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A variational inequality arising from European option pricing with transaction costs

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Abstract In this paper we present a method which can transform a variational inequality with gradient constraints into a usual two obstacles problem in one dimensional case. The prototype of the problem is a parabolic variational inequality with the constraints of two first order differential inequalities arising from a two-dimensional model of European call option pricing with transaction costs. We obtain the monotonicity and smoothness of two free boundaries.

Keywords: **option pricing, transaction costs, free boundary, variational inequality**

MSC(2000): 35R35

1 Introduction

In this paper we consider a parabolic variational inequality with constraints of two first order differential inequalities arising from the model of European call option pricing with transaction costs[1]. More precisely, we will find $Q(y, S, t)$ satisfying

$$\begin{cases} \min\Big\{\partial_y Q + \gamma(1+\lambda)SQe^{r(T-t)},\ -(\partial_y Q + \gamma(1-\mu)SQe^{r(T-t)}), \\ \qquad \partial_t Q + \dfrac{\sigma^2}{2}S^2\partial_{SS}Q + \alpha S\partial_S Q\Big\} = 0, \quad y \in \mathbb{R},\ \ S > 0,\ \ 0 \leqslant t < T, