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Fundamental cycles and graph embeddings

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Abstract In this paper, we investigate fundamental cycles in a graph G and their relations with graph embeddings. We show that a graph G may be embedded in an orientable surface with genus at least g if and only if for any spanning tree T, there exists a sequence of fundamental cycles $C_1, C_2, \ldots, C_{2g}$ with $C_{2i-1} \cap C_{2i} \neq \emptyset$ for $1 \leqslant i \leqslant g$. In particular, among $\beta(G)$ fundamental cycles of any spanning tree T of a graph G, there are exactly $2\gamma_M(G)$ cycles $C_1, C_2, \ldots, C_{2\gamma_M(G)}$ such that $C_{2i-1} \cap C_{2i} \neq \emptyset$ for $1 \leqslant i \leqslant \gamma_M(G)$, where $\beta(G)$ and $\gamma_M(G)$ are the Betti number and the maximum genus of G, respectively. This implies that it is possible to construct an orientable embedding with large genus of a graph G from an arbitrary spanning tree T (which may have very large number of odd components in $G \backslash E(T)$). This is different from the earlier work of Xuong and Liu, where spanning trees with small odd components are needed. In fact, this makes a common generalization of Xuong, Liu and Fu et al. Furthermore, we show that (1) this result is useful for locating the maximum genus of a graph having a specific edge-cut. Some known results for embedded graphs are also concluded; (2) the maximum genus problem may be reduced to the maximum matching problem. Based on this result and the algorithm of Micali-Vazirani, we present a new efficient algorithm to determine the maximum genus of a graph in $O((\beta(G))^{\frac{5}{2}})$ steps. Our method is straight and quite different from the algorithm of Furst, Gross and McGeoch which depends on a result of Giles where matroid parity method is needed.

Keywords: fundamental cycle, maximum genus, upper-embedded

MSC(2000): 05C10, 05C70

1 Introduction

The graphs considered here are finite and undirected and, furthermore, are connected unless they are stated otherwise. In general, multiple edges and loops are allowed. Terminologies and notations without explicit explanation follow from [1−3].

By a surface, denoted by S, we mean a compact and connected 2-manifold without boundary. It is well known from elementary topology that surfaces can be divided into two classes: orientable and nonorientable ones. An orientable surface can be viewed as a sphere attached h handles, while a nonorientable surface as a sphere attached k crosscaps. The number h or k is called the genus of the surface. A cellular embedding of a graph G into a surface S is a continuous one-to-one mapping ϕ: $G \to S$ such that each component of $S \backslash \phi(G)$ is homeomorphic to an open disc, called a face of G (with respect to this embedding ϕ) and ϕ is called a cellular

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embedding (or embedding). A cycle (curve) C in an embedded graph in a surface S is called surface separating if $S - C$ is disconnected. In particular, if $S - C$ has an open disc, denoted by int(C), then C is called contractible (otherwise, C is noncontractible), and int(C) + C =Int(C) is the inner part of C. The other part of $S - C$ is called the exterior of C and is denoted by Ext(C).

Recall that the maximum genus $\gamma_M(G)$ of a graph G is the largest integer k such that G has an embedding in an orientable surface with genus k. Since any graph G embedded in a surface has at least one face, Euler's formula shows that $\gamma_M(G) \leqslant \lfloor \frac{\beta(G)}{2} \rfloor$, where $\beta(G) = |E(G)| - |V(G)| + 1$ is known as Betti number of G (which is equal to the cycle rank number of G). A graph is upper-embeddable if $\gamma_M(G) = \lfloor \frac{\beta(G)}{2} \rfloor$.

Let G be a graph and T be a spanning tree of G. It is clear that for any edge $e \in E(G) \backslash E(T)$, $T + e$ contains a unique cycle of G, denoted by $C_T(e)$, which is called a fundamental cycle of G (with respect to the spanning tree T of G). If a pair of edges e_1 and e_2 have a common end vertex in a graph G, then we say that the pair $\langle e_1, e_2 \rangle$ is an adjacent-edge pair in G. Let G_1 and G_2 be a pair of disjoint subgraphs of G. Then $E[G_1, G_2]$ is the set of edges with their ends in G_1 and G_2, respectively.

