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A distribution space for Hilbert transform and its applications

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Abstract In this paper, a new distribution space $\mathscr{D}'_H$ is constructed and the definition of the classical Hilbert transform is extended to it. It is shown that $\mathscr{D}'_H$ is the biggest subspace of $\mathscr{D}'$ on which the extended Hilbert transform is a homeomorphism and both the classical Hilbert transform for L^p functions and the circular Hilbert transform for periodic functions are special cases of the extension. Some characterizations of the space $\mathscr{D}_H$ are given and a class of useful nonlinear phase signals is shown to be in $\mathscr{D}'_H$. Finally, the applications of the extended Hilbert transform are discussed.

Keywords: Hilbert transform, distribution, time-frequency analysis

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1 Introduction

Fourier and Hilbert transforms are two most important transforms for many subjects such as physics, mathematics and engineering. Fourier transform is the indisputable hegemony for the frequency spectrum analysis in classical signal processing[1, Chapter 1]. Likewise, Hilbert transform, by carrying the instantaneous frequency information of a signal, provides a solid foundation for non-stationary signal analysis. By setting the Hilbert transform of a real-valued signal as the imaginary part, an analytic signal is produced, with which the commonly accepted definitions for instantaneous amplitude and frequency are obtained for any given signal[2]. Commonly encountered signals in reality include mainly those of finite energy (L^2-functions in mathematics), periodic signals and Dirac impulses (generalized functions). Thus, to analyze signals mathematically, it is a very important task to establish a function space containing the above signals such that Fourier or Hilbert transform is closed in it. As one knows, the ideal space for Fourier transform is that of tempered distributions, which includes all the signals mentioned above and the Fourier transform is a homeomorphism on it. However, we have no such a space for Hilbert transform yet. As the development of information science, Hilbert transform plays a more and more important role in nonstationary signal processing. The recent proposed technique, Hilbert-Huang transform[3], employs Hilbert transform to produce Hilbert spectrum based on the so-called empirical mode decomposition, which stimulates some novel researches on Hilbert transform and its relevant topics such as Bedrosian identity[4–7].

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The classical Hilbert transform is defined as

$$Hf(t) := \frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}} \frac{f(t-\tau)}{\tau} d\tau, \tag{1.1}$$

where, p.v. is the Cauchy principal. It is a continuous linear operator from $L^p(\mathbb{R})$ to $L^p(\mathbb{R})$ for any $1 < p < \infty$ and satisfies the following basic properties (see [8, 9]): (i) $H^{-1} = -H : L^2(\mathbb{R}) \to L^2(\mathbb{R})$; (ii) $\|Hf\|_2 = \|f\|_2$ ($\forall f \in L^2(\mathbb{R})$), and (iii) $(Hf)\wedge(\omega) = -i(\text{sgn}\omega)\hat{f}(\omega)$ a.e. $\omega \in \mathbb{R}$ ($\forall f \in L^2(\mathbb{R})$), where $\hat{f}$ is the Fourier transform of f defined by

$$\hat{f}(\omega) := \int_{\mathbb{R}} f(x) e^{-i\omega x} dx$$

for $f \in L^1(\mathbb{R})$ and by the density of $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ in $L^2(\mathbb{R})$ for $f \in L^2(\mathbb{R})$ (see [10]). Similarly the circular Hilbert transform is defined as

$$\tilde{H}f(x) := \frac{1}{\pi}\text{p.v.}\int_{-T/2}^{T/2} \frac{f(t-\tau)}{2\tan\frac{\tau}{2}} d\tau \tag{1.2}$$

for any T-periodic function f. Both H and $\tilde{H}$ defined respectively by (1.1) and (1.2) are called Hilbert transforms. A natural and interesting question is: what is the relation between them? If $s(t) = f(t) + \cos t$ with $f \in L^2(\mathbb{R})$, what is the Hilbert transform of s and how to find its instantaneous frequency?

No literature is reported on the research of the above questions. Up till now, many achievements have been made to extend the classical Hilbert transform to some generalized function spaces[11–16]. Most of them (cf. [11–13]) on this topic is to extend the Hilbert transform to a preexistent distribution space by using the analytic representation of distributions. The notable one among them is [13] by Orton, in which, Hilbert transform is extended to $\mathscr{D}'$, the space of Schwartz distributions. Her extension depends on the analytic representation, which is unique up to an entire analytic function, namely, the Hilbert transform of $f \in \mathscr{D}'$ is essentially an equivalent class. In [14] Hilbert transform is extended to $\mathscr{D}'$ directly with conjugate operator by introducing the topology on $H(\mathscr{D})$. With this extension, for any $f \in \mathscr{D}'$, its Hilbert transform Hf is in $H'(\mathscr{D})$, which is called a space of ultradistributions[14]. It can be verified that $H'(\mathscr{D})$ is not a subspace of $\mathscr{D}'$ since $H\phi \notin \mathscr{D}$ for $\phi \in \mathscr{D}$ unless $\phi = 0$. Let us recall that, the similar case occurs for Fourier transform since the Fourier transform $\hat{\phi}$ of $\phi \in \mathscr{D}$ is not in $\mathscr{D}$ unless $\phi \equiv 0$. To extend Fourier transform to distributions, the Schwartz space $\mathscr{S}$ of rapidly decreasing functions is considered. It is well-known that $\mathscr{D} \subsetneq \mathscr{S}$ (see [17] for the exact meaning of embeding '$\subsetneq$') and the Fourier transform is a homeomorphism on $\mathscr{S}$. Therefore, the dual space of $\mathscr{S}$ satisfies $\mathscr{S}' \subsetneq \mathscr{D}'$ and Fourier transform is extended to $\mathscr{S}'$ successfully. Following this idea, this paper will establish a new space of distributions and extend the classical Hilbert transform to it such that Hilbert transform is a homeomorphism. It is also shown that the Hilbert transforms defined respectively by (1.1) and (1.2) are special cases of the extended Hilbert transform.

For clarification, let us denote some commonly used notations as follows: Let $\mathbb{N}$ be the set of all the natural numbers, $\mathbb{Z}_+$ be the set of all the nonnegative integers, $\mathbb{R}$ be the set of real numbers. For a Lebesgue measurable set $E \subset \mathbb{R}$, let $L^p(E)$ be the space of p-power Lebesgue integrable functions with the well-known $L^p(E)$ norm for $1 \leqslant p \leqslant \infty$, $L_{\text{loc}}(\mathbb{R})$ be the space of all the locally integrable functions on $\mathbb{R}$, $C^k(\mathbb{R})$ ($k \in \mathbb{Z}_+$) be that of all the k-times differentiable functions on $\mathbb{R}$, $C(\mathbb{R}) := C^0(\mathbb{R})$, $C^\infty(\mathbb{R}) := \cap_{k\in\mathbb{N}} C^k(\mathbb{R})$, and $\mathscr{D}$ be the test function space of all the compactly supported $C^\infty(\mathbb{R})$ functions endowed with the usual topology such that its dual space $\mathscr{D}'$ is the space of (Schwartz) distributions[17].

In the rest of the paper, a distribution space $\mathscr{D}_H$ is constructed and some characterizations are given in Section 2. It is also shown in this section that the $\mathscr{D}_H$ is the smallest space with our desired properties and correspondingly its dual space $\mathscr{D}_H'$ is the biggest distribution space such that $\mathscr{D}_H' \subsetneq \mathscr{D}'$. In Sections 3, it is shown that two classical function spaces are continuously embedded into $\mathscr{D}_H'$ and a class of nonlinear phase signals is in $\mathscr{D}_H'$. Then in Section 4, the classical Hilbert transform is extended to $\mathscr{D}_H'$ and it is shown that the circular Hilbert transform is also a special case of the extension. Finally, Section 5 shows two simple applications.

2 Space $\mathscr{D}_H$

The typical method for extending the classical Hilbert transform to a distribution space $\mathscr{X}'$, where $\mathscr{X}$ is a function space, is using the conjugate operator. We denote the space to be constructed by $\mathscr{D}_H$ and assume it satisfies $\mathscr{D} \subset \mathscr{D}_H$ and $H(\mathscr{D}_H) = \mathscr{D}_H$ such that $\mathscr{D}_H' \subset \mathscr{D}'$ and H maps $\mathscr{D}_H$ into itself. The properties imply that $\mathscr{D}, H(\mathscr{D}) \subset \mathscr{D}_H$, consequently $\mathscr{D} + H(\mathscr{D}) \subset \mathscr{D}_H$. In general, the smaller $\mathscr{D}_H$ is, the bigger $\mathscr{D}_H'$ is. In this paper, we define

$$\mathscr{D}_H := \mathscr{D} + H(\mathscr{D}), \tag{2.1}$$

and will show that it is what we desire.

Through this paper, we always use $C(A)$ to denote a nonnegative constant depending only on A but not necessarily the same at different occurrences, where A may be a set of some given numbers, functions, and sets.

2.1 Direct sum $\mathscr{D} \dot{+} H(\mathscr{D})$

Lemma 2.1. *Let $f \in C[-a, a]$, $a > 0$, and $f(x) := 0$ for all $x \in \mathbb{R}\backslash[-a, a]$. Suppose f has $n-1$ ($n \in \mathbb{Z}_+$) vanishing moments, i.e.,*

$$\int_{\mathbb{R}} t^k f(t)dt = 0 \qquad (\forall\, k \in \mathbb{Z}_+,\ 0 \leqslant k \leqslant n-1). \tag{2.2}$$

Then

$$\left| x^{n+1} Hf(x) - \frac{1}{\pi}\int_{\mathbb{R}} t^n f(t)dt \right| \leqslant \frac{1}{\pi}\frac{a^{n+1}}{|x|-a}\|f\|_1 \qquad (\forall\, |x| > a),$$

where $\|f\|_1$ stands for the well-known $L^1(\mathbb{R})$-norm.

Proof. Using supp$f \subset [-a, a]$, we have

$$Hf(x) = \frac{1}{\pi}\lim_{A^{-1},\epsilon\to 0}\int_{\epsilon<|t-x|<A,|t|\leqslant a}\frac{f(t)}{x-t}dt.$$

For any $|x| > a$, since $\epsilon < |t-x| < A$ holds for all $|t| \leqslant a$ if A^{-1}, ϵ are small enough, it is followed that $\{t \in \mathbb{R} \,|\, |t| \leqslant a\} = \{t \in \mathbb{R} \,|\, \epsilon < |t-x| < A, |t| \leqslant a\}$, which implies that

$$Hf(x) = \frac{1}{\pi}\int_{|t|\leqslant a}\frac{f(t)}{x-t}dt = \frac{1}{\pi}\int_{-a}^{a}\frac{f(t)}{x}\sum_{k=0}^{\infty}\left(\frac{t}{x}\right)^k dt = \frac{1}{\pi}\int_{-a}^{a}\sum_{k=0}^{\infty}\frac{t^k f(t)}{x^{k+1}}dt$$

for any $|x| > a$. Since

$$\sum_{k=0}^{\infty}\left|\frac{t^k f(t)}{x^{k+1}}\right| = \frac{|f(t)|}{|x|-|t|} \leqslant \frac{1}{|x|-a}|f(t)| \in L^1([-a,a]),$$

we conclude by Fubini-Tonelli's theorem (see [18]) that

$$Hf(x) = \frac{1}{\pi}\int_{-a}^{a}\sum_{k=0}^{\infty}\frac{t^k f(t)}{x^{k+1}}dt = \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{1}{x^{k+1}}\int_{-a}^{a}t^k f(t)dt.$$

Consequencly,

$$x^{n+1}Hf(x) = \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{1}{x^k}\int_{-a}^{a}t^{n+k}f(t)dt.$$

Denoting

$$y := \frac{1}{x}, \qquad c_k := \int_{-a}^{a}t^{n+k}f(t)dt,$$

we get

$$\left|x^{n+1}Hf(x) - \frac{1}{\pi}\int_{-a}^{a}t^n f(t)dt\right| \leqslant \frac{1}{\pi}\sum_{k=1}^{\infty}|c_k y^k| \leqslant \frac{1}{\pi}\frac{a^{n+1}}{|x|-a}\int_{-a}^{a}|f(t)|dt \quad (\forall\,|x|>a).$$

This ends the proof of the lemma.

Note 1. Every function has -1 vanishing moment since no integer k satisfies $0 \leqslant k \leqslant -1$ in (2.2).

Note 2. Let $n \in \mathbb{Z}_+$. Function $f(x)$ is said to have exactly $n-1$ vanishing moments if it has $n-1$ vanishing moments and $\int_{\mathbb{R}} t^n\psi(t)dt \neq 0$.

The following theorem shows that the sum of linear spaces $\mathscr{D} + H(\mathscr{D})$ is a direct sum and therefore is denoted as $\mathscr{D}_H = \mathscr{D}\dot{+}H(\mathscr{D})$.

Theorem 2.2. $\mathscr{D} \cap H(\mathscr{D}) = \{0\}$.

Proof. If $\mathscr{D}\cap H(\mathscr{D}) \neq \{0\}$, there must exist $\phi \in \mathscr{D}\cap H(\mathscr{D})$ satisfying $\phi \not\equiv 0$. Let $\psi \in \mathscr{D}$, $\psi \not\equiv 0$ satisfy $\phi = H\psi$. There must exist $n \in \mathbb{Z}_+$ such that ψ has exactly $n-1$ vanishing moments.

In fact, if ψ has arbitrary vanishing moments, let supp$\psi \subset [-a,a]$ for some $a > 0$, then

$$\int_{-a}^{a}p(t)\psi(t)dt = 0$$

for any polynomial $p(t)$. Due to the density of polynomials in $C[-a,a]$, it is yielded that $\psi \equiv 0$, consequently, $\phi = H\psi \equiv 0$, which contradicts $\phi \not\equiv 0$.

For this n, using Lemma 2.1 we have

$$\lim_{x\to\infty}x^{n+1}\phi(x) = \lim_{x\to\infty}x^{n+1}H\psi(x) = \frac{1}{\pi}\int_{-a}^{a}t^n\psi(t)dt \neq 0.$$

It contradicts $\phi \in \mathscr{D}$.

Corollary 2.3. *Let $g \in \mathscr{D}_H\backslash\mathscr{D}$. Then there exist $n \in \mathbb{Z}_+$ and a constant $c \neq 0$ satisfying $\lim_{x\to\infty}x^{n+1}g(x) = c$.*

Proof. $g \in \mathscr{D}_H \backslash \mathscr{D}$ implies that there are $\phi, \psi \in \mathscr{D}$ such that $g = \phi + H\psi$ and $\psi \not\equiv 0$. Then ψ has exactly $n-1$ ($n \in \mathbb{Z}_+$) vanishing moments for some $n \in \mathbb{Z}_+$. By Lemma 2.1, there exists a constant $c \neq 0$ such that $\lim_{x\to\infty} x^{n+1} H\psi(x) = c$, which implies that

$$\lim_{x\to\infty} x^{n+1} g(x) = \lim_{x\to\infty} x^{n+1}[\phi(x) + H\psi(x)] = c.$$

2.2 The topology in $\mathscr{D}_H$

Since $\mathscr{D}_H := \mathscr{D} \dot{+} H(\mathscr{D})$ is a direct sum, we define the convergence in $\mathscr{D}_H$ as follows:

$$\forall \{\phi_n + H\psi_n\} \subset \mathscr{D}_H, \text{ define } \phi_n + H\psi_n \to 0 \text{ (in } \mathscr{D}_H) \text{ if } \phi_n, \psi_n \to 0 \text{ (in } \mathscr{D}). \tag{2.3}$$

Endowed with this topology, $\mathscr{D}_H$ becomes a topological vector space satisfying $H(\mathscr{D}_H) = \mathscr{D}_H$ and $H : \mathscr{D}_H \to \mathscr{D}_H$ is a continuous linear operator. Accordingly, H is a homeomorphism on $\mathscr{D}_H$ since $H^{-1} = -H$.

Let $\mathscr{X}$ and $\mathscr{Y}$ be two topological vector spaces satisfying $\mathscr{X} \subset \mathscr{Y}$. Space $\mathscr{X}$ is continuously embeded in $\mathscr{Y}$ and denoted as $\mathscr{X} \hookrightarrow \mathscr{Y}$ if $\forall\{x_n\} \subset \mathscr{X}$, $\{x_n\}$ converging to 0 in $\mathscr{X}$ implies that $\{x_n\}$ converges to 0 in $\mathscr{Y}$ (see [17]). One knowns that $\mathscr{X}'$, the dual of $\mathscr{X}$, is also a topological vector space with the usual addition, scale multiplication and the following convergence: f_n is said to converges to 0 if $f_n(x) \to 0$ for any $x \in \mathscr{X}$. It is easy to show that $\mathscr{X} \hookrightarrow \mathscr{Y}$ implies $\mathscr{Y}' \hookrightarrow \mathscr{X}'$.

With the topology of $\mathscr{D}_H$ defined above, it is easy to see that $\mathscr{D} \hookrightarrow \mathscr{D}_H$. Therefore $\mathscr{D}'_H \hookrightarrow \mathscr{D}'$. Moreover, the following theorem shows that $\mathscr{D}_H$ is the smallest space such that $\mathscr{D} \hookrightarrow \mathscr{D}_H$ and H is a homeomorphism from $\mathscr{D}_H$ to itself.

Theorem 2.4. *Let $\mathscr{X}$ be a topological vector space such that $\mathscr{D} \hookrightarrow \mathscr{X} \subset L^2(\mathbb{R})$ and the Hilbert transform $H : \mathscr{X} \to \mathscr{X}$ be a continuous linear operator. Then $\mathscr{D}_H \hookrightarrow \mathscr{X}$, consequently $\mathscr{X}' \hookrightarrow \mathscr{D}'_H$.*

Proof. Embedding $\mathscr{D} \hookrightarrow \mathscr{X}$ implies $H(\mathscr{D}) \subset H(\mathscr{X}) \subset \mathscr{X}$. Therefore $\mathscr{D} + H(\mathscr{D}) \subset \mathscr{X}$, i.e., $\mathscr{D}_H \subset \mathscr{X}$.

For any $\{\phi_n + H\psi_n\} \subset \mathscr{D}_H$, $\phi_n + H\psi_n \to 0$ (in $\mathscr{D}_H$), we have $\phi_n, \psi_n \to 0$ (in $\mathscr{D}$). Then, $\phi_n, \psi_n \to 0$ (in $\mathscr{X}$) and consequently, $H\psi_n \to 0$ (in $\mathscr{X}$). Hence, $\phi_n + H\psi_n \to 0$ (in $\mathscr{X}$), which shows that $\mathscr{D}_H \hookrightarrow \mathscr{X}$.

For $1 < p < \infty$, denote $\mathscr{D}_{L^p} := \{f | f \in C^\infty(\mathbb{R}) \text{ and } f^{(k)} \in L^p(\mathbb{R})\ (\forall k \in \mathbb{Z}_+)\}$. It is proved in [14, 15] that $H : \mathscr{D}_{L^p} \to \mathscr{D}_{L^p}$ is a continuous linear operator. By Theorem 2.4 we have $\mathscr{D}_H \hookrightarrow \mathscr{D}_{L^p}$, which implies $\mathscr{D}'_{L^p} \hookrightarrow \mathscr{D}'_H$ and $\mathscr{D}_H \subset \bigcap_{1<p<\infty} \mathscr{D}_{L^p}$. Let us show that $\mathscr{D}_H \subsetneqq \bigcap_{1<p<\infty} \mathscr{D}_{L^p}$. In fact, it is easy to see that Gaussian function $g(x) := e^{-|x|^2}$ is in $\bigcap_{1<p<\infty} \mathscr{D}_{L^p}$ and $\lim_{x\to\infty} x^n g(x) = 0$ for any $n \in \mathbb{N}$, which shows $g \notin \mathscr{D}_H \setminus \mathscr{D}$. On the other hand, it is obvious that $g \notin \mathscr{D}$. Hence, $g \notin \mathscr{D}_H$.

A typical example of the distribution in $\mathscr{D}'_H$ is the Dirac impulse.

Example 1. Let $x \in \mathbb{R}$. Then the Dirac impulse function defined by $\delta_x(\phi) := \phi(x)$ ($\forall \phi \in \mathscr{D}_H$) is in $\mathscr{D}'_H$.

Proof. For any $g = \phi + H\psi \in \mathscr{D}_H$, $\phi, \psi \in \mathscr{D}$, it is easy to see that $\langle \delta_x, g\rangle := g(x) = \phi(x) + (H\psi)(x)$ is a linear functional on $\mathscr{D}_H$. To show its continuity, we need only to prove that $\delta_x(H\psi_n) = (H\psi_n)(x) \to 0$ for any $\{\psi_n\} \subset \mathscr{D}$ satisfying $\psi_n \to 0$ (in $\mathscr{D}$).

For any $\{\psi_n\} \subset \mathscr{D}$ satisfying $\psi_n \to 0$ (in $\mathscr{D}$), there exists $a > 0$ such that $\mathrm{supp}\psi_n \subset [-a, a]$. By Lemma 3.1 in the next section we have

$$\begin{aligned}|H\psi_n(x)| &\leqslant (1+|x|)|H\psi_n(x)| \leqslant C(a)(\|\psi_n\|_2 + \|\psi_n'\|_2)\\ &\leqslant C(a)\sqrt{2a}(\|\psi_n'\|_{C(\mathbb{R})} + \|\psi_n\|_{C(\mathbb{R})}) \to 0 \quad (n\to\infty),\end{aligned}$$

which shows that $\delta_x \in \mathscr{D}_H'$.

