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Several splittings for non-Hermitian linear systems

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Abstract For large sparse non-Hermitian positive definite system of linear equations, we present several variants of the Hermitian and skew-Hermitian splitting (HSS) about the coefficient matrix and establish correspondingly several HSS-based iterative schemes. Theoretical analyses show that these methods are convergent unconditionally to the exact solution of the referred system of linear equations, and they may show advantages on problems that the HSS method is ineffective.

Keywords: Hermitian and skew-Hermitian splitting, non-Hermitian linear system, splitting iterative scheme, convergence

MSC(2000): 65F10, 65F15, 65F50, 65N22

1 Introduction

Many problems in the area of scientific computing require the solution of a sequence of large linear systems, usually written in the form

$$A\boldsymbol{x} = \boldsymbol{b} \ \text{ or } \ A\boldsymbol{u} = \boldsymbol{f}, \quad \boldsymbol{x}, \boldsymbol{b}, \boldsymbol{u}, \boldsymbol{f} \in \mathbb{C}^n, \quad A \in \mathbb{C}^{n\times n}. \tag{1}$$

Often we may assume that A is invertible. The form $A\boldsymbol{u} = \boldsymbol{f}$ usually signals that the authors are really thinking about discretized versions of partial differential equations whereas those who work with more diverse applications in optimization, or approximation or signal processing seem to prefer $A\boldsymbol{x} = \boldsymbol{b}$.

The crucial feature in discussion of methods for solving (1) is the extent to which A is known. At one extreme is the case when A is given only as a black box which can return the vector $A\boldsymbol{v}$ for any $\boldsymbol{v} \in \mathbb{C}^n$, even A^*, the adjoint of A for the standard Euclidean inner product, is not available. At the other extreme are discretizations of known differential operators on nice tensor-product domains with preconditioners that are spectrally equivalent to the positive definite Hermitian part of A.

The long enduring tension between experts in direct methods and experts in iterative methods began to dissipate in the 1990's when it was perceived that direct methods play a decisive role

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in fashioning good preconditioners without which Krylov subspace methods and truly iterative methods are ineffective on challenging problems.

In other words the demands placed on numerical analysts in the twenty-first century have enabled the two groups of experts to see that they need each other.

Our results are situated nearer the ignorant end of the scale of knowledge of A. However we do assume that both $A\boldsymbol{v}$ and $A^*\boldsymbol{v}$ may be computed for any $\boldsymbol{v}$ and this permits us to write (1) as

$$(H - S)\boldsymbol{x} = \boldsymbol{b} \tag{2}$$

with

$$H = \frac{1}{2}(A^* + A), \quad S = \frac{1}{2}(A^* - A).$$

The negative sign in (2) leads to nicer formulations later. The great appeal of (2) is that H and S are normal matrices even though the nature of A is unknown. $H - S$ is called a splitting of A provided that at least one of the terms is invertible. If S is invertible then $S^2 = -S^*S$ is negative definite and we again recover the Hermitian case. We will make use of S^2 later.

In general we will not know whether H dominates S or vice versa. If A is a discretization of a second-order partial differential equation, then, as the mesh size goes to zero, H will dominate S but in convection-diffusion equations with strong convection the reverse may hold for practical values of the mesh size h[1−6]. What we want is that when A is a function of some parameter then, as the balance between H and S changes, our iterative method should adapt automatically to the change.

We accept that so-called inner iterations will be needed to provide approximate solutions to the outer iterations that we will discuss. In this paper we focus on obtaining satisfactory outer iterations given by rather limited information about A. Note that much has changed concerning iterative methods since the 1960's and 1970's. In those days users were willing to take a huge number of iterations, far exceeding n, and consequently the asymptotic rate of convergence, governed by the spectral radius of the iteration matrix for stationary methods, was of prime importance.

By the late 1970's Krylov subspace methods were widely appreciated and the scene changed dramatically; see [7, 8]. In exact arithmetic these methods terminate after at most n steps and even $\frac{1}{2}n$ iterations was regarded as excessive. (Multigrid methods strengthened this trend.) These are all considered as iterative methods. Now, for outer iteration, it is the norm of the iteration matrix that matters and values too close to 1 are not acceptable.

The arrival of GMRES, by Saad and Schultz[9], defined a turning point in the development of iterative methods. Here is a viable method that makes minimal demands on A. Any new iteration must outperform GMRES, or a variant such as GMRES(20), in order to be taken seriously.

With these remarks in mind we examine the use of (2), and an extension, to provide powerful outer iterative schemes in this paper.

