{\alpha/2}|_D$ that satisfies zero *exterior* condition on D^c. Though the transition density function of Brownian motion has been known for quite a long time, due to the complication near the boundary, a complete sharp two-sided estimates on the transition density of killed Brownian motion in bounded $C^{1,1}$ domains D (equivalently, the

Dirichlet heat kernel) have been established only recently in 2002, see [18] and the references therein. Very recently in [19], we have obtained sharp two-sided heat kernel estimates for Dirichlet fractional Laplacian operator in $C^{1,1}$ open sets, while in [20, 21], we derived sharp two-sided estimates for transition density functions of censored stable processes and of relativistic α-stable processes in $C^{1,1}$ open sets, respectively.

The rest of the paper is organized as follows. Heat kernel estimates, PHI and a priori Hölder estimates for stable-like processes and mixed stable-like processes on n-sets in $\mathbb{R}^n$ are discussed in Sections 2 and 3, respectively. In Section 4, we deal with finite range stable-like processes on $\mathbb{R}^n$, while results for diffusions with jumps are surveyed in Section 5. Sections 6 and 7 are devoted to sharp heat kernel estimates for symmetric stable processes and censored stable processes in $C^{1,1}$-open sets. To give a glimpse of our approach to the DeGiorgi-Nash-Moser-Aronson type theory for non-local operators using probabilistic means, we give an outline of the main ideas in our investigation for the following three classes of processes: symmetric stable-like processes on open n-sets in $\mathbb{R}^n$ in Section 2, diffusions with jumps on $\mathbb{R}^n$ in Section 5 and symmetric stable processes in open subsets of $\mathbb{R}^n$ in Section 6. This paper surveys some recent research that the author is involved. See [22] for a survey for related topics on SDEs with jumps, Harnack inequalities and Hölder continuity of harmonic functions for non-local operators, and [23] for a survey (prior to 2000) on potential theory of symmetric stable processes in open sets.

Throughout this paper, $n \geqslant 1$ is an integer. We denote by m or dx the n-dimensional Lebesgue measure in $\mathbb{R}^n$, and $C_c^1(\mathbb{R}^n)$ the space of C^1-functions on $\mathbb{R}^n$ with compact support. For a closed subset F of $\mathbb{R}^n$, $C_c(F)$ denotes the space of continuous functions with compact support in F. For a Markov process X on a state space E and a subset $K \subset E$, we let $\sigma_K := \inf\{t \geqslant 0 : X_t \in K\}$ and $\tau_K := \inf\{t \geqslant 0 : X_t \notin K\}$ to denote the first entering and exiting time of K by X.

2 Stable-like processes

A Borel subset F in $\mathbb{R}^n$ with $n \geqslant 1$ is said to be an n-set if there exist constants $r_0 > 0$, $C_2 > C_1 > 0$ so that

$$C_1 r^n \leqslant m(B(x,r)) \leqslant C_2 r^n \qquad \text{for all } \ x \in F, \ 0 < r \leqslant r_0. \tag{2.1}$$

In this section and the next, $B(x,r) := \{y \in F : |x-y| < r\}$ and $|\cdot|$ is the Euclidean metric in $\mathbb{R}^n$. Every uniformly Lipschitz domain in $\mathbb{R}^n$ is an n-set, so is its Euclidean closure. It is easy to check that the classical von Koch snowflake domain in $\mathbb{R}^2$ is an open 2-set. An n-set can have very rough boundary since every n-set with a subset having zero Lebesgue measure removed is still an n-set.

For a closed n-set $F \subset \mathbb{R}^n$ and $0 < \alpha < 2$, define

$$\mathcal{F} = \left\{ u \in L^2(F,m) : \int_{F\times F} \frac{(u(x)-u(y))^2}{|x-y|^{n+\alpha}} m(dx)m(dy) < \infty \right\}, \tag{2.2}$$

$$\mathcal{E}(u,v) = \frac{1}{2}\int_{F\times F} (u(x)-u(y))(v(x)-v(y)) \frac{c(x,y)}{|x-y|^{n+\alpha}} m(dx)m(dy) \tag{2.3}$$

for u, $v \in \mathcal{F}$, where $c(x,y)$ is a symmetric function on $F \times F$ that is bounded between two strictly positive constants $C_4 > C_3 > 0$, that is,

$$C_3 \leqslant c(x,y) \leqslant C_4 \qquad \text{for } m\text{-a.e. } x, y \in F. \tag{2.4}$$

It is easy to check that $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form on $L^2(F,m)$ and therefore there is an associated m-symmetric Hunt process X on F starting from every point in F except for an exceptional set that has zero capacity. We call such kind of process a α-stable-like process on F. Note that when $F=\mathbb{R}^n$ and $c(x,y)$ is a constant function, then X is nothing but a rotationally symmetric α-stable process on $\mathbb{R}^n$.

Theorem 2.1 ([11, Theorem 1.1]). *Suppose that $F\subset\mathbb{R}^n$ is a closed n-set and $0<\alpha<2$. Then X has a Hölder continuous transition density function $p(t,x,y)$ with respect to m. This in particular implies that X can be modified to start from every point in F as a Feller process. Moreover, there are constants $c_2>c_1>0$ that depend only on n, α, and the constants C_k, $k=1,\dots,4$ in* (2.1) *and* (2.4)*, respectively, such that*

$$c_1\min\left\{t^{-n/\alpha},\ \frac{t}{|x-y|^{n+\alpha}}\right\}\leqslant p(t,x,y)\leqslant c_2\min\left\{t^{-n/\alpha},\ \frac{t}{|x-y|^{n+\alpha}}\right\}, \tag{2.5}$$

for all $x,y\in F$ and $0<t\leqslant 1$.

If F is a global n-set in the sense that (2.1) holds for every $0<t\leqslant 0$, then the heat kernel estimates in (2.5) holds for every $t>0$.

Note that in [11, Theorem 1.1], the dependence of c_1,c_2 on $(C_1,\dots,C_4)$ in Theorem 2.1 is stated for every α except for the case of $0<\alpha=n<2$. The reason is that in [11], the n-diagonal estimate (Nash's inequality) for the case of $\alpha=n<2$ was established by using an interpolation method. This restriction can be removed by an alternative way to establish Nash's inequality, see [13, Theorem 3.1].

The detailed heat kernel estimates such as those in (2.5) are very useful in the study of sample path properties of the processes. For example, the following is proved in [11].

Theorem 2.2 ([11, Theorem 1.2]). *Under the assumption of Theorem* 2.1*, for every $x\in F$, $\mathbb{P}_x$-a.s., the Hausdorff dimension of $X[0,1]:=\{X_t:\ 0\leqslant t\leqslant 1\}$ is $\alpha\wedge n$.*

In fact, a much stronger result can be derived from Theorem 2.1. The following uniform Hausdorff dimensional result and boundary trace result are established in Remarks 3.10 and 4.4 of [24], respectively.

Theorem 2.3. *Let D be an open n-set in $\mathbb{R}^n$ with $n\geqslant 2$ and X be an α-stable-like process on $\overline{D}$. Then for every $x\in\overline{D}$,*

$$\mathbb{P}_x(\dim_H X(E)=\alpha\dim_H E\ \ \textit{for all Borel sets } E\subset\mathbb{R}_+)=1$$

and $\mathbb{P}_x$-a.s.

$$\dim_H(X[0,\infty)\cap\partial D)=\max\left\{1-\frac{n-\dim_H\partial D}{\alpha},\ 0\right\}.$$

Here for a time set $E\subset\mathbb{R}_+$, $X(E):=\{X_t:t\in E\}$ and $\dim_H(A)$ is the Hausdorff dimension of a set A.

The approach to Theorem 2.1 in [11] is probabilistic in nature and is motivated by the work of Bass and Levin[10, 25] on stable-like processes on $\mathbb{Z}^n$ and on $\mathbb{R}^n$. However there are new challenges for stable-like processes on n-sets, as [25] deals with (possibly non-symmetric) semimartingale stable-like processes on $\mathbb{R}^n$, when restricted to the symmetric processes case, requiring $c(x,y) = f(x, y-x)$ and $f(x,h)$ be an even function in h, while [10] is concerned about the transition density function estimates for discrete time stable-like Markov chains on $\mathbb{Z}^n$.

By Nash's inequality and [16, Theorems 3.1 and 3.2], there is a properly exceptional set $\mathcal{N} \subset F$ and a positive symmetric function $p(t,x,y)$ defined on $(0,\infty)\times(F\setminus\mathcal{N})\times(F\times\mathcal{N})$ so that $p(t,x,y)$ is the density function for X_t under $\mathbb{P}_x$ for every $x \in F\setminus\mathcal{N}$,

$$p(t+s,x,y) = \int_F p(s,x,z)p(t,z,y)m(dz) \quad \text{for every } x,y \in F\setminus\mathcal{N} \text{ and } t>0,$$

and

$$p(t,x,y) \leqslant ct^{-n/\alpha} \quad \text{for every } t>0 \text{ and } x,y\in F\setminus\mathcal{N}.$$

Moreover, there is an $\mathcal{E}$-nest $\{F_k, k\geqslant 1\}$ of compact sets so that $\mathcal{N} = E\setminus\bigcup_{k\geqslant 1}F_k$ and that for every $t>0$ and $y\in F\setminus\mathcal{N}$, $x\mapsto p(t,x,y)$ is continuous on each F_k. The proof of Theorem 2.1 given in [11] relies on the following three key propositions. The first proposition is a tightness result for X.