3 Embedment theorems in distribution space $\mathscr{D}_H'$

As shown in Theorem 2.4, $\mathscr{D}_H$ is the smallest space such that $\mathscr{D} \subsetneq \mathscr{D}_H$ and H is a homeomorphism from $\mathscr{D}_H$ to itself. In this section, many frequently used spaces are shown to be continuously embedded into $\mathscr{D}_H'$.

3.1 General embedment theorems

Lemma 3.1. *Let $\psi \in \mathscr{D}$ have $n-1$ vanishing moments and satisfy* $\mathrm{supp}\psi \subset [-a, a]$ *for some $n \in \mathbb{Z}_+$ and $a > 0$. Then there exists a constant $C(a, n) > 0$ such that*

$$(1+|x|^{n+1})|H\psi(x)| \leqslant C(a,n)\|\psi\|_{1,2} \quad (\forall\, x\in\mathbb{R}),$$

where $\|\psi\|_{1,2} := \|\psi\|_2 + \|\psi'\|_2$ and $\|\cdot\|_2$ is the $L^2(\mathbb{R})$-norm.

Proof. We assume $\psi \not\equiv 0$ without losing generality. By Lemma 2.1, there holds

$$\left|x^{n+1}H\psi(x) - \frac{1}{\pi}\int_{-a}^{a} t^n\psi(t)dt\right| \leqslant \frac{1}{\pi}\frac{a^{n+1}}{|x|-a}\|\psi\|_1 \quad (\forall\,|x|>a), \tag{3.1}$$

which yields that

$$|x^{n+1}H\psi(x)| \leqslant \frac{1}{\pi}a^n\left(1+\frac{a}{|x|-a}\right)\|\psi\|_1 \leqslant C(a,n)\|\psi\|_2 \quad (\forall\,|x|>2a).$$

To estimate $|H\psi(x)|$ for $|x| \leqslant 2a$, let $H\psi$ reach its minimum over $[-2a, 2a]$ at $\xi \in [-2a, 2a]$, then (see [15])

$$|H\psi(x) - H\psi(\xi)| = \left|\int_\xi^x (H\psi)'(t)dt\right| = \left|\int_\xi^x H\psi'(t)dt\right| \leqslant \sqrt{4a}\|H\psi'\|_2 = 2\sqrt{a}\|\psi'\|_2$$

holds for $x \in [-2a, 2a]$. However,

$$|H\psi(\xi)| \leqslant \left(\frac{1}{4a}\int_{-2a}^{2a}|H\psi(t)|^2dt\right)^{1/2} \leqslant \frac{1}{2\sqrt{a}}\|H\psi\|_2 = \frac{1}{2\sqrt{a}}\|\psi\|_2.$$

Thus,

$$|H\psi(x)| \leqslant |H\psi(x) - H\psi(\xi)| + |H\psi(\xi)| \leqslant C(a)(\|\psi'\|_2 + \|\psi\|_2) \quad (\forall\,|x|\leqslant 2a).$$

Theorem 3.2. *Let $\frac{f}{1+|\cdot|^2} \in L^1(\mathbb{R})$, and*

$$\lim_{A\to\infty}\int_1^A \frac{f(x)}{x}dx, \qquad \lim_{A\to\infty}\int_{-A}^{-1}\frac{f(x)}{x}dx$$

exist. Then, the following functional F_f defined by

$$F_f(\phi) := \langle f, \phi\rangle := \lim_{A,B\to\infty}\int_{-A}^{B} f(x)\phi(x)dx \quad (\forall\,\phi\in\mathscr{D}_H). \tag{3.2}$$

is in $\mathscr{D}'_H$, f *is determined uniquely by* F_f, *and for any* $a \geqslant 1$ *there holds*

$$\begin{cases} |F_f(\phi)| \leqslant C(a)\|\phi\|_{1,2} \displaystyle\int_{\mathbb{R}} \frac{|f(x)|}{1+x^2}dx, & \forall\, \phi \in \mathscr{D},\ \operatorname{supp}\phi \subset [-a,a], \\ |F_f(H\psi)| \leqslant C(a)\|\psi\|_{1,2} \left[|\lambda_f(2a)| + \displaystyle\int_{\mathbb{R}} \frac{|f(x)|}{1+x^2}dx\right], & \forall\, \psi \in \mathscr{D},\ \operatorname{supp}\psi \subset [-a,a], \end{cases} \tag{3.3}$$

where

$$\lambda_f(2a) := \lim_{A,B\to\infty} \int_{[-A,B]\setminus[-2a,2a]} \frac{f(x)}{x}dx. \tag{3.4}$$

Proof. It is easy to see that f is Lebesgue integrable on any bounded interval, which implies $f\phi \in L^1(\mathbb{R})$ and consequently the limit on the right hand of (3.2) exists for any $\phi \in \mathscr{D}$. It is easy to deduce that

$$|F_f(\phi)| \leqslant \left(\int_{\mathbb{R}} \frac{|f(x)|}{1+x^2}dx\right) \max_{x\in\operatorname{supp}\phi} [(1+x^2)|\phi(x)|] \quad (\forall\, \phi \in \mathscr{D}).$$

For any $a \geqslant 1$, let $\phi \in \mathscr{D}, \operatorname{supp}\phi \subset [-a,a]$ and $|\phi|$ arrive at the minimum at $\xi \in [-a,a]$. Then

$$|\phi(\xi)| \leqslant C(a)\|\phi\|_2, \quad |\phi(x) - \phi(\xi)| = \left|\int_x^\xi \phi'(t)dt\right| \leqslant C(a)\|\phi'\|_2 \quad (\forall\, |x| \leqslant a),$$

which implies that $|\phi(x)| \leqslant C(a)(\|\phi\|_2 + \|\phi'\|_2)$ $(\forall\, |x| \leqslant a)$. Therefore

$$|F_f(\phi)| \leqslant C(a)\|\phi\|_{1,2} \int_{\mathbb{R}} \frac{|f(x)|}{1+x^2}dx \quad (\forall\, \phi \in \mathscr{D}).$$

The first inequality of (3.3) is proved.

Let us consider the existence of the limit on the right hand of (3.2) for any $\phi = H\psi \in H(\mathscr{D})$ with $\operatorname{supp}\psi \subset [-a,a]$. By Lemma 2.1, we have

$$|xH\psi(x) - c_\psi| \leqslant \frac{1}{\pi}\frac{a}{|x|-a}\|\psi\|_1 \leqslant \frac{C(a)}{|x|}\|\psi\|_2 \quad (\forall\, |x| \geqslant 2a),$$

where $c_\psi := \frac{1}{\pi}\int_{-a}^{a}\psi(t)dt$. It is followed that

$$\left|\frac{f(x)}{x}[xH\psi(x) - c_\psi]\right| \leqslant C(a)\frac{|f(x)|}{x^2}\|\psi\|_2 \in L^1(\mathbb{R}\setminus[-2a,2a]).$$

Therefore,

$$\lim_{A,B\to\infty} \int_{[-A,B]\setminus[-2a,2a]} \frac{f(x)}{x}[xH\psi(x) - c_\psi]dx = \int_{\mathbb{R}\setminus[-2a,2a]} \frac{f(x)}{x}[xH\psi(x) - c_\psi]dx,$$

which concludes that

$$\lim_{A,B\to\infty} \int_{[-A,B]\setminus[-2a,2a]} f(x)H\psi(x)dx = c_\psi\lambda_f(2a) + \int_{\mathbb{R}\setminus[-2a,2a]} \frac{f(x)}{x}[xH\psi(x) - c_\psi]dx.$$

Hence the limit on the right hand of (3.2) exists and

$$F_f(H\psi) = c_\psi\lambda_f(2a) + \int_{\mathbb{R}\setminus[-2a,2a]} \frac{f(x)}{x}[xH\psi(x) - c_\psi]dx + \int_{-2a}^{2a} f(x)H\psi(x)dx.$$

Using Lemma 3.1 we have

$$\int_{-2a}^{2a} |f(x)H\psi(x)|dx \leqslant C(a)\|\psi\|_{1,2} \int_{\mathbb{R}} \frac{|f(x)|}{1+x^2}dx.$$

Therefore

$$|F_f(H\psi)| \leqslant |c_\psi \lambda_f(2a)| + C(a)\|\psi\|_{1,2} \int_{\mathbb{R}\setminus[-2a,2a]} \frac{|f(x)|}{1+x^2}dx,$$

which together with $|c_\psi| \leqslant C(a)\|\psi\|_2$ shows the second inequality of (3.3).

It is easy to see that F_f is a linear functional on $\mathscr{D}_H$. Using (3.3) one can show the continuity of F_f over $\mathscr{D}_H$ without difficulty. $F_f \in \mathscr{D}_H'$ has been proved.

Finally, using the facts that $f \in L_{\text{loc}}(\mathbb{R})$ and $\mathscr{D} \subset \mathscr{D}_H$ we can show easily that f is determined by F_f uniquely.

Note 1. Usually, the functional F_f defined by (3.2) is denoted as f and $F_f(\phi)$ is rewritten as $\langle f, \phi\rangle$ if no confusion occurs. Accordingly, we have $f \in \mathscr{D}_H'$.

Note 2. A typical case is that $\frac{f}{1+|\cdot|} \in L^1(\mathbb{R})$. In this case all the conditions of Theorem 3.2 are satisfied and the functional F_f defined by (3.2) can be written as

$$F_f(\phi) := \langle f, \phi\rangle = \int_{\mathbb{R}} f(x)\phi(x)dx \quad (\forall\, \phi \in \mathscr{D}_H). \tag{3.5}$$

Let us turn to another sufficient condition of Theorem 3.2. We first extend the second mean value theorem of Riemann's integral calculus to Lebegues' integral.

Lemma 3.3. *Let $f \in L^1([a,b])$ and g be a monotone function on $[a,b]$. Then there exists a $\xi \in [a,b]$ such that*

$$\int_a^b f(x)g(x)dx = g(a)\int_a^\xi f(x)dx + g(b)\int_\xi^b f(x)dx. \tag{3.6}$$

Proof. The lemma can be proved easily according to the density of $C[a,b]$ in $L^1([a,b])$ and the second mean value theorem of Riemann's integral calculus. We omit the details here.

Corollary 3.4. *Let $\frac{f}{1+|\cdot|^2} \in L^1(\mathbb{R})$. If there exists a constant $C \geqslant 0$ such that*

$$\left|\int_1^A f(x)dx\right| \leqslant C, \qquad \left|\int_{-A}^{-1} f(x)dx\right| \leqslant C \quad (\forall\, A \geqslant 1), \tag{3.7}$$

then the results of Theorem 3.2 hold.

Proof. For any $A_2 \geqslant A_1 \geqslant 1$, by Lemma 3.3 there exists $\xi \in [A_1, A_2]$ such that

$$\left|\int_{A_1}^{A_2} \frac{f(x)}{x}dx\right| = \left|\frac{1}{A_1}\int_{A_1}^{\xi} f(x)dx + \frac{1}{A_2}\int_\xi^{A_2} f(x)dx\right| \leqslant C\left(\frac{1}{A_1} + \frac{1}{A_2}\right) \to 0 \ (A_1, A_2 \to \infty).$$

Therefore $\lim_{A\to\infty}\int_1^A \frac{f(x)}{x}dx$ exists. Similarly, $\lim_{A\to\infty}\int_{-A}^{-1}\frac{f(x)}{x}dx$ exists. By Theorem 3.2, the corollary is proved.

3.2 Embeding: $L^p(\mathbb{R}) \subsetneq \mathscr{D}_H'$ $(1 \leqslant p < \infty)$

The following corollary shows that $L^p(\mathbb{R})$ is continuously embedded into $\mathscr{D}_H'$.

Corollary 3.5. $L^p(\mathbb{R}) \subsetneq \mathscr{D}_H'$ $(1 \leqslant p < \infty)$.

Proof. Any $f \in L^p(\mathbb{R})$ satisfies $\frac{f}{1+|\cdot|} \in L^1(\mathbb{R})$. Thus, $L^p(\mathbb{R}) \subset \mathscr{D}'_H$.

Let $\{f_n\} \subset L^p(\mathbb{R})$ satisfy $\|f_n\|_p \to 0$ $(n \to \infty)$.

(i) For any $\phi \in \mathscr{D}$, assume supp$\phi \subset [-a, a]$, by (3.3) we have

$$|\langle f_n, \phi\rangle| \leqslant C(a)\|\phi\|_{1,2} \int_{\mathbb{R}} \frac{|f_n(x)|}{1+x^2} dx \leqslant C(a, p, \phi)\|f_n\|_p \to 0 \quad (n \to \infty).$$

(ii) For any $\psi \in \mathscr{D}$, assume supp$\psi \subset [-a, a]$, by (3.3) similarly we have

$$|\langle f_n, H\psi\rangle| \leqslant C(a, p, \psi)[|\lambda_{f_n}(2a)| + \|f_n\|_p] \to 0 \quad (n \to \infty),$$

where

$$|\lambda_{f_n}(2a)| = \lim_{A,B\to\infty} \left| \int_{[-A,B]\setminus[-2a,2a]} \frac{f_n(x)}{x} dx \right| \leqslant C(a, p)\|f_n\|_p.$$

3.3 Embeding: $\dot{L}^1_T \subsetneq \mathscr{D}'_H$

Let L^p_T $(1 \leqslant p < \infty, T > 0)$ be the space of all the T-periodic Lebesgue's measurable functions f satisfying

$$\|f\|_{L^p_T} := \left(\int_0^T |f(x)|^p dx\right)^{1/p} < \infty.$$

Denote

$$\dot{L}^p_T := \left\{ f \in L^p_T \,\middle|\, \int_0^T f(x)dx = 0 \right\}.$$

Then we have the following embedment corollary.

Corollary 3.6. $\dot{L}^1_T \subsetneq \mathscr{D}'_H$.

Proof. For any $f \in \dot{L}^1_T$, it is easy to verify that $\frac{f}{1+|\cdot|^2} \in L^1(\mathbb{R})$. For any $A \geqslant 1$, let $k \in \mathbb{Z}_+$ satisfy $1 + kT \leqslant A < 1 + (k+1)T$. Then

$$\left|\int_1^A f(x)dx\right| = \left|\left(\int_1^{1+kT} + \int_{1+kT}^A\right) f(x)dx\right| \leqslant \int_{1+kT}^A |f(x)|dx \leqslant \|f\|_{L^1_T}. \tag{3.8}$$

Similarly, we have $|\int_{-A}^{-1} f(x)dx| \leqslant \|f\|_{L^1_T}$. By Corollary 3.4 we have $f \in \mathscr{D}'_H$, which concludes that $\dot{L}^1_T \subset \mathscr{D}'_H$.

To show that $\dot{L}^1_T$ is embedded continuously into $\mathscr{D}'_H$, let $\{f_n\} \subset \dot{L}^1_T$ satisfy $\|f_n\|_{L^1_T} \to 0$ $(n \to \infty)$. $\forall\, \phi, \psi \in \mathscr{D}$ satisfying suppϕ, supp$\psi \subset [-a, a]$, using (3.3), we have

$$|\langle f_n, \phi\rangle| \leqslant C(a)\|\phi\|_{1,2}\|f_n\|_{L^1_T} \to 0 \quad (n \to \infty)$$

and

$$|\langle f_n, H\psi\rangle| \leqslant C(a)\|\psi\|_{1,2}[|\lambda_{f_n}(2a)| + \|f_n\|_{L^1_T}], \tag{3.9}$$

where

$$|\lambda_{f_n}(2a)| = \lim_{A,B\to\infty} \left| \int_{[-A,B]\setminus[-2a,2a]} \frac{f_n(x)}{x} dx \right|.$$

For any $B \geqslant 2a$, according to Lemma 3.3 there exists $\xi_n \in [2a, B]$ such that

$$\int_{2a}^B \frac{f_n(x)}{x} dx = \frac{1}{2a} \int_{\xi_n}^B f_n(x)dx + \frac{1}{B} \int_{\xi_n}^B f_n(x)dx.$$

Similar to the proof of (3.8), we have $|\int_{2a}^{B}\frac{f_n(x)}{x}dx| \leqslant \frac{1}{a}\|f_n\|_{L^1_T}$. Similarly, $\forall\, A \geqslant 2a$, there holds $|\int_{-A}^{-2a}\frac{f_n(x)}{x}dx| \leqslant \frac{1}{a}\|f_n\|_{L^1_T}$. Hence $|\lambda_{f_n}(2a)| \leqslant \frac{2}{a}\|f_n\|_{L^1_T}$, which together with (3.9) implies that $|\langle f_n, H\psi\rangle| \leqslant C(a)\|\psi\|_{1,2}\|f_n\|_{L^1_T} \to 0$ $(n\to\infty)$. The proof is complete.

Example 2. Let $f(x) = \cos\omega x$ or $\sin\omega x$, where $\omega \in \mathbb{R}\setminus\{0\}$. Then $f \in \mathscr{D}'_H$.

3.4 Nonlinear phase signals $\{\cos\theta(x),\ \sin\theta(x)\} \subset \mathscr{D}'_H$

The last two subsections show that the classical function spaces $L^p(\mathbb{R})$ and $\dot{L}^1_T$ are subsets of $\mathscr{D}'_H$. However, some typical signals encountered frequently in signal processing such as the linear chirp $s(t) = \cos(bt^2 + ct)$ with $b, c \in \mathbb{R}$ are not in these two classes. In this subsection, it is shown that $\mathscr{D}'_H$ contains a class of nonlinear phase signals of the form $\cos\theta(x), \sin\theta(x)$, which covers almost all the signals used in time-frequency analysis and signal processing.

Theorem 3.7. *Let $\theta \in C^1(\mathbb{R})$ be strictly monotone on $(-\infty, -A)$ and (A, ∞) respectively for some $A > 0$ and $\lim_{|x|\to\infty}|\theta(x)| = \infty$. Then $\cos\theta(\cdot), \sin\theta(\cdot) \in \mathscr{D}'_H$.*

Proof. It is obvious that $\frac{\cos\theta(\cdot)}{1+|\cdot|^2} \in L^1(\mathbb{R})$. Assume θ is strictly increasing on $\mathbb{R}$ without losing generality. Let $\{x_k\}_{k=1}^{\infty}$ satisfy $\theta(x_k) = k\pi + \frac{\pi}{2}$ $(\forall\, k \in \mathbb{N})$. Then $\forall\, n, m \in \mathbb{N}, n > m$, we have

$$\int_{x_m}^{x_n}\frac{\cos\theta(x)}{x}dx = \int_{m\pi+\pi/2}^{n\pi+\pi/2}[\ln\theta^{-1}(t)]'\cos t dt = \int_{m\pi+\pi/2}^{n\pi+\pi/2}[\ln\theta^{-1}(t)]\sin t dt$$
$$= \sum_{k=m+1}^{n}(-1)^k\int_0^{\pi/2}\ln\frac{\theta^{-1}(k\pi+t)}{\theta^{-1}(k\pi-t)}\sin t dt.$$

It can be deduced that $\ln\theta^{-1}(t)$ is strictly increasing and $\lim_{|t|\to\infty}|\theta^{-1}(t)| = \infty$. Hence

$$\left|\int_{x_m}^{x_n}\frac{\cos\theta(x)}{x}dx\right| \leqslant \sum_{k=m+1}^{n}\int_0^{\pi/2}\ln\frac{\theta^{-1}(k\pi+t)}{\theta^{-1}(k\pi-t)}\sin t dt$$
$$\leqslant \sum_{k=m+1}^{n}\ln\frac{\theta^{-1}(k\pi+\frac{1}{2}\pi)}{\theta^{-1}(k\pi-\frac{1}{2}\pi)}$$
$$= \ln\theta^{-1}\left(n\pi + \frac{1}{2}\pi\right) - \ln\theta^{-1}\left(m\pi + \frac{1}{2}\pi\right) \to 0 \quad (n, m\to\infty),$$

which implies the existence of $\lim_{A\to\infty}\int_1^A\frac{\cos\theta(x)}{x}dx$. It can be shown similarly that $\lim_{A\to\infty}\int_{-A}^{-1}\frac{\cos\theta(x)}{x}dx$ exists. By Theorem 3.2 we have $\cos\theta(x) \in \mathscr{D}'_H$. Similarly we have $\sin\theta(x) \in \mathscr{D}'_H$.

With the results obtained in this section it is easy to verify that almost all the signals used in time-frequency analysis and signal processing are in $\mathscr{D}'_H$. The signals below are from [2].

Example 3. All the following signals are in $\mathscr{D}'_H$:

(1) The linear chirp $s(t) = \exp(i\beta t^2 + i\gamma t)$ with $\beta \neq 0$;

(2) Gaussian envelope signal: $s(t) = \exp(-\alpha t^2 + i\beta t^2 + i\gamma t)$ with $\alpha > 0$;

(3) $s(t) = \exp(-\alpha t^2 + i\beta t^2 + im\sin(\omega_m t) + i\omega_0 t)$ with $\alpha > 0$;

(4) $s(t) = \exp(i\beta t^2 + im\sin(\omega_m t) + i\omega_0 t)$ with $\beta \neq 0$.

Proof. (1) and (4) are easily to be verified according to Theorem 3.7; (2) and (3) are obviously true since $s \in L^2(\mathbb{R})$ for $\alpha > 0$.

4 Hilbert transform of distribution

4.1 Extension of Hilbert transform by conjugate operator

In this section, the classical Hilbert transform H will be extended to $\mathscr{D}'_H$ by using the conjugate operator. Before doing this, let us recall the following equality (see [19, p. 132]):

$$\int_{\mathbb{R}}(Hf)(x)\phi(x)dx = -\int_{\mathbb{R}} f(x)(H\phi)(x)dx \quad (\forall\, f, \phi \in L^2(\mathbb{R})). \tag{4.1}$$

Considering the constraint of H on $\mathscr{D}_H$, we know that $H : \mathscr{D}_H \to \mathscr{D}_H$ is a continuous and linear operator, which implies that its conjugate operator $H^* : \mathscr{D}'_H \to \mathscr{D}'_H$, which is defined as $\langle H^*f, \phi\rangle := \langle f, H\phi\rangle$ $(\forall\, f \in \mathscr{D}'_H,\ \phi \in \mathscr{D}_H)$, is a continuous and linear operator. For any $f \in \mathscr{D}_H$, using (4.1), we have

$$\langle Hf, \phi\rangle := \int_{\mathbb{R}}(Hf)(x)\phi(x)dx = -\int_{\mathbb{R}} f(x)(H\phi)(x)dx = -\langle f, H\phi\rangle = \langle -H^*f, \phi\rangle \quad (\forall\, \phi \in \mathscr{D}_H),$$

i.e., $Hf = -H^*f$ (in $\mathscr{D}'_H$). Moreover, if $S : \mathscr{D}_H \to \mathscr{D}_H$ is also a continuous and linear operator satisfying $Hf = S^*f$ $(\forall\, f \in \mathscr{D}_H)$, then

$$\int_{\mathbb{R}} f(x)(H\phi)(x)dx = -\int_{\mathbb{R}} f(x)(S\phi)(x)dx \quad (\forall\, f, \phi \in \mathscr{D}_H),$$

which yields that $S\phi = -H\phi$ $(\forall\, \phi \in \mathscr{D}_H)$. Therefore, $-H^*$ can be defined as the extension of H to the distribution space $\mathscr{D}'_H$, namely, we have

Definition 4.1. *Let $H^* : \mathscr{D}'_H \to \mathscr{D}'_H$ be the conjugate operator of the classical Hilbert transform $H : \mathscr{D}_H \to \mathscr{D}_H$. Then $-H^* : \mathscr{D}'_H \to \mathscr{D}'_H$ is defined as the extension of H to the distribution space $\mathscr{D}'_H$, and denoted as H still if no confusion occurs.*

It is easy to see that the extended Hilbert transform is a homeomorphism from $\mathscr{D}'_H$ to itself.

Theorem 4.2. *$H : \mathscr{D}'_H \to \mathscr{D}'_H$ satisfies $-H^2 = I$ (the identity operator).*

Proof. For any $f \in \mathscr{D}'_H$, we have $\langle H^2 f, \phi\rangle = -\langle Hf, H\phi\rangle = \langle f, H^2\phi\rangle = -\langle f, \phi\rangle$ $(\forall\, \phi \in \mathscr{D}_H)$, which concludes that $-H^2 = I$.

4.2 The coincidence of the extension with the classical Hilbert transform

Theorem 4.3. *Let $f \in L^p(\mathbb{R})$ $(1 < p < \infty)$. Then, Hf, as the classical Hilbert transform, coincides with the extended one defined by Definition* 4.1.

Proof. The classical Hilbert transfrom Hf can be regarded as a distribution on $\mathscr{D}_H$ according to (3.5), whose function on $\phi \in \mathscr{D}_H$ is

$$\langle Hf, \phi\rangle = \int_{\mathbb{R}}(Hf)(x)\phi(x)dx. \tag{4.2}$$

On the other hand, as the extended Hilbert transform, Hf is also a distribution on $\mathscr{D}_H$, whose function on $\phi \in \mathscr{D}_H$ is

$$\langle Hf, \phi\rangle = -\langle H^*f, \phi\rangle = -\langle f, H\phi\rangle = -\int_{\mathbb{R}} f(x)(H\phi)(x)dx. \tag{4.3}$$

It can be verified that

$$\int_{\mathbb{R}}(Hf)(x)\phi(x)dx = -\int_{\mathbb{R}} f(x)(H\phi)(x)dx \quad (\forall\, f \in L^p(\mathbb{R}),\ \phi \in \mathscr{D}_H),$$

which means that (4.2) and (4.3) give the same result.

4.3 Circular Hilbert transform

The circular Hilbert transform defined by (1.2) can be expressed equivalently as

$$\tilde{H}f(x) = \sum_{k\in\mathbb{Z}}[-i(\mathrm{sgn}(k))]c_k e^{ik\frac{2\pi}{T}x} \tag{4.4}$$

for any $f(x) = \sum_{k\in\mathbb{Z}} c_k e^{ik\frac{2\pi}{T}x} \in L^2_T$, where sgn is the signum function defined by $\mathrm{sgn}(0) := 0$ and $\mathrm{sgn}(x) := x/|x|$ for $x \neq 0$ (see [20, Chapter 1] and [21, p. 15]). Since $\dot{L}^2_T \subsetneq \mathscr{D}'_H$ we will show that $\tilde{H}f = Hf$ for any $f \in \dot{L}^2_T$. Before this, let us calculate the Hilbert transforms of $\sin(\omega x)$ and $\cos(\omega x)$ for $\omega \in \mathbb{R}\setminus\{0\}$.

Lemma 4.4. *Let $\phi \in \mathscr{D}_H$. Then $\hat{\phi} \in C^\infty(\mathbb{R}\setminus\{0\})$.*

Proof. It is enough to show $\hat{\phi} \in C(\mathbb{R}\setminus\{0\})$ for any $\phi = H\psi \in H(\mathscr{D})$. Since $\hat{\phi}(\omega) = (H\psi)\hat{}(\omega) = -i\mathrm{sgn}(\omega)\hat{\psi}(\omega)$ and $\psi \in \mathscr{D}$, the proof is complete.

Theorem 4.5. *Let $\omega \in \mathbb{R}\setminus\{0\}$. Then $He^{i\omega x} = -i\mathrm{sgn}(\omega)e^{i\omega x}$ a.e. $x \in \mathbb{R}$, i.e.,*

$$H\cos(\omega x) = \mathrm{sgn}(\omega)\sin(\omega x), \quad H\sin(\omega x) = -\mathrm{sgn}(\omega)\cos(\omega x) \quad a.e.\ x \in \mathbb{R}.$$

Proof. For $\omega \in \mathbb{R}\setminus\{0\}$, we have

$$\begin{aligned}\langle He^{-i\omega\cdot}, \phi\rangle &= -\langle e^{-i\omega\cdot}, H\phi\rangle = -\lim_{A,B\to\infty}\int_{[-A,B]} e^{-i\omega x}H\phi(x)dx \\ &= -(H\phi)\hat{}(\omega) = i\mathrm{sgn}(\omega)\hat{\phi}(\omega) \\ &= i\mathrm{sgn}(\omega)\langle e^{-i\omega\cdot}, \phi\rangle \quad (\forall\, \phi \in \mathscr{D}_H),\end{aligned}$$

which concludes that $He^{-i\omega\cdot} = i(\mathrm{sgn}\omega)e^{-i\omega\cdot}$ $(\forall\, \omega \in \mathbb{R}\setminus\{0\})$.

Theorem 4.6. *For any $f = \sum_{k\in\mathbb{Z}\setminus\{0\}} c_k e^{ik\frac{2\pi}{T}\cdot} \in \dot{L}^2_T$, there holds*

$$Hf = \sum_{k\in\mathbb{Z}\setminus\{0\}}[-i\mathrm{sgn}(k)]c_k e^{ik\frac{2\pi}{T}\cdot}. \tag{4.5}$$

Proof. Based on the fact $\dot{L}^2_T \subsetneq \mathscr{D}'_H$ and Theorem 4.5 we have

$$Hf = \sum_{k\in\mathbb{Z}\setminus\{0\}} c_k He^{ik\frac{2\pi}{T}\cdot} = \sum_{k\in\mathbb{Z}\setminus\{0\}}[-i\mathrm{sgn}(k)]c_k e^{ik\frac{2\pi}{T}\cdot}.$$

5 Application of the extension of Hilbert transform

5.1 Signal demodulation and Bedrosian identity

Let $s \in L^2(\mathbb{R})$ be a real-valued signal. The classical way to define its instantaneous amplitude and phase without ambiguity is to write $s(t)$ as $s(t) = \rho(t)\cos\theta(t)$ with $\rho(t) \geqslant 0$ such that $a(t) = \rho(t)e^{i\theta(t)}$ is an analytic signal (c.f. [2, 22, 23]). That is

$$H(\rho(t)\cos\theta(t)) = \rho(t)\sin\theta(t), \tag{5.1}$$

where $Hs(t)$ is the classical Hilbert transform defined by (1.1). The left side of (5.1) is the Hilbert transform of a product. As many authors said[22], it is appropriate to make use of the so-called Bedrosian identity:

$$H(fg) = fHg, \tag{5.2}$$

which is shown to hold for $f, g \in L^2(\mathbb{R})$ satisfying some conditions such as supp$\hat{f} \subset [-B, B]$ and supp$\hat{g} \subset \mathbb{R}\backslash[-B, B]$ for some $B > 0$ (c.f. [5, 7, 24–26]). This identity is used without doubt to conclude (5.1) by setting $f(t) = \rho(t)$ and $g(t) = \cos\theta(t)$ (c.f. [22]):

$$H(\rho(t)\cos\theta(t)) = \rho(t)H\cos\theta(t), \tag{5.3}$$

However, we point out that since $\cos\theta(t)$ is not in $L^2(\mathbb{R})$ usually, Bedrosian identity (5.2) cannot be used to deduce (5.3). In fact, for $s(t) = \rho(t)\cos\theta(t) \in L^2(\mathbb{R})$, the Hilbert transform on the left side of (5.3) is associated with definition (1.1) but $H\cos\theta(t)$ on its right side does not make sense with this definition.

If $\theta(t)$ satisfies the conditions of Theorem 3.7, then $\cos\theta(t) \in \mathscr{D}'_H$ and both sides of (5.3) make sense. Let A be the analytic signal operator defined by $A\phi = \phi + iH\phi$ for any $\phi \in \mathscr{D}_H$ (see [2, 23]). Denote $\mathscr{D}'_A := \{f \in \mathscr{D}'_H | \langle f, A\phi\rangle = 0\ (\forall\, \phi \in \mathscr{D}_H)\}$. By means of Therorem 3.7 we have the following Bedrosian theorem.

Theorem 5.1. *Let $\rho \in L^2(\mathbb{R})$ satisfy $\rho(x) \geqslant 0$ and θ satisfy the conditions of Theorem* 3.7. *Then*

$$H(\rho(x)\cos\theta(x)) = \rho(x)H\cos\theta(x) = \rho(x)\sin\theta(x) \tag{5.4}$$

if and only if $\rho e^{i\theta}, e^{i\theta} \in \mathscr{D}'_A$.

Proof. It is easy to see that $H\cos\theta(x) = \sin\theta(x)$ if and only if $He^{i\theta(x)} = -ie^{i\theta(x)}$, which is equivalent to $-\langle e^{i\theta}, H\phi\rangle = -\langle e^{i\theta}, i\phi\rangle$ for any $\phi \in \mathscr{D}_H$, i.e., $e^{i\theta} \in \mathscr{D}'_A$.

Similarly, it can be shown that $H\cos\theta(x) = \sin\theta(x)$ if and only if $\rho e^{i\theta} \in \mathscr{D}'_A$. The proof of the theorem is complete.

As a special case, we have the following

Coroally 5.2. *Let $\rho \in L^2(\mathbb{R})$ satisfy $\rho(x) \geqslant 0$ and $\omega > 0$, $\theta \in \mathbb{R}$. Then*

$$H(\rho(x)\cos(\omega x + \theta)) = \rho(x)H\cos(\omega x + \theta) = \rho(x)\sin(\omega x + \theta) \tag{5.5}$$

if and only if supp$\hat{\rho} \subset [-\omega, \omega]$.

Proof. It is obvious that $\theta(x) = \omega x + \theta$ satisfies the conditions of Theorem 3.7. Equality $\langle e^{i(\omega x+\theta)}, A\phi\rangle = e^{i\theta}(A\phi)\hat{}(-\omega) = 0\ (\forall\, \phi \in \mathscr{D}_H)$ implies $e^{i(\omega\cdot+\theta)} \in \mathscr{D}'_A$. Therefore (5.5) holds if and only if $\rho e^{i(\omega\cdot+\theta)} \in \mathscr{D}'_A$, that is $\langle \rho e^{i\omega\cdot}, \phi + iH\phi\rangle = 0\ (\forall\, \phi \in \mathscr{D}_H)$, which is equivalent to $\rho(x)e^{i\omega x} + iH[\rho(x)e^{i\omega x}] = 0$. Operating by Fourier transform we get $[\mathrm{sgn}(\xi) - 1]\hat{\rho}(\xi - \omega) = 0$. Using the Hermit symmetry of $\hat{\rho}$ we conclude $\hat{\rho}(\xi) = 0$ for all $|\xi| > |\omega|$.

5.2 Operator equations

The fact that $H : \mathscr{D}'_H \to \mathscr{D}'_H$ is a homeomorphism can be used to solve operator equation. As an example, we consider the following equation

$$y + Hy = f, \tag{5.6}$$

where f is a given distribution. This equation is solved for $f \in \mathscr{D}'$ in [14] where the classical Hilbert transform is extended as an operator $H:\ \mathscr{D}' \to H'(\mathscr{D})$ and $H'(\mathscr{D})$ is defined as the so-called ultradistributiona space. In [14] the author does not clarify in what distribution space to solve the solution y. Since the left side of (5.6) is a sum of y and Hy, which makes sense only if y and Hy are in the same space, it is very suitable to solve it in $\mathscr{D}'_H$. In fact $\mathscr{D}'_H$ is the most largest space for this equation. Hence, (5.6) can be understood as: for a given $f \in \mathscr{D}'_H$, finding $y \in \mathscr{D}'_H$ such that (5.6) holds.

Operating on both sides of (5.6) by H, the extended Hilbert transform defined by Definition 4.1, we have $Hy - y = Hf$, which together with (5.6) concludes the solution $y = \frac{1}{2}(f - Hf)$.

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