2 Hermitian and skew-Hermitian splitting (HSS)

By analogy with the alternating direction implicit (ADI) iteration[10] for elliptic and separable partial differential equations, Bai, Golub, and Ng[11] considered a two-step scheme derived

from (2).

HSS method. Given an initial approximate solution $\boldsymbol{x}^{(0)}$ to (2), for $k = 0, 1, 2, \ldots$ until convergence solve

$$(\alpha I + H)\boldsymbol{x}^{(k+1/2)} = (\alpha I + S)\boldsymbol{x}^{(k)} + \boldsymbol{b}, \tag{3a}$$

$$(\alpha I - S)\boldsymbol{x}^{(k+1)} = (\alpha I - H)\boldsymbol{x}^{(k+1/2)} + \boldsymbol{b}, \tag{3b}$$

where α is a given positive constant.

Theorem 2.2 in [11] shows that if the coefficient matrix $A \in \mathbb{C}^{n\times n}$ is positive definite[1)], i.e., its Hermitian part H is positive definite, then $\boldsymbol{x}^{(k)} \longrightarrow A^{-1}\boldsymbol{b}$ and the contraction factor at each step is bounded by

$$\sigma(\alpha) := \max_i \left|\frac{\alpha - \lambda_i[H]}{\alpha + \lambda_i[H]}\right| < 1,$$

where $\lambda_i[H]$ denotes the i-th eigenvalue of the matrix H. Moreover, in Corollary 2.3 in [11] it is shown that

$$\min_{\alpha>0} \max_{\lambda_{\min}[H]\leqslant\mu\leqslant\lambda_{\max}[H]} \left|\frac{\alpha - \mu}{\alpha + \mu}\right|$$

occurs when $\alpha = \alpha_* = \sqrt{\lambda_{\min}[H] \cdot \lambda_{\max}[H]}$ and then

$$\sigma(\alpha_*) = \frac{\sqrt{\kappa(H)} - 1}{\sqrt{\kappa(H)} + 1}.$$

Here, $\lambda_{\min}[H]$ and $\lambda_{\max}[H]$ are the minimum and the maximum eigenvalues, and $\kappa(H) = \lambda_{\max}[H]/\lambda_{\min}[H]$ denotes the spectral condition number of the Hermitian positive definite matrix H. The result just quoted suggests that for HSS to achieve a performance level similar to the conjugate gradient (CG) method applied to H it would be necessary to know $\lambda_{\min}[H]$ and $\lambda_{\max}[H]$ explicitly in order to use α_*. However that message is misleading because $\sigma(\alpha)$ is only a bound on the contraction factor and a very pessimistic one when S dominates H. Numerical experiments in [11] show spectral radii as small as 0.75 for a wide range of α values when solving convection-diffusion equations despite $\sigma(\alpha_*) \geqslant 0.95$, see [11, Figures 5.1 and 5.2]. Thus HSS is more effective when S dominates H than vice versa. Nevertheless HSS requires the solution of two nice systems and should be compared with the solution of (1) by using GMRES on A. The positive definite system in HSS (3a) is suitable for CG and the other system (3b) can be solved by an efficient Krylov subspace method that exploits the fact that S is skew-Hermitian. However when S is very small compared to H then (3b) contributes little to convergence and the inner iterations must be designed to terminate promptly since

$$(I - S/\alpha)^{-1} = I + S/\alpha + (S/\alpha)^2 + \cdots.$$

A blemish on HSS is that the iteration matrix

$$L(\alpha) := (\alpha I - S)^{-1}(\alpha I - H)(\alpha I + H)^{-1}(\alpha I + S)$$

is similar to $W(\alpha)Q(\alpha)$, where $W(\alpha) := (I - H/\alpha)(I + H/\alpha)^{-1}$ is Hermitian, indefinite, and $\|W(\alpha)\| < 1$, and $Q(\alpha) := (I + S/\alpha)(I - S/\alpha)^{-1}$ is unitary for all α. Here, $\|\cdot\|$ denotes the

[1)] A non-Hermitian matrix $A \in \mathbb{C}^{n\times n}$ is called positive (negative) definite if for all $\boldsymbol{x} \in \mathbb{C}^n\backslash\{\boldsymbol{o}\}$, the real part of $\boldsymbol{x}^* A\boldsymbol{x}$ is positive (negative), where $\boldsymbol{o}$ denotes the zero vector.

spectral norm. In [11, Corollary 2.3] it is made clear how to choose a good α when S is small compared to H, using approximations to $\lambda_{\min}[H]$ and $\lambda_{\max}[H]$, but it is not clear how to choose α when S dominates H; see [12] for some further results. Yet it is in these skew-dominated cases that HSS is effective for a large range of α.