Proposition 2.4 ([11, Proposition 4.1]). *For each $r_0>0$, $A>0$ and $0<B<1$, there exists $0<\gamma<1$ such that for every $0<r\leqslant r_0$,*

$$\mathbb{P}_x(\tau_{B(x,\,Ar)} < \gamma\, r^\alpha) \leqslant B \qquad \textit{for every } x\in F\setminus\mathcal{N}.$$

Moreover, the constant γ can be chosen to depend only on (r_0,A,B,n,α) and the constants (C_1,C_2,C_3,C_4) in (2.1) *and* (2.4) *respectively.*

Proposition 2.5 ([11, Proposition 4.2]). (i) *For each $a>0$, there exists $c_1>0$ such that for every $x\in F\setminus\mathcal{N}$,*

$$\mathbb{P}_x(\sigma_{B(y,\,ar)} < r^\alpha) \leqslant c_1\left(\frac{r}{|x-y|}\right)^{d+\alpha} \qquad \textit{for every } r\in(0,\,2^{1/\alpha}]. \tag{2.6}$$

Moreover, the constant c_1 above can be chosen to depend only on (a,n,α) and on the constants (C_1,C_2,C_3,C_4) in (2.1) *and* (2.4)*, respectively.*

(ii) *For each $a,b>0$, there exists $c_2>0$ such that*

$$\mathbb{P}_x(\sigma_{B(y,\,ar)} < r^\alpha) \geqslant c_2\left(\frac{r}{|x-y|}\right)^{d+\alpha}, \tag{2.7}$$

for every $r\in(0,\,2^{1/\alpha}]$ and such that $|x-y|\geqslant b\,r$. Moreover, the constant c_2 above can be chosen to depend only on (a,b,n,α) and on the constants (C_1,C_2,C_3,C_4) in (2.1) *and* (2.4)*, respectively.*

The last key proposition is a parabolic Harnack inequality. For this we need to introduce space-time process $Z_s := (V_s, X_s)$, where $V_s = V_0 + s$. The filtration generated by Z satisfying the usual condition will be denoted as $\{\widetilde{\mathcal{F}}_s;\ s\geqslant 0\}$. The law of the space-time process $s\mapsto Z_s$ starting from (t,x) will be denoted as $\mathbb{P}^{(t,x)}$. We say that a non-negative Borel measurable function $q(t,x)$ on $[0,\infty)\times F$ is *parabolic* in a relatively open subset D of $(0,\infty)\times F$ if for every relatively compact open subset D_1 of D, $q(t,x) = \mathbb{E}^{(t,x)}[q(Z_{\tau_{D_1}})]$ for every $(t,x)\in D_1\cap$

$(0,\infty)\times(F\setminus\mathcal{N})$, where $\tau_{D_1}=\inf\{s>0:\ Z_s\notin D_1\}$. It is easy to see that for each $t_0>0$ and $x_0\in F\setminus\mathcal{N}$, $q(t,x):=p(t_0-t,x,x_0)$ is parabolic on $[0,t_0)\times F$.

For each $R_0>0$, we denote $\gamma_{R_0}:=\gamma(R_0,1/2,1/2)<1$ the constant in Proposition 2.4 corresponding to $r_0=R_0$ and $A=B=1/2$. For $t\leqslant 1$ and $r\leqslant R_0$, we define

$$Q_{R_0}(t,x,r):=[t,t+\gamma_{R_0}r^{\alpha}]\times(B(x,r)\cap F\setminus\mathcal{N}).$$

Proposition 2.6 ([11, Proposition 4.3]). *For every $R_0>0$, $0<\delta\leqslant\gamma_{R_0}$, there exists $c>0$ such that for every $z\in F$, $0<R\leqslant R_0$ and every non-negative function q on $[0,\infty)\times F$ that is parabolic and bounded on $[0,3\gamma_{R_0}R^{\alpha}]\times B(z,R)$,*

$$\sup_{(t,y)\in Q_{R_0}(\delta R^{\alpha},z,R/3)} q(t,y)\leqslant c\inf_{y\in B(z,R/3)} q(0,y).$$

Moreover, the constant c above can be chosen to depend only on (R_0,δ,n,α) and on the constants (C_1,C_2,C_3,C_4) in (2.1) and (2.4) respectively.

Note that the parabolic Harnack inequality implies the elliptic Harnack inequality.

With the above three propositions, the heat kernel estimates for $p(t,x,y)$ in Theorem 2.1 can be established for every $0<t\leqslant 1$ and $x,y\in F\setminus\mathcal{N}$. That $p(t,x,y)$ is jointly continuous and hence the heat kernel estimates hold for every $0<t\leqslant 0$ and $x,y\in F$ comes from the following theorem.

Theorem 2.7 ([11, Theorem 4.14]). *For every $R_0>0$, there is a constant $c=c(R_0)>0$ such that for every $0<R\leqslant R_0$ and every bounded parabolic function q in $Q_{R_0}(0,x_0,\max\{4,\,4^{1/\alpha}\}R)$,*

$$|q(s,x)-q(t,y)|\leqslant c\,\|q\|_{\infty,R}\,R^{-\beta}\,(|t-s|^{1/\alpha}+|x-y|)^{\beta}$$

holds for (s,x), $(t,y)\in Q_{R_0}(0,x_0,R)$, where $\|q\|_{\infty,R}:=\sup_{(t,y)\in[0,\,\gamma_{R_0}\max\{4,\,4^{\alpha}\}R^{\alpha}]\times(F\setminus\mathcal{N})}|q(t,y)|$. In particular, for the transition density function $p(t,x,y)$ of X, there are constants $c>0$ and $\beta>0$ such that for any $0<t_0<1$, $t,\,s\in[t_0,\,2]$ and $(x_i,y_i)\in(F\setminus\mathcal{N})\times(F\setminus\mathcal{N})$ with $i=1,2$,

$$|p(s,x_1,y_1)-p(t,x_2,y_2)|\leqslant c\,t_0^{-(d+\beta)/\alpha}(|t-s|^{1/\alpha}+|x_1-x_2|+|y_1-y_2|)^{\beta}.$$

Moreover, the constant c above can be chosen to depend only on (R_0,t_0,n,α) and on the constants (C_1,C_2,C_3,C_4) in (2.1) and (2.4).

3 Mixed stable-like processes

In applications, the stochastic model may have more than one type of noises. So it is natural to consider mixed stable-like processes and a mixture of diffusion and jump-type processes.

Let F be a closed global n-set in $\mathbb{R}^n$. Let $\phi=\phi_1\psi$ be a strictly increasing continuous functions on $\mathbb{R}_+$, where ψ is non-decreasing function on $[0,\infty)$ with $\psi(r)=1$ for $0<r\leqslant 1$ that is either the constant function 1 on $\mathbb{R}_+$ or there are constants $c_0>0$, $c_2\geqslant c_1>0$ and $\gamma_2\geqslant\gamma_1>0$ so that

$$c_1e^{\gamma_1 r}\leqslant\psi(r)\leqslant c_2e^{\gamma_2 r}\qquad\text{for every }\ 1<r<\infty,\tag{3.1}$$

with

$$\psi(r+1)\leqslant c_0\psi(r)\qquad\text{for every } r\geqslant 1,\tag{3.2}$$

and ϕ_1 is a strictly increasing function on $[0,\infty)$ with $\phi_1(0)=0$, $\phi_1(1)=1$ and satisfies the following: there exist constants $c_2>c_1>0$, $c_3>0$, and $\beta_2\geqslant\beta_1>0$ such that

$$c_1\left(\frac{R}{r}\right)^{\beta_1}\leqslant\frac{\phi_1(R)}{\phi_1(r)}\leqslant c_2\left(\frac{R}{r}\right)^{\beta_2}\qquad\text{for every } 0<r<R<\infty, \tag{3.3}$$

$$\int_0^r\frac{s}{\phi_1(s)}\,ds\leqslant c_3\,\frac{r^2}{\phi_1(r)}\qquad\text{for every } r>0. \tag{3.4}$$

Remark 3.1. Note that condition (3.3) is equivalent to the existence of constants $c_4,c_5>1$ and $L_0>1$ such that for every $r>0$,

$$c_4\phi_1(r)\leqslant\phi_1(L_0r)\leqslant c_5\,\phi_1(r).$$

Denote by d the diagonal of $F\times F$ and J be a symmetric measurable function on $F\times F\setminus d$ such that for every $(x,y)\in F\times F\setminus d$,

$$\frac{c_1}{|x-y|^n\,\phi(c_2|x-y|)}\leqslant J(x,y)\leqslant\frac{c_3}{|x-y|^n\,\phi(c_4|x-y|)}. \tag{3.5}$$

For $u\in L^2(F,m)$, define $\mathcal{F}:=\{u\in L^2(F;m):\ \int_{F\times F}(u(x)-u(y))^2J(x,y)m(dx)m(dy)<\infty\}$ and

$$\mathcal{E}(u,u):=\int_{F\times F}(u(x)-u(y))(v(x)-v(y))J(x,y)m(dx)m(dy)\quad\text{for } u,v\in\mathcal{F}. \tag{3.6}$$

For $\beta>0$,

$$\mathcal{E}_\beta(u,u):=\mathcal{E}(u,u)+\beta\int_F u(x)^2m(dx).$$

It is not difficult to show that $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form on $L^2(F,m)$ (see [12, Proposition 2.2 and Remark 4.10 (ii)]. So there is a symmetric Hunt process Y associated with it, starting from quasi-every point in F. However the next theorem, which is a special case of [12, Theorem 1.2] (cf. [13, Remark 4.4 (iv)]), says that X can be refined to start from every point in F. Moreover, it has a jointly continuous transition density function $p(t,x,y)$ with respect to the Lebesgue measure on F. The inverse function of the strictly increasing function $t\mapsto\phi(t)$ is denoted by $\phi^{-1}(t)$.

Theorem 3.2 ([12, Theorem 1.2]). *Under the above conditions, there is a conservative Feller process Y associated with $(\mathcal{E},\mathcal{F})$ that starts from every point in F. Moreover the process Y has a continuous transition density function on $(0,\infty)\times F\times F$ with respect to the measure m, which has the following estimates. There are positive constants $c_1>0$, $c_2>0$ and $C\geqslant1$ such that*

$$C^{-1}\left(\frac{1}{\phi^{-1}(t)^n}\wedge\frac{t}{|x-y|^n\,\phi(c_1|x-y|)}\right)\leqslant p(t,x,y)\leqslant C\left(\frac{1}{\phi^{-1}(t)^n}\wedge\frac{t}{|x-y|^n\phi(c_2|x-y|)}\right)$$

for every $t\in(0,1]$ and $x,y\in F$. Moreover, when $\psi\equiv1$, the above heat kernel estimates hold for every $t>0$ and $x,y\in F$.

We now give some examples such that Theorem 3.2 applies.