Denote by $\xi(G, T)$ the number of components of $G \backslash E(T)$ with an odd number of edges. Then the Betti deficiency of G denoted by $\xi(G)$ is defined as the value $\min_T \xi(G, T)$, where the minimum is taken over all spanning trees T of G. A spanning tree T of G is said to be an optimal spanning tree if $\xi(G, T) = \xi(G)$.

2 A good characterization

Lemma 1[2,4]**.** *Let G be a graph, then*

(1) $\gamma_M(G) = \frac{1}{2}(\beta(G) - \xi(G))$;

(2) *G is upper-embeddable if and only if $\xi(G) \leqslant 1$.*

Theorem 1. *If a graph G contains a spanning tree T such that there exist $2g$ fundamental cycles $C_1, C_2, \ldots, C_{2g}$ with $C_{2i-1} \cap C_{2i} \neq \emptyset$, for $i = 1, 2, \ldots, g$, then G may be embedded in an orientable surface with genus at least g.*

Proof. Let $e_1, e_2, \ldots, e_{2g}$ be edges in $E(G) \backslash E(T)$ such that C_i is the unique cycle in $T + e_i$ $(1 \leqslant i \leqslant 2g)$. We may suppose further that $G = T + \{e_1, e_2, \ldots, e_{2g}\}$ by Xuong's constructive proof of maximum genus formula[4]. Let $G_0 = T$, and $G_1 = G_0 + \{e_1, e_2\}$. Then we have the following:

Claim 1. $\xi(G_1) \leqslant \xi(G_0)$ $(\Longleftrightarrow \gamma_M(G_1) \geqslant \gamma_M(G_0) + 1)$. To see this, we observe that $\beta(G_0) = \beta(G_1) - 2$. And so, $\xi(G_1) \equiv \xi(G_0) (\mathrm{mod}\ 2)$.

If $\xi(G_1) \geqslant \xi(G_0) + 2$, then we have one of the following situations:

(a) Both e_1 and e_2 have, respectively, their ends in distinct even components in $E(G_1) \backslash E(T)$ (see Figure 1(a)).

(b) Both e_1 and e_2 have, respectively, their ends in the same even components in $E(G_1) \backslash E(T)$ (see Figure 1(b)).

(c) Exactly, one of e_1 and e_2, say e_1, joins two even components of $E(G_1) \backslash E(T)$, while e_2 has two ends in the same even components in $E(G_1) \backslash E(T)$ (see Figure 1(c)).

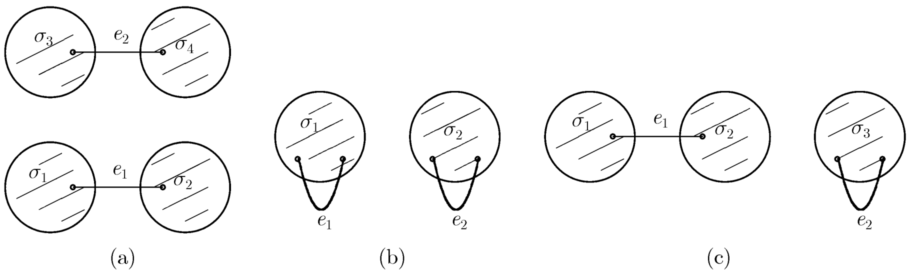

Figure 1

Without loss of generality, we may suppose that $e_1 \cap e_2 = \emptyset$, and consider case (a).

Let $e_1 \in E[\sigma_1, \sigma_2]$, $e_2 \in E[\sigma_3, \sigma_4]$, and C_i be the fundamental cycle in $T + e_i$ $(1 \leqslant i \leqslant 2)$.

Subcase A. $C_1 \cap C_2$ is a path. Let $P = C_1 \cap C_2$ be a path with an end vertex x in $C_1 \cap C_2$. Let e_1' and e_2' be two edges such that $e_1', e_2' \in E(T)$, and $x \in e_1' \cap e_2'$. Now $e_1', e_2' \notin E(P)$ (see the left-hand side of Figure 2). Consider a new spanning tree $T' = T + \{e_1, e_2\} - \{e_1', e_2'\}$.

