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On mean curvatures in submanifolds geometry

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Abstract By using moving frame theory, first we introduce $2p$-th mean curvatures and $(2p+1)$-th mean curvature vector fields for a submanifold. We then give an integral expression of them that characterizes them as mean values of symmetric functions of principle curvatures. Next we apply it to derive directly the celebrated Weyl-Gray tube formula in terms of integrals of the $2p$-th mean curvatures and some Minkowski-type integral formulas.

Keywords: moving frame, mean curvature, tube

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1 Mean curvatures and integral expressions

Let $f: M^n \to N^{n+m}$ be an isometric immersion of an n-dimensional submanifold M into an $(n+m)$-dimensional Riemannian manifold N. Around each point p in M, we take a local frame $\{e_1, \dots, e_{n+m}\}$ of N such that $\{e_1, \dots, e_n\}$ are orthonormal vectors tangent to M and $\{e_{n+1}, \dots, e_{n+m}\}$ are orthonormal vectors normal to M, and we let $\omega_1, \dots, \omega_{n+m}$ be the dual co-frame. Here we identify tangent vectors with its image under f_*. We assume the indices to be $1 \leqslant i, j, k \leqslant n; n+1 \leqslant \alpha, \beta, \gamma \leqslant n+m; 1 \leqslant A, B, C \leqslant n+m; 1 \leqslant a, b, c \leqslant m, \bar{a} = n+a$. Then the structure equations of N are given by

$$\begin{cases} d\omega_A = \sum_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \\ d\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB} - \Omega^N_{AB}. \end{cases} \tag{1.1}$$

If we restrict these forms on M, then $\omega_\alpha = 0$. Now we obtain the structure equations of M:

$$\begin{cases} d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \\ d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \Omega^M_{ij}. \end{cases} \tag{1.2}$$

From (1.1) and (1.2), we can see that

$$\Omega_{ij} =: \Omega^M_{ij} - \Omega^N_{ij} = \sum_\alpha \omega_{i\alpha} \wedge \omega_{j\alpha}.$$

Since $0 = d\omega_\alpha = -\sum_i \omega_{i\alpha} \wedge \omega_i$, by Cartan's lemma we can write $\omega_{i\alpha} = \sum_j h^\alpha_{ij} \omega_j,\ h^\alpha_{ij} = h^\alpha_{ji}$. The matrices $(\omega_{AB}), (\omega_{ij}), (\Omega^N_{AB}), (\Omega^M_{ij})$ are the connection forms and curvature forms of N,

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M respectively. The matrix (Ω_{ij}) is called the relative curvature form of the immersion f and $h = \sum_{\alpha,i,j} h^{\alpha}_{ij}\omega_i\omega_j e_\alpha$ is the second fundamental form of the immersion f.

Letting $I_k = (i_1, \dots, i_k)$ be k integers in $\{1, \dots, n\}$, $0 \leqslant p \leqslant [\frac{n}{2}]$, we define

$$\Omega(I_{2p}) =: \frac{1}{(2p)!}\sum_{J_{2p}} \delta^{j_1,\dots,j_{2p}}_{i_1,\dots,i_{2p}} \Omega_{j_1j_2} \wedge \cdots \wedge \Omega_{j_{2p-1}j_{2p}},$$

$$\Omega^{\alpha}(I_{2p+1}) =: \frac{1}{(2p+1)!}\sum_{J_{2p+1}} \delta^{j_1,\dots,j_{2p+1}}_{i_1,\dots,i_{2p+1}} \Omega_{j_1j_2} \wedge \cdots \wedge \Omega_{j_{2p-1}j_{2p}} \wedge \omega_{j_{2p+1}\alpha},$$

$$K^f_{2p} =: \frac{1}{\binom{n}{2p}}\frac{1}{(2p)!}\sum_{I_{2p}} \Omega(I_{2p})(e_{i_1}, \dots, e_{i_{2p}}), \qquad (K^f_0 =: 1),$$

$$H^f_{2p+1} =: \frac{1}{\binom{n}{2p+1}}\frac{1}{(2p+1)!}\sum_{\alpha}\sum_{I_{2p+1}} \Omega^{\alpha}(I_{2p+1})(e_{i_1}, \dots, e_{i_{2p+1}})e_\alpha =: \sum_\alpha H^{\alpha}_{2p+1}e_\alpha, \ (H^f_{n+1} =: 0),$$

where the sums take over all selections of the indices and $\delta^{j_1,\dots,j_k}_{i_1,\dots,i_k} = \det((\delta_{i_s j_t}))$ is the generalized Kronecker symbol.

