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On the existence and the uniqueness theorem for fractional differential equations with bounded delay within Caputo derivatives

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Abstract Local and global existence and uniqueness theorems for a functional delay fractional differential equation with bounded delay are investigated. The continuity with respect to the initial function is proved under Lipschitz and the continuity kind conditions are analyzed.

Keywords: fractional Caputo derivative, C_r-condition, delay, Lipschitz

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1 Introduction

Fractional calculus is an emerging field[1, 2] and it represents an alternative tool to solve several problems from various fields.

Many applications of fractional calculus amount to replacing the time derivative in a given evolution equation by a derivative of fractional order. The results of several studies clearly stated that the fractional derivatives seem to arise generally and universally from important mathematical reasons. Recently, an interesting attempt to give the physical meaning to the initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives was proposed in [3].

The fractional derivatives are non-local objects and they have less properties than the classical ones. This property makes these derivatives very useful to describe the anomalous phenomena[4−9]. For example the dissipative phenomena have been analyzed from the fractional Lagrangian and Hamiltonian point of view[10−16] and by replacing the classical derivatives with the fractional ones and keeping the classical potential local.

This leads us to the idea to replace the classical potential by a non-local one and implicitly to use the combination of fractional and delay[17, 18] techniques.

Most of the terminologies for the theory of delay differential equations and fractional differential equations used in this article are inherited from [2, 17]. For an interval $J = [a, b]$, we mean by $C[a, b]$ the space of all continuous functions defined on J with values in the m-Euclidean space $\mathbb{R}^m$. This is a Banach space with the norm $\|x\|_C = \max_{t\in J} \|x(t)\|,\ x \in C[a, b]$, where

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$\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^m$. For a natural number r, $C^r[a,b]$ will mean the space of all functions defined on J, which are r-times continuously differentiable, where $C^0[a,b]=C[a,b]$. This is also a Banach space with the norm $\|x\|_r=\sum_{k=0}^{r}\|x^{(k)}\|_C$. If D is a subset of $\mathbb{R}^m$, then $C^r([a,b],D)$ means the complete metric of all functions in $C^r[a,b]$ with values in D with the induced metric from $\|\cdot\|_r$.

If z is a function defined at least on $[t-\tau,t]\to\mathbb{R}^m$, then the new function $z_t:[-\tau,0]\to\mathbb{R}^m$, is defined by

$$z_t(s)=z(t+s)\quad\text{for}\quad -\tau\leqslant s\leqslant 0. \tag{1.1}$$

Clearly if z is an r-times continuously differentiable function, then $z_t\in C^r[-\tau,0]$. Throughout, for an interval $I=[-\tau,0]$, $\tau>0$, B and B_D stand for our state spaces $C^r[-\tau,0]$ and $C^r([-\tau,0],D)$, respectively.

A function $F:[a,\beta)\times B_D\to\mathbb{R}^m$ is said to satisfy the C_r-condition if $F(t,y_t)$ is continuous with respect to t in $[a,\beta)$ for each given r-times continuously differentiable function $y:[a-\tau,\beta)\to D$.

The map $F:[a,\beta)\times B_D\to\mathbb{R}^m$ is said to be r-Lipschitzian ($(r-1)$-Lipschitzian, respectively) with Lipschitz constant K if

$$\|F(t,\phi_1)-F(t,\phi_2)\|\leqslant K\|\phi_1-\phi_2\|_r,\quad\forall\,\phi_1,\phi_2\in B_D \tag{1.2}$$

and

$$\|F(t,\phi_1)-F(t,\phi_2)\|\leqslant K\|\phi_1-\phi_2\|_{r-1},\quad\forall\,\phi_1,\phi_2\in B_D, \tag{1.3}$$

respectively. F is said to be locally r-Lipschchzian (locally $(r-1)$-Lipschitzian, respectively) with Lipschitz constant K if for each given $(t,\phi)\in[a,\beta)\times B_D$ there exist $b_1>0$ and $b_2>0$ such that the set $A=[a-b_1,a+b_1]\times\{\psi\in B:\|\psi-\varphi\|_r\leqslant b_2\}$ is a subset of $[a,\beta)\times B_D$ and F is r-Lipschitian ($(r-1)$-Lipschitzian, respectively) on A.

For a function f defined on an interval $J=[a,\beta]$, the Riemann-Liouville integral $I^{\alpha}_{a+}f$ of order $\alpha>0$ is defined by [2]

$$(I^{\alpha}_{a+}f)(t)=\frac{1}{\Gamma(\alpha)}\int_a^t\frac{f(s)ds}{(t-s)^{1-\alpha}} \tag{1.4}$$

and Riemann-Liouville fractional derivative $\mathbf{D}^{\alpha}_{a+}f$ of order $\alpha>0$ is defined by

$$(\mathbf{D}^{\alpha}_{a+}f)(t)=\left(\frac{d}{dt}\right)^n(I^{n-\alpha}_{a+}f)(t), \tag{1.5}$$

where, $n=-[-\alpha]$ while Caputo-fractional derivative ${}^C\mathbf{D}^{\alpha}_{a+}f$ is defined by

$$({}^C\mathbf{D}^{\alpha}_{a+}f)(t)=(I^{n-\alpha}_{a+}f^{(n)})(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t\frac{f^{(n)}(s)ds}{(t-s)^{1+\alpha-n}}. \tag{1.6}$$

Consider the differential equation

$$({}^C\mathbf{D}^{\alpha}_{a+}y)=f[x,y(x)],\quad \alpha>0,\quad a\leqslant x\leqslant b \tag{1.7}$$

with the initial condition

$$y^{(k)}(a)=b_k,b_k\in\mathbb{R},\qquad k=0,1,\ldots,r. \tag{1.8}$$

Consider, also the Volterra integral equation

$$y(x) = \sum_{j=0}^{n-1} \frac{b_j}{j!}(x-a)^j + \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f[t, y(t)]}{(x-t)^{1-\alpha}} dt, \quad a \leqslant x \leqslant b. \tag{1.9}$$

Theorem 1.1. *Let $\alpha > 0$ and $n = -[-\alpha]$. Let G be an open set in $\mathbb{C}$ and let $f : (a, b] \times G \to \mathbb{C}$ be a function such that, for any $y \in G$, $f[x, y] \in C\gamma[a, b]$ with $0 \leqslant \gamma < 1$ and $\gamma \leqslant \alpha$. Let $l = n$ for $\alpha \in \mathbb{N}$ and $l = n - 1$ for $\alpha \notin \mathbb{N}$. If $y(x) \in C^r[a, b]$, then $y(x)$ satisfies the relations* (1.7) *and* (1.8) *if and only if $y(x)$ satisfies the Volterra integral equation* (1.9)*, where $C\gamma[a, b]$ is the weighted space of functions f on $(a, b]$ such that the function $(x - a)^\gamma f(x) \in C[a, b]$.*

In this paper, we will prove existence and uniqueness theorems for the following α-order Caputo fractional functional differential equation, with bounded delay:

$$({}^C\mathbf{D}^\alpha_{a+} x)(t) = F(t, x_t), \quad t \in [a, \beta), \quad \alpha > 0, \tag{1.10}$$

$$x_a = \phi, \tag{1.11}$$

where ϕ is an element of the state space $B_D = C^r([-\tau, 0], D)$, with D being any subset of $\mathbb{R}^m$. Throughout this paper, $n = -[-\alpha]$, $r = n - 1$.

We mean, by a solution of (1.10) and (1.11), a function $x \in C^r[a - \tau, \beta]$ such that

$$\begin{aligned} &\text{(i)} \quad x(t) \equiv \phi(t-a) \text{ in } [a-\tau, a], \\ &\text{(ii)} \quad x(t) \text{ satisfies (1.10) in } [a, \beta_1), \text{ for some } \beta_1 \leqslant \beta. \end{aligned} \tag{1.12}$$

These properties are important from the numerical methods point of view[19], especially by the matrix approach to discrete fractional calculus which has been initiated very recently by Podlubny[20].

2 Main results

In view of Theorem 1.1 of [2] the following lemma is valid for the representation of solution of (1.10) and (1.11).

Lemma 2.1. *If $F : [a, \beta) \times B_D \to \mathbb{R}^m$ satisfies the C_r-condition, then an r-times continuously differentiable function x mapping $[a, \beta_1) \to D$, for some $\beta_1 \in (a, \beta]$, is a solution of* (1.10) *and* (1.11) *if and only if*

$$x(t) = \begin{cases} \phi(t-a), & \textit{for} \quad a - \tau \leqslant t \leqslant a, \\ y_0(t) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^t \frac{F(s, x_s)}{(t-s)^{1-\alpha}} ds, & \textit{for} \quad a \leqslant t < \beta_1, \end{cases} \tag{2.1}$$

where

$$y_0(t) = \sum_{j=0}^{n-1} \frac{\phi^{(j)}(0)}{j!}(t-a)^j.$$

Lemma 2.2. *Let $x : [a - \tau, \beta) \to \mathbb{R}^m$ be continuous. Then, given any $\overline{t} \in [a, \beta)$ and any $\epsilon > 0$, there exists $\delta > 0$ such that $\|x_t - x_{\overline{t}}\|_C < \epsilon$ whenever $t \in [a, \beta)$ and $|t - \overline{t}| < \delta$.*

The proof can be done by contradiction and using Bolzano-Weierstrass theorem.

Corollary 2.3. *Let $x \in C^r[a-\tau,\beta]$. Then, given any $\overline{t} \in [a,\beta)$ and any $\epsilon > 0$, there exists $\delta > 0$ such that $\|x_t - x_{\overline{t}}\|_r < \epsilon$ whenever $t \in [a,\beta)$ and $|t-\overline{t}| < \delta$.*

Proof. It is proved by applying Lemma 2.2 to the continuous functions $x^{(i)}, i = 0,1,\ldots,r$.

Lemma 2.4 *Let $x \in C^r[a,\beta)$ be such that $\|({}^C\mathbf{D}^\alpha_{a+}x)(t)\| \leqslant M, \forall\, t \in [a,\beta)$ and $({}^C\mathbf{D}^\alpha_{a+}x)(t)$ is continuous in $[a,\beta)$. Then $\lim_{t\to\beta} x(t)$ exists.*