Example 3.3. If there is $0<\alpha_1<\alpha_2<2$ and a probability measure ν on $[\alpha_1,\alpha_2]$ such that

$$\phi(r):=\left(\int_{\alpha_1}^{\alpha_2}r^{-\alpha}\,\nu(d\alpha)\right)^{-1},$$

then conditions (3.3)–(3.4) are satisfied with $\psi \equiv 1$. Clearly, ϕ is a continuous strictly increasing function with $\phi(0) = 0$ and $\phi(1) = 1$. The condition (3.3) is satisfied with $\gamma \equiv 1$ because

$$\frac{1}{2^{\alpha_1}} \leqslant \frac{\phi(r)}{\phi(2r)} \leqslant \frac{1}{2^{\alpha_2}} \qquad \text{for any } r > 0.$$

For $r > 0$, by Fubini's theorem,

$$\int_0^r \frac{s}{\phi(s)} ds = \int_0^r \int_{\alpha_1}^{\alpha_2} r^{1-\alpha} \nu(d\alpha) ds = \int_{\alpha_1}^{\alpha_2} \frac{1}{2-\alpha} r^{2-\alpha} \nu(d\alpha) \leqslant \frac{1}{2-\alpha_2} \frac{r^2}{\phi(r)},$$

and so condition (3.4) is satisfied. In this case,

$$J(x,y) \asymp \int_{\alpha_1}^{\alpha_2} \frac{1}{|x-y|^{n+\alpha}} \, \nu(d\alpha).$$

A particular case is when ν is a discrete measure. For example, ν is a discrete measure concentrate on $\alpha, \beta \in (0, 2)$. In this case, $J(x,y) = \frac{c_1(x,y)}{|x-y|^{n+\alpha}} + \frac{c_2(x,y)}{|x-y|^{n+\beta}}$, where $c_i(x,y)$ are two symmetric functions that are bounded between two positive constants, and

$$\phi(r) \asymp \min\{r^\alpha, \, r^\beta\}, \qquad \phi^{-1}(r) \asymp \max\{r^{1/\alpha}, \, r^{1/\beta}\}.$$

Theorem 3.2 gives the precise heat kernel estimates for mixed stable-like processes on F. When $F = \mathbb{R}^n$, Theorem 3.2 in particular gives the heat kernel estimate for Lévy processes on $\mathbb{R}^n$ which are linear combinations of independent symmetric α-stable processes. Of course, Theorem 3.2 holds much more generally, even in the case of $F = \mathbb{R}^n$.

Example 3.4. Let $Y = \{Y_t, t \geqslant 0\}$ be the relativistic α-stable processes on $\mathbb{R}^n$ with mass $m_0 > 0$. That is, $\{Y_t, t \geqslant 0\}$ is a Lévy process on $\mathbb{R}^n$ with

$$\mathbb{E}[\exp(i\langle \xi, Y_t - Y_0 \rangle)] = \exp(t(m_0^\alpha - (|\xi|^2 + m_0^2)^{\alpha/2})).$$

where $\alpha \in (0, 2)$. It is shown in [26] that the corresponding jumping intensity satisfies

$$J(x,y) \asymp \frac{\Psi(m_0|x-y|)}{|x-y|^{n+\alpha}},$$

where $\Psi(r) \asymp e^{-r}(1 + r^{(n+\alpha-1)/2})$ near $r = \infty$, and $\Psi(r) = 1 + \Psi''(0)r^2/2 + o(r^4)$ near $r = 0$. So the conditions (3.1)–(3.4) are satisfied with $\gamma_1 > 0$ for the jumping intensity kernel for every relativistic α-stable processes on $\mathbb{R}^n$.

When $\alpha = 1$, the process is called a relativistic Hamiltonian process. In this case, the heat kernel can be written as

$$p(t,x,y) = \frac{t}{(2\pi)^n \sqrt{|x-y|^2 + t^2}} \int_{\mathbb{R}^n} e^{m_0 t} e^{-\sqrt{(|x-y|^2+t^2)(|z|^2+m_0^2)}} dz,$$

see [27]. For simplicity, take $m_0 = 1$. It can be shown that for every $t > 0$ and $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$,

$$\begin{aligned} &\frac{c_1 t}{(|x-y|+t)^{n+1}} (1 \vee (|x-y|+t)^{d/2}) e^{-c_2 \frac{|x-y|^2}{\sqrt{|x-y|^2+t^2}}} \\ &\qquad \leqslant p(t,x,y) \leqslant \frac{c_3 t}{(|x-y|+t)^{n+1}} (1 \vee (|x-y|+t)^{d/2}) e^{-c_4 \frac{|x-y|^2}{\sqrt{|x-y|^2+t^2}}}. \end{aligned}$$

This in particular implies that for every fixed $t_0 > 0$, there exist $c_1, \ldots, c_4 > 0$ which depend on t_0 such that

$$c_1\left(t^{-n} \wedge \frac{t}{|x-y|^{n+1}}\right)e^{-c_2|x-y|} \leqslant p(t,x,y) \leqslant c_3\left(t^{-n} \wedge \frac{t}{|x-y|^{n+1}}\right)e^{-c_4|x-y|}$$

for every $t \in (0, t_0]$ and $x, y \in \mathbb{R}^n$, which is a special case of Theorem 3.2.

The following construction of Meyer[28] for jump processes played an important role in our approach in [12]. Suppose we have two jump intensity kernels $J(x,y)$ and $J_0(x,y)$ on $F \times F$ such that their corresponding pure jump Dirichlet forms given in terms of (3.6) with $\mathcal{F} = \overline{\mathcal{D}(\mathcal{E})}^{\mathcal{E}_1}$ are regular on F. Let $Y = \{Y_t, t \geqslant 0, \mathbb{P}_x, x \in F\backslash\mathcal{N}\}$ and $Y^{(0)} = \{Y_t^{(0)}, t \geqslant 0, \mathbb{P}_x, x \in F\backslash\mathcal{N}_0\}$ be the processes corresponding to the Dirichlet forms whose Lévy densities are $J(x,y)$ and $J_0(x,y)$, respectively. Here $\mathcal{N}$ and $\mathcal{N}_0$ are the properly exceptional sets of Y and $Y^{(0)}$, respectively. Suppose that $J_0(x,y) \leqslant J(x,y)$ and

$$\mathcal{J}(x) := \int_F (J(x,y) - J_0(x,y))m(dy) \leqslant c,$$

for all $x \in F$. Let

$$J_1(x,y) := J(x,y) - J_0(x,y) \qquad \text{and} \qquad q(x,y) = \frac{J_1(x,y)}{\mathcal{J}(x)}. \tag{3.7}$$

Then we can construct a process Y corresponding to the jump kernel J from $Y^{(0)}$ as follows. Let S_1 be an exponential random variable of parameter 1 independent of $Y^{(0)}$, let $C_t = \int_0^t \mathcal{J}(Y_s^{(0)})\, ds$, and let U_1 be the first time that C_t exceeds S_1. We let $Y_s = Y_s^{(0)}$ for $0 \leqslant s < U_1$.

At time U_1 we introduce a jump from Y_{U_1-} to Z_1, where Z_1 is chosen at random according to the distribution $q(Y_{U_1-}, y)$. We set $Y_{U_1} = Z_1$, and repeat, using an independent exponential S_2, etc. Since $\mathcal{J}(x)$ is bounded, only finitely many new jumps are introduced in any bounded time interval. In [28] it is proved that the resulting process corresponds to the kernel J. See also [29]. Note that if $\mathcal{N}_0$ is the properly exceptional set corresponding to $Y^{(0)}$, then this construction gives that the properly exceptional set $\mathcal{N}$ for Y can be chosen to be a subset of $\mathcal{N}_0$.

Conversely, we can also remove a finite number of jumps from a process Y to obtain a new process $Y^{(0)}$. For simplicity, assume that $J_0(x,y)J_1(x,y) = 0$. Suppose one starts with the process Y (associated with J), runs it until the stopping time $S_1 = \inf\{t : J_1(Y_{t-}, Y_t) > 0\}$, and at that time restarts Y at the point Y_{S_1-}. Suppose one then repeats this procedure over and over. Meyer[28] proves that the resulting process $Y^{(0)}$ will correspond to the jump kernel J_0. In this case $\mathcal{N}_0 \subset \mathcal{N}$.

Assume that the processes Y and $Y^{(0)}$ have transition density functions $p(t,x,y)$ and $p^{(0)}(t,x,y)$, respectively. Let $\{\mathcal{F}_t\}_{t\geqslant 0}$ be the filtration generated by the process $Y^{(0)}$. The following lemma is shown in [16, Lemma 2.4] and in [30, Lemma 3.2].

Lemma 3.5. (i) *For any $A \in \mathcal{F}_t$,*

$$\mathbb{P}_x(\{Y_s = Y_s^{(0)} \text{ for all } 0 \leqslant s \leqslant t\} \cap A) \geqslant e^{-t\|\mathcal{J}\|_\infty}\, \mathbb{P}_x(A).$$

(ii) *If $\|J_1\|_\infty < \infty$, then*

$$p(t,x,y) \leqslant p^{(0)}(t,x,y) + t\|J_1\|_\infty.$$

The use of Lemma 3.5 can also significantly simplify the proofs in [11] for results in the last section. The relation between Y and Y_0 can be viewed as the probabilistic counterpart of the Trotter's semigroup perturbation method. For example, the proof for Propositions 2.5 and 2.6 can be simplified by using Lemma 3.5. See the proof of Propositions 4.9 and 4.11 of [12] in this regard.

Comparing with Theorem 2.1, Theorem 3.2 says that the rate function for stable processes of mixed type associated with (3.5)–(3.6) is ϕ. Parabolic Harnack inequality and a prior Hölder estimate also hold for parabolic functions of X, with this rate function ϕ. For each $r, t > 0$, we define

$$Q(t, x, r) := [t, t + \gamma\phi(r)] \times (B(x, r) \cap F).$$

Theorem 3.6 ([12, Theorem 4.12]). *For every $0 < \delta \leqslant \gamma$, there exists $c_1 > 0$ such that for every $z \in F$, $R \in (0, 1]$ (resp. $R > 0$ when $\gamma_1 = \gamma_2 = 0$) and every non-negative function h on $[0, \infty) \times F$ that is parabolic and bounded on $[0, \gamma\phi(2R)] \times B(z, 2R)$,*

$$\sup_{(t,y)\in Q(\delta\phi(R),z,R)} h(t, y) \leqslant c_1 \inf_{y\in B(z,R)} h(0, y).$$

In particular, the following holds for $t \leqslant 1$ (resp. $t > 0$ when $\gamma_1 = \gamma_2 = 0$).

$$\sup_{(s,y)\in Q((1-\gamma)t,z,\phi^{-1}(t))} p(s, x, y) \leqslant c \inf_{y\in B(z,\phi^{-1}(t))} p((1 + \gamma)t, x, y). \tag{3.8}$$

Proposition 3.7 ([12, Proposition 4.14]). *For every $R_0 \in (0, 1]$ (resp. $R_0 > 0$ when $\gamma_1 = \gamma_2 = 0$), there are constants $c = c(R_0) > 0$ and $\kappa > 0$ such that for every $0 < R \leqslant R_0$ and every bounded parabolic function h in $Q(0, x_0, 2R)$,*

$$|h(s, x) - h(t, y)| \leqslant c\, \|h\|_{\infty,F}\, R^{-\kappa} \left(\phi^{-1}(|t - s|) + \rho(x, y)\right)^{\kappa}$$

holds for (s, x), $(t, y) \in Q(0, x_0, R)$, where $\|h\|_{\infty,F} := \sup_{(t,y)\in[0,\, \gamma\phi(2R)]\times F} |h(t, y)|$. In particular, for the transition density function $p(t, x, y)$ of X, for any $t_0 \in (0, 1)$ (resp. any $T > 0$ and any $t_0 \in (0, T)$ when $\gamma_1 = \gamma_2 = 0$), there are constants $c = c(t_0) > 0$ and $\kappa > 0$ such that for any t, $s \in [t_0, 1]$ (resp. $t, s \in [t_0, T]$) and $(x_i, y_i) \in F \times F$ with $i = 1, 2$,

$$|p(s, x_1, y_1) - p(t, x_2, y_2)| \leqslant c \frac{1}{\phi^{-1}(t_0)^n\, \phi^{-1}(t_0)^{\kappa}} (\phi^{-1}(|t - s|) + \rho(x_1, x_2) + \rho(y_1, y_2))^{\kappa}.$$

4 Finite range stable-like processes

A finite range α-stable-like process X on $\mathbb{R}^n$ is a symmetric Hunt process on $\mathbb{R}^n$ of purely discontinuous type whose jumping kernel is $J(x, y) = \frac{c(x,y)}{|x-y|^{n+\alpha}} \mathbb{1}_{\{|x-y|\leqslant\kappa\}}$, where $\alpha \in (0, 2)$, $\kappa > 0$ and $c(x, y)$ is a symmetric function on $\mathbb{R}^n \times \mathbb{R}^n$ that is bounded between two positive constants. The Dirichlet form $(\mathcal{E}, \mathcal{F})$ associated with X on $L^2(\mathbb{R}^n, m)$ is given by

$$\begin{aligned}\mathcal{F} &= \left\{u \in L^2(\mathbb{R}^n; m) : \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(u(x) - u(y))^2}{|x - y|^{n+\alpha}} \mathbb{1}_{\{|x-y|\leqslant\kappa\}} m(dx)m(dy) < \infty\right\} \\ &= \left\{u \in L^2(\mathbb{R}^n; m) : \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(u(x) - u(y))^2}{|x - y|^{n+\alpha}} m(dx)m(dy) < \infty\right\},\end{aligned} \tag{4.1}$$

$$\mathcal{E}(u,v)=\frac{1}{2}\int_{F\times F}(u(x)-u(y))(v(x)-v(y))\frac{c(x,y)}{|x-y|^{n+\alpha}}\mathbb{1}_{\{|x-y|\leqslant\kappa\}}m(dx)m(dy) \tag{4.2}$$

for u, $v\in\mathcal{F}$. The L^2-infinitesimal generator of X and $(\mathcal{E},\mathcal{F})$ is a non-local (integro-differential) operators $\mathcal{L}$ on $\mathbb{R}^n$ with measurable symmetric kernel $J(x,y)=\frac{c(x,y)}{|x-y|^{n+\alpha}}\mathbb{1}_{\{|x-y|\leqslant\kappa\}}$:

$$\mathcal{L}u(x)=\lim_{\varepsilon\downarrow 0}\int_{\{y\in\mathbb{R}^n:\,|y-x|>\varepsilon\}}(u(y)-u(x))J(x,y)dy.$$

Theorem 4.1 ([13, Proposition 2.1 and Theorems 2.3 and 3.6]). *The finite range stable-like process X has a jointly continuous transition density function $p(t,x,y)$ and so X can be refined to start from every point on $\mathbb{R}^n$. Moreover the following sharp two-sided heat kernel estimates hold.*

(i) *There is $R_*\in(0,1)$ so that for every $t\in(0,R_*^\alpha]$ and $x,y\in\mathbb{R}^n$ with $|x-y|\leqslant R_*$*

$$p(t,x,y)\asymp\left(t^{-n/\alpha}\wedge\frac{t}{|x-y|^{n+\alpha}}\right).$$

(ii) *There exists $C_*\in(0,1)$ such that for $x,y\in\mathbb{R}^n$ with $|x-y|\geqslant\max\{t/C_*,R_*\}$,*

$$p(t,x,y)\asymp\left(\frac{t}{|x-y|}\right)^{c|x-y|}=\exp\left(-c|x-y|\log\frac{|x-y|}{t}\right).$$

(iii) *For $t\geqslant R_*^\alpha$ or $x,y\in\mathbb{R}^n$ with $|x-y|\in[R_*,t/C_*]$,*

$$p(t,x,y)\asymp t^{-n/2}\exp\left(-\frac{c|x-y|^2}{t}\right).$$

The following weighted Poincaré inequality for non-local operators together with Lemma 3.5 played a crucial role in our proof of Theorem 4.1 in [13]. In the remainder of this paper, $B(x,r)$ denotes the Euclidean ball in $\mathbb{R}^n$ with radius r centered at x.

Theorem 4.2 ([13, Proposition 3.2]). *Suppose that $J(x,y)$ is a symmetric non-negative kernel on $\mathbb{R}^n\times\mathbb{R}^n$ such that $J(x,y)=0$ when $|x-y|\geqslant 1$ and*

$$\kappa_1|x-y|^{-n-\alpha}\leqslant J(x,y)\leqslant\kappa_2|x-y|^{-n-\beta}\qquad \text{when } |x-y|<1$$

for some constants $\kappa_1,\kappa_2>0$ and $0<\alpha<\beta<2$. Let $\phi(x):=c(1-|x|^2)^{12/(2-\beta)}\mathbb{1}_{B(0,1)}(x)$, where $c>0$ is the normalizing constant so that $\int_{\mathbb{R}^n}\phi(x)dx=1$. Then there is a positive constant $c_1=c_1(n,\alpha,\beta)$ independent of $r>1$, such that for every $u\in L^1(B(0,1),\phi dx)$,

$$\int_{B(0,1)}(u(x)-u_\phi)^2\phi(x)dx\leqslant c_1\int_{B(0,1)\times B(0,1)}(u(x)-u(y))^2\,r^{n+2}J(rx,ry)\sqrt{\phi(x)\phi(y)}\,dxdy.$$

Here $u_\phi:=\int_{B(0,1)}u(x)\phi(x)dx$.

5 Diffusions with jumps

In this section, we consider symmetric Markov processes on $\mathbb{R}^n$ that have both the diffusion and pure jumping components. More precisely, consider the following regular Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(\mathbb{R}^n;m)$ given by

$$\begin{cases}\mathcal{E}(u,v)=\dfrac{1}{2}\displaystyle\int_{\mathbb{R}^n}\nabla u(x)\cdot A(x)\nabla v(x)dx+\int_{\mathbb{R}^n}(u(x)-u(y))(v(x)-v(y))J(x,y)dxdy,\\ \mathcal{F}\;=\overline{C_c^1(\mathbb{R}^n)}^{\mathcal{E}_1},\end{cases} \tag{5.1}$$

where $A(x) = (a_{ij}(x))_{1\leqslant i,j\leqslant n}$ is a measurable $n \times n$ matrix-valued function on $\mathbb{R}^n$ that is uniform elliptic and bounded in the sense that there exists a constant $c \geqslant 1$ such that

$$c^{-1}\sum_{i=1}^n \xi_i^2 \leqslant \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \leqslant c\sum_{i=1}^n \xi_i^2 \qquad \text{for every } x,\ (\xi_1,\dots,\xi_d)\in\mathbb{R}^n, \tag{5.2}$$

and J is a symmetric non-negative measurable kernel on $\mathbb{R}^n \times \mathbb{R}^n$ such that there are positive constants $\kappa_0 > 0$, and $\beta \in (0,2)$ so that

$$J(x,y) \leqslant \kappa_0|x-y|^{-n-\beta} \qquad \text{for } |x-y| \leqslant \delta_0, \tag{5.3}$$

and that

$$\sup_{x\in\mathbb{R}^n}\int_{\mathbb{R}^n}(|x-y|^2\wedge 1)J(x,y)\,dy < \infty.. \tag{5.4}$$

Clearly under Condition (5.3), Condition (5.4) is equivalent to

$$\sup_{x\in\mathbb{R}^n}\int_{\{y\in\mathbb{R}^n:|y-x|\geqslant 1\}} J(x,y)\,dy < \infty.$$

By the Dirichlet form theory, there is an $\mathbb{R}^n$-valued symmetric Hunt process X associated with $(\mathcal{E},\mathcal{F})$. The L^2-infinitesimal generator of X ia a non-local (pseudo-differential) operators $\mathcal{L}$ on $\mathbb{R}^n$:

$$\mathcal{L}u(x) = \frac{1}{2}\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u(x)}{\partial x_j}\right) + \lim_{\varepsilon\downarrow 0}\int_{\{y\in\mathbb{R}^n:\,|y-x|>\varepsilon\}}(u(y)-u(x))J(x,y)dy. \tag{5.5}$$

When the jumping kernel $J\equiv 0$ in (5.5) and (5.1), $\mathcal{L}$ is a uniform elliptic operator of divergence form and X is a symmetric diffusion on $\mathbb{R}^n$. It is well-known that X has a joint Hölder continuous transition density function $p(t,x,y)$, which enjoys the celebrated Aronson's two-sided heat kernel estimate (1.1).

When $A(x)\equiv 0$ in (5.1) and J is given by

$$J(x,y) \asymp \frac{1}{|x-y|^d\,\phi(|x-y|)}, \tag{5.6}$$

where ϕ a strictly increasing continuous function $\phi:\mathbb{R}_+\to\mathbb{R}_+$ with $\phi(0)=0$, and $\phi(1)=1$ that satisfies the conditions (3.3)–(3.4) with ϕ in place of ϕ_1 there, the corresponding process X is a mixed stable-like process on $\mathbb{R}^n$ appeared in the previous section. We know from Theorem 3.2 that there are positive constants $0<c_1<c_2$ so that

$$c_1 p^j(t,|x-y|) \leqslant p(t,x,y) \leqslant c_2 p^j(t,|x-y|) \qquad \text{for } t>0, x,y\in\mathbb{R}^n,$$

where

$$p^j(t,r) := \left(\phi^{-1}(t)^{-n}\wedge\frac{t}{r^n\phi(r)}\right) \tag{5.7}$$

with ϕ^{-1} being the inverse function of ϕ.

In this section, we consider the case where both A and J are non-trivial in (5.5) and (5.1). Clearly such non-local operators and diffusions with jumps take up an important place both in theory and in applications. However, there are very limited work in literature for this mixture case on the topics of this paper until very recently. One of the difficulties in obtaining fine properties for such an operator $\mathcal{L}$ and process X is that they exhibit different scales: the diffusion part has Brownian scaling $r \mapsto r^2$ while the pure jump part has a different type of scaling. Nevertheless, there is a folklore which says that with the presence of the diffusion part corresponding to $\frac{1}{2}\sum_{i,j=1}^n \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j})$, better results can be expected under weaker assumptions on the jumping kernel J as the diffusion part helps to smooth things out. Our investigation in [17] confirms such an intuition. In fact we can establish a priori Hölder estimate and parabolic Harnack inequality under weaker conditions than (5.6). We now present the main results of [17]. Let $W^{1,2}(\mathbb{R}^n)$ denote the Sobolev space of order $(1,2)$ on $\mathbb{R}^n$; that is, $W^{1,2}(\mathbb{R}^n) := \{f \in L^2(\mathbb{R}^n;m) : \nabla f \in L^2(\mathbb{R}^n;m)\}$. It is not difficult (see Proposition 1.1 of [17]) to show that under the conditions (5.2)–(5.4), the domain of the Dirichlet form of (5.1) is characterized by

$$\mathcal{F} = W^{1,2}(\mathbb{R}^n)$$

and that ([17, Theorem 2.2]) the corresponding process X has infinite lifetime. Let $Z = \{Z_t := (V_0 - t, X_t), t \geqslant 0\}$ denote the space-time process of X. We say that a non-negative real valued Borel measurable function $h(t,x)$ on $[0,\infty)\times\mathbb{R}^n$ is *parabolic* (or *caloric*) on $D = (a,b)\times B(x_0,r)$ if there is a properly exceptional set $\mathcal{N} \subset \mathbb{R}^n$ such that for every relatively compact open subset D_1 of D,

$$h(t,x) = \mathbb{E}^{(t,x)}[h(Z_{\tau_{D_1}})]$$

for every $(t,x) \in D_1 \cap ([0,\infty)\times(\mathbb{R}^n \setminus \mathcal{N}))$, where $\tau_{D_1} = \inf\{s > 0 : Z_s \notin D_1\}$. We remark that in Sections 2 and 3 the space-time process is defined to be $(V_0 + t, X_t)$ but this is merely a notational difference. (For reader's convenience, we keep the notations same as those in the references [11, 12, 17].)

Theorem 5.1 ([17, Theorem 1.2]). *Assume that the Dirichlet form $(\mathcal{E},\mathcal{F})$ given by* (5.1) *satisfies the conditions* (5.2)–(5.4) *and that for every $0 < r < \delta_0$,*

$$\inf_{\substack{x_0,y_0\in\mathbb{R}^n \\ |x_0-y_0|=r}} \inf_{x\in B(x_0,r/16)} \int_{B(y_0,r/16)} J(x,z)dz > 0. \tag{5.8}$$

Then for every $R_0 \in (0,1]$, there are constants $c = c(R_0) > 0$ and $\kappa > 0$ such that for every $0 < R \leqslant R_0$ and every bounded parabolic function h in $Q(0,x_0,2R) := (0,4R^2)\times B(x_0,2R)$,

$$|h(s,x) - h(t,y)| \leqslant c\,\|h\|_{\infty,R}\,R^{-\kappa}\,(|t-s|^{1/2} + |x-y|)^\kappa$$

holds for (s,x), $(t,y) \in (3R^2,4R^2)\times B(x_0,R)$, where $\|h\|_{\infty,R} := \sup_{(t,y)\in[0,4R^2]\times\mathbb{R}^n\setminus\mathcal{N}} |h(t,y)|$. In particular, X has a jointly continuous transition density function $p(t,x,y)$ with respect to the Lebesgue measure. Moreover, for every $t_0 \in (0,1)$ there are constants $c > 0$ and $\kappa > 0$ such that for any t, $s \in (t_0, 1]$ and $(x_i,y_i) \in \mathbb{R}^n\times\mathbb{R}^n$ with $i = 1,2$,

$$|p(s,x_1,y_1) - p(t,x_2,y_2)| \leqslant c\,t_0^{-(n+\kappa)/2}(|t-s|^{1/2} + |x_1-x_2| + |y_1-y_2|)^\kappa.$$

In addition to (5.2)–(5.4) and (5.8), if there is a constant $c > 0$ such that

$$J(x,y) \leqslant \frac{c}{r^n}\int_{B(x,r)} J(z,y)dz \quad \text{whenever} r \leqslant \frac{1}{2}|x-y| \wedge 1,\ x,y\in\mathbb{R}^n, \tag{5.9}$$

then the following parabolic Harnack principle holds for non-negative parabolic functions of X.

Theorem 5.2 ([17, Theorem 1.3]). *Suppose that the Dirichlet form* $(\mathcal{E},\mathcal{F})$ *given by* (5.1) *satisfies the condition* (5.2)–(5.4), (5.8) *and* (5.9). *For every* $\delta \in (0,1)$, *there exist constants* $c_1 = c_1(\delta)$ *and* $c_2 = c_2(\delta) > 0$ *such that for every* $z \in \mathbb{R}^n$, $t_0 \geqslant 0$, $0 < R \leqslant c_1$ *and every non-negative function* u *on* $[0,\infty) \times \mathbb{R}^n$ *that is parabolic on* $(t_0, t_0 + 6\delta R^2) \times B(z, 4R)$,

$$\sup_{(t_1,y_1)\in Q_-} u(t_1,y_1) \leqslant c_2 \inf_{(t_2,y_2)\in Q_+} u(t_2,y_2), \tag{5.10}$$

where $Q_- = (t_0 + \delta R^2, t_0 + 2\delta R^2) \times B(x_0,R)$ *and* $Q_+ = (t_0 + 3\delta R^2, t_0 + 4\delta R^2) \times B(x_0, R)$.

We next present a two-sided heat kernel estimate for X when $J(x,y)$ satisfies the condition (5.6). Clearly (5.3)–(5.4), (5.8) and (5.9) are satisfied when (5.6) holds. Recall that functions $p^c(t,x,y)$ and $p^j(t,x,y)$ are defined by (1.2) and (5.7), respectively.

Theorem 5.3 ([17, Theorem 1.4]). *Suppose that* (5.2) *holds and that the jumping kernel* J *of the Dirichlet form* $(\mathcal{E},\mathcal{F})$ *given by* (5.1) *satisfies the condition* (5.6). *Denote by* $p(t,x,y)$ *the continuous transition density function of the symmetric Hunt process* X *associated with the regular Dirichlet form* $(\mathcal{E},\mathcal{F})$ *of* (5.1) *with the jumping kernel* J *given by* (5.6). *There are positive constants* c_i, $i = 1,2,3,4$ *such that for every* $t > 0$ *and* $x, y \in \mathbb{R}^n$,

$$\begin{aligned} &c_1\, (t^{-n/2} \wedge \phi^{-1}(t)^{-n}) \wedge (p^c(t, c_2|x-y|) + p^j(t,|x-y|)) \\ &\quad \leqslant p(t,x,y) \leqslant c_3\, (t^{-n/2} \wedge \phi^{-1}(t)^{-n}) \wedge (p^c(t, c_4|x-y|) + p^j(t,|x-y|)). \end{aligned} \tag{5.11}$$

Here p^c *and* p^j *are the functions given by* (1.2) *and* (5.7), *respectively.*

When $A(x) \equiv I_{n\times n}$, the $n \times n$ identity matrix, and $J(x,y) = c|x-y|^{-n-\alpha}$ for some $\alpha \in (0,2)$ in (5.1), that is, when X is the independent sum of a Brownian motion W on $\mathbb{R}^n$ and an isotropically symmetric α-stable process Y on $\mathbb{R}^n$, the transition density function $p(t,x,y)$ can be expressed as the convolution of the transition density functions of W and Y, whose two-sided estimates are known. In [31], heat kernel estimates for this Lévy process X are carried out by computing the convolution and the estimates are given in a form that depends on which region the point (t,x,y) falls into. Subsequently, the parabolic Harnack inequality (5.10) for such a Lévy process X is derived in [31] by using the two-sided heat kernel estimate. Clearly such an approach is not applicable in our setting even when $\phi(r) = r^\alpha$, since in our case, the diffusion and jumping part of X are typically not independent. The two-sided estimate in this simple form of (5.11) is a new observation of [17] even in the independent sum of a Brownian motion and an isotropically symmetric α-stable process case considered in [31].

The approach in [17] employs methods from both probability theory and analysis, but it is mainly probabilistic. It uses some ideas previously developed in [11–13, 16, 30]. To get a priori Hölder estimates for parabolic functions of X, we establish the following three key ingredients.

(i) Exit time upper bound estimate:

$$\mathbb{E}_x[\tau_{B(x_0,r)}] \leqslant c_1 r^2 \qquad \text{for } x \in B(x_0,r),$$

where $\tau_{B(x_0,r)} := \inf\{t > 0 : X_t \notin B(x_0,r)\}$ is the first exit time from $B(x_0,r)$ by X.

(ii) Hitting probability estimate:

$$\mathbb{P}_x(X_{\tau_{B(x,r)}} \notin B(x,s)) \leqslant \frac{c_2 r^2}{(s\wedge 1)^2} \qquad \text{for every } r \in (0,1] \text{ and } s \geqslant 2r.$$

(iii) Hitting probability estimate for space-time process $Z_t = (V_0 - t, X_t)$: for every $x \in \mathbb{R}^n$, $r \in (0, 1]$ and any compact subset $A \subset Q(x, r) := (0, r^2) \times B(x, r)$,

$$\mathbb{P}^{(r^2,x)}(\sigma_A < \tau_r) \geqslant c_3 \frac{m_{n+1}(A)}{r^{n+2}},$$

where by slightly abusing the notation, $\sigma_A := \{t > 0 : Z_t \in A\}$ is the first hitting time of A, $\tau_r := \inf\{t > 0 : Z_t \notin Q(x, r)\}$ is the first exit time from $Q(x, r)$ by Z and m_{n+1} is the Lebesgue measure on $\mathbb{R}^{n+1}$.