Since by a change of local frame: $e'_i = \sum_j a_{ij}e_j$, $e'_\alpha = \sum_\beta a_{\alpha\beta}e_\beta$, $\sum_k a_{ik}a_{jk} = \delta_{ij}$, $\sum_\gamma a_{\alpha\gamma}a_{\beta\gamma} = \delta_{\alpha\beta}$; we know from (1.1) and (1.2) the corresponding connection and curvature forms change as: $\omega'_{i\alpha} = \sum_{j\beta} a_{ij}a_{\alpha\beta}\omega_{j\beta}$, $\Omega'_{ij} = \sum_{k,l} a_{ik}a_{jl}\Omega_{kl}$. Thus we can easily find that K^f_{2p}, H^f_{2p+1} are independent of the choice of the local frame, so they define respectively a sequence of functions and normal vector fields on M, called $2p$-th mean curvature and $(2p+1)$-th mean curvature vector field of f. Besides [1, 2], recently there are several studies about them and related variational problems, such as [3, 4], where the authors defined them assuming the ambient manifold N to be of constant sectional curvature.

Remark 1.1. Note that H^f_1 is just the usually mean curvature vector field H_1. When M is a hypersurface with unit normal vector field ν, then K^f_{2p} and $\langle H^f_{2p+1}, \nu\rangle$ are just the usually k-th mean curvatures.

Remark 1.2. Note that if we define K^f_{2p} by Ω^M_{ij} instead of Ω_{ij}, then they are just the $2p$-th normalized scalar curvature K^M_{2p} in the sense of [5]. They are defined only by curvature forms of M and thus are intrinsic, i.e. depend only on the Riemannian metric of M. For example, K^M_2 is just the normalized scalar curvature , when n is even, K^M_n is just the Lipschitz-Killing curvature. When the ambient manifold N is of constant sectional curvature λ, i.e. $\Omega^N_{AB} = \lambda\omega_A \wedge \omega_B$, $\Omega_{ij} = \Omega^M_{ij} - \lambda\omega_i \wedge \omega_j$, then $K^f_{2p} = \sum_{k=0}^{P}(-\lambda)^{p-k}\binom{P}{k}K^M_{2k}$ and thus they are intrinsic. In this case, we write K^λ_{2p} instead of K^f_{2p} and define $S^\lambda_{2p}(M) = \int_M K^\lambda_{2p}\, dV_M$.

Now we prepare two important integral formulas over a unit sphere whose first one was given in matrix form in [6].

Lemma 1.3.

$$\int_{S^{m-1}(1)}\Big(\sum_a x_a\omega_{i_1\bar{a}}\Big)\wedge\cdots\wedge\Big(\sum_a x_a\omega_{i_k\bar{a}}\Big)dx = \begin{cases} \dfrac{C_{m+2p-1}}{2^{2p}\pi^p p!}k!\Omega(I_k), & k=2p; \\ 0, & k=2p+1, \end{cases} \tag{1.3}$$

$$\int_{S^{m-1}(1)} x_b\Big(\sum_a x_a\omega_{i_1\bar{a}}\Big)\wedge\cdots\wedge\Big(\sum_a x_a\omega_{i_k\bar{a}}\Big)dx = \begin{cases} \dfrac{C_{m+2p-1}}{2^{2p}\pi^p p!}\dfrac{k!}{m+2p}\Omega^{\bar{b}}(I_k), & k=2p+1; \\ 0, & k=2p. \end{cases} \tag{1.4}$$

where $(x_1, \ldots, x_m)$ is the coordinate and dx is the volume element of $S^{m-1}(1)$, $C_{m+2p-1} = \frac{2\pi^{\frac{m+2p}{2}}}{\Gamma(\frac{m+2p}{2})}$ is the volume of $S^{m+2p-1}(1)$.

Proof. The proof is similar to that of [6] (see also [7]), so we just give a brief proof of the second formula.

By the symmetry of sphere, the zero case is obvious. It suffices to prove the case when $k = 2p+1$. For $I_k = (i_1, \ldots, i_k)$ and $T_k = (\tau_1, \ldots, \tau_m)$ with $\sum_a \tau_a = k$, we put

$$P_{I_k}(T_k) =: \left(\bigwedge_{s=1}^{\tau_1} \omega_{i_s\bar{1}}\right) \wedge \cdots \wedge \left(\bigwedge_{s=k-\tau_m+1}^{k} \omega_{i_s\bar{m}}\right).$$

Then it is clear that

$$\begin{aligned} LHS &= \int_{S^{m-1}(1)} x_b \frac{1}{k!} \sum_{J_k} \delta^{i_1,\ldots,i_k}_{j_1,\ldots,j_k} \left(\sum_a x_a \omega_{j_1\bar{a}}\right) \wedge \cdots \wedge \left(\sum_a x_a \omega_{j_k\bar{a}}\right) dx \\ &= \int_{S^{m-1}(1)} \left\{ x_b \frac{1}{k!} \sum_{T_k=(2\lambda_1,..,2\lambda_b+1,..,2\lambda_m)} x_1^{\tau_1} \cdots x_m^{\tau_m} \frac{k!}{\tau_1! \cdots \tau_m!} \sum_{J_k} \delta^{i_1,\ldots,i_k}_{j_1,\ldots,j_k} P_{J_k}(T_k) + \delta \right\} dx \\ &= \sum_{T_k=(2\lambda_1,..,2\lambda_b+1,..,2\lambda_m)} \sum_{J_k} \delta^{i_1,\ldots,i_k}_{j_1,\ldots,j_k} P_{J_k}(T_k) \frac{\int_{S^{m-1}(1)} x_b x_1^{\tau_1} \cdots x_m^{\tau_m} dx}{\tau_1! \cdots \tau_m!} \\ &= \sum_{T_k=(2\lambda_1,..,2\lambda_b+1,..,2\lambda_m)} \sum_{J_k} \delta^{i_1,\ldots,i_k}_{j_1,\ldots,j_k} P_{J_k}(T_k) \frac{1}{\lambda_1! \cdots \lambda_m!} \frac{C_{m+2p-1}}{2^{2p}\pi^p(m+2p)}, \end{aligned}$$

where δ contains the terms with some x_a having odd order and so integrates to zero, the last equality has applied the moments formula (see [8]) for $(a_1, \ldots, a_m)$ with $\sum_b a_b = p+1$:

$$\int_{S^{m-1}(1)} x_1^{2a_1} \cdots x_m^{2a_m} dx = \frac{(2a_1-1)!! \cdots (2a_m-1)!!}{m(m+2)\cdots(m+2p)} C_{m-1}.$$

$$\begin{aligned} RHS &= \frac{C_{m+2p-1}}{2^{2p}\pi^p p!} \frac{1}{m+2p} \sum_{J_k} \delta^{i_1,\ldots,i_k}_{j_1,\ldots,j_k} \Omega_{j_1j_2} \wedge \cdots \wedge \Omega_{j_{2p-1}j_{2p}} \wedge \omega_{j_k\alpha} \\ &= \frac{C_{m+2p-1}}{2^{2p}\pi^p p!} \frac{1}{m+2p} \sum_{J_k} \delta^{i_1,\ldots,i_k}_{j_1,\ldots,j_k} \left(\sum_a \omega_{j_1\bar{a}}\omega_{j_2\bar{a}}\right) \wedge \cdots \wedge \left(\sum_a \omega_{j_{2p-1}\bar{a}}\omega_{j_{2p}\bar{a}}\right) \omega_{j_k\bar{b}} \\ &= \frac{C_{m+2p-1}}{2^{2p}\pi^p p!} \frac{1}{m+2p} \sum_{J_k} \delta^{i_1,\ldots,i_k}_{j_1,\ldots,j_k} \sum_{T_{2p}=(2\lambda_1,..,2\lambda_m)} \frac{p!}{\lambda_1! \cdots \lambda_m!} P_{J_{2p}}(T_{2p}) \omega_{j_k\bar{b}} \\ &= \sum_{T_k=(2\lambda_1,..,2\lambda_b+1,..,2\lambda_m)} \sum_{J_k} \delta^{i_1,\ldots,i_k}_{j_1,\ldots,j_k} P_{J_k}(T_k) \frac{1}{\lambda_1! \cdots \lambda_m!} \frac{C_{m+2p-1}}{2^{2p}\pi^p(m+2p)}. \end{aligned}$$