So we look for different iterations that combine H and S^*S more naturally.

3 The skew-normal equations

In general the normal equations $A^*A\boldsymbol{x} = A^*\boldsymbol{b}$, although attractive theoretically, are not an efficient approach to solving $A\boldsymbol{x} = \boldsymbol{b}$. However, when A is orthogonal we use them unconsciously when writing $\boldsymbol{x} = A^*A\boldsymbol{x} = A^*\boldsymbol{b}$. The prejudice against the normal equations grew from the prevalence of systems that were positive definite or nearly so.

Since $-S^2 = S^*S$ is Hermitian positive semi-definite, there is a strong attraction to making use of it. When S is invertible, we may, without loss of generality, multiply (2) on the left by S to obtain a splitting of what we call the skew-normal equations

$$(SH - S^2)\boldsymbol{x} = S\boldsymbol{b}. \tag{4}$$

In order to derive a two-step iteration, we add αH to each side and write down two fixed-point equations

$$(\alpha H + SH)\boldsymbol{x} \equiv (\alpha I + S)H\boldsymbol{x} = (\alpha H + S^2)\boldsymbol{x} + S\boldsymbol{b}, \tag{5a}$$

$$(\alpha H - S^2)\boldsymbol{x} = (\alpha H - SH)\boldsymbol{x} + S\boldsymbol{b} \equiv (\alpha I - S)H\boldsymbol{x} + S\boldsymbol{b}, \tag{5b}$$

where α is a free positive parameter.

The new extra term $H\boldsymbol{x}$ causes no extra work provided that we solve (5a) before (5b). In analogy to the HSS method we present the Skew-Normal Splitting (SNS) method.

SNS method. Given an initial approximate solution $\boldsymbol{x}^{(0)}$ to (1), for $k = 0, 1, 2, \ldots$ until convergence solve

$$(\alpha I + S)\boldsymbol{x}^{(k+1/2)} = (\alpha H + S^2)\boldsymbol{x}^{(k)} + S\boldsymbol{b}, \tag{6a}$$

$$(\alpha H - S^2)\boldsymbol{x}^{(k+1)} = (\alpha I - S)\boldsymbol{x}^{(k+1/2)} + S\boldsymbol{b}, \tag{6b}$$

with given $\alpha > 0$.

Let us compare the coefficient matrices of SNS to those in HSS. Our $\alpha I + S$ is similar to the $\alpha I - S$ in HSS but the Hermitian system (6b) changes from $\alpha I + H$ in HSS to $\alpha H - S^2 = \alpha H + S^*S$ for us. Note that we do not need to assume that H is positive definite, only

$$S \text{ is invertible}, \quad \alpha H + S^*S \text{ is positive definite}.$$

However, we can obtain stronger results when H is positive definite.

Convergence theory

From (6a) and (6b) we form the iteration matrix

$$M(\alpha) := (\alpha H - S^2)^{-1}(\alpha I - S)(\alpha I + S)^{-1}(\alpha H + S^2).$$

Then we can demonstrate that the iteration matrix $M(\alpha)$ has the following two properties.

Property 1. *Given invertible $A \in \mathbb{C}^{n\times n}$. Let $H = \frac{1}{2}(A^* + A)$ and $S = \frac{1}{2}(A^* - A)$. If*

$$H \textit{ is positive definite}, \quad S \textit{ is invertible},$$

then, for all $\alpha > 0$, $M(\alpha)$ is similar to $Q(\alpha)W(\alpha)$, where $Q(\alpha)$ is unitary and all eigenvalues of $W(\alpha)$ are included in the interval $(-1, 1)$.