Here we use the following notations. The probability law of the process X starting from x is denoted as $\mathbb{P}_x$ and the mathematical expectation under it is denoted as $\mathbb{E}_x$, while probability law of the space-time process $Z = (V, X)$ starting from (t, x), i.e. $(V_0, X_0) = (t, x)$, is denoted as $\mathbb{P}^{(t,x)}$ and the mathematical expectation under it is denoted as $\mathbb{E}^{(t,x)}$. To establish parabolic Harnack inequality, we need in addition the following

(iv) Short time near-diagonal heat kernel estimate: for every $t_0 > 0$, there is $c_4 = c_4(t_0) > 0$ such that for every $x_0 \in \mathbb{R}^n$ and $t \in (0, t_0]$,

$$p^{B(x_0,\sqrt{t})}(t, x, y) \geqslant c_4 t^{-n/2} \qquad \text{for } x, y \in B(x_0, \sqrt{t}/2).$$

Here $p^{B(x_0,\sqrt{t})}$ is the transition density function for the part process $X^{B(x_0,\sqrt{t})}$ of X killed upon leaving the ball $B(x_0, \sqrt{t})$.

(v) Let $R \leqslant 1$ and $\delta < 1$. $Q_1 = [t_0 + 2\delta R^2/3,\ t_0 + 5\delta R^2] \times B(x_0, 3R/2)$, $Q_2 = [t_0 + \delta R^2/3,\ t_0 + 11\delta R^2/2] \times B(x_0, 2R)$ and define Q_- and Q_+ as in Theorem 5.2. Let $h : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}_+$ be bounded and supported in $[0, \infty) \times B(x_0, 3R)^c$. Then there exists $c_5 = c_5(\delta) > 0$ such that

$$\mathbb{E}^{(t_1,y_1)}[h(Z_{\tau_{Q_1}})] \leqslant c_5 \mathbb{E}^{(t_2,y_2)}[h(Z_{\tau_{Q_2}})] \qquad \text{for } (t_1, y_1) \in Q_- \text{ and } (t_2, y_2) \in Q_+.$$

The proof of (iv) uses ideas from [16], where a similar inequality is established for finite range pure jump process. However, some difficulties arise due to the presence of the diffusion part.

The upper bound heat kernel estimate in Theorem 5.3 is established by using method of scaling, by Meyer's construction of the process X based on finite range process $X^{(\lambda)}$, where the jumping kernel J is replaced by $J(x, y)\mathbb{1}_{\{|x-y| \leqslant \lambda\}}$, and by Davies' method from [32] to derive an upper bound estimate for the transition density function of $X^{(\lambda)}$ through carefully chosen testing functions. Here we need to select the value of λ in a very careful way that depends on the values of t and $|x - y|$.

To get the lower bound heat kernel estimate in Theorem 5.3, we need a full scale parabolic Harnack principle that extends Theorem 5.2 to all $R > 0$ with the scale function $\widetilde{\phi}(R) := R^2 \wedge \phi(R)$ in place of $R \mapsto R^2$ there. To establish such a full scale parabolic Harnack principle, we show the following

(iii′) Strengthened version of (iii): for every $x \in \mathbb{R}^n$, $r > 0$ and any compact subset $A \subset Q(0, x, r) := [0, \gamma_0 \widetilde{\phi}(r)] \times B(x, r)$,

$$\mathbb{P}^{(\gamma_0 \widetilde{\phi}(r), x)}(\sigma_A < \tau_r) \geqslant c_3 \frac{m_{n+1}(A)}{r^n \widetilde{\phi}(r)}.$$

Here γ_0 denotes the constant $\gamma(1/2, 1/2)$ in Proposition 6.2 of [17].

(vi) For every $\delta \in (0, \gamma_0]$, there is a constant $c_6 = c_6(\gamma)$ so that for every $0 < R \leqslant 1$, $r \in (0, R/4]$ and $(t, x) \in Q(0, z, R/3)$ with $0 < t \leqslant \gamma_0 \widetilde{\phi}(R/3) - \delta \widetilde{\phi}(r)$,

$$\mathbb{P}^{(\gamma_0 \widetilde{\phi}(R/3), z)}(\sigma_{U(t,x,r)} < \tau_{Q(0,z,R)}) \geqslant c_6 \frac{r^n \widetilde{\phi}(r)}{R^n \widetilde{\phi}(R)},$$

where $U(t, x, r) := \{t\} \times B(x, r)$.

With the full scale parabolic Harnack inequality, the lower bound heat kernel estimate can then be derived once the following estimate is obtained.

(vii) Tightness result: there are constants $c_7 \geqslant 2$ and $c_8 > 0$ such that for every $t > 0$ and $x, y \in \mathbb{R}^n$ with $|x - y| \geqslant c_7 \widetilde{\phi}(t)$,

$$\mathbb{P}_x(X_t \in B(y, c_7 \widetilde{\phi}^{-1}(t))) \geqslant c_8 \frac{t(\widetilde{\phi}^{-1}(t))^n}{|x - y|^n \widetilde{\phi}(|x - y|)}.$$

6 Dirichlet heat kernel estimates for symmetric stable processes

Many times one encounters part process X^D of X killed upon exiting a open set D. The infinitesimal generator $\mathcal{L}^D$ of X^D is the infinitesimal generator $\mathcal{L}$ of X satisfying Dirichlet boundary or zero exterior condition. It is a fundamental problem both in analysis and in probability theory to study precise estimate for the transition density function of X^D (or equivalently, the Dirichlet heat kernel of $\mathcal{L}^D$). However due to the complication near the boundary, two-sided estimates on the transition density of killed Brownian motion in bounded $C^{1,1}$ domains D (equivalently, the Dirichlet heat kernel) have been established only recently in 2002, see [18] and the references therein. In this section, we survey the recent result from [19] on sharp two-sided estimates on the transition density function $p_D(t, x, y)$ of part process X^D of a rotationally symmetric α-stable process killed upon leaving a $C^{1,1}$ open set D. The infinitesimal generator of X^D is the fractional Laplacian $c\,\Delta^{\alpha/2}|_D$ satisfying zero exterior condition on D^c.

Recall that an open set D in $\mathbb{R}^n$ (when $n \geqslant 2$) is said to be a $C^{1,1}$ open set if there exist a localization radius $R_0 > 0$ and a constant $\Lambda_0 > 0$ such that for every $z \in \partial D$, there is a $C^{1,1}$-function $\phi = \phi_z : \mathbb{R}^{n-1} \to \mathbb{R}$ satisfying $\phi(0) = \nabla\phi(0) = 0$, $\|\nabla\phi\|_\infty \leqslant \Lambda_0$, $|\nabla\phi(x) - \nabla\phi(z)| \leqslant \Lambda_0|x - z|$, and an orthonormal coordinate system CS_z: $y = (y_1, \ldots, y_{n-1}, y_n) := (\widetilde{y}, y_n)$ with its origin at z such that

$$B(z, R_0) \cap D = \{y \in B(0, R_0) : y_n > \phi(\widetilde{y})\},$$

where the ball $B(0, R_0)$ on the right hand side is in the coordinate system CS_z. The pair (R_0, Λ_0) is called the characteristics of the $C^{1,1}$ open set D. We remark that in some literatures, the $C^{1,1}$ open set defined above is called a uniform $C^{1,1}$ open set as (R_0, Λ_0) is universal for every $z \in \partial D$. For $x \in \mathbb{R}^n$, let $\delta_D(x)$ denote the Euclidean distance between x and D^c. By a $C^{1,1}$ open set in $\mathbb{R}$ we mean an open set which can be written as the union of disjoint intervals so that the minimum of the lengths of all these intervals is positive and the minimum of the distances between these intervals is positive. Note that a $C^{1,1}$ open set can be unbounded and disconnected.

Theorem 6.1 ([19, Theorem 1.1])**.** *Let D be a $C^{1,1}$ open subset of $\mathbb{R}^n$ with $n \geqslant 1$ and $\delta_D(x)$ the Euclidean distance between x and D^c.*

(i) *For every $T>0$, on $(0,T]\times D\times D$,*

$$p_D(t,x,y)\asymp t^{-n/\alpha}\left(1\wedge\frac{t^{1/\alpha}}{|x-y|}\right)^{n+\alpha}\left(1\wedge\frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha/2}\left(1\wedge\frac{\delta_D(y)}{t^{1/\alpha}}\right)^{\alpha/2}.$$

(ii) *Suppose in addition that D is bounded. For every $T>0$, there are positive constants $c_1<c_2$ so that on $[T,\infty)\times D\times D$,*

$$c_1\,e^{-\lambda_1 t}\,\delta_D(x)^{\alpha/2}\,\delta_D(y)^{\alpha/2}\leqslant p_D(t,x,y)\leqslant c_2\,e^{-\lambda_1 t}\,\delta_D(x)^{\alpha/2}\,\delta_D(y)^{\alpha/2},$$

where $\lambda_1>0$ is the smallest eigenvalue of the Dirichlet fractional Laplacian $(-\Delta)^{\alpha/2}|_D$.

By integrating the two-sided heat kernel estimates in Theorem 6.1 with respect to t, one can easily recover the following estimate of the Green function $G_D(x,y)=\int_0^\infty p_D(t,x,y)dt$, initially obtained independently in [33, 34] when $n\geqslant 2$.

Corollary 6.2 ([19, Corollary 1.2]). *Let D be a bounded $C^{1,1}$-open set in $\mathbb{R}^n$ with $n\geqslant 1$. Then on $D\times D$,*

$$G_D(x,y)\asymp\begin{cases}\dfrac{1}{|x-y|^{n-\alpha}}\left(1\wedge\dfrac{\delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2}}{|x-y|^\alpha}\right) & \textit{when } n>\alpha,\\ \log\left(1+\dfrac{\delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2}}{|x-y|^\alpha}\right) & \textit{when } n=1=\alpha,\\ (\delta_D(x)\delta_D(y))^{(\alpha-1)/2}\wedge\dfrac{\delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2}}{|x-y|} & \textit{when } n=1<\alpha.\end{cases}$$

Theorem 6.1 (i) is established in [19] through Theorems 6.3 and 6.4, which give the upper bound and lower bound estimates, respectively. Theorem 6.1 (ii) is an easy consequence of the intrinsic ultracontractivity of the symmetric α-stable process in a bounded $C^{1,1}$ open set. In fact, the upper bound estimates in both Theorem 6.1 and Corollary 6.2 hold for any domain D with (a weak version of) the uniform exterior ball condition in place of the $C^{1,1}$ condition, while the lower bound estimates in both Theorem 6.1 and Corollary 6.2 hold for any domain D with the uniform interior ball condition in place of the $C^{1,1}$ condition.