Proof. Let t_i be a sequence in $[a,\beta)$ such that $\lim_{i\to\infty} t_i = \beta$. Then for any two naturals $i \geqslant j$ the assumptions lead to

$$\begin{aligned}\|x(t_i)-x(t_j)\| &\leqslant \frac{1}{\Gamma(\alpha)}\left\|\int_a^{t_i}\frac{({}^CD^\alpha_{a+}x)(s)}{(t_i-s)^{1-\alpha}}ds - \int_a^{t_j}\frac{({}^CD^\alpha_{a+}x)(s)}{(t_j-s)^{1-\alpha}}ds\right\| \\ &\quad + \sum_{k=0}^{n-1}|(t_i-a)^k-(t_j-a)^k|\frac{\|x^{(k)}(a)\|}{k!} \\ &\leqslant \frac{M}{\Gamma(\alpha)}\left|\frac{(t_i-a)^k}{\alpha}-\frac{(t_j-a)^k}{\alpha}\right| + \sum_{k=0}^{n-1}|(t_i-a)^k-(t_j-a)^k|\frac{\|x^{(k)}(a)\|}{k!},\end{aligned}$$

from which it follows that $x(t_i)$ is a Cauchy sequence in $\mathbb{R}^m$, hence convergent. Let $\lim_{t\to\infty} x(t_i) = z = (z_1,z_2,\ldots,z_m)$. We have not yet shown that $\lim_{t\to\beta} x(t) = z$. To establish this, let $\epsilon > 0$ be given and consider any $t \in (\beta-\delta,\beta)$, where $\delta = \frac{\epsilon}{2M}$. Then from $\lim_{t\to\infty} x(t_i) = z = (z_1,z_2,\ldots,z_m)$, we can choose some $t_i \in (\beta-\delta,\beta)$ such that $|z_k - x_k(t_i)| < \frac{\epsilon}{2}$ for each $k = 1,2,\ldots,m$. Then by the help of mean value theorem we obtain

$$|z_k - x_k(t)| \leqslant |z_k - x_k(t_i)| + |x_k(t_i) - x_k(t)| \leqslant \frac{\epsilon}{2} + M\frac{\epsilon}{2M} = \epsilon.$$

This shows that $\lim_{t\to\beta} x(t) = z$.

The following theorem asserts, under the assumption of a global Lipschitzian condition, that the solutions of (1.10), with $\alpha \geqslant 1$, together with their $r-1$ derivatives if $\alpha \in \mathbb{R}-\mathbb{N}$ and r derivatives if $\alpha \in \mathbb{N}$ depend continuously on the initial function.

Theorem 2.5. *Let $F : [a,\beta) \times B_D \to \mathbb{R}^m$ satisfy the (C_r)-condition. Let ϕ and ψ be given functions in B_D and let x and y be unique solutions of (10) with $x_a = \phi$ and $y_a = \psi$. If both x and y are valid on $[a-\tau,\beta_1)$, then*

(a) *if $\alpha \in (1,\infty)-\mathbb{N}$ and F is $(r-1)$-Lipschitz condition, then for $k = 0,1,\ldots,r-1$ we have*

$$\|x^{(k)}(t) - y^{(k)}(t)\| \leqslant C_1\|\phi-\psi\|_r e^{C_2(t-a)} \quad \text{for} \quad a \leqslant t < \beta_1, \tag{2.2}$$

where

$$C_1 = r \max_{0\leqslant k\leqslant r} \max\{(\beta_1-a)^{j-k} : k \leqslant j \leqslant r\}$$

and

$$C_2 = r\max\left\{\frac{K(\beta_1-a)^{\alpha-k-1}}{\Gamma(\alpha-k)} : k = 0,1,\ldots,r-1\right\}.$$

(b) *if $\alpha \in \mathbb{N}$ and F is r-Lipschitzian, then for $k = 0,1,\ldots,r$*

$$\|x^{(k)}(t) - y^{(k)}(t)\| \leqslant C_1'\|\phi-\psi\|_r e^{C_2'(t-a)} \quad \text{for} \quad a \leqslant t < \beta_1, \tag{2.3}$$

where

$$C_1' = (r+1) \max_{0\leqslant k\leqslant r} \max\{(\beta_1-a)^{j-k} : k \leqslant j \leqslant r\}$$

and

$$C_2' = (r+1)\max\left\{\frac{K(\beta_1-a)^{\alpha-k-1}}{\Gamma(\alpha-k)} : k = 0,1,\dots,r\right\}.$$

Proof. First note that for each $k = 0,1,\dots,r$ and for $a \leqslant t < \beta_1$ the solution x of (1.10) satisfies

$$x^{(k)}(t) = y_0^{(k)}(t) + \frac{1}{\Gamma(\alpha-k)}\int_a^t \frac{F(s,x_s)}{(t-s)^{(1-\alpha+k)}}ds, \tag{2.4}$$

where

$$y_0^k(t) = \sum_{j=k}^{r} \frac{\phi^{(k)}(0)}{(j-k)!}(t-a)^{j-k}.$$

We will do (a) and (b) in a similar way. Indeed, for $t \in [a,\beta)$ and $k = 0,1,\dots,r-1$, we have

$$\begin{aligned}\|x^{(k)}(t) - y^{(k)}(t)\| &\leqslant \sum_{j=k}^{r} \frac{\|\phi^{(k)}(0) - \psi^{(k)}(0)\|}{(j-k)!}|t-a|^{j-k} \\ &\quad + \frac{1}{\Gamma(\alpha-k)}\int_a^t \|F(s,x_s) - F(s,y_s)\| |t-s|^{\alpha-k-1}ds \\ &\leqslant \max\{(\beta_1-a)^{j-k} : k \leqslant j \leqslant r\}\|\phi-\psi\|_r \\ &\quad + \frac{K}{\Gamma(\alpha-k)}(\beta_1-a)^{\alpha-k-1}\int_a^t \|x_s-y_s\|_{r-1}ds. \end{aligned} \tag{2.5}$$

Therefore, noting that the right-hand side of the above inequality is an increasing function of t, for $a \leqslant t < \beta_1$, we get

$$\begin{aligned}\|x_t^{(k)} - y_t^{(k)}\|_C &\leqslant \max\{(\beta_1-a)^{j-k} : k \leqslant j \leqslant r\}\|\phi-\psi\|_r \\ &\quad + \frac{K}{\Gamma(\alpha-k)}(\beta_1-a)^{\alpha-k-1}\int_a^t \|x_s-y_s\|_r ds\end{aligned}$$

and by summing up $k = 0,1,\dots,r-1$, we get

$$\|x_t - y_t\|_{r-1} \leqslant C_1\|\phi-\psi\|_{r-1} + C_2\int_a^t \|x_s-y_s\|_{r-1}ds. \tag{2.6}$$

Then, by using Grownwall's lemma, (2.19) implies, for $a \leqslant t < \beta_1$,

$$\|x_t - y_t\|_{r-1} \leqslant C_1\|\phi-\psi\|_r e^{C_2(t-a)}, \tag{2.7}$$

from which (2.2) follows with the desired C_1 and C_2.