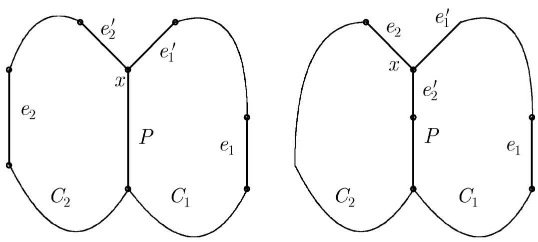

Figure 2

Subcase B. $x \in e_1$ or $x \in e_2$, say $x \in e_2$. (See the right-hand side of Figure 2).

If $|E(P)| \geqslant 1$, then we take edges $e_1' \in C_1 \backslash E(T)$, $x \in e_1'$, $e_2' \in E(T)$, $x \in e_2'$. We construct a new spanning tree $T' = T + \{e_1, e_2\} - \{e_1', e_2'\}$. If $|E(P)| = 0$, then this may be a special case of A.

Let T' be the spanning tree as defined in either Subcase A or B. It is easy to see that $E(G_1) \backslash E(T')$ has at most $\xi(G_0)$ odd components. It is contradictory to our suppose. Therefore $\xi(G_1) \leqslant \xi(G_0)$.

Similarly, we may prove the claim in the cases of (b) and (c).

Repeat this procedure for $G_2 = G_1 + \{e_3, e_4\}, \ldots, G_g = G_{g-1} + \{e_{2g-1}, e_{2g}\}$ until we get $\xi(G_g) \leqslant \xi(G_{g-1}) \leqslant \cdots \leqslant \xi(G_0)$, so $\gamma_M(G_g) \geqslant \gamma_M(G_0) + g = g$.

Theorem 2. *Let G be a connected graph embedded in an orientable surface S_g and T be a spanning tree of G. Then there are at least $2g$ noncontractible fundamental cycles $C_1, C_2, \ldots, C_{2g}$, such that $C_{2i-1} \cap C_{2i} \neq \emptyset$ for $1 \leqslant i \leqslant g$. In particular, if G is a one-face-embedded graph in S_g, then for any spanning tree T of G, there are $2g$ edges in $G \backslash E(T)$ such that the corresponding $2g$ fundamental cycles $C_1, C_2, \ldots, C_{2g}$ satisfy $C_{2i-1} \cap C_{2i} \neq \emptyset$ for $1 \leqslant i \leqslant g$.*

Proof. We contract T into a single vertex v_T and delete all the possible edges on distinct faces. Then we get a vertex-graph G_T with exactly one vertex v_T and one face in S_g. There are two *crossed loops*, say e_α, e_β, such that the local rotation of semi-edges incident to v_T is

$e_\alpha \cdots e_\beta \cdots e_\alpha \cdots e_\beta$ (see Figure 3). Furthermore, e_β is the only possible edge crossing e_α (otherwise G_T would have at least two faces!). Hence, all edges of G_T may be listed as follows: $e_1, e_2, \ldots, e_{2g-1}, e_{2g}$ such that e_{2i-1} crossing e_{2i} for $i = 1, 2, \ldots, g$. It is easy to see that e_{2i-1} and e_{2i} determine two fundamental cycles C_{2i-1} and C_{2i} with a vertex in common.

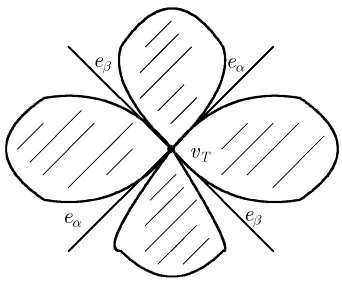

Figure 3

Remark. Theorems 1 and 2 give a good characterization of maximum genus of a graph (i.e., they imply the existence of a polynomially bounded algorithm to find the maximum genus of a graph).

