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Exact traveling wave solutions and dynamical behavior for the $(n+1)$-dimensional multiple sine-Gordon equation

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Abstract Using the methods of dynamical systems for the $(n+1)$-dimensional multiple sine-Gordon equation, the existences of uncountably infinite many periodic wave solutions and breaking bounded wave solutions are obtained. For the double sine-Gordon equation, the exact explicit parametric representations of the bounded traveling solutions are given. To guarantee the existence of the above solutions, all parameter conditions are determined.

Keywords: nonlinear wave, bifurcation, exact explicit traveling wave solution, double sine-Gordon equation, multiple sine-Gordon equation.

MSC(2000): 35B10, 35L70, 35Q51

1 Introduction

In this paper, we consider the following generalized forms of the double sine-Gordon equation:

$$k\sum_{j=1}^{n}\frac{\partial^2 u}{\partial x_j^2}-\frac{\partial^2 u}{\partial t^2}=2\alpha\sin(mu)+\beta\sin(2mu),\quad m\geqslant 1 \tag{1}$$

and the multiple sine-Gordon equation

$$k\sum_{j=1}^{n}\frac{\partial^2 u}{\partial x_j^2}-\frac{\partial^2 u}{\partial t^2}=\sum_{l=1}^{p}\alpha_l\sin(lmu), \tag{1_{mul}}$$

where m, n and p are positive integers.

When $\beta=0$ and $m=1$, (1) is the $(n+1)$-dimensional sine-Gordon equation (see [1, 2] and references therein). Recently, by using the tanh method and a variable separated ODE method, for the case $n=1$, Wazwaz[2] derived several exact travelling wave solutions of (1). There are some interesting problems: does an exact travelling wave solution obtained by the computer algebraic method really satisfy the given travelling equation? What is the dynamical behavior of the known exact travelling wave solutions? How do the travelling wave solutions depend on the parameters of the system? Are there the dynamics of the so-called compacton and peakon solutions for (1)? As we know, these problems have not been considered before for

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(1) and (1_{mul}). In this paper, we shall consider the existence and dynamical behavior of the bounded travelling wave solutions of (1) in different regions of the parametric space, by using the methods of dynamical systems (see [3, 4]). We shall give the possible exact explicit parametric representations for these bounded travelling wave solutions of (1). The more generalized form (1_{mul}) of (1) will be briefly considered. The results of this paper will more completely answer the above problems and improve the conclusions of [2].

To find the travelling wave solutions for (1) and (1_{mul}), we use the wave variable $\xi = \sum_{j=1}^n \mu_j x_j - ct$, where c is the propagating wave velocity. Then, (1) and (1_{mul}) can become the ordinary differential equations

$$\left(c^2 - k\sum_{j=1}^n \mu_j^2\right)u_{\xi\xi} + 2\alpha\sin(mu) + \beta\sin(2mu) = 0 \tag{2}$$

and

$$\left(c^2 - k\sum_{j=1}^n \mu_j^2\right)u_{\xi\xi} + \sum_{l=1}^p \alpha_l\sin(lmu) = 0. \tag{2_{mul}}$$

Making the transformation $v = e^{imu}$, $i = \sqrt{-1}$, we have

$$\sin(mu) = \frac{v - v^{-1}}{2i}, \quad \cos(mu) = \frac{v + v^{-1}}{2}, \quad \sin(2mu) = \frac{v^2 - v^{-2}}{2i}. \tag{3}$$

Denote that

$$A = c^2 - k\sum_{j=1}^n \mu_j^2. \tag{4}$$

We always assume $A \neq 0$. Substituting (3) for (2), we have

$$2Avv_{\xi\xi} - 2Av_\xi^2 + m(v^2 - 1)(\beta v^2 + 2\alpha v + \beta) = 0, \tag{5}$$

which is equivalent to the system

$$\frac{dv}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{2Ay^2 - m(v^2 - 1)(\beta v^2 + 2\alpha v + \beta)}{2Av}. \tag{6}$$

This system has the first integral

$$H(v, y) = \frac{y^2}{v^2} + \frac{m}{2Av^2}(\beta v^4 + 4\alpha v^3 + 4\alpha v + \beta) = h. \tag{7}$$

Notice that

$$\sin(3mu) = 3\sin(mu) - 4\sin^3(mu), \quad \sin(4mu) = 8\cos^3(mu)\sin(mu) - 4\cos(mu)\sin(mu), \ldots.$$

Hence,

$$\sum_{l=1}^p \alpha_l \sin(lmu) = P\left(\frac{v - v^{-1}}{2i}, \frac{v + v^{-1}}{2}\right),$$

where $P(\cdot, \cdot)$ is a p-degree polynomial of two variables. Thus, (2_{mul}) can become the system

$$\frac{dv}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{2Av^{p-2}y^2 - mQ(v)}{2Av^{p-1}}, \tag{6_{mul}}$$

where

$$Q(v) = \alpha_p v^{2p} + \alpha_{p-1} v^{2p-1} + \cdots + \alpha_1 v^{p+1} - \alpha_1 v^{p-1} - \alpha_2^{p-2} - \cdots - \alpha_{p-1} v - \alpha_p = (v^2-1)Q_1(v).$$

(6_{mul}) has the first integral

$$H_p(v,y) = \frac{y^2}{v^2} + \frac{m}{Av^2}\int \frac{(v^2-1)Q_1(v)}{v^{p+1}}dv. \tag{7_{mul}}$$

The system (6_{mul}) abounds in a dynamical bifurcation behavior due to the high order nonlinearity of $Q(v)$.

We see from (3) that

$$u = \frac{1}{m}\arccos\frac{v^2+1}{2v}. \tag{I}$$

We emphasize that when $v = 0$, the right hands of the second equations of systems (6) and (6_{mul}) are discontinuous. We call such systems as the singular travelling wave systems. The straight line $v = 0$ in the $v-y$-phase plane is called a singular straight line. It derives the existence of some non-smooth behavior and breaking properties of travelling wave solutions of systems (6) and (6_{mul}) (see [3, 4]).

In next two sections, we shall use the "three-step method" posed by the author to discuss system (6). Namely, (i) Making a transformation of the variable, such that a singular travelling wave system becomes a "regular system"; (ii) Investigating the dynamical bifurcation behavior for the "regular system"; (ii) By using the fact of different scales of "new (fast time scale)" variable and "old (slow time scale)" variable near the singular straight line, we determine the profiles of the travelling wave solutions and give all possible exact explicit parametric representations for the bounded travelling wave solutions.

2 Bifurcations of the phase portraits of (6)

We first make the transformation $d\xi = v d\zeta$ for $v \neq 0$, such that the system (6) becomes

$$\frac{dv}{d\zeta} = vy, \quad \frac{dy}{d\zeta} = y^2 - \frac{m}{2A}(v^2-1)(\beta v^2 + 2\alpha v + \beta). \tag{8}$$

Clearly, (8) has the same phase portraits as (6). But, the phase orbits of two systems have different parametric representations. Denote that

$$q(v) = (v^2-1)(\beta v^2 + 2\alpha v + \beta), \tag{9}$$

$$q'(v) = (v^2-1)(2\beta v + \alpha) + 2v(\beta v^2 + 2\alpha v + \beta). \tag{10}$$

Clearly, when $|\alpha| > |\beta|$, system (8) has four equilibrium points $A(-1,0), B(1,0), C(v_1,0)$ and $D(v_2,0)$ on the v-axis, where

$$v_1 = \frac{1}{\beta}(-\alpha - \sqrt{\alpha^2-\beta^2}), \qquad v_2 = \frac{1}{\beta}(-\alpha + \sqrt{\alpha^2-\beta^2}).$$

When $|\alpha| = |\beta|$, the equilibrium points $C(v_1,0)$ and $D(v_2,0)$ coincide with each other. When $|\alpha| < |\beta|$, (8) has two equilibrium points $A(-1,0)$ and $B(1,0)$ on the v-axis. When $\beta A < 0$, (8) has two equilibrium points $S_-(0,-Y)$ and $S_+(0,Y)$ on the y-axis, where $Y = \sqrt{\frac{-m\beta}{2A}}$. Especially, when $\alpha \neq 0$ and $\beta = 0$, the origin $O(0,0)$ is an high order equilibrium point of (8).

Let $M(v_e, y_e)$ be the coefficient matrix of the linearized system of (8) at an equilibrium point (v_e, y_e) and $J(v_e, y_e)$ be its Jacobin determinant. Then, we have $\text{Trace}(M(v_e, 0)) = 0$ and $J(v_e, 0) = -\frac{mv_e}{A} q'(v_e)$.

By the theory of planar dynamical systems, we know that for an equilibrium point of a planar integrable system, if $J < 0$ then the equilibrium point is a saddle point; if $J > 0$ and $\text{Trace}(M(\phi_i, 0)) = 0$ then it is a center point; if $J > 0$ and $(\text{Trace}(M(\phi_i, 0)))^2 - 4J(\phi_i, 0) > 0$ then it is a node; if $J = 0$ and the Poincare index of the equilibrium point is 0 then it is a cusp.

Now, we have

$$J(-1, 0) = -\frac{2m}{A}(\alpha - \beta), \qquad J(1, 0) = \frac{2m}{A}(\alpha + \beta),$$

$$J(v_{1,2}, 0) = \frac{4m(-v_{1,2})}{A}(\alpha^2 - \beta^2)(\alpha \pm \sqrt{\alpha^2 - \beta^2}), \qquad J(0, \pm Y) = 2Y^2 > 0.$$

So that, the equilibrium points $S_\pm(0, \pm Y)$ of (8) are node points. When $|\alpha| > |\beta|$, the equilibrium points C and D have the same types (center points or saddle points), while the equilibrium points A and B have different types. When $|\alpha| < |\beta|$, the equilibrium points A and B have the same types (center points or saddle points).

For the invariant function $H(v, y)$ given by (7), we have

$$h_A = H(-1, 0) = -\frac{m}{A}(4\alpha - \beta), \qquad h_B = H(1, 0) = \frac{m}{A}(4\alpha + \beta),$$

$$h_C = H(v_1, 0) = \frac{m[(\beta^4 - 4\alpha^4) - (4\alpha^3 + 2\alpha\beta^2)\sqrt{\alpha^2 - \beta^2}]}{A\beta(\alpha + \sqrt{\alpha^2 - \beta^2})^2} = -\frac{m(2\alpha^2 + \beta^2)}{A\beta},$$

$$h_D = H(v_2, 0) = \frac{m[(\beta^4 - 4\alpha^4) + (4\alpha^3 + 2\alpha\beta^2)\sqrt{\alpha^2 - \beta^2}]}{A\beta(\alpha - \sqrt{\alpha^2 - \beta^2})^2} = -\frac{m(2\alpha^2 + \beta^2)}{A\beta} = h_C.$$

By using the above fact to do a qualitative analysis, we obtain the bifurcations of phase portraits of (8) shown in Fig. 1.

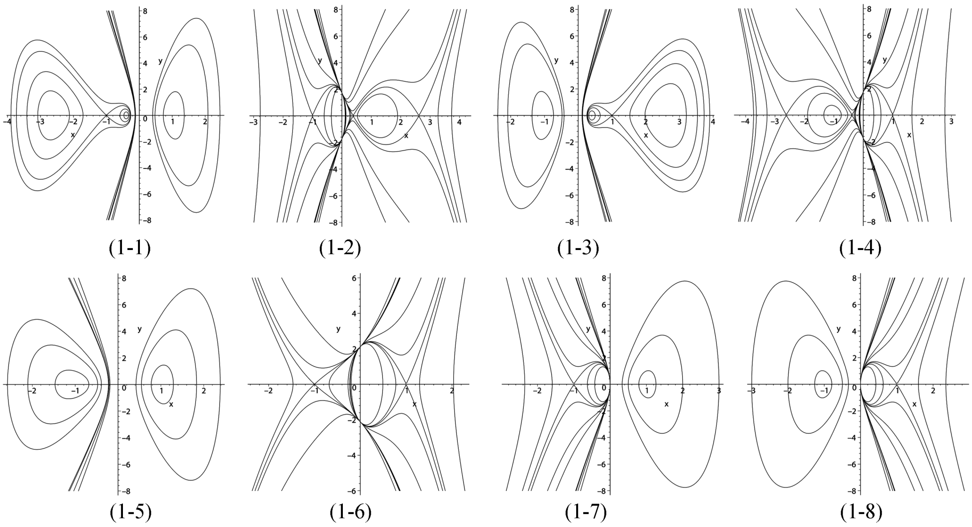

Fig. 1. The bifurcations of phase portraits of (8).

(1-1) $|\alpha| > |\beta|,\ \alpha A > 0,\ \beta A > 0$; (1-2) $|\alpha| > |\beta|,\ \alpha A > 0,\ \beta A < 0$; (1-3) $|\alpha| > |\beta|,\ \alpha A < 0,\ \beta A > 0$; (1-4) $|\alpha| > |\beta|,\ \alpha A < 0,\ \beta A < 0$; (1-5) $|\alpha| \leqslant |\beta|, \alpha A > 0,\ \beta A > 0$; (1-6) $|\alpha| \leqslant |\beta|, \alpha A < 0,\ \beta A < 0$; (1-7) $\alpha A > 0,\ \beta = 0$; (1-8) $\alpha A < 0,\ \beta = 0$.

3 The parametric representations of the bounded travelling wave solutions of system (6)

In this section, we use the results given by sec. 2 to determine the parametric representations of the bounded phase orbits of (6) in its parameter space. Then, we give the exact explicit travelling wave solutions of (1). Because the phase portraits (1-3), (1-4) and (1-8) are the reflections of the phase portraits (1-1), (1-2) and (1-7) in Fig. 1 with respect to the y-axis, respectively, we only discuss the last three phase portraits, and Fig. 1 (1-5) and (1-6).

3.1 Solitary wave solutions and periodic wave solutions

Suppose that $|\alpha| > |\beta|,\ \alpha A > 0,\ \beta A > 0$. For example, $\alpha > 0,\ \beta > 0, A > 0$. We consider the case of Fig. 1 (1-1).

(i) Corresponding to $H(v, y) = h_A$, we have two homoclinic orbits of (8). The function (7) can be written as

$$y^2 = -\frac{m}{2A}(v+1)^2(\beta v^2 + 2(2\alpha-\beta)v + \beta) = \frac{m\beta}{2A}(v+1)^2(v_M - v)(v - v_m), \tag{11}$$

where $v_{M,m} = \frac{1}{\beta}[-(2\alpha-\beta) \pm 2\sqrt{\alpha(\alpha-\beta)}]$. By using (11) and the first equation of (6), we obtain the following two parametric representations:

$$v(\xi) = 1 + \frac{2a_1}{e_M \cosh(\omega_1 \xi) - b_1}, \tag{12}$$

$$v(\xi) = 1 + \frac{2a_1}{e_m \cosh(\omega_1 \xi) - b_1}, \tag{13}$$

where $a_1 = -(v_M+1)(v_m+1),\ b_1 = v_M + v_m = -\frac{2}{\beta}(2\alpha-\beta),\ e_M = b_1 + \frac{2a_1}{v_M+1},\ e_m = b_1 + \frac{2a_1}{v_m+1},\ \omega_1 = \sqrt{\frac{m a_1 \beta}{2A}}$. From the relationship (I), we obtain two solitary wave solutions of (1) with the peak type and valley type, respectively, as follows:

$$u\left(\sum_{j=i}^{n} \mu_j x_j - ct\right) = \frac{1}{m}\arccos\left(\frac{\left(1 + \frac{2a_1}{e_M \cosh\left(\omega_1\left(\sum_{j=1}^{n}\mu_j x_j - ct\right)\right) - b_1}\right)^2 + 1}{2\left(1 + \frac{2a_1}{e_M \cosh\left(\omega_1\left(\sum_{j=1}^{n}\mu_j x_j - ct\right)\right) - b_1}\right)}\right) \tag{14}$$

and

$$u\left(\sum_{j=i}^{n} \mu_j x_j - ct\right) = \frac{1}{m}\arccos\left(\frac{\left(1 + \frac{2a_1}{e_m \cosh\left(\omega_1\left(\sum_{j=1}^{n}\mu_j x_j - ct\right)\right) - b_1}\right)^2 + 1}{2\left(1 + \frac{2a_1}{e_m \cosh\left(\omega_1\left(\sum_{j=1}^{n}\mu_j x_j - ct\right)\right) - b_1}\right)}\right). \tag{15}$$

(ii) Corresponding to $H(v, y) = h,\ h \in (h_C, h_A)$, we have two families of periodic orbits of (8) for which the function (7) can be respectively written as

$$y^2 = -\frac{m}{2A}(\beta v^4 + 4\alpha v^3 + 4\alpha v + \beta) + hv^2 = \frac{m\beta}{2A}(r_1 - v)(v - r_2)(v - r_3)(v - r_4)$$

and

$$y^2 = -\frac{m}{2Av^2}(\beta v^4 + 4\alpha v^3 + 4\alpha v + \beta) + hv^2 = \frac{m\beta}{2A}(r_1 - v)(r_2 - v)(r_3 - v)(v - r_4).$$

By using these formulas and the first equation of (6), we obtain the following two parametric representations:

$$v(\xi) = r_2 + \frac{(r_1 - r_2)(r_2 - r_3)\mathrm{sn}^2(\omega_2\xi, k_1)}{(r_1 - r_3) - (r_1 - r_2)\mathrm{sn}^2(\omega_2\xi, k_1)} \tag{16}$$

and

$$v(\xi) = r_4 + \frac{(r_1 - r_4)(r_3 - r_4)\mathrm{sn}^2(\omega_2\xi, k_1)}{(r_1 - r_3) + (r_3 - r_4)\mathrm{sn}^2(\omega_2\xi, k_1)}, \tag{17}$$

where $\omega_2 = \sqrt{\frac{m\beta(r_1-r_3)(r_2-r_4)}{8A}}$, $k_1 = \sqrt{\frac{(r_1-r_2)(r_3-r_4)}{(r_1-r_3)(r_2-r_4)}}$. Hence, there exist the following periodic travelling wave solutions of (1):

$$u(\xi) = \frac{1}{m}\arccos\left(\frac{\left(r_2 + \frac{(r_1-r_2)(r_2-r_3)\mathrm{sn}^2(\omega_2\xi,k_1)}{(r_1-r_3)-(r_1-r_2)\mathrm{sn}^2(\omega_2\xi,k_1)}\right)^2 + 1}{2\left(r_2 + \frac{(r_1-r_2)(r_2-r_3)\mathrm{sn}^2(\omega_2\xi,k_1)}{(r_1-r_3)-(r_1-r_2)\mathrm{sn}^2(\omega_2\xi,k_1)}\right)}\right) \tag{18}$$

and

$$u(\xi) = \frac{1}{m}\arccos\left(\frac{\left(r_4 + \frac{(r_1-r_4)(r_3-r_4)\mathrm{sn}^2(\omega_2\xi,k_1)}{(r_1-r_3)+(r_3-r_4)\mathrm{sn}^2(\omega_2\xi,k_1)}\right)^2 + 1}{2\left(r_4 + \frac{(r_1-r_4)(r_3-r_4)\mathrm{sn}^2(\omega_2\xi,k_1)}{(r_1-r_3)+(r_3-r_4)\mathrm{sn}^2(\omega_2\xi,k_1)}\right)}\right). \tag{19}$$

(iii) Corresponding to $H(v, y) = h,\ h \in (h_A, h_B)$, we have a family of periodic orbits of (8) enclosing three equilibrium points A, C and D, for which the function (7) can be written as

$$y^2 = -\frac{m}{2A}(\beta v^4 + 4\alpha v^3 + 4\alpha v + \beta) + hv^2 = \frac{m\beta}{2A}(r_1 - v)(v - r_2)[(v - g_2)^2 + g_1^2].$$

By using this formula and the first equation of (6), we obtain the following parametric representations:

$$v(\xi) = \frac{(r_1B_1 + r_2A_1) - (r_1B_1 - r_2A_1)\mathrm{cn}(\omega_3\xi, k_2)}{(A_1 + B_1) + (A_1 - B_1)\mathrm{cn}(\omega_3\xi, k_2)}, \tag{20}$$

where $A_1^2 = (r_1 - g_2)^2 + g_1^2$, $B_1^2 = (r_2 - g_2)^2 + g_1^2$, $k_2^2 = \frac{(r_1-r_2)^2-(A_1-B_1)^2}{4A_1B_1}$, $\omega_3 = \sqrt{\frac{m\beta A_1B_1}{2A}}$. Thus, we have the periodic traveling wave solutions of (1):

$$u(\xi) = \frac{1}{m}\arccos\left(\frac{\left(\frac{(r_1B_1+r_2A_1)-(r_1B_1-r_2A_1)\mathrm{cn}(\omega_3\xi,k_2)}{(A_1+B_1)+(A_1-B_1)\mathrm{cn}(\omega_3\xi,k_2)}\right) + 1}{2\left(\frac{(r_1B_1+r_2A_1)-(r_1B_1-r_2A_1)\mathrm{cn}(\omega_3\xi,k_2)}{(A_1+B_1)+(A_1-B_1)\mathrm{cn}(\omega_3\xi,k_2)}\right)}\right). \tag{21}$$

(iv) Corresponding to $H(v, y) = h,\ h \in (h_B, \infty)$, we have two families of periodic orbits of (8). One family encloses three equilibrium points A, C and D, while another family encloses the equilibrium point B. Now, the function (7) has the same representation as the case (ii). Therefore, we have the same periodic travelling wave solutions of (1) as (18) and (19).

3.2 Kink wave solutions, anti-kink wave solutions and periodic wave solutions

Suppose that $|\alpha| > |\beta|,\ \alpha A > 0,\ \beta A < 0$. For example, $\alpha > 0,\ \beta < 0, A > 0$. We consider the case of Fig. 1(1-2).

(i) Corresponding to $H(v, y) = h_C$, we have two heteroclinic orbits of (8) connecting the equilibrium points C and D. The function (7) can be written as

$$y^2 = -\frac{m}{2A}\left(\beta v^4 + 4\alpha v^3 + \frac{2(2\alpha^2 + \beta^2)}{\beta}v^2 + 4\alpha v + \beta\right) = \frac{-m\beta}{2A}(v_2 - v)^2(v - v_1)^2, \tag{22}$$

where v_1 and v_2 were given in sec. 2. By using this formula and the first equation of (6), we obtain the following two parametric representations:

$$v(\xi) = \pm\left[-\frac{\alpha}{\beta} + \frac{\sqrt{\alpha^2-\beta^2}}{\beta}\tanh(\omega_4\xi)\right], \tag{22}$$

where $\omega_4 = \sqrt{\frac{-m(\alpha^2-\beta^2)}{2A\beta}}$. Hence, we obtain the kink wave solution and anti-kink wave solution of (1) as follows:

$$u(\xi) = \frac{1}{m}\arccos\left(\frac{\Big(-\frac{\alpha}{\beta} + \frac{\sqrt{\alpha^2-\beta^2}}{\beta}\tanh(\omega_4\xi)\Big)^2 + 1}{\pm 2\Big(-\frac{\alpha}{\beta} + \frac{\sqrt{\alpha^2-\beta^2}}{\beta}\tanh(\omega_4\xi)\Big)}\right). \tag{23}$$

(ii) Corresponding to $H(v,y) = h,\ h \in (h_B, h_C)$, we have a family of periodic orbits of (8) enclosing the equilibrium point B, for which the function (7) can be respectively written as

$$y^2 = -\frac{m}{2A}(\beta v^4 + 4\alpha v^3 + 4\alpha v + \beta) + hv^2 = -\frac{m\beta}{2A}(r_1 - v)(r_2 - v)(v - r_3)(v - r_4).$$

By using this formula and the first equation of (6), we obtain the following parametric representation:

$$v(\xi) = r_3 + \frac{(r_3 - r_4)(r_2 - r_3)\mathrm{sn}^2(\omega_5\xi, k_3)}{(r_2 - r_4) - (r_2 - r_3)\mathrm{sn}^2(\omega_5\xi, k_3)}, \tag{24}$$

where $\omega_5 = \sqrt{\frac{-m\beta(r_1-r_3)(r_2-r_4)}{8A}}$, $k_5 = \sqrt{\frac{(r_2-r_3)(r_1-r_4)}{(r_1-r_3)(r_2-r_4)}}$. Thus, we have the periodic travelling wave solutions of (1) as follows:

$$u(\xi) = \frac{1}{m}\arccos\left(\frac{\Big(r_3 + \frac{(r_3-r_4)(r_2-r_3)\mathrm{sn}^2(\omega_5\xi,k_3)}{(r_2-r_4)-(r_2-r_3)\mathrm{sn}^2(\omega_5\xi,k_3)}\Big)^2 + 1}{2\Big(r_3 + \frac{(r_3-r_4)(r_2-r_3)\mathrm{sn}^2(\omega_5\xi,k_3)}{(r_2-r_4)-(r_2-r_3)\mathrm{sn}^2(\omega_5\xi,k_3)}\Big)}\right). \tag{25}$$

3.3 Two families of periodic travelling wave solutions

Suppose that $|\alpha| \leqslant |\beta|,\ \alpha A > 0,\ \beta A > 0$. For example, $\alpha > 0,\ \beta > 0, A > 0$. We consider the case of Fig. 1 (1-5).

(i) Corresponding to $H(v,y) = h,\ h \in (h_A, h_B)$, we have a family of periodic orbits of (8) enclosing the equilibrium point A, for which the function (7) has the same representation as in 3.1 (iii), so that we have the same family of periodic travelling wave solutions as (21).

(ii) Corresponding to $H(v,y) = h,\ h \in (h_B, \infty)$, we have two families of periodic orbits of (8) enclosing the equilibrium point A and B, respectively, for which the function (7) has the same representation as in 3.1 (ii), so that we have the same two families of periodic travelling wave solutions as (18) and (19).

3.4 The breaking kink (or anti-kink) wave solutions in the "half-time intervals" for the existence of the solutions

We next assume that $|\alpha| \leqslant |\beta|,\ \alpha A < 0,\ \beta A < 0$. Namely, we consider the case of Fig. 1 (1-6). In addition, we continuously consider the case of Fig. 1 (1-2).

(i) Corresponding to $H(v,y) = h_A$, we have two heteroclinic orbits of (8) connecting the equilibrium points $A(-1,0)$ and $S_\pm(0,\pm Y)$. For (6), we have the equilibrium points $S_\pm(0,\pm Y)$

on the singular straight line $v=0$. These two connecting orbits determine two breaking wave solutions of (6) in their "half-time intervals" for the existence of the solutions. In fact, now, the function (7) can be written as

$$y^2=-\frac{m\beta}{2A}(v+1)^2\left(v^2+\frac{2(2\alpha-\beta)}{\beta}v+1\right).$$

By using this formula and the first equation of (6), we obtain

$$v(\xi)=-1\pm\frac{2a_2}{\sqrt{q}\sinh(\omega_6\xi+\xi_0)-a_2},\quad \xi\in(-\infty,\xi_f), \tag{26}$$

where

$$a_2=\frac{4(\beta-\alpha)}{\beta},\ q=\frac{16\alpha(\beta-\alpha)}{\beta^2},\ \omega_6=\sqrt{\frac{-2m(\beta-\alpha)}{A}},$$
$$\xi_f=\sinh^{-1}\left(\frac{3\sqrt{\frac{\beta-\alpha}{\alpha}}-\xi_0}{\omega_6}\right),\ \xi_0=-\frac{1}{\sqrt{a_2}}\sinh^{-1}\left(\frac{2a_2+b(v_0+1)}{(v_0+1)\sqrt{q}}\right),$$

for any $v_0\in(-1,0)$.

Clearly, when the "time variable ξ" takes a finite value $\xi=\xi_f=\sinh^{-1}\left(\frac{3\sqrt{\frac{\beta-\alpha}{\alpha}}-\xi_0}{\omega_6}\right)$, the solution $v(\xi)$ has arrived the state $S_\pm(0,\pm Y)$. Thus, the parametric representations

$$u(\xi)=\frac{1}{m}\arccos\left(\frac{\left(-1\pm\frac{2a_2}{\sqrt{q}\sinh(\omega_6\xi+\xi_0)-a_2}\right)^2+1}{2\left(-1\pm\frac{2a_2}{\sqrt{q}\sinh(\omega_6\xi+\xi_0)-a_2}\right)}\right),\quad \xi\in(-\infty,\xi_f) \tag{27}$$

give rise to a breaking kink wave solution and a breaking anti-kink wave solution of (1).

Similarly, corresponding to $H(v,y)=h_B$, we have two heteroclinic orbits of (8) connecting the equilibrium points $B(1,0)$ and $S_\pm(0,\pm Y)$. We also have the similar breaking kink wave solution and breaking anti-kink wave solution of (1) as (27).

3.5 The existence of uncountably infinite many bounded breaking wave solutions of (1)

We first assume that $|\alpha|>|\beta|,\ \alpha A>0,\ \beta A<0$ (see Fig. 1 (1-2)).

(i) Corresponding to $H(v,y)=h,\ h\in(-\infty,h_A)$, there exist two families of heteroclinic orbits of (8) connecting two equilibrium points $S_\pm(0,\pm Y)$ inside the two curve triangles consisting of the segment S_+S_- and four heteroclinic orbits $AS_+,\ AS_-$ and $BS_+,\ BS_-$, respectively. For (6), these orbits are close to the singular straight line $v=0$ and connecting to two points $S_\pm$. Due to the different "time scales" of ξ and ζ near the singular straight line $v=0$, two connecting orbits give rise to the uncountably infinite many breaking wave solutions of (1). In fact, the function (7) has the form

$$y^2=-\frac{m}{2A}(\beta v^4+4\alpha v^3+4\alpha v+\beta)+hv^2=\frac{m\beta}{2A}(r_1-v)(r_2-v)(v-r_3)(v-r_4),\quad h\in(-\infty,h_A).$$

By using this formula and the first equation of (6), we obtain the following two parametric representations, for the orbits in the left side of S_+S_-:

$$v(\xi)=r_3+\frac{(r_3-r_4)(r_2-r_3)\mathrm{sn}^2(\omega_5\xi,k_3)}{(r_2-r_4)-(r_2-r_3)\mathrm{sn}^2(\omega_5\xi,k_3)},\quad \xi\in(0,\xi_{lf}) \tag{28}$$

and for the orbits in the right side of S_+S_-:

$$v(\xi) = r_2 - \frac{(r_1 - r_2)(r_2 - r_3)\mathrm{sn}^2(\omega_5\xi, k_3)}{(r_1 - r_3) - (r_2 - r_3)\mathrm{sn}^2(\omega_5\xi, k_3)}, \quad \xi \in (0, \xi_{rf}), \tag{29}$$

where

$$\omega_5 = \sqrt{\frac{-m\beta(r_1 - r_3)(r_2 - r_4)}{8A}}, \qquad k_5 = \sqrt{\frac{(r_2 - r_3)(r_1 - r_4)}{(r_1 - r_3)(r_2 - r_4)}},$$

$$\xi_{lf} = \frac{1}{\omega_5}\mathrm{sn}^{-1}\left(\sqrt{\frac{r_3(r_2 - r_4)}{r_4(r_2 - r_3)}}, k_5\right), \qquad \xi_{rf} = \frac{1}{\omega_5}\mathrm{sn}^{-1}\left(\sqrt{\frac{r_2(r_1 - r_3)}{r_1(r_2 - r_3)}}, k_5\right).$$

Because the above two solutions given by (32) and (33) are only determined in two finite existence "time intervals". So they are the breaking solutions. Therefore, we have the uncountably infinite many bounded breaking wave solutions of (1) as follows:

$$u(\xi) = \frac{1}{m}\arccos\left(\frac{\left(r_3 + \frac{(r_3-r_4)(r_2-r_3)\mathrm{sn}^2(\omega_5\xi,k_3)}{(r_2-r_4)-(r_2-r_3)\mathrm{sn}^2(\omega_5\xi,k_3)}\right)^2 + 1}{2\left(r_3 + \frac{(r_3-r_4)(r_2-r_3)\mathrm{sn}^2(\omega_5\xi,k_3)}{(r_2-r_4)-(r_2-r_3)\mathrm{sn}^2(\omega_5\xi,k_3)}\right)}\right), \qquad \xi \in (0, \xi_{lf}) \tag{30}$$

$$u(\xi) = \frac{1}{m}\arccos\left(\frac{\left(r_2 - \frac{(r_1-r_2)(r_2-r_3)\mathrm{sn}^2(\omega_5\xi,k_3)}{(r_1-r_3)-(r_2-r_3)\mathrm{sn}^2(\omega_5\xi,k_3)}\right)^2 + 1}{2\left(r_2 - \frac{(r_1-r_2)(r_2-r_3)\mathrm{sn}^2(\omega_5\xi,k_3)}{(r_1-r_3)-(r_2-r_3)\mathrm{sn}^2(\omega_5\xi,k_3)}\right)}\right), \qquad \xi \in (0, \xi_{rf}). \tag{31}$$

Second, we assume that $|\alpha| < |\beta|,\ \alpha A < 0,\ \beta A < 0$ (see Fig. 1 (1-6)).

(ii) Corresponding to $H(v, y) = h,\ h \in (-\infty, h_B)$, there exist two families of heteroclinic orbits of (8) connecting two equilibrium points $S_\pm(0, \pm Y)$ inside the two curve triangles consisting of the segment S_+S_- and four heteroclinic orbits $AS_+,\ AS_-$ and $BS_+,\ BS_-$, respectively. We have the same results as (30) and (31).

3.6 The bounded travelling wave solutions of $(1)_{\beta=0}$ (i.e. sine-Gordon equation)

We finally assume that $\alpha A > 0,\ \beta = 0$. That is, we consider the case of Fig. 1 (1–7). Under these conditions, eq. (1) is the original $(n+1)$-dimensional sine-Gordon equation. In this case, we have $h_A = -\frac{4m\alpha}{A} = -h_B$.

(i) Corresponding to $H(v, y) = h_A$, we have two heteroclinic orbits of (8) connecting the equilibrium points $A(-1, 0)$ and the origin $O(0, 0)$. In this case, (7) becomes

$$y^2 = -\frac{2m\alpha}{A}v(v + 1)^2.$$

By using this formula and the first equation of (6), we obtain the following two parametric representations:

$$v(\xi) = -\tanh^2\left(\sqrt{\frac{m\alpha}{2A}}\xi\right), \quad \xi \in [0, \infty) \ \text{ and } \ \xi \in (-\infty, 0), \ \text{ respectively.} \tag{32}$$

Thus, we have a breaking kink wave solution and a breaking anti-kink wave solution of (1) as follows:

$$u(\xi) = \frac{1}{m}\arccos\left(\frac{\tanh^4\left(\frac{m\alpha}{2A}\xi\right) + 1}{-2\tanh^2\left(\sqrt{\frac{m\alpha}{2A}}\xi\right)}\right), \quad \xi \in [0, \infty) \ \text{ and } \ \xi \in (-\infty, 0), \ \text{ respectively.} \tag{33}$$

(ii) Corresponding to $H(v,y)=h,\ h\in(h_B,\infty)$, there exists a family of periodic orbits of (8) enclosing the equilibrium point $B(1,0)$. The function (7) has the form

$$y^2=-\frac{2m\alpha}{A}(v^3+v)+hv^2=\frac{2m\alpha}{A}v(v_M-v)(v-v_m),$$

where

$$v_{M,m}=\frac{1}{4m\alpha}(hA\pm\sqrt{h^2A^2-16\alpha^2m^2})=\frac{1}{4m\alpha}(hA\pm\sqrt{\Delta_1(h)}).$$

By using this formula and the first equation of (6), we obtain the following parametric representation:

$$v(\xi)=v_m+\frac{v_m\sqrt{\Delta_1(h)}\mathrm{sn}^2(\omega_7\xi,k_4)}{2m\alpha v_M-\sqrt{\Delta_1(h)}\mathrm{sn}^2(\omega_7\xi,k_4)},\quad h\in(h_B,\infty), \tag{34}$$

where

$$\omega_7=\sqrt{\frac{m\alpha v_M}{2A}},\qquad k_4=\sqrt{\frac{\sqrt{\Delta_1(h)}}{v_M}}.$$

Hence, we have the family of periodic wave solutions of $(1)_{\beta=0}$ as follows:

$$u(\xi)=\frac{1}{m}\arccos\left(\frac{\left(v_m+\frac{v_m\sqrt{\Delta_1(h)}\mathrm{sn}^2(\omega_7\xi,k_4)}{2m\alpha v_M-\sqrt{\Delta_1(h)}\mathrm{sn}^2(\omega_7\xi,k_4)}\right)^2+1}{2\left(v_m+\frac{v_m\sqrt{\Delta_1(h)}\mathrm{sn}^2(\omega_7\xi,k_4)}{2m\alpha v_M-\sqrt{\Delta_1(h)}\mathrm{sn}^2(\omega_7\xi,k_4)}\right)}\right). \tag{35}$$

(iii) Corresponding to $H(v,y)=h,\ h\in(-\infty,h_A)$, there exists a family of homoclinic orbits of (8) connecting to the origin $O(0,0)$ inside the two heteroclinic orbits defined by $H(v,y)=h_A$. Since these orbits are tangent to the singular line $v=0$ of (6) at the origin $(0,0)$, they give rise to the uncountably infinite many periodic orbits of (6). In fact, in this case, the function (7) becomes

$$y^2=\frac{-2m\alpha}{A}v(v^2-hv+1)=\frac{-2m\alpha}{A}v(v-v_M)(v-v_m).$$

By using this formula and the first equation of (6), we obtain the following parametric representation:

$$v(\xi)=v_M+\frac{(-v_M)\sqrt{\Delta_1(h)}\mathrm{sn}^2(\omega_7\xi,k_7)}{2m\alpha[(-v_M)+v_m\mathrm{sn}^2(\omega_7,k_7)]}, \tag{36}$$

where $v_{M,m},\ \omega_5,\ k_5$ and $\Delta_1(h)$ are the same as (ii).

Thus, we have the family of periodic wave solutions of $(1)_{\beta=0}$ as follows:

$$u(\xi)=\frac{1}{m}\arccos\left(\frac{\left(v_M+\frac{(-v_M)\sqrt{\Delta_1(h)}\mathrm{sn}^2(\omega_7\xi,k_4)}{2m\alpha[(-v_M)+v_m\mathrm{sn}^2(\omega_7\xi,k_4)]}\right)^2+1}{2\left(v_M+\frac{(-v_M)\sqrt{\Delta_1(h)}\mathrm{sn}^2(\omega_7\xi,k_4)}{2m\alpha[(-v_M)+v_m\mathrm{sn}^2(\omega_7\xi,k_4)]}\right)}\right). \tag{37}$$

For all $h\in(-\infty,h_A)$, the solutions given by (37) are just so called compacton solutions.

4 Brief discussion of bounded travelling wave solutions for system (6_{mul})

In this section, we discuss the system

$$\frac{dv}{d\xi}=y,\quad \frac{dy}{d\xi}=\frac{2Av^{p-2}y^2-m(v^2-1)(\alpha_pv^{2(p-1)}+\cdots+\alpha_p)}{2Av^{p-1}}, \tag{6_{mul}}$$

where $p > 2$. For example, when $p = 3$ and $p = 4$, (6_{mul}) has the equivalent forms

$$\frac{dv}{d\zeta} = v^2 y, \quad \frac{dy}{d\zeta} = vy^2 - \frac{m}{2A}(v^2 - 1)(\alpha_3 v^4 + \alpha_2 v^3 + (\alpha_1 + \alpha_3)v^2 + \alpha_2 v + \alpha_3) \tag{38}$$

with the first integral

$$H_3(v, y) = \frac{y^2}{v^2} + \frac{m}{6Av^3}(2\alpha_3 v^6 + 3\alpha_2 v^5 + 6\alpha_1 v^4 + 6\alpha_1 v^2 + 3\alpha_2 v + 2\alpha_3); \tag{39}$$

and

$$\begin{aligned} \frac{dv}{d\zeta} &= v^3 y, \\ \frac{dy}{d\zeta} &= v^2 y^2 - \tfrac{m}{2A}(v^2 - 1)(\alpha_4 v^6 + \alpha_3 v^5 + (\alpha_2 + \alpha_4)v^4 + (\alpha_1 + \alpha_3)v^3 + (\alpha_2 + \alpha_4)v^2 + \alpha_3 v + \alpha_4) \end{aligned} \tag{40}$$

with the first integral

$$H_4(v, y) = \frac{y^2}{v^2} + \frac{m}{12Av^4}(3\alpha_4 v^8 + 4\alpha_3 v^7 + 6\alpha_2 v^6 + 12\alpha_1 v^5 + 12\alpha_1 v^3 + 6\alpha_2 v^2 + 4\alpha_3 v + 3\alpha_4). \tag{41}$$

System (6_{mul}) is a $(p + 1)$-parameter system consisting of the parameter group $(A, \alpha_1, \alpha_2, \ldots, \alpha_p)$. If the polynomial $Q(u) = \alpha_p v^{2(p-1)} + \cdots - \alpha_p$ has more real zeros, then, (6_{mul}) has a very complicated dynamical bifurcation behavior.

We again consider the system

$$\frac{dv}{d\zeta} = v^{p-1} y, \quad \frac{dy}{d\zeta} = v^{p-2} y^2 - \frac{m}{2A}(v^2 - 1)(\alpha_p v^{2(p-1)} + \cdots + \alpha_p). \tag{42}$$

Obviously, for any $p > 2$, system (42) always has two equilibrium points $A(-1, 0), B(1, 0)$. There is no equilibrium point on the y-axis. Therefore, for $p > 2$, system (6_{mul}) has no equilibrium point (node) on the singular line $v = 0$. Unlike system (6), it has no breaking kink and anti-kink solutions.

For the linearized system of (42), we have the Jacobin determinants

$$J(-1, 0) = \frac{2m}{A}(-1)^p Q_1(-1), \quad J(1, 0) = \frac{2m}{A} Q_1(1).$$

And from (39) and (41), we obtain

$$\begin{aligned} &h_{A3} = H_3(-1, 0) = -\frac{m}{3A}(2\alpha_3 - 3\alpha_2 + 6\alpha_1), \quad h_{B3} = H_3(1, 0) = \frac{m}{3A}(2\alpha_3 + 3\alpha_2 + 6\alpha_1); \\ &h_{A4} = H_4(-1, 0) = \frac{m}{6A}(3\alpha_4 - 4\alpha_3 + 6\alpha_2 - 12\alpha_1), h_{B4} = H_4(1, 0) = \frac{m}{6A}(3\alpha_4 + 4\alpha_3 + 6\alpha_2 - 12\alpha_1). \end{aligned}$$

Generally, we denote that $h_{Ap} = H_p(-1, 0), h_{Bp} = H_p(1, 0)$.

By choosing the parameters in the parameter space $S(A, \alpha_1, \alpha_2, \ldots, \alpha_p)$, we can make $Q_1(v)$ not have real zero. Then, there exist three cases: (i) both equilibrium points A and B are centers; (ii) both equilibrium points A and B are saddle points; (iii) one is a center, another is a saddle point (see Fig. 2).

We see from Fig. 2 that there is a region of the parameter space S such that the following fact holds.

(1) Corresponding to the orbit family defined by $H_p(v,y)=h,\ h\in(-\infty,h_{Ap})$ with the initial condition $(v_0,0),\ -1<v_0<0$, there is a family of bounded solutions of (6_{mul}) (for which $v(\xi)$ is bounded, but $v'(\xi)$ is unbounded). This family gives rise to the uncountably finite many breaking wave solutions of (1_{mul}).

(2) Corresponding to the orbit family defined by $H_p(v,y)=h,\ h\in(h_{Bp},\infty)$ with the initial condition $(v_0,0),\ 0<v_0<\infty$, there is a family of periodic solutions of (6_{mul}). This family gives rise to the uncountably finite many periodic wave solutions of (1_{mul}).

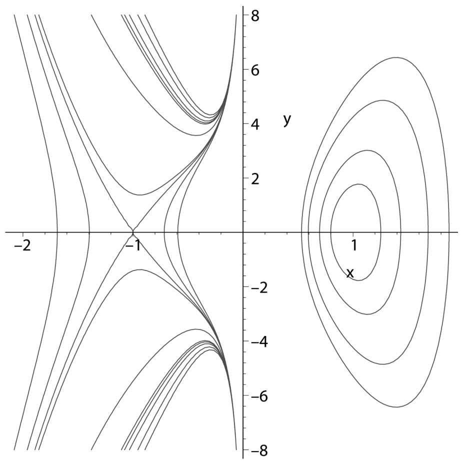

Fig. 2. A phase portrait of (42) with a center and a saddle point.

To sum up, we have the following conclusion:

Theorem A. *There exists a region of the parameter space $S(A,\alpha_1,\ldots,\alpha_p)$ such that the multiple sine-Gordon equation (1_{mul}) has the uncountably finite many periodic wave solutions and the uncountably finite many bounded breaking wave solutions.*

When the polynomial $Q_1(v)$ has real zeros, then, (1_{mul}) has the smooth solitary wave solutions, the kink wave solutions and the anti-kink wave solutions. Because the bifurcation behavior is too complicated in this case, we do not generally consider the travelling wave solution problem.

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