Corollary 2.6. *Let $F : [a,\beta) \times B_D \to \mathbb{R}^m$ satisfy the C_r-condition and let it be r-Lipschitzian if $\alpha \in \mathbb{N}$ and $(r-1)$-Lipschitzian if $\alpha \in (1,\infty) - \mathbb{N}$ (with Lipschitz constant K). Then, given any $\phi \in B_D$, (1.10) and (1.11) have at most one solution on $[a-\tau,\beta_1)$ for any $\beta_1 \in (a,\beta]$.*

We can also obtain a uniqueness theorem by replacing Lipschitzian by locally one. This can be done by the help of the corollary of Lemma 2.2.

Theorem 2.7. *Let $F : [a,\beta) \times B_D \to \mathbb{R}^m$ satisfy the (C_r)-condition and let it be r-locally Lipschitzian if $\alpha \in \mathbb{N}$ and $(r-1)$-locally Lipschitzian if $\alpha \in (1,\infty) - \mathbb{N}$ (with Lipschitz constant K). Then, given any $\phi \in B_D$, (1.10) and (1.11), $\alpha \geqslant 1$ have at most one solution on $[a-\tau,\beta_1)$ for any $\beta_1 \in (a,\beta]$.*

Proof. Suppose (for contradiction) that for some $\beta_1 \in (a,\beta]$ there are two solutions x and y mapping $[a-\tau,\beta_1)$ into D with $x \neq y$. Let $t_1 = \inf\{t \in (a,\beta_1) : x(t) \neq y(t)\}$. Then, $a \leqslant t_1 < \beta_1$ and $x(t) = y(t)$, for $t \in [a-\tau, t_1]$.

Since $(t_1, x_{t_1}) \in [a,\beta_1) \times B_D$, there exist numbers $b_1 > 0$ and $b_2 > 0$ such that the set

$$A = [t_1, t_1 + b_1] \times \{\psi \in B : \|\psi - x_{t_1}\|_r \leqslant b_2\}$$

is contained in $[a,\beta_1)\times B_D$ and F is $(r-1)$-Lipschitzian on A if $\alpha \in (1,\infty)-\mathbb{N}$ and r-Lipschitzian on A if $\alpha \in \mathbb{N}$ with Lipschitz constant K.

By Lemma 2.2, there exists $\delta \in (0, b_1]$ such that $(t, x_t) \in A$ and $(t, y_t) \in A$, for $t_1 \leqslant t < t_1+\delta$. Then, as we did in the proof of Theorem 2.5 above and by using that $\|x(t) - y(t)\| = 0$ for $t_1-\tau \leqslant t \leqslant t_1$, we obtain, $t \in [t_1, t_1+\delta)$, $\|x_t-y_t\|_{r-1} \leqslant C_2 \int_{t_1}^{t} \|x_s-y_s\|_{r-1} ds$ for $\alpha \in (1,\infty)-\mathbb{N}$ and $\|x_t - y_t\|_r \leqslant C_2 \int_{t_1}^{t} \|x_s - y_s\|_r ds$ for $\alpha \in \mathbb{N}$. From this and Grownwall's lemma, it follows that $x(t) = y(t)$ on $[t_1, t_1 + \delta)$, contradicting the definition of t_1.

Theorem 2.8. *Let $F : [a,\beta) \times B_D \to \mathbb{R}^m$ satisfy the (C_r)-condition and let it be locally r-Lipschitzian. Then, for each $\phi \in B_D$, (1.10) and (1.11), with $\alpha \in (0,\infty)$, have a unique solution $y \in C^r[a-\tau, a+\Delta)$ for some $\Delta > 0$. Moreover, $({}^C\mathbf{D}^{\alpha}_{a+}y)(t) \in C[a, a+\Delta]$.*

Proof. Choose $b_1 > 0$ and $b_2 > 0$ such that $A = [a_1, a+b_1] \times \{\psi \in B : \|\psi - \varphi\|_r \leqslant b_2\}$ is a subset of $[a,\beta) \times B_D$ and F is r-Lipscitzian on A. Define an r-times continuously differentiable function $y : [a-\tau, a+b_1] \to \mathbb{R}^m$ by

$$y(t) = \begin{cases} \phi(t-a) & a-\tau \leqslant t \leqslant a; \\ y_0(t) & a \leqslant t \leqslant a+b_1, \end{cases} \tag{2.8}$$

where

$$y_0(t) = \sum_{j=0}^{n-1} \frac{\phi^{(j)}(0)}{j!}(t-a)^j.$$

Then, the C_r-condition implies that $F(t, y_t)$ depends continuously on t, and hence there exists $M_1 > 0$ such that for all $t \in [a, a+b_1]$ we have $\|F(t,y_t)\| \leqslant M_1$. Let $M = \frac{Kb_2}{2} + M_1$. By Corollary 2.3, choose $b_1' \in (0, b_1)$ such that $\|y_t - \phi\|_r = \|y_t - y_a\|_r \leqslant \frac{b_2}{2}$, $\forall\, t \in [a, a+b_1']$.