Let T be a spanning tree in G with a group of fundamental cycles $C_1, C_2, \ldots, C_{2g}$. If $C_{2i-1} \cap C_{2i} \neq \emptyset$ for $1 \leqslant i \leqslant g$, then we say $\langle C_{2i-1}, C_{2i} \rangle$ is an adjacent fundamental cycle pairs ($1 \leqslant i \leqslant g$). If g is chosen as the largest number meeting above condition, then we call g the maximum number of adjacent fundamental cycle pairs of T. Hence Theorem 2 implies the following:

Theorem 3. *Any two spanning trees T_1 and T_2 in a graph G have the same maximum number of adjacent fundamental cycle pairs.* (*In fact, this unique number is $\gamma_M(G)$, the maximum genus of G.*)

This generalizes a result of Fu et al.[5] where they introduced the concept intersecting graph which is determined by bases of cycle space of a graph to describe the maximum genus of a graph. In fact, our result stands for any spanning tree's fundamental cycles.

Corollary 1. *If a connected graph G has a spanning tree T such that any two fundamental cycles have a vertex in common. Then G is upper-embeddable.*

However sometimes, we need a refined form of Theorems 1 and 2 in practice. The following result gives us a recursive relation between the maximum genera of a graph and its subgraph(s).

Theorem 4. *Let G be a connected graph and T be an arbitrary spanning tree in G. If e_1, e_2 are two edges not in T and the two cycles $C_T(e_1)$ and $C_T(e_2)$ have a vertex in common. Then $\gamma_M(G) = \gamma_M(G + e_1 + e_2) - 1$. In particular, G is upper-embeddable if and only if $G + e_1 + e_2$ is upper-embeddable.*

One may easily see that this generalizes a recursive relation for maximum genus of Xuong[4] and (see in the next section) is much more practical.

3 Applications

Now in this section, we begin to apply Theorems 1 and 2 to determining the maximum genus of some type of graphs.

Let us recall that the essence of Xuong's method[4] consists of two parts: one is to find an optimal tree T in a graph G having the smallest number of odd components; the other is to organize edges of $E(G)\backslash E(T)$ into adjacent pairs such as

$$E(G)\backslash E(T) = \{e_1, e_2, \ldots, e_{2s}\} \cup \{f_1, f_2, \ldots, f_m\},$$

where $e_{2i-1} \cap e_{2i} \neq \emptyset$ $(1 \leqslant i \leqslant s)$ and $C_T(f_i) \cap C_T(f_j) = \emptyset$, for $1 \leqslant i < j \leqslant m$ and $s = \gamma_M(G), m = \xi(G)$. Compared with the above procedure, Theorems 1 and 2 consider adjacent fundamental cycle pairs (rather than adjacent pairs of edges). We may construct large genus embedding from any spanning tree T, although it may have very large number of odd components in $G\backslash E(T)$. This greatly releases the conditions of Xuong. Of course, an optimal tree is also valid in our constructions. Hence, Theorems 1 and 2 generalize Xuong's characterization of maximum genus. Based on this idea, we may construct a large orientable genus as follows: Take a specific spanning tree T in graph G and first organize some non-tree edges into adjacent pairs (as Xuong did) and then match other possible non-tree edges into pairs such that their fundamental cycles also become adjacent fundamental cycle pairs. It is easy for one to see that the second part of non-tree edges may be chosen as an edge-cut of G. Therefore, Theorems 1 and 2 may be useful to determine a maximum genus of a graph G with a specific edge-cut. Now, the following result is easy to be verified.

Theorem 5. *Let $A = \{e_1, e_2, \ldots, e_k\}$ be an edge-cut of G such that $G - A$ has exactly two components G_1 and G_2. If both G_1 and G_2 are upper-embeddable, then $\gamma_M(G) \geqslant \lfloor \frac{\beta(G)}{2} \rfloor - 1$. Furthermore, if G satisfies one of the following conditions, then G is upper-embeddable:*

(a) $\beta(G_1) \equiv \beta(G_2) \equiv 0 (\text{mod } 2)$,

(b) $|A| \equiv 1 (\text{mod } 2)$ *and* $\beta(G_1) + \beta(G_2) \equiv 1$ (mod 2).