The proof is now complete.

Let $\{u_1(x), \ldots, u_n(x)\}$ be the n (real) eigenvalues of the real matrix $(\sum_a x_a h^{\bar{a}}_{ij}) = (\sum_a x_a \omega_{i\bar{a}}(e_j))$, i.e. they are the principle curvatures with respect to the shape operator $A_{v(x)}$ where $v(x) = \sum_a x_a e_{\bar{a}}$. The normalized symmetric functions are defined by

$$\prod_{i=1}^{n} (1 + t\, u_i(x)) = \sum_{k=0}^{n} \binom{n}{k} M_k(x)\, t^k.$$

Taking values on $(e_{i_1}, \ldots, e_{i_k})$ in both sides of the preceding lemma and summing over all distinct selections of the indices, we can get the following integral expressions of the $2p$-th mean curvatures and the $(2p+1)$-th mean curvature vector fields:

Theorem 1.4. *Notations as above,*

$$\int_{S^{m-1}(1)} M_k(x)dx = \begin{cases} \dfrac{C_{m+2p-1}}{2^{2p}\pi^p p!} k! K_k^f, & k = 2p; \\ 0, & k = 2p+1, \end{cases}$$

$$\int_{S^{m-1}(1)} v(x)M_k(x)dx = \begin{cases} \dfrac{C_{m+2p-1}}{2^{2p}\pi^p p!} \frac{k!}{m+2p} H_k^f, & k = 2p+1; \\ 0, & k = 2p. \end{cases}$$

Remark 1.5. Comparing the case when M is a hypersurface, these two formulas allow us to name K_{2p}^f, H_{2p+1}^f the mean curvatures logically. When $k = n$ is even, the first formula can be applied to prove directly a theorem of Chern and a theorem of Lashof and Smale (see [6]). Using the Newton's inequality and formulas above, we can easily find that $K_2^f \leqslant |H_1^f|^2$ which can be also derived by Gauss equation and Cauchy inequality.

If $K_2^f > 0$, then from the above remark we can know that H_1^f is a nowhere vanishing normal vector field. This implies the following

Corollary 1.6. *When M^n is an immersed compact oriented submanifold of N^{2n} such that $K_2^f > 0$, and if the tangent bundle is isomorphic to the normal bundle, then the Euler characteristic of M is zero.*

The following standard embeddings

$$S^3 \hookrightarrow \mathbb{R}^4 \hookrightarrow \mathbb{R}^6, \quad S^2 \times S^1 \times S^1 \hookrightarrow \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^2 \hookrightarrow \mathbb{R}^8$$

have positive (relative) scalar curvature and their tangent bundles and normal bundles are easily seen to be trivial and thus are isomorphic. Meanwhile, from this corollary we can see that every even dimensional whitney sphere S^{2n} lagrangian immersed in $\mathbb{R}^{4n}$ must have its scalar curvature nonpositve somewhere.

2 Applications to submanifolds of real space forms

In this section we will apply the integral expressions of mean curvatures to derive directly the celebrated Weyl-Gray tube formula and some Minkowski-type integral formulas for submanifolds of the real space forms.