Choose $\Delta > 0$ such that

$$\sum_{k=0}^{n-1} \frac{\Delta^{\alpha-k}}{\Gamma(\alpha-k+1)} < \min\left\{\frac{1}{K}, \frac{b_2}{2M\rho}\right\}, \quad \Delta < b_1',$$

where, $\rho = \max\{\Gamma(\alpha), \Gamma(\alpha+1-n)\}$.

Let S be the set of all r-times continuously differentiable functions $z : [a-\tau, a+\Delta] \to \mathbb{R}^m$ such that

$$\begin{cases} \phi(t-a), & a-\tau \leqslant t \leqslant a; \\ \|y-y_0\|_r \leqslant \dfrac{b_2}{2}, & a \leqslant t \leqslant a+\Delta. \end{cases} \tag{2.9}$$

Note that if $x \in S$ and $t \in [a, a+\Delta]$, then $\|x_t - y_t\| \leqslant \frac{b_2}{2}$; so that $\|x_t - \phi\|_r \leqslant b_2$, $\|y_t - \phi\|_r \leqslant b_2$ and

$$\|F(t,x_t)\| \leqslant \|F(t,x_t) - F(t,y_t)\| + \|F(t,y_t)\| \leqslant K\|x_t - y_t\| + M_1 \leqslant \frac{Kb_2}{2} + M_1 = M. \tag{2.10}$$

Define $T : S \to S$ by

$$(Tx)(t) = \begin{cases} \phi(t-a), & a-\tau \leqslant t \leqslant a, \\ y_0(t) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^t \dfrac{F(s,x_s)}{(t-s)^{1-\alpha}} ds, & a \leqslant t \leqslant a+\Delta. \end{cases}$$

For $x \in S$, we have

$$\begin{aligned} \|Tx - y_0\|_r &= \sum_{k=0}^{r} \left\| \frac{1}{\Gamma(\alpha)} \int_a^t \frac{F(s,x_s)}{(t-s)^{1-\alpha+k}} ds \right\|_C \leqslant M \sum_{k=0}^{r} \frac{\Delta^{\alpha-k}}{\alpha-k} \\ &\leqslant M\rho \sum_{k=0}^{r} \frac{\Delta^{\alpha-k}}{\Gamma(\alpha-k+1)} \leqslant \frac{b_2}{2}. \end{aligned} \tag{2.11}$$

By the help of the identity

$$\|I_{a+}^{\alpha} g\|_C \leqslant \frac{1}{\Gamma(\alpha+1)} \|g\|_C, \quad g \in C[a, a+\Delta], \tag{2.12}$$

we conclude that each $(Tx)^{(k)}$ is continuous. Hence, $Tx \in S$, for each $x \in S$. It remains to show that T is contraction and then apply Banach-Fixed Point Theorem to obtaining a unique solution in $C^r[a-\tau, a+\Delta]$. Indeed, for any $x, y \in S$ we have

$$\|Tx - Ty\|_r = \sum_{k=0}^{r} \|(Tx)^k - (Ty)^k\|_C. \tag{2.13}$$

But then, by the definition of T, (2.12), the Lipschitzian of F on A and using that $\max_{s\in[a,a+\Delta]} \|x_s - y_s\|_r = \|x-y\|_r$, we obtain

$$\|Tx - Ty\|_r \leqslant K \sum_{k=0}^{r} \frac{\Delta^{\alpha-k}}{\Gamma(\alpha-k+1)} \|x-y\|_r < \|x-y\|_r. \tag{2.14}$$

Hence, by Banach-Fixed Point Theorem, there exists a unique solution $y = \lim_{m\to\infty} y_m$ in $C^r[a-\tau, a+\Delta]$.

$$\lim_{m\to\infty} \|y_m - y\|_r = 0, \tag{2.15}$$

where $y_m = T^m z_0$, for some $z_0 \in C^r[a, a+\Delta]$. But then, taking into account (1.10), the C_r-condition, and the r-Lipschitz condition we have

$$\begin{aligned} &\|({}^C\mathbf{D}_{a+}^{\alpha} y_m)(t) - ({}^C\mathbf{D}_{a+}^{\alpha} y)(t)\|_C \\ &\quad = \|F(t, y_{m_t}) - F(t, y_t)\|_C = \max_{t\in[a,a+\Delta]} \|F(t, y_{m_t}) - F(t, y_t)\| \\ &\quad \leqslant K \max_{t\in[a,a+\Delta]} \|y_{m_t} - y_t\|_r = K\|y_m - y\|_r. \end{aligned} \tag{2.16}$$

From this and (2.15), we obtain

$$\lim_{m\to\infty} \|({}^C\mathbf{D}_{a+}^{\alpha} y_m)(t) - ({}^C\mathbf{D}_{a+}^{\alpha} y)(t)\|_C = 0. \tag{2.17}$$

Thus, $({}^C\mathbf{D}_{a+}^{\alpha} y)(t) \in C[a, a+\Delta]$.

Definition 2.9. *The functional $F : [a, \beta) \times B_D \to \mathbb{R}^m$ is said to be quasi-bounded if it is bounded on every set of the form $[a, \beta_1] \times B_A$, where $a < \beta_1 < \beta$ and A is a closed bounded subset of D.*

The following preliminary assertion will be useful in extending the solution.