The next result is from Huang. As a consequence of the above results, we will give another proof.

Theorem 6[5]**.** *Let G be a strongly embedded graph in an orientable surface S_g (i.e., all facial walks are cycles). If the dual graph G^* of G has a surface separating Hamiltonian cycle, then G is upper-embeddable.*

Proof. We will show the existence of a spanning tree T of G satisfying the conditions in Theorems 1 and 2. Let $\mathcal{F} = \{f_1,\ f_2, \ldots, f_\varphi\}$ be the face-set of G and C^* be a surface separating Hamiltonian cycle in G^*. Let $E(C^*) = \{e_1^*, e_2^*, \ldots, e_\varphi^*\}$ and $e_i^* = (f_i, f_{i+1})$ for $1 \leqslant i \leqslant \varphi$. Let e_i be the edge in $\partial f_i \cap \partial f_{i+1}$ corresponding to e_i^* for $1 \leqslant i \leqslant \varphi$ (where ∂f_i denotes the boundary of f_i).

Claim 2. $G - \{e_1, e_2, \ldots, e_{\varphi-1}\}$ is a one-face embedded subgraph of G in S_g, Furthermore, $G - \{e_1, e_2, \ldots, e_\varphi\}$ has exactly two components G_1 and G_2.

Now $E[G_1,\ G_2] = \{e_1,\ e_2, \ldots, e_\varphi\}$. Let $G_1 \subset \text{Int}(C^*)$ and $G_2 \subset \text{Ext}(C^*)$ and ∂f_i denotes the boundary cycle of f_i for $1 \leqslant i \leqslant \varphi$. Then we may construct a graph as follows. $H_0 = (\partial f_1 \cup \partial f_2 \cup \cdots \cup \partial f_{\varphi-1}) \setminus \{e_1, e_2, \ldots, e_{\varphi-1}\}$. It is easy to see that H_0 is a connected spanning subgraph of $G - \{e_1,\ e_2, \ldots, e_{\varphi-1}\}$. (Hence, a spanning subgraph of G). Let $e_\varphi = (\alpha, \beta)$ with $\alpha \in\ V(G_1),\ \beta \in\ V(G_2)$. Then $H_0 - e_\varphi$ has exactly two components $H',\ H_1$ with $H' = G_1$.

Claim 3. If H_1 has a cycle C, then C must be a noncontractible cycle.

This follows from the fact that $S_g - H_0$ has only one component. If H_1 has a cycle C_1, then delete an edge $e'_1 \in C_1$ and get a subgraph H_2 of H_1 with $V(H_2) = V(H_1)$. Repeat this procedure until we arrive at a connected subgraph H_k of H_1 with $V(H_k) = V(H_1)$ and H_k has no cycle.

Claim 4. $T = H' \cup H_k \cup \{e_\varphi\}$ is a spanning tree of G, such that each fundamental cycle $C_T(e_i)$ in $T + e_i$ has an edge e_φ in common for $i = 1, 2, \ldots, \varphi - 1$.

To see this, we consider an edge $e_i = (x_i, y_i) \in [H', H_k] \subseteq [G_1, G_2]$, such that $x_i \in H', y_i \in H_k$. Since $H'(H_k)$ is connected, there is a path $P_i(Q_i)$ in $H'(H_k)$ joining $\alpha(\beta)$ and $x_i(y_i)$. Hence, $C_T(e_i) = \{e_\varphi\} \cup P_i \cup Q_i \cup \{e_i\}$ is a cycle containing e_φ for $1 \leqslant i \leqslant \varphi$.

Now we find a spanning tree T of G such that

(i) all the fundamental cycles $C_T(e_1),\ C_T(e_2), \ldots, C_T(e_{\varphi-1})$ have an edge in common;

(ii) by Theorems 1, 2, and the fact that T is also a spanning tree in $G - \{e_1, e_2, \ldots, e_{\varphi-1}\}$, there are another group of fundamental (noncontractible) cycles $C_1, C_2, \ldots, C_{2g}$ such that $C_{2i-1} \cap C_{2i} \neq \emptyset$ for $1 \leqslant i \leqslant g$. By Theorem 1, G is upper-embeddable.