Proof.

$$(\alpha H - S^2)M(\alpha)(\alpha H - S^2)^{-1} = Q(\alpha)W(\alpha),$$
$$Q(\alpha) := (\alpha I - S)(\alpha I + S)^{-1},$$
$$W(\alpha) := (\alpha H + S^2)(\alpha H - S^2)^{-1}.$$

$Q(\alpha)$ is the Cayley transform of S and thus is unitary for all $\alpha > 0$. The matrix $W(\alpha)$ is not, in general, Hermitian. Nevertheless, since αH and S^*S are positive definite there is an invertible matrix Z such that $H = ZZ^*$ and $S^*S = Z\Lambda Z^*$ with Λ diagonal and positive definite. Thus

$$W(\alpha) = Z(\alpha I - \Lambda)Z^*[Z(\alpha I + \Lambda)Z^*]^{-1} = Z(\alpha I - \Lambda)(\alpha I + \Lambda)^{-1}Z^{-1},$$
$$\|(\alpha I - \Lambda)(\alpha I + \Lambda)^{-1}\| = \max_i \left|\frac{\alpha - \lambda_i}{\alpha + \lambda_i}\right| < 1, \quad \text{for } \alpha > 0.$$

Property 2. *With the notation of Property 1, let $V := V(\alpha) = (\alpha H + S^*S)$. Then, for all $\alpha > 0, \|M(\alpha)\| = \|Q(I - 2J)\|_{V^{-2}}$ with $Q := Q(\alpha)$ unitary and the spectrum of $I - 2J$ lying in $(-1, 1)$.*

Proof.

$$M := M(\alpha) = V^{-1}Q[V - 2S^*S] = V^{-1}Q[I - 2J]V,$$

with $J := J(\alpha) = S^*SV^{-1} = S^*S(\alpha H + S^*S)^{-1}$. Hence

$$\begin{aligned}\max_{\boldsymbol{x}\neq\boldsymbol{o}} \frac{\boldsymbol{x}^*M^*M\boldsymbol{x}}{\boldsymbol{x}^*\boldsymbol{x}} &= \max_{\boldsymbol{x}\neq\boldsymbol{o}} \frac{\boldsymbol{x}^*V(I - 2J^*)Q^*V^{-2}Q(I - 2J)V\boldsymbol{x}}{\boldsymbol{x}^*\boldsymbol{x}} \\ &= \max_{\boldsymbol{y}\neq\boldsymbol{o}} \frac{\boldsymbol{y}^*(I - 2J^*)Q^*V^{-2}Q(I - 2J)\boldsymbol{y}}{\boldsymbol{y}^*V^{-2}\boldsymbol{y}} \\ &= \|Q(I - 2J)\|^2_{V^{-2}},\end{aligned}$$

where $\|\cdot\|_B$ is the matrix norm subordinate to the vector norm $\|\boldsymbol{x}\|^2_B := \max_{\boldsymbol{x}\neq\boldsymbol{o}} \frac{\boldsymbol{x}^*B\boldsymbol{x}}{\boldsymbol{x}^*\boldsymbol{x}}$. Since

$$V^{-1/2}JV^{1/2} = V^{-1/2}S^*SV^{-1/2} =: G^*G,$$

we have

$$\begin{aligned}\|J\|^2_{V^{-1}} &= \max_{\boldsymbol{x}\neq\boldsymbol{o}} \frac{\boldsymbol{x}^*J^*V^{-1}J\boldsymbol{x}}{\boldsymbol{x}^*V^{-1}\boldsymbol{x}} = \max_{\boldsymbol{y}\neq\boldsymbol{o}} \frac{\boldsymbol{y}^*V^{1/2}J^*V^{-1}JV^{1/2}\boldsymbol{y}}{\boldsymbol{y}^*\boldsymbol{y}} \\ &= \max_{\boldsymbol{y}\neq\boldsymbol{o}} \frac{\boldsymbol{y}^*(G^*G)^2\boldsymbol{y}}{\boldsymbol{y}^*\boldsymbol{y}} = \|G^*G\|^2.\end{aligned}$$

Since $V - S^*S = \alpha H$ is positive definite, $I - G^*G = \alpha V^{-1/2}HV^{-1/2}$ is also positive definite. Thus, as a quadratic from, $0 < G^*G < I$ and the singular values of G lie in $(0, 1)$. Note also that $V^{-1/2}(I - 2J)V^{1/2} = I - 2G^*G$ has eigenvalues in the open interval $(-1, 1)$. Thus $\|M\| = \|Q(I - 2J)\|_{V^{-2}}$.

Theorem 1. *Given invertible $A \in \mathbb{C}^{n\times n}$. Let $H = \frac{1}{2}(A^* + A)$ and $S = \frac{1}{2}(A^* - A)$. If*

$$H \text{ is positive definite}, \quad S \text{ is invertible},$$

then, for all $\alpha > 0$, the spectral radius of the iteration matrix $M(\alpha)$ of the SNS *method is bounded by*

$$\varsigma(\alpha) = \|(I + \alpha S^{-1}HS^{-1})(I - \alpha S^{-1}HS^{-1})^{-1}\|,$$

i.e., $\rho(M(\alpha)) \leqslant \varsigma(\alpha)$. Moreover, it holds that $\rho(M(\alpha)) \leqslant \varsigma(\alpha) < 1, \forall\, \alpha > 0$. That is to say, the SNS method converges to the unique solution of the system of linear equations (1).