Lemma 2.10. *Let $a < c < b$, $g \in C^r[a, c]$, and $g \in C^r[c, b]$. Then $g \in C^r[a, b]$ and*

$$\|g\|_r^{[a,b]} \leqslant \max\{\|g\|_r^{[a,c]}, \|g\|_r^{[c,b]}\}. \tag{2.18}$$

Theorem 2.11. *Let $F : [a, \beta) \times B_D \to \mathbb{R}^m$ satisfy the C_r-condition and let it be locally r-Lipschitzian and quasi-bounded. Then for each $\phi \in B_D$,* (1.10) *and* (1.11) *have a unique r-times continuously differentiable solution $x : [a - \tau, \beta_1) \to D$ and if $\{x(t) : t \in [a - \tau, \beta_1)\}$ is contained in some closed bounded subset of D, then $\beta_1 = \beta$.*

Proof. Define $\beta_1 = \sup\{s : \text{a unique solution exists on } [a - \tau, s)\}$. Then by Theorem 2.8 such a β_1 exists and for every $s \in (a, \beta_1)$ a unique solution $y_{(s)}$ exists on $[a - \tau, s)$. Define a function $x : [a-\tau, \beta_1) \to D$ as follows. For each $t \in [a-\tau, \beta_1)$, let $x(t) = y_{(s)}(t)$. Then such x is a unique noncontinuable solution for (1.10) and (1.11). Now the proof will be done by contradiction. Assume $\beta_1 < \beta$. Let $A' = A \bigcup \{\phi(t) : t \in [-\tau, a]\}$. Then A' is a closed and bounded subset of D with $\{x(t) : t \in [a - \tau, \beta_1]\} \subseteq A'$. Therefore, by quasi-boundedness of F find $M > 0$ such that

$$\|F(t, \psi)\| \leqslant M, \quad \text{for all } (t, \psi) \in [a, \beta_1] \times B_{A'}. \tag{2.19}$$

Hence,

$$\|({}^C\mathbf{D}_{a+}^{\alpha} x(t))\| = \|F(t, x_t)\| \leqslant M, \quad \text{for all } t \in [a, \beta_1), \tag{2.20}$$

and by Lemma 2.4, $\lim_{t\to\beta_1} x(t) = z \in A \subseteq D$. Actually, Lemma 2.4 can be improved by using (2.4) to guarantee the existence of $\lim_{t\to\beta_1} x^{(k)}(t)$, for all $k = 0, 1, \ldots, r$. Rewrite (2.1), for $t \geqslant a$, in the form

$$y(t) = y_1(t) + \frac{1}{\Gamma(\alpha)} \int_{\beta_1}^{t} \frac{F(s, y_s)}{(t - s)^{1-\alpha}} ds, \tag{2.21}$$

where the function

$$y_1(t) = y_0(t) + \frac{1}{\Gamma(\alpha)} \int_{a}^{\beta_1} \frac{F(s, y_s)}{(t - s)^{1-\alpha}} ds \tag{2.22}$$

is uniquely determined in $[a, \beta_1]$. Using the same argument as in Theorem 2.8 we find a unique solution in $x_1 \in C^r[\beta_1 - \tau, \beta_1 + \Delta]$, for some $\Delta > 0$. Assume also that $x_1(t) \equiv x(t)$ on $[a - \tau, \beta_1 - \tau]$. Then by using Lemma 2.10 we get a unique solution to (1.10) and (1.11) on $[a - \tau, \beta_1 + \Delta)$, which contradicts the definition of β_1.

Corollary 2.12. *Let $F : [a, \beta) \times B \to \mathbb{R}^m$ satisfy the C_r-condition and let it be r-locally Lipschitzian. Let $\alpha \geqslant 1$ and $M(t)$, $N(t)$ are continuous positive functions on $[a, \beta)$.*

(a) *If $\alpha \in \mathbb{N}$ and*

$$\|F(t, \varphi)\| \leqslant M(t) + N(t)\|\varphi\|_r, \quad \textit{for all } (t, \varphi) \in [a, \beta) \times B, \tag{2.23}$$

then the unique noncontinuable solution of (1.10) *and* (1.11) *exists on the entire interval $[a, \beta)$.*

(b) *If $\alpha \in (1, \infty) - \mathbb{N}$ and*

$$\|F(t, \varphi)\| \leqslant M(t) + N(t)\|\varphi\|_{r-1}, \quad \textit{for all } (t, \varphi) \in [a, \beta) \times B, \tag{2.24}$$

then also the unique noncontinuable solution of (1.10) *and* (1.11) *exists on the entire interval* $[a, \beta)$.

Proof. Either of the conditions (2.23) and (2.24) implies that F is quasi-bounded. Let x be the unique noncontinuable solution of (1.10) and (1.11), and suppose that $\beta_1 < \beta$. Then, there exist $M_1, N_1 > 0$ such that $M(t) \leqslant M_1$ and $N(t) \leqslant N_1$ for all $t \in [a, \beta_1]$.