We say that D is an open set satisfying (a weak version of) the uniform exterior ball condition with radius $r_0>0$ if for every $z\in\partial D$ and $r\in(0,r_0)$, there is a ball B^z of radius r such that $B^z\subset\mathbb{R}^n\setminus\overline{D}$ and $\partial B^z\cap\partial D=\{z\}$.

Theorem 6.3 ([19, Theorem 2.4]). *Let D be an open set in $\mathbb{R}^n$ that satisfies the uniform exterior ball condition with radius $r_0>0$. For every $T>0$, there exists a positive constant $c=c(T,r_0,\alpha)$ such that for $t\in(0,T]$ and $x,y\in D$,*

$$p_D(t,x,y)\leqslant c\,t^{-n/\alpha}\left(1\wedge\frac{t^{1/\alpha}}{|x-y|}\right)^{n+\alpha}\left(1\wedge\frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha/2}\left(1\wedge\frac{\delta_D(y)}{t^{1/\alpha}}\right)^{\alpha/2}. \tag{6.1}$$

An open set D is said to satisfy the uniform interior ball condition with radius $r_0>0$ in the following sense: For every $x\in D$ with $\delta_D(x)<r_0$, there is $z_x\in\partial D$ so that $|x-z_x|=\delta_D(x)$ and $B(x_0,r_0)\subset D$ for $x_0:=z_x+r_0(x-z_x)/|x-z_x|$. It is well-known that any (uniform) $C^{1,1}$ open set D satisfies both the uniform interior ball condition and the uniform exterior ball condition.

Theorem 6.4 ([19, Theorem 3.1]). *Assume that D is an open set in $\mathbb{R}^n$ satisfying the uniform interior ball condition. Then for every $T > 0$ there exists a positive constant $c = c(r_0, \alpha, T)$ such that for all $(t, x, y) \in (0, T] \times D \times D$,*

$$p_D(t,x,y) \geqslant c\, t^{-n/\alpha}\left(1 \wedge \frac{t^{1/\alpha}}{|x-y|}\right)^{n+\alpha}\left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha/2}\left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\right)^{\alpha/2}.$$

There are significant differences between obtaining two-sided Dirichlet heat kernel estimates for the Laplacian and the fractional Laplacian, as the latter is a non-local operator. Our approach in [19] is mainly probabilistic. It uses only the following five ingredients:

(i) the upper bound heat kernel estimate for the rotationally symmetric α-stable process X in $\mathbb{R}^n$ and the stable-scaling property of X;

(ii) the Lévy system of X that describes how the process jumps;

(iii) the mean exit time estimates from annuli and from balls;

(iv) the boundary Harnack inequality of X in annuli (when $n \geqslant 2$) and in intervals (when $n = 1$), and the parabolic Harnack inequality of X;

(v) the intrinsic ultracontractivity of X in bounded open sets.

The upper bound heat kernel estimate of X on $\mathbb{R}^n$ gives an upper bound for $p_D(t, x, y)$, while the Lévy system is the basic tool used throughout our argument as the symmetric stable process moves by "pure jumping". To get the boundary decay rate of $p_D(t, x, y)$, we use the boundary Harnack inequality and the domain monotonicity of the killed stable process X^D in D by comparing it with certain truncated exterior balls (i.e. annulus) as well as interior balls. The mean exit time estimate for an annulus is applied with the help of the boundary Harnack inequality to get the boundary decay rate in the upper bound heat kernel estimates. The two-sided estimates in the ball $B = B(0, 1)$: $\mathbb{E}_x[\tau_B] \asymp \delta_B(x)^{\alpha/2}$ is used to get the two-sided estimate on the first eigenfunction in balls. The latter is then used to get the boundary decay rate for the lower bound estimate in $p_D(t, x, y)$. The parabolic Harnack inequality allows us to get pointwise lower bound on $p_D(t, x, y)$ from the integral of $w \mapsto p_D(t/2, x, w)$ over some suitable region. When X^D is intrinsic ultracontractive, $p_D(t, x, y)$ is comparable to $c_t\phi_D(x)\phi_D(y)$ for some $c_t > 0$ and a good control is known for c_t when t is above a certain large t_0, where ϕ_D is the positive first eigenfunction of $(-\Delta)^{\alpha/2}|_D$, the infinitesimal generator of X^D.

Note that the large time heat kernel estimate in Theorem 6.1(ii) requires D to be bounded. See [35] for recent results on large time sharp heat kernel estimates for symmetric stable processes in certain unbounded $C^{1,1}$ open sets.

The approach developed in [19] is quite general in principle and can be adapted to study heat kernel estimates for other types of jump processes in open subsets and their perturbations, such as censored stable processes to be discussed in next section.

7 Dirichlet heat kernel estimates for censored stable processes

Censored α-stable processes in an open subset of $\mathbb{R}^n$ were introduced and studied by Bogdan, Burdzy and Chen in [36]. Fix an open set D in $\mathbb{R}^n$ with $n \geqslant 1$. Define a bilinear form $\mathcal{E}$ on $C_c^\infty(D)$ by

$$\mathcal{E}(u,v) := \frac{1}{2}\int_D\int_D (u(x)-u(y))(v(x)-v(y))\frac{c}{|x-y|^{n+\alpha}}dxdy, \quad u, v \in C_c^\infty(D), \tag{7.1}$$

where $c > 0$ is a constant. Using Fatou's lemma, it is easy to check that the bilinear form $(\mathcal{E}, C_c^\infty(D))$ is closable in $L^2(D, dx)$. Let $\mathcal{F}$ be the closure of $C_c^\infty(D)$ under the Hilbert inner product $\mathcal{E}_1 := \mathcal{E} + (\,\cdot\,,\,\cdot\,)_{L^2(D,dx)}$. As is noted in [36], $(\mathcal{E}, \mathcal{F})$ is Markovian and hence a regular symmetric Dirichlet form on $L^2(D, dx)$, and therefore there is an associated symmetric Hunt process $X = \{X_t, t \geqslant 0, \mathbb{P}_x, x \in D\}$ taking values in D. The process X is called a censored (or resurrected) α-stable process in D.

Let Y be a rotationally symmetric α-stable process in $\mathbb{R}^n$ with jumping kernel $c|x-y|^{-n-\alpha}$. For any open subset D of $\mathbb{R}^n$, we use Y^D to denote the subprocess of Y killed upon exiting from D. The following result gives two other ways of constructing a censored α-stable process.

Theorem 7.1 ([36, Theorem 2.1 and Remark 2.4])**.** *The following processes have the same distribution*:

(i) *the symmetric Hunt process X associated with the regular symmetric Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(D, dx)$*;

(ii) *the strong Markov process X obtained from the killed symmetric α-stable-like process Y^D in D through the Ikeda-Nagasawa-Watanabe piecing together procedure*;

(iii) *the process X obtained from Y^D through the Feynman-Kac transform $e^{\int_0^t \kappa_D(Y_s^D)ds}$ with*

$$\kappa_D(x) := \int_{D^c} \frac{c}{|x-y|^{n+\alpha}} dy.$$

The Ikeda-Nagasawa-Watanabe piecing together procedure mentioned in (ii) goes as follows. Let $X_t(\omega) = Y_t^D(\omega)$ for $t < \tau_D(\omega)$. If $Y^D_{\tau_D -}(\omega) \notin D$, set $X_t(\omega) = \partial$ for $t \geqslant \tau_D(\omega)$. If $Y^D_{\tau_D -}(\omega) \in D$, let $X_{\tau_D}(\omega) = Y^D_{\tau_D -}(\omega)$ and glue an independent copy of Y^D starting from $Y^D_{\tau_D -}(\omega)$ to $X_{\tau_D}(\omega)$. Iterating this procedure countably many times, we obtain a process on D which is a version of the strong Markov process X; the procedure works for every starting point in D.

For any open n-set D in $\mathbb{R}^n$, define

$$\mathcal{F}^{\mathrm{ref}} := \left\{ u \in L^2(D) : \int_D \int_D \frac{(u(x)-u(y))^2}{|x-y|^{n+\alpha}}\, dxdy < \infty \right\},$$

and

$$\mathcal{E}^{\mathrm{ref}}(u,v) := \frac{1}{2} \int_D \int_D (u(x)-u(y))(v(x)-v(y)) \frac{c}{|x-y|^{n+\alpha}}\, dxdy, \quad u, v \in \mathcal{F}^{\mathrm{ref}}.$$

As we see from Section 2, the bilinear form $(\mathcal{E}^{\mathrm{ref}}, \mathcal{F}^{\mathrm{ref}})$ is a regular symmetric Dirichlet form on $L^2(\overline{D}, dx)$. The process $\overline{X}$ on $\overline{D}$ associated with $(\mathcal{E}^{\mathrm{ref}}, \mathcal{F}^{\mathrm{ref}})$ is called in [36] a reflected α-stable process on $\overline{D}$. By Theorem 2.1, $\overline{X}$ has a Hölder continuous transition density function $\overline{p}(t,x,y)$ on $(0,\infty) \times \overline{D} \times \overline{D}$ and for every $T_0 > 0$, there are positive constants c_1, c_2 so that for $t \in (0, T_0]$ and $x, y \in \overline{D}$,

$$c_1\, t^{-n/\alpha} \left(1 \wedge \frac{t^{1/\alpha}}{|x-y|} \right)^{n+\alpha} \leqslant \overline{p}(t,x,y) \leqslant c_2\, t^{-n/\alpha} \left(1 \wedge \frac{t^{1/\alpha}}{|x-y|} \right)^{n+\alpha}. \tag{7.2}$$

This in particular implies that $\overline{X}$ can start from every point in $\overline{D}$. When D is an open n-set in $\mathbb{R}^n$, the censored α-stable-like process X can be realized as a subprocess of $\overline{X}$ killed upon leaving D (see [36, Remark 2.1]). It is proved in [36] that when D is a global Lipschitz domain and $\alpha \in (0, 1]$, then X and $\overline{X}$ are the same and so X has a sharp two-sided heat kernel estimate (7.2) in this case. Hence in the following we will concentrate on the case of $\alpha \in (1, 2)$. The next theorem gives a sharp two-sided heat kernel estimate for the transition density function $p_D(t, x, y)$ of censored α-stable process in an $C^{1,1}$ open set with $\alpha \in (1, 2)$.