Proof. By the similarity invariance of the matrix spectrum and the nonsingularity of the matrix S, we have

$$\begin{aligned}
\rho(M(\alpha)) &= \rho((\alpha I - S)(\alpha I + S)^{-1}(\alpha H + S^2)(\alpha H - S^2)^{-1}) \\
&= \rho((\alpha I - S)(\alpha I + S)^{-1}S(\alpha S^{-1}HS^{-1} + I)(\alpha S^{-1}HS^{-1} - I)^{-1}S^{-1}) \\
&= \rho(S^{-1}(\alpha I - S)(\alpha I + S)^{-1}S(\alpha S^{-1}HS^{-1} + I)(\alpha S^{-1}HS^{-1} - I)^{-1}) \\
&= \rho((I - \alpha S^{-1})(I + \alpha S^{-1})^{-1}(I + \alpha S^{-1}HS^{-1})(I - \alpha S^{-1}HS^{-1})^{-1}). \qquad (7)
\end{aligned}$$

Because S is skew-Hermitian, we easily know that S^{-1} is skew-Hermitian, too. It then follows that $\widehat{Q}(\alpha) := (I-\alpha S^{-1})(I+\alpha S^{-1})^{-1}$ is a Cayley transform and is, thus, orthogonal. Therefore, for $\forall \alpha > 0$ it holds that $\|\widehat{Q}(\alpha)\| = 1$.

Now, from (7) we can further obtain

$$\begin{aligned}
\rho(M(\alpha)) &\leqslant \|(I - \alpha S^{-1})(I + \alpha S^{-1})^{-1}(I + \alpha S^{-1}HS^{-1})(I - \alpha S^{-1}HS^{-1})^{-1}\| \\
&\leqslant \|(I - \alpha S^{-1})(I + \alpha S^{-1})^{-1}\| \cdot \|(I + \alpha S^{-1}HS^{-1})(I - \alpha S^{-1}HS^{-1})^{-1}\| \\
&= \|(I + \alpha S^{-1}HS^{-1})(I - \alpha S^{-1}HS^{-1})^{-1}\| \\
&= \varsigma(\alpha).
\end{aligned}$$

Let $P = -S^{-1}HS^{-1} \equiv S^{-*}HS^{-1}$. Because H is Hermitian positive definite and S is invertible, we see that P is a Hermitian positive definite matrix. Therefore, when $\alpha > 0$, it holds that

$$\begin{aligned}
\varsigma(\alpha)^2 &= \|(I - \alpha P)(I + \alpha P)^{-1}\|^2 \\
&= \rho([(I - \alpha P)(I + \alpha P)^{-1}]^*[(I - \alpha P)(I + \alpha P)^{-1}]) \\
&= \max_{\|\boldsymbol{x}\|=1} \boldsymbol{x}^*(I + \alpha P^*)^{-1}(I - \alpha P^*)(I - \alpha P)(I + \alpha P)^{-1}\boldsymbol{x} \\
&= \max_{\|\boldsymbol{x}\|=1} \frac{\boldsymbol{x}^*(I - \alpha P)^2\boldsymbol{x}}{\boldsymbol{x}^*(I + \alpha P)^2\boldsymbol{x}} = \max_{\|\boldsymbol{x}\|=1} \frac{1 - 2\alpha \cdot \boldsymbol{x}^*P\boldsymbol{x} + \boldsymbol{x}^*P^2\boldsymbol{x}}{1 + 2\alpha \cdot \boldsymbol{x}^*P\boldsymbol{x} + \boldsymbol{x}^*P^2\boldsymbol{x}} < 1.
\end{aligned}$$

That is to say, $\rho(M(\alpha)) \leqslant \varsigma(\alpha) < 1,\ \forall\, \alpha > 0$.

If the lower and the upper bounds of the eigenvalues of the Hermitian positive definite matrix P are known, then the optimal parameter α for $\varsigma(\alpha)$ (or the upper bound of $\rho(M(\alpha))$) can be obtained. This fact is precisely stated as the following theorem.

Theorem 2. *Given invertible $A \in \mathbb{C}^{n\times n}$. Let $H = \frac{1}{2}(A^* + A)$ and $S = \frac{1}{2}(A^* - A)$. Assume that*

$$H \text{ is positive definite, } S \text{ is invertible.}$$

Let $\lambda_{\min}[P]$ and $\lambda_{\max}[P]$ be the minimum and the maximum eigenvalues of the matrix $P = -S^{-1}HS^{-1}$, respectively, and α be a positive constant. Then

$$\alpha_* \equiv \arg\min_{\alpha} \left\{ \max_{\lambda_{\min}[P] \leqslant \lambda \leqslant \lambda_{\max}[P]} \left| \frac{\alpha - \lambda}{\alpha + \lambda} \right| \right\} = \sqrt{\lambda_{\min}[P] \cdot \lambda_{\max}[P]},$$

and

$$\varsigma(\alpha_*) = \frac{\sqrt{\lambda_{\max}[P]} - \sqrt{\lambda_{\min}[P]}}{\sqrt{\lambda_{\max}[P]} + \sqrt{\lambda_{\min}[P]}} = \frac{\sqrt{\kappa(P)} - 1}{\sqrt{\kappa(P)} + 1},$$

where $\kappa(P)$ is the spectral condition number of P.

Proof. Now,

$$\varsigma(\alpha) = \max\left\{\left|\frac{\alpha - \lambda_{\min}[P]}{\alpha + \lambda_{\min}[P]}\right|, \left|\frac{\alpha - \lambda_{\max}[P]}{\alpha + \lambda_{\max}[P]}\right|\right\}.$$

To compute an approximate optimal $\alpha > 0$ such that the convergence factor $\rho(M(\alpha))$ of the SNS iteration is minimized, we can minimize the upper bound $\varsigma(\alpha)$ of $\rho(M(\alpha))$ instead. If α_* is such a minimum point, then it must satisfy $\alpha_* - \lambda_{\min}[P] > 0$, $\alpha_* - \lambda_{\max}[P] < 0$, and

$$\frac{\alpha_* - \lambda_{\min}[P]}{\alpha_* + \lambda_{\min}[P]} = \frac{\lambda_{\max}[P] - \alpha_*}{\lambda_{\max}[P] + \alpha_*}.$$

Therefore,

$$\alpha_* = \sqrt{\lambda_{\min}[P] \cdot \lambda_{\max}[P]},$$

and the result follows.

We remark that in the above discussions, the condition that S is invertible is not intrinsic and is removable by a slight modification of the original system of linear equations (1). More precisely, when the skew-Hermitian part S is singular, we can choose a small positive constant $\epsilon \in (0, \lambda_{\min}[H])$ such that $H_\epsilon = H - \epsilon I$ is still Hermitian positive definite and $S_\epsilon = S - \epsilon I$ is nonsingular. Then, we can equivalently transform the system of linear equations $A\boldsymbol{x} = \boldsymbol{b}$ into the one $(H_\epsilon - S_\epsilon)\boldsymbol{x} = \boldsymbol{b}$ and the skew-normal equations (4) into the one $(S_\epsilon H_\epsilon - S_\epsilon^2)\boldsymbol{x} = S_\epsilon \boldsymbol{b}$. Now, by directly applying the SNS iteration technique to this new skew-normal equations we can obtain the corresponding iteration scheme and theoretical results for this case.

4 The skew-scaling equations

Another way of using the skew-Hermitian matrix S is to employ it to scale the linear system (1). When $S = 0$, we use some appropriate standard methods. Otherwise, let $\alpha > 0$ be a scalar parameter, and we first add and then subtract $\frac{1}{\alpha}S^2\boldsymbol{x}$ to the fixed-point equation

$$H\boldsymbol{x} = S\boldsymbol{x} + \boldsymbol{b}, \tag{8}$$

see (2), to obtain

$$\left(H + \frac{1}{\alpha}S^2\right)\boldsymbol{x} = \left(I + \frac{1}{\alpha}S\right)S\boldsymbol{x} + \boldsymbol{b}, \quad \left(H - \frac{1}{\alpha}S^2\right)\boldsymbol{x} = \left(I - \frac{1}{\alpha}S\right)S\boldsymbol{x} + \boldsymbol{b}.$$

After rearranging these equalities and choosing $\boldsymbol{x}^{(0)}$ wisely, we can straightforwardly get the following iteration scheme:

$$(\alpha I + S)S\boldsymbol{x}^{(k+1/2)} = (\alpha H + S^2)\boldsymbol{x}^{(k)} - \alpha\boldsymbol{b}, \tag{9a}$$

$$(\alpha H - S^2)\boldsymbol{x}^{(k+1)} = (\alpha I - S)S\boldsymbol{x}^{(k+1/2)} + \alpha\boldsymbol{b}. \tag{9b}$$

The new extra term $S\boldsymbol{x}^{(k+1/2)}$ causes no extra work provided that we let $\boldsymbol{x}^{(k+1/2)} := S\boldsymbol{x}^{(k+1/2)}$ in (9a) and (9b). Therefore, in analogy to the HSS method we present the Skew-Scaling Splitting (SSS) method.

SSS method. Given an initial approximate solution $\boldsymbol{x}^{(0)}$ to (1), for $k = 0, 1, 2, \ldots$ until convergence solve

$$(\alpha I + S)\boldsymbol{x}^{(k+1/2)} = (\alpha H + S^2)\boldsymbol{x}^{(k)} - \alpha \boldsymbol{b}, \tag{10a}$$
$$(\alpha H - S^2)\boldsymbol{x}^{(k+1)} = (\alpha I - S)\boldsymbol{x}^{(k+1/2)} + \alpha \boldsymbol{b}, \tag{10b}$$

with given $\alpha > 0$.

Clearly, the coefficient matrices of SSS and SNS are exactly the same. The only difference between these two methods is on the constant vector terms which change from $S\boldsymbol{b}$ in SNS to $\alpha\boldsymbol{b}$ in SSS. Hence, SSS has the same convergence speed but is much cheaper than SNS. Note that for the SSS method we do not need to assume that H is positive definite and S is invertible, only $\alpha H + S^*S$ is positive definite. However, just like the SNS method, we can obtain stronger results about the SSS method when H is positive definite.

Convergence theory

From (10a) and (10b) we form the iteration matrix of the SSS method as

$$M(\alpha) := (\alpha H - S^2)^{-1}(\alpha I - S)(\alpha I + S)^{-1}(\alpha H + S^2).$$

Note that this matrix is exactly the same as the iteration matrix of the SNS method. Therefore, exactly following the demonstrations in Section 3, we can prove that Properties 1 and 2 for the SNS method also hold for the SSS method, and the convergence of the SSS method can be described by the following theorem.

Theorem 3. *Given invertible $A \in \mathbb{C}^{n\times n}$. Let $H = \frac{1}{2}(A^* + A)$ and $S = \frac{1}{2}(A^* - A)$. If*

$$H \text{ is positive definite}, \quad S \text{ is invertible},$$

then, for all $\alpha > 0$, the spectral radius of the iteration matrix $M(\alpha)$ of the SSS method is bounded by

$$\varsigma(\alpha) = \|(I + \alpha S^{-1}HS^{-1})(I - \alpha S^{-1}HS^{-1})^{-1}\|,$$

i.e., $\rho(M(\alpha)) \leqslant \varsigma(\alpha)$. Moreover, it holds that $\rho(M(\alpha)) \leqslant \varsigma(\alpha) < 1$, $\forall\, \alpha > 0$. That is to say, the SSS method converges to the unique solution of the system of linear equations (1).

Analogously to Theorem 2, if the lower and the upper bounds of the eigenvalues of the Hermitian positive definite matrix P are known, then the optimal parameter α for $\varsigma(\alpha)$ (or the upper bound of $\rho(M(\alpha))$) can be obtained. This fact is precisely stated as the following theorem.

Theorem 4. *Given invertible $A \in \mathbb{C}^{n\times n}$. Let $H = \frac{1}{2}(A^* + A)$ and $S = \frac{1}{2}(A^* - A)$. Assume that*

$$H \text{ is positive definite}, \quad S \text{ is invertible}.$$

Let $\lambda_{\min}[P]$ and $\lambda_{\max}[P]$ be the minimum and the maximum eigenvalues of the matrix $P = -S^{-1}HS^{-1}$, respectively, and α be a positive constant. Then

$$\alpha_* \equiv \arg\min_{\alpha} \left\{ \max_{\lambda_{\min}[P] \leqslant \lambda \leqslant \lambda_{\max}[P]} \left| \frac{\alpha - \lambda}{\alpha + \lambda} \right| \right\} = \sqrt{\lambda_{\min}[P] \cdot \lambda_{\max}[P]},$$

and

$$\varsigma(\alpha_*) = \frac{\sqrt{\lambda_{\max}[P]} - \sqrt{\lambda_{\min}[P]}}{\sqrt{\lambda_{\max}[P]} + \sqrt{\lambda_{\min}[P]}} = \frac{\sqrt{\kappa(P)} - 1}{\sqrt{\kappa(P)} + 1},$$

where $\kappa(P)$ is the spectral condition number of P.