The Weyl-Gray tube formula calculates the volume of a tube of small radius r around a submanifold in the real space forms, which was given first by Weyl[8] for the Euclidean and Spherical cases and then by Gray[9] (see also [10]) for the general case. By using moving frame theory and the integral expressions of the $2p$-th mean curvatures, we can give a simple proof of it with its coefficients being integrals of the $2p$-th mean curvatures and thus the intrinsic property of the coefficients is obvious by Remark 1.2. Since any two simply connected complete Riemannian manifolds of constant sectional curvature λ are isometric to each other, we need only to prove the formula in the three standard space forms $\mathbb{R}^n(\lambda)$ defined as follows (see [11]): $\mathbb{R}^n(0)$ is just the Euclidean space $\mathbb{R}^n$ with flat metric: $g_0 = dx_1^2 + \cdots + dx_n^2$.

For $\lambda \neq 0$, $\mathbb{R}^n(\lambda) = \{x = (x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} | \, |x|^2 = \frac{1}{\lambda}, \, x_{n+1} \geqslant \frac{-\mathrm{sgn}\lambda}{\sqrt{|\lambda|}}\}$, where the norm $|\cdot|$ is defined by the metric: $g_\lambda = dx_1^2 + \cdots + dx_n^2 + \mathrm{sgn}\lambda \, dx_{n+1}^2$.

The Riemannian connection induced by g_λ on $\mathbb{R}^{n+1}$ is the ordinary Euclidean connection, and the metric induced on $\mathbb{R}^n(\lambda)$ is complete and of constant curvature λ.

Let $f: M^n \to \mathbb{R}^{n+m}(\lambda)$ be an isometric immersion. Choosing a local frame $\{e_1, \dots, e_{n+m}\}$ on $\mathbb{R}^{n+m}(\lambda)$ as in Section 1, then the structure equations are given by (1.1) and (1.2) and the moving equations are given by

$$\begin{cases} df = \sum_i \omega_i e_i, \\ De_A = \sum_B \omega_{AB} e_B, \end{cases} \tag{2.1}$$

where D denotes the covariant derivation of the Levi-Civita connection on $\mathbb{R}^n(\lambda)$.

Denoting by $S\nu$ the unit normal sphere bundle of f, the tubular hypersurface $M(r)$ and (when f is an embedding) the tube $\tilde{M}(r)$ of small radius r can be characterized by

$$\begin{aligned} f_r: \quad & S\nu \longrightarrow M(r) \subset \mathbb{R}^{n+m}(\lambda), \\ & (p,v) \longmapsto \cos(r\sqrt{\lambda}) f(p) + \frac{\sin(r\sqrt{\lambda})}{\sqrt{\lambda}} v, \end{aligned} \tag{2.2}$$

$$\begin{aligned} \tilde{f}: \quad & S\nu \times [0,r] \longrightarrow \tilde{M}(r) \subset \mathbb{R}^{n+m}(\lambda), \\ & (p,v), \quad t \longmapsto \cos(t\sqrt{\lambda}) f(p) + \frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}} v, \end{aligned} \tag{2.3}$$

where $\cos(z) = \frac{e^{\sqrt{-1}z} + e^{-\sqrt{-1}z}}{2}$, $\sin(z) = \frac{e^{\sqrt{-1}z} - e^{-\sqrt{-1}z}}{2\sqrt{-1}}$, and for $\lambda = 0$, $\frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}} = t$.

In fact, since $g_\lambda(q, X) = 0$ for any $X \in T_q R^{n+m}(\lambda), (\lambda \neq 0)$, $\tilde{f}_{(p,v)}(t) = \tilde{f}((p,v),t)$ is the geodesic along v at p in $\mathbb{R}^{n+m}(\lambda)$. The unit normal vector field of $M(r)$ in $\mathbb{R}^{n+m}(\lambda)$ is

$$v(r) = -\sqrt{\lambda} \sin(r\sqrt{\lambda}) f + \cos(r\sqrt{\lambda}) v. \tag{2.4}$$

We can assume that $v = \sum_a x_a e_{\bar{a}}$ with $x = (x_1, \dots, x_m) \in S^{m-1}(1)$. Choosing another local normal frame $\{v_1, \dots, v_m\}$ such that $v_m = v$, as in (2.1) we have

$$Dv_m = \sum_i -\theta_{im} e_i + \sum_b \theta_{mb} v_b, \qquad \theta_{im} = \sum_a x_a \omega_{i\bar{a}}. \tag{2.5}$$