Theorem 7.2 ([20, Theorem 1.1]). *Suppose that $n \geqslant 1$, $\alpha \in (1, 2)$ and D is a $C^{1,1}$ open subset of $\mathbb{R}^n$. Let $\delta_D(x)$ be the Euclidean distance between x and D^c.*

(i) *For every $T > 0$, on $(0, T] \times D \times D$*

$$p_D(t, x, y) \asymp t^{-n/\alpha}\left(1 \wedge \frac{t^{1/\alpha}}{|x-y|}\right)^{n+\alpha}\left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha-1}\left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\right)^{\alpha-1}.$$

(ii) *Suppose in addition that D is bounded. For every $T > 0$, there exist positive constants $c_1 < c_2$ such that for all $(t, x, y) \in [T, \infty) \times D \times D$,*

$$c_1 e^{-\lambda_1 t}\delta_D(x)^{\alpha-1}\delta_D(y)^{\alpha-1} \leqslant p_D(t, x, y) \leqslant c_2 e^{-\lambda_1 t}\delta_D(x)^{\alpha-1}\delta_D(y)^{\alpha-1},$$

where $-\lambda_1 < 0$ is the largest eigenvalue of the L^2-generator of X.

By integrating the above two-sided heat kernel estimates in Theorem 7.2 with respect to t, one can easily obtain the following sharp two-sided estimate of the Green function $G_D(x, y) = \int_0^\infty p_D(t, x, y)dt$ of a censored stable process in a bounded $C^{1,1}$ open set D.

Corollary 7.3 ([20, Corollary 1.2]). *Suppose that $n \geqslant 1$, $\alpha \in (1, 2)$ and D is a bounded $C^{1,1}$ open set in $\mathbb{R}^n$. Then on $D \times D$, we have*

$$G_D(x, y) \asymp \begin{cases} \dfrac{1}{|x-y|^{n-\alpha}}\left(1 \wedge \dfrac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right)^{\alpha-1} & \text{when } n \geqslant 2, \\ \big(\delta_D(x)\delta_D(y)\big)^{(\alpha-1)/2} \wedge \left(\dfrac{\delta_D(x)\delta_D(y)}{|x-y|}\right)^{\alpha-1} & \text{when } n = 1. \end{cases}$$

Sharp two-sided estimates on the Green function are very important in understanding deep potential theoretic properties of Markov processes. When D is a bounded $C^{1,1}$ connected open sets in $\mathbb{R}^n$ and $n \geqslant 2$, estimates in Corollary 7.3 had been obtained in [37].

Our approach in [20] is adapted from that of [19]. In [19], the following domain monotonicity for the killed symmetric stable processes is used in a crucial way. Let Z be a symmetric α-stable process and Z^D be the subprocess of Z killed upon leaving an open set D. If U is an open subset of D, then Z^U is a subprocess of Z^D killed upon leaving U. However censored stable-like processes do not have this kind of domain monotonicity. So there are new challenges to overcome when studying heat kernel estimates for censored stable processes. A quantitative version of the intrinsic ultracontractivity, a crucial use of boundary Harnack inequality for censored stable process and the reflected stable process $\overline{X}$ all played an important role in our approach in [20].

8 Concluding remarks

In this paper, we surveyed some recent progress in the study of fine potential theoretic properties of various models of symmetric discontinuous Markov processes that the author is involved. To keep the exposition as transparent as possible, sometimes we did not state the results in its most general form. For example, results in Sections 2 and 3 hold for general d-sets F in $\mathbb{R}^n$ and for F being a measure-metric space satisfying certain conditions, see [11, 12]; and the Dirichlet heat kernel estimates in Section 7 in fact holds also for censored stable-like processes, see [20]. Two-sided transition density function estimates for relativistic stable processes in $C^{1,1}$ open sets have recently been established in [21]. The study of sharp two-sided heat kernel estimates for discontinuous Markov processes is in its early stage and is currently a very active research area. There are many questions waiting to be answered and several active studies are currently underway. For instance, it is natural to study the large time estimate for $p(t, x, y)$ of the processes considered in Section 3 for the case of $\gamma_2 > \gamma_1 > 0$. It is also nature and important to investigate the sharp two-sided heat kernel estimates for stable processes of mixed type in $C^{1,1}$-open sets, and for Lévy processes that is the independent sum of a Brownian motion and a symmetric stable process in $C^{1,1}$-domains. Some promising progress has already been made in these studies.

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References

1 Stroock D W. Diffusion semigroup corresponding to uniformly elliptic divergence form operator. *Lecture Notes in Math*, **1321**: 316–347 (1988)

2 Bertoin J. Lévy Processes. Cambridge: Cambridge University Press, 1996

3 Janicki A, Weron A. Simulation and Chaotic Behavior of α-Stable Processes. New York: Dekker, 1994

4 Klafter J, Shlesinger M F, Zumofen G. Beyond Brownian motion. *Physics Today*, **49**: 33–39 (1996)

5 Samorodnitsky G, Taqqu M S. Stable Non-Gaussian Random Processes. New York-London: Chapman & Hall, 1994

6 Caffarelli L A, Salsa S, Silvestre Luis. Regularity estimates for the solution and the free boundary to the obstacle problem for the fractional Laplacian. *Invent Math*, **171**(1): 425–461 (2008)

7 Silvestre L. Hölder estimates for solutions of integro differential equations like the fractional Laplace. *Indiana Univ Math J*, **55**: 1155–1174 (2006)

8 Blumenthal R M, Getoor R K. Some theorems on stable processes. *Trans Amer Math Soc*, **95**: 263–273 (1960)

9 Kolokoltsov V. Symmetric stable laws and stable-like jump-diffusions. *Proc London Math Soc*, **80**: 725–768 (2000)

10 Bass R F, Levin D A. Transition probabilities for symmetric jump processes. *Trans Amer Math Soc*, **354**: 2933–2953 (2002)

11 Chen Z-Q, Kumagai T. Heat kernel estimates for stable-like processes on d-sets. *Stochastic Process Appl*, **108**: 27–62 (2003)

12 Chen Z-Q, Kumagai T. Heat kernel estimates for jump processes of mixed types on metric measure spaces. *Probab Theory Related Fields*, **140**: 277–317 (2008)

13 Chen Z-Q, Kim P, Kumagai T. Weighted Poincaré inequality and heat kernel estimates for finite range jump processes. *Math Ann*, **342**: 833–883 (2008)

14 Hurst S R, Platen E, Rachev S T. Option pricing for a logstable asset price model. *Math Comput Modelling*, **29**: 105–119 (1999)

15 Matacz A. Financial modeling and option theory with the truncated Lévy process. *Int J Theor Appl Finance*, **3**(1): 143–160 (2000)

16 Barlow M T, Bass R F, Chen Z-Q, et al. Non-local Dirichlet forms and symmetric jump processes. *Trans Amer Math Soc*, **361**: 1963–1999 (2009)

17 Chen Z-Q, Kumagai T. A priori Hölder estimate, parabolic Harnack inequality and heat kernel estimates for diffusions with jumps. *Revista Mathemática Iberoamericana*, to appear (2009)

18 Zhang Q S. The boundary behavior of heat kernels of Dirichlet Laplacians. *J Differential Equations*, **182**: 416–430 (2002)

19 Chen Z-Q, Kim P, Song R. Heat kernel estimates for Dirichlet fractional Laplacian. *J European Math Soc*,

to appear (2009)

20 Chen Z-Q, Kim P, Song R. Two-sided heat kernel estimates for censored stable-like processes. *Probab Theory Related Fields*, to appear (2009)

21 Chen Z-Q, Kim P, Song R. Sharp heat kernel estimates for relativistic stable processes. Preprint, 2009

22 Bass R F. SDEs with jumps. Notes for Cornell Summer School 2007. http://www.math.uconn.edu/~bass/cornell.pdf

23 Chen Z-Q. Multidimensional symmetric stable processes. *Korean J Comput Appl Math*, **6**: 227–266 (1999)

24 Benjamini I, Chen Z-Q, Rohde S. Boundary trace of reflecting Brownian motions. *Probab Theory Relat Fields*, **129**: 1–17 (2004)

25 Bass R F, Levin D A. Harnack inequalities for jump processes. *Potential Anal*, **17**: 375–388 (2002)

26 Chen Z-Q, Song R. Drift transforms and Green function estimates for discontinuous processes. *J Funct Anal*, **201**: 262–281 (2003)

27 Herbst I W, Sloan D. Perturbation of translation invariant positive preserving semigroups on $L^2(\mathbb{R}^N)$. *Trans Amer Math Soc*, **236**: 325–360 (1978)

28 Meyer P A. Renaissance, recollements, mélanges, ralentissement de processus de Markov. *Ann Inst Fourier*, **25**: 464–497 (1975)

29 Ikeda N, Nagasawa M, Watanabe S. A construction of Markov process by piecing out. *Proc Japan Acad Ser A Math Sci*, **42**: 370–375 (1966)

30 Barlow M T, Grigor'yan A, Kumagai T. Heat kernel upper bounds for jump processes and the first exit time. *J Reine Angew Math*, **626**: 135–157 (2009)

31 Song R, Vondraček Z. Parabolic Harnack inequality for the mixture of Brownian motion and stable process. *Tohoku Math J*, **59**: 1–19 (2007)

32 Carlen E A, Kusuoka S, Stroock D W. Upper bounds for symmetric Markov transition functions. *Ann Inst Heri Poincaré-Probab Statist*, **23**: 245–287 (1987)

33 Chen Z-Q, Song R. Estimates on Green functions and Poisson kernels of symmetric stable processes. *Math Ann*, **312**: 465–601 (1998)

34 Kulczycki T. Properties of Green function of symmetric stable processes. *Probab Math Stat*, **17**: 381–406 (1997)

35 Chen Z-Q, Tokle J. Global heat kernel estimates for fractional Laplacians in un bounded open sets. Preprint, 2009

36 Bogdan K, Burdzy K, Chen Z Q. Censored stable processes. *Probab Theory Related Fields*, **127**: 89–152 (2003)

37 Chen Z-Q, Kim P. Green function estimate for censored stable processes. *Probab Theory Related Fields*, **124**: 595–610 (2002)