One may readily see that a surface separating cycle may not be Hamiltonian and the hosting surface on which a graph is embedded may not be orientable. Thus, Theorem 5 can be extended to a much more generalized form.

Theorem 7. *Let G be an embedded graph in a surface $\sum$ such that the dual graph G^* of G contains a surface separating cycle C^* such that both of the left subgraph $G_L(C^*)$ and right subgraph $G_R(C^*)$ of G are upper embeddable. Then $\gamma_M(G) \geqslant \lfloor \frac{\beta(G)}{2} \rfloor - 1$. In particular, if $\beta(G_L(C^*)) \equiv \beta(G_R(C^*)) \equiv 0 \pmod 2$, then G is upper-embeddable.*

Remark. The term "left (right) subgraph" follows from [3].

Corollary 2. *If G is an embedded graph on the Klein bottle such that the dual graph G^* has a surface separating Hamiltonian cycle. Hence $\gamma_M(G) \geqslant \lfloor \frac{\beta(G)}{2} \rfloor - 1$.*

In practical use, our attention need not to be restricted to graphs with an edge-cut. Theorems 1−4 provide us a tool to evaluate large genus embeddings in more extended range of graphs. The following results show us how to do so (we omit the proof of them).

Theorem 8. *The following graphs are upper-embeddable*:

(1) *The cartesian product $G \times P_n$ of a simple connected graph G and a path P_n with n ($\geqslant 1$) edges*;

(2) *the composition of two disjoint Halin graphs H_1 and H_2 with some edges $e_1, e_2, \ldots, e_k$ ($k \geqslant 2$) connecting them*;

(3) *the n-cube Q_n which is composed of two $(n-1)$-cube Q_{n-1} together with some edges joining the two copies of vertice in Q_{n-1}*;

(4) *the generalized Petersen graphs $P(n,k)$ which is determined by n-cycle $(u_1, u_2, \ldots, u_n)$ and vertices $v_1, v_2, \ldots, v_n$ such that* (i) *each $(u_i, v_i) \in E, 1 \leqslant i \leqslant n$;* (ii) $(u_i, v_{i+k}) \in E, 1 \leqslant i \leqslant n$.

Note. A graph $G = (V,\ E)$ is a Halin graph if G is obtained by joining the leaves (1-valent vertices) of a plane tree T with a cycle in this orientation and the definition of cartesian product of two graphs may be found in any textbook of graph theory.

4 A polynomially bounded algorithm

In this section we shall present a polynomially bounded algorithm to find the maximum genus of a given graph. A basic fact is that Theorems 1 and 2 present a good characterization of maximum genus problem, i.e., we have the following

Theorem 9. *To determine the maximum genus of a graph G is equivalent to determine a maximum matching of the graph $G_M = (V_M, E_M)$, called fundamental intersecting graph of G, where V_M is the set of fundamental cycles of a spanning tree T of G and any two cycles in V_M are adjacent if and only if they have at least a vertex in common.*

We observe that the fastest algorithm to find a maximum matching in a graph G is due to Micali-Vazirani[7] which will end in $O(m\sqrt{n})$ steps, where m and n are, respectively, the number of edges and vertices of G. Based on this fact and Theorems 1 and 2 we may construct a new algorithm to determine the maximum genus of a graph G.

Fundamental cycle algorithm

Step 1. Input the date of the graph G and then searching for a spanning tree T and the set V_M of fundamental cycles in G.

Step 2. For cycles in V_M we build the graph G_M.

Step 3. Perform Micali-Vazirani algorithm to find a maximum matching in G_M and then terminate.

Remark. Since the number of fundamental cycles in a graph G of order n is $\beta(G)$, this algorithm will end in at most $O((\beta(G))^{\frac{5}{2}})$ steps. Although Furst, Gross and McGeoch had already construct the first polynomially bounded algorithm[8] which depends on a result[9] of Giles where matroid parity method is needed, this result is a new approach to do so.

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