We remark that in the above discussions, the condition that S is invertible is not intrinsic and is removable by a slight modification of the original system of linear equations (1). More precisely, when the skew-Hermitian part S is singular, we can choose a small positive constant $\epsilon \in (0, \lambda_{\min}[H])$ such that $H_\epsilon = H - \epsilon I$ is still Hermitian positive definite and $S_\epsilon = S - \epsilon I$ is nonsingular. Then, we can equivalently transform the system of linear equations $A\boldsymbol{x} = \boldsymbol{b}$ into the one $(H_\epsilon - S_\epsilon)\boldsymbol{x} = \boldsymbol{b}$ and the skew-scaling equations (8) into the one $S_\epsilon^{-*} H_\epsilon S_\epsilon^{-1} \widetilde{\boldsymbol{x}} = S_\epsilon^{-*} \widetilde{\boldsymbol{x}} + \widetilde{\boldsymbol{b}}$, with $\widetilde{\boldsymbol{x}} := S_\epsilon \boldsymbol{x}$ and $\widetilde{\boldsymbol{b}} := S_\epsilon^{-*} \boldsymbol{b}$. Now, by directly applying the SSS iteration technique to this new equations we can obtain the corresponding iteration scheme and theoretical results for this case.

5 Remarks

Theorems 1 and 3 show that the convergence speeds of both SNS and SSS iterations are bounded by $\varsigma(\alpha)$, which only depends on the spectrum of the Hermitian positive definite matrix $P := -S^{-1}HS^{-1}$, but does not depend on the spectrum of the coefficient matrix A, and neither on the eigenvectors of the matrices H, S, P and A. When the matrix P has a more clustered spectrum than the matrix H, both SNS and SSS iterations may show faster convergence speeds than the HSS iteration.

We emphasize that in Theorems 2 and 4 the optimal parameter α_* only minimizes the upper bound $\varsigma(\alpha)$ of the spectral radius of the iteration matrix, but does not minimize the spectral radius itself.

Theorems 2 and 4 also show that when the so-called optimal parameter α_* is employed, the upper bounds of the convergence rates of both SNS and SSS iterations are about the same as that of the conjugate gradient method associated with the Hermitian positive definite linear system $P\boldsymbol{x} = \boldsymbol{b}$.

Moreover, if we introduce a vector norm $|||\boldsymbol{x}||| = \|S^{-1}(\alpha H - S^2)\boldsymbol{x}\|$, $\forall\, \boldsymbol{x} \in \mathbb{C}^n$, and represent the induced matrix norm by

$$|||X||| = \|S^{-1}(\alpha H - S^2)X(\alpha H - S^2)^{-1}S\|, \qquad \forall\, X \in \mathbb{C}^{n\times n},$$

then from Theorems 1 and 3 we see that

$$\begin{aligned} |||M(\alpha)||| &= \|S^{-1}(\alpha I - S)(\alpha I + S)^{-1}(\alpha H + S^2)(\alpha H - S^2)^{-1}S\| \\ &= \|(I - \alpha S^{-1})(I + \alpha S^{-1})^{-1}(I + \alpha S^{-1}HS^{-1})(I - \alpha S^{-1}HS^{-1})^{-1}\| \\ &\leqslant \|(I + \alpha S^{-1}HS^{-1})(I - \alpha S^{-1}HS^{-1})^{-1}\| \\ &= \varsigma(\alpha), \end{aligned}$$

and it follows that

$$|||\boldsymbol{x}^{(k+1)} - \widehat{\boldsymbol{x}}||| \leqslant \varsigma(\alpha)|||\boldsymbol{x}^{(k)} - \widehat{\boldsymbol{x}}|||, \qquad k = 0, 1, 2, \ldots,$$

where $\widehat{\boldsymbol{x}}$ is the exact solution of the system of linear equations (1). Therefore, $\varsigma(\alpha)$ is also an upper bound of the contraction factors of both SNS and SSS iterations in the sense of $|||\cdot|||$-norm. It should be mentioned that when the coefficient matrix A is normal, we have $HS = SH$

and, therefore, $\rho(M(\alpha)) = |||M(\alpha)||| = \varsigma(\alpha)$. The optimal parameter α_* then minimizes all of these three quantities.

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