(a) Equation (2.4) implies that for $t \in [a, \beta_1)$ and $k = 0, 1, \ldots, r-1$, we have

$$\begin{aligned}\|x^{(k)}(t)\| &\leqslant \sum_{j=k}^{r} \frac{\|\phi^{(k)}(0)\|}{(j-k)!}|t-a|^{j-k} + \frac{1}{\Gamma(\alpha-k)}\int_a^t \|F(s, x_s)\| \cdot |t-s|^{\alpha-k-1} ds \\ &\leqslant \max\{(\beta_1 - a)^{j-k} : k \leqslant j \leqslant r\}\|\phi\|_r + \frac{(\beta_1-a)^{\alpha-k-1}}{\Gamma(\alpha-k)}\int_a^t (M_1 + N_1\|x_s\|_{r-1}) ds.\end{aligned}$$

Therefore, noting that the right-hand side of the above inequality is an increasing function of t for $a \leqslant t < \beta_1$, we get

$$\begin{aligned}\|x_t^{(k)}\|_C &\leqslant \max\{(\beta_1 - a)^{j-k} : k \leqslant j \leqslant r\}\|\phi\|_{r-1} + \frac{M_1\beta_1(\beta_1-a)^{\alpha-k-1}}{\Gamma(\alpha-k)} \\ &\quad + \frac{N_1}{\Gamma(\alpha-k)}(\beta_1-a)^{\alpha-k-1}\int_a^t \|x_s\|_{r-1} ds \end{aligned} \tag{2.25}$$

and by summing up $k = 0, 1, \ldots, r-1$, we get

$$\|x_t\|_{r-1} \leqslant H_1 + L_1 \int_a^t \|x_s\|_{r-1} ds, \tag{2.26}$$

where

$$\begin{aligned}H_1 &= r \max_{0 \leqslant k \leqslant r-1} \max\{(\beta_1 - a)^{j-k} : k \leqslant j \leqslant r\}\|\phi\|_{r-1} \\ &\quad + r\beta_1 M_1 \max\left\{\frac{(\beta_1-a)^{\alpha-k-1}}{\Gamma(\alpha-k)} : 0 \leqslant k \leqslant r-1\right\}\end{aligned} \tag{2.27}$$

and

$$L_1 = rN_1 \max\left\{\frac{(\beta_1-a)^{\alpha-k-1}}{\Gamma(\alpha-k)} : 0 \leqslant k \leqslant r-1\right\}.$$

Then, by using Grownwall's lemma, (2.26) implies, for $a \leqslant t < \beta_1$

$$\|x_t\|_{r-1} \leqslant H_1 e^{L_1(\beta_1 - a)}, \tag{2.28}$$

from which we obtain

$$\|x(t)\| \leqslant H_1 e^{L_1(\beta_1 - a)} \quad \text{for all } t \in [a, \beta_1). \tag{2.29}$$

This shows that the solution $x(t)$ remains in a closed bounded set, contradicting that $\beta_1 < \beta$. Hence, $\beta_1 = \beta$.

(b) can be done in a similar way as (a). Indeed, we can show that

$$\|x(t)\| \leqslant \|x_t\|_r \leqslant H_2 e^{L_2(\beta_1 - a)} \quad \text{for all } t \in [a, \beta_1), \tag{2.30}$$

where

$$H_2 = (r+1) \max_{0 \leqslant k \leqslant r} \max\{(\beta_1 - a)^{j-k} : k \leqslant j \leqslant r\}\|\phi\|_r$$

$$+ (r+1)\beta_1 M_1 \max\left\{\frac{(\beta_1 - a)^{\alpha-k-1}}{\Gamma(\alpha-k)} : 0 \leqslant k \leqslant r\right\} \tag{2.31}$$

and

$$L_2 = (r+1)N_1 \max\left\{\frac{(\beta_1 - a)^{\alpha-k-1}}{\Gamma(\alpha-k)} : 0 \leqslant k \leqslant r\right\}. \tag{2.32}$$

Corollary 2.13. *If $F : [a,\beta) \times B \to \mathbb{R}^m$ satisfies the C_r-condition. Then*

(a) *If F is $(r-1)$-Lipschitzian, then for $\alpha \in (1,\infty) - \mathbb{N}$, and for each $\phi \in B$,* (1.10) *and* (1.11) *have a unique solution on $[a,\beta)$.*

(b) *If F is r-Lipschitzian, then for $\alpha \in \mathbb{N}$, and for each $\phi \in B$,* (1.10) *and* (1.11) *have a unique solution on $[a,\beta)$.*

Proof. The $(r-1)$-Lipschitzian (r-Lipscitzian, respectively) implies the condition (2.23) ((2.24) respectively). For example, when F is r-Lipscitzian with constant K, we have

$$\|F(t,\varphi)\| \leqslant \|F(t,0)\| + \|F(t,\varphi) - F(t,0)\| \leqslant \|F(t,0)\| + K\|\phi\|_r. \tag{2.33}$$

Note that although F satisfies the C_r-condition, we could not use that it is bounded as a function of t on the half open interval $[a,\beta)$, so that in the proof of Corollary 2.12 it is not possible to put the solution inside a closed bounded set for the case, $\alpha \in (0,1)$. However, if we assume that the functional F is defined on $J \times B_D$, where J is the closed bounded interval $[a,\beta]$, then under the assumption it is r-Lipschitzian with the C_r-condition satisfied, we can benefit by its boundedness and obtain a unique solution in the whole interval J. Namely, we obtain the following result.

Theorem 2.14. *Let $[a,\beta] \times B_D \to \mathbb{R}^m$ be r-Lipschitzian with Lipschitz constant K and satisfy the C_r-condition. Then for each $\phi \in B$ there exists a unique solution $x \in C^r[a,\beta]$ for* (1.10) *and* (1.11). *Moreover, $({}^C\mathbf{D}^{\alpha}_{a+}x)(t) \in C[a,\beta]$.*

Proof. Choose $a < t_1 < \beta$ such that

$$K \sum_{k=0}^{r} \frac{(t_1 - a)^{\alpha-k}}{\Gamma(\alpha-k+1)} < 1. \tag{2.34}$$

Then, define $T : C^r[a-\tau, t_1] \to C^r[a-\tau, t_1]$ by

$$(Tx)(t) = \begin{cases} \phi(t-a), & a-\tau \leqslant t \leqslant a, \\ y_0(t) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^t \frac{F(s,x_s)}{(t-s)^{1-\alpha}} ds, & a \leqslant t \leqslant t_1. \end{cases}$$