Taking differentials of (2.2)–(2.4), from (2.1), (2.5) we can get

$$df_r = \sum_i \Big(\cos(r\sqrt{\lambda}) \omega_i - \frac{\sin(r\sqrt{\lambda})}{\sqrt{\lambda}} \sum_a x_a \omega_{i\bar{a}} \Big) e_i + \frac{\sin(r\sqrt{\lambda})}{\sqrt{\lambda}} \sum_a \theta_{ma} v_a, \tag{2.6}$$

$$d\tilde{f} = df_t + dt\, v(t), \tag{2.7}$$

$$dv(r) = \sum_i \Big(-\sqrt{\lambda} \sin(r\sqrt{\lambda}) \omega_i - \cos(r\sqrt{\lambda}) \sum_a x_a \omega_{i\bar{a}} \Big) e_i + \cos(r\sqrt{\lambda}) \sum_a \theta_{ma} v_a. \tag{2.8}$$

Letting $s = s(r) =: \frac{\sin(r\sqrt{\lambda})}{\cos(r\sqrt{\lambda})\sqrt{\lambda}}$, from (2.6), (2.8), we can get the principle curvatures of the tubular hypersurface $M(r)$ with respect to $v(r)$,

$$\kappa_i = \frac{\lambda s + u_i(x)}{1 - s u_i(x)}, \quad \kappa_\alpha = -\frac{1}{s}, \qquad 1 \leqslant i \leqslant n, \ \ n+1 \leqslant \alpha \leqslant n+m-1. \tag{2.9}$$

The volume element of $M(r)$ is easily computed from (2.6) as

$$dV_r = (\cos(r\sqrt{\lambda}))^n \Big(\frac{\sin(r\sqrt{\lambda})}{\sqrt{\lambda}} \Big)^{m-1} \bigwedge_i \Big(\omega_i - s \sum_a x_a \omega_{i\bar{a}} \Big) \wedge \Big(\bigwedge_{a=1}^{m-1} \theta_{ma} \Big). \tag{2.10}$$

Since $\bigwedge_{a=1}^{m-1}\theta_{ma} = dx + \Box$, (see [12]), where $\Box$ will vanish when restricted to a fibre, we get from (2.7), (2.10) the volume element of $M(r), \tilde{M}(r)$:

$$dV_r = (\cos(r\sqrt{\lambda}))^n \left(\frac{\sin(r\sqrt{\lambda})}{\sqrt{\lambda}}\right)^{m-1} \bigwedge_i \left(\omega_i - s\sum_a x_a\omega_{i\bar{a}}\right)\wedge dx, \tag{2.11}$$

$$d\tilde{V} = (\cos(t\sqrt{\lambda}))^n \left(\frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}}\right)^{m-1} \bigwedge_i \left(\omega_i - s(t)\sum_a x_a\omega_{i\bar{a}}\right)\wedge dx\wedge dt. \tag{2.12}$$

Now we can calculate the volume of the tube directly by (1.3), (2.11), (2.12).

Theorem 2.1. *Suppose that M^n is an isometric embedded n-dimensional submanifold with compact closure in an $n+m$-dimensional manifold $\mathbb{K}^{n+m}(\lambda)$ with constant sectional curvature λ. Then for small r, we have*

$$V(\tilde{M}(r)) = \sum_{p=0}^{[\frac{n}{2}]} \frac{C_{m+2p-1}}{2^{2p}\pi^p p!}\binom{n}{2p}(2p)!S_{2p}^{\lambda}(M)\int_0^r (\cos(t\sqrt{\lambda}))^{n-2p}\left(\frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}}\right)^{m+2p-1} dt.$$

Proof. It suffices to integrate (2.12) over $\tilde{M}(r)$.

Let $\mathfrak{S}_n$ be the permutation group of $(1,\ldots,n)$, $\mathfrak{S}_{k,n} = \{I = (i_1,\ldots,i_n)\in\mathfrak{S}_n | 1\leqslant i_1 < \cdots < i_k \leqslant n,\ 1\leqslant i_{k+1} < \cdots < i_n \leqslant n\}$. It follows from (1.3) that

$$\begin{aligned}
&\int_{S^{m-1}(1)} \bigwedge_i(\omega_i - s\sum_a x_a\omega_{i\bar{a}})\wedge dx\\
&=\sum_{p=0}^{[\frac{n}{2}]} s^{2p}\sum_{I\in\mathfrak{S}_{2p,n}}\delta(I)\left\{\int_{S^{m-1}(1)}\left(\sum_a x_a\omega_{i_1\bar{a}}\right)\wedge\cdots\wedge\left(\sum_a x_a\omega_{i_{2p}\bar{a}}\right)dx\right\}\wedge(\omega_{i_{2p+1}}\wedge\cdots\wedge\omega_{i_n})\\
&=\sum_{p=0}^{[\frac{n}{2}]} s^{2p}\sum_{I\in\mathfrak{S}_{2p,n}}\frac{C_{m+2p-1}}{2^{2p}\pi^p p!}(2p)!\Omega(I_{2p})(e_{i_1},\ldots,e_{i_{2p}})\, dV_M\\
&=\sum_{p=0}^{[\frac{n}{2}]} s^{2p}\frac{C_{m+2p-1}}{2^{2p}\pi^p p!}(2p)!\binom{n}{2p}K_{2p}^{\lambda}\, dV_M.
\end{aligned}$$

Recalling the definition of $S_{2p}^{\lambda}(M)$, we have completed the proof.

Remark 2.2. We can find that the coefficients $S_{2p}^{\lambda}(M)$ are related to Weyl-Gray's $k_{2p}(R^M - R^{\mathbb{K}^{n+m}(\lambda)})$ (see [9]) by $S_{2p}^{\lambda}(M) = \frac{2^p p!}{\binom{n}{2p}(2p)!}k_{2p}(R^M - R^{\mathbb{K}^{n+m}(\lambda)})$.

Now we apply the integral expressions of mean curvatures to prove the following Minkowski-type integral formulas for submanifolds in space forms, where the first formula was given in [2] and the second formula for the case of $\lambda\neq 0$ appears in [4] as a corollary of an application of the L_r operator.

Theorem 2.3. *Let $f: M^n \to \mathbb{R}^{n+m}(\lambda)$ be an isometric immersion of a compact orientable n-dimensional submanifold into the real space form $\mathbb{R}^{n+m}(\lambda)$, let $\langle\cdot,\cdot\rangle$ denote the metric g_λ. Then we have for $0\leqslant p\leqslant[\frac{n-1}{2}]$,*

(1) *for* $\lambda = 0$, $$\int_M (\langle H_{2p+1}^f,\ f\rangle + K_{2p}^{\lambda})\, dV_M = 0;$$

(2) $\forall\lambda, \forall\xi\in\mathbb{R}^{n+m+1}$, $$\int_M (\langle H_{2p+1}^f,\ \xi\rangle - \lambda K_{2p}^{\lambda}\langle f\ ,\ \xi\rangle)\, dV_M = 0.$$

Remark 2.4. As in Remark 1.2, if we define H^f_{2p+1} by Ω^M_{ij} instead of Ω_{ij}, they are also well-defined normal vector fields over M, denoted as H^M_{2p+1}. Then these formulas are also correct when H^f_{2p+1}, K^λ_{2p} are replaced by H^M_{2p+1}, K^M_{2p} respectively.

Proof. (1) Let $\psi : \Sigma \to \mathbb{R}^{n+m}$ be an immersed compact orientable hypersurface with unit normal vector field $\mathcal{N}$ and mean curvature H, let Δ denote the Laplacian of Σ. We can easily compute that $\Delta|\psi|^2 = 2(n+m-1)(1 + H\langle\psi, \mathcal{N}\rangle)$, which implies

$$\int_\Sigma (1 + H\ \langle\ \psi,\ \mathcal{N}\ \rangle)\ dV_\Sigma = 0. \tag{2.13}$$

Now for small r, we know from (2.11) that the tubular hypersurface $M(r)$ of M is an immersed compact orientable hypersurface of $\mathbb{R}^{n+m}$. Its volume element is

$$dV_r = P(r)\ dV_M\ dx; \qquad P(r) = \prod_i (1 - ru_i(x))\ r^{m-1}. \tag{2.14}$$

Its unit normal vector field is $v(r)$ given by (2.4), the corresponding mean curvature can be got from (2.9)

$$H(r) = \frac{1}{n+m-1}\left(\sum_i \frac{u_i(x)}{1 - ru_i(x)} - \frac{m-1}{r}\right) = \frac{-1}{n+m-1}\frac{P'(r)}{P(r)}. \tag{2.15}$$

Substituting (2.2), (2.4), (2.14), (2.15) into (2.13), we have

$$\int_M \int_{S^{m-1}(1)} \left\{ - P'(r)\left\langle f,\ \sum_b x_b e_{\bar{b}} \right\rangle + [(n+m-1)P(r) - rP'(r)] \right\} dx dV_M = 0. \tag{2.16}$$

Integrating (2.16) over $S^{m-1}(1)$ by applying Theorem 1.4 and comparing the coefficients of r^{m+2p-1}, we can get the formula of (1).

(2) Let $\psi : \Sigma \to \mathbb{R}^{n+m}(\lambda)$ be an immersed compact orientable hypersurface, ξ be a fixed vector in $\mathbb{R}^{n+m+1}$. Then by a straightforward computation, we can get $\Delta\langle\psi, \xi\rangle = (n+m-1)(H\langle\mathcal{N}, \xi\rangle - \lambda\langle\psi, \xi\rangle)$, and thus

$$\int_\Sigma (H\ \langle\mathcal{N}, \xi\rangle - \lambda\ \langle\psi, \xi\rangle)\ dV_\Sigma = 0. \tag{2.17}$$

As in (1), we can apply formula (2.17) to $M(r)$. From (2.11), its volume element is

$$dV_r = (\cos(r\sqrt{\lambda}))^{n+m-1} P(s)\ dV_M dx. \tag{2.18}$$

The unit normal vector field is still given by (2.4), the corresponding mean curvature can be got from (2.9)

$$\begin{aligned} H(r) &= \frac{1}{n+m-1}\left(\sum_i \frac{\lambda s + u_i(x)}{1 - su_i(x)} - \frac{m-1}{s}\right) \\ &= \frac{\lambda(n+m-1)\cos(r\sqrt{\lambda})^2 sP(s) - P'(s)}{(n+m-1)\cos(r\sqrt{\lambda})^2 P(s)}. \end{aligned} \tag{2.19}$$

Substituting (2.2), (2.4), (2.18), (2.19) into (2.17), we have

$$\int_M \int_{S^{m-1}(1)} \left\{ - P'(s)\left\langle \sum_b x_b e_{\bar{b}}, \xi \right\rangle - \lambda[(n+m-1)P(s) - sP'(s)]\ \langle f,\ \xi\rangle \right\} dx dV_M = 0. \tag{2.20}$$

Then as in (2.16), integrating (2.20) over $S^{m-1}(1)$ and comparing the coefficients of s^{m+2p-1}, we can get the formula of (2).

Remark 2.5. Recall Remark 1.1, when the codimension is 1, Theorem 2.3 reduces to a "half" of the Minkowski-type integral formulas of hypersurface. The other "half" can also be obtained by this method if one only integrates (2.13), (2.17) over the upper tubular hypersurface (see [13], where they use these formulas successfully in proving the well-known Aleksandroff-Reilly-Ros theorem).

Given any fixed $\xi \in \mathbb{R}^{n+1}$, let D_ξ denote the open half space of $\mathbb{R}^n(\lambda)$ in the direction of ξ, i.e. $D_\xi = \{\eta \in \mathbb{R}^n(\lambda) | \langle \eta, \xi \rangle > 0\}$. Then it follows from Theorem 2.3 that

Corollary 2.6. *There exist no compact minimal submanifolds immersed in any open half space of* $\mathbb{R}^n(\lambda)$.

When K_{2p}^{λ} (resp. K_{2p}^{M}) is nonzero and nonnegative, the preceding assertion is also correct for $2p$-minimal submanifolds, where $2p$-minimal was defined in [4] as $H_{2p+1}^{f} = 0$ (resp. in [14] as $H_{2p+1}^{M} = 0$).

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