Then

$$\|Tx - Ty\|_r \leqslant K \sum_{k=0}^{r} \frac{(t_1 - a)^{\alpha-k}}{\Gamma(\alpha-k+1)} \|x-y\|_r.$$

Hence, by the choice of t_1, we conclude that T is contraction and the result follows from Banach-Fixed Point Theorem. The solution can be continued to another subinterval $[a-\tau, t_2]$, $t_2 > t_1$ and then to whole $[a-\tau,\beta]$ by rewriting (2.1) in the form

$$y(t) = y_1(t) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t \frac{F(s,y_s)}{(t-s)^{1-\alpha}} ds, \tag{2.35}$$

where the function

$$y_1(t) = y_0(t) + \frac{1}{\Gamma(\alpha)} \int_a^{t_1} \frac{F(s, y_s)}{(t-s)^{1-\alpha}} ds, \tag{2.36}$$

and proceed as done in the proof of Theorem 2.11. Continuity of $({}^C\mathbf{D}_{a+}^{\alpha} x)(t)$ on $[a, \beta]$ also can be proved as in the proof of Theorem 2.8.

Recall that a function $f : J \times D^{r+1} \to \mathbb{R}^m$, where D is a subset of $\mathbb{R}^m$, is called Lipschitz, if there exists $K > 0$ such that for all $z = (z_0, z_1, z_2, \ldots, z_r), w = (w_0, w_1, w_2, \ldots, w_r) \in D^{r+1}$, we have

$$\|f(t, z_0, z_1, z_2, \ldots, z_r) - f(t, w_0, w_1, w_2, \ldots, w_r)\| \leqslant K \sum_{i=0}^{r} \|z_i - w_i\|, \quad \text{for all } t \in J.$$

Also recall that f is called locally Lipschitz if for each $(t_0, z_0, z_1, z_2, \ldots, z_r) \in J \times D^{r+1}$ there exist $b_1 > 0, b_2 > 0$ such that each

$$A_i \triangleq \{z \in \mathbb{R}^m : \|z - z_i\| \leqslant b_2\}, \quad i = 0, 1, 2, \ldots, r \tag{2.37}$$

is a subset of D and f is Lipschitz on the set

$$A \triangleq ([t_0 - b_1, t_0 + b_1] \cap J) \times A_0 \times A_1 \times A_2 \times \cdots \times A_r.$$

The following lemma which is an improved version of Lemma 25.B in [17], will be used in the next example

Lemma 2.15. *Let $\phi \in B_D$ be given. Then there exists $\delta > 0$ such that*

$$\left\{ z \in \mathbb{R}^m : \sum_{i=0}^{r} \|z - \phi^{(i)}(s)\| \leqslant \delta \text{ for some } s \in [-\tau, 0] \right\}$$

is a subset of D, and hence in particular $\{\psi \in B : \|\psi - \phi\|_r \leqslant \delta\} \subset B_D$.

Example 2.16. Consider the α- order fractional differential equation (1.10) with

$$F(t, \psi) = f(t, \psi(-\tau), \psi'(-\tau), \ldots, \psi^{(r)}(-\tau)), \quad \text{for all } (t, \psi) \in J \times B_D. \tag{2.38}$$

Then, $F(t, x_t) = f(t, x(t-\tau), x'(t-\tau), \ldots, x^{(r)}(t-\tau))$, and we have

(a) If f is continuous then F satisfies the C_r-condition. Also if f is Lipschitzian, then F is r-Lipschitzian. This is clear, since for $(t, \psi), (t, \varphi) \in J \times B_D$ we have

$$\begin{aligned} \|F(t, \psi) - F(t, \varphi)\| &= \|f(t, \psi(-\tau), \psi'(-\tau), \ldots, \psi^{(r)}(-\tau)) - f(t, \varphi(-\tau), \varphi'(-\tau), \ldots, \varphi^{(r)}(-\tau))\| \\ &\leqslant K \sum_{i=0}^{r} \|\psi^{(i)}(-\tau) - \varphi^{(i)}(-\tau)\| \leqslant K \|\psi - \varphi\|_r. \end{aligned} \tag{2.39}$$

(b) If f is locally Lipschitz, then F is locally r-Lipschitzian. Namely, if $(t_0, \phi) \in J \times B_D$ is given. Then, by Lemma 2.15, there exists a number $\delta > 0$ such that the set $\{z \in \mathbb{R}^m : \|z - \phi(s)\| \leqslant \delta$ for some $s \in [-\tau, 0]\}$ is contained in D. Now choose $b_1 > 0$ and $0 < b_2 < \delta$ sufficiently small so that f is Lipschitzian (say with Lipschitz constant K) on $([t_0 - a, t_0 + a] \cap J) \times A_0 \times A_1 \times \cdots \times A_r$, where

$$A_i = \left\{ z \in \mathbb{R}^m : \sum_{i=0}^{r} \|z - \phi^{(i)}(-\tau)\| \leqslant b_2 \right\}, \quad i = 0, 1, \ldots, r. \tag{2.40}$$

Then $A = ([t_0 - a, t_0 + a] \cap J) \times \{\psi \in B : \|\psi - \phi\|_r \leqslant b_2\}$ is a subset of $J \times B_D$ and as in (a), if $(t, \psi), (t, \varphi) \in A$ we obtain

$$\|F(t,\psi) - F(t,\varphi)\| \leqslant K\|\psi - \varphi\|_r. \tag{2.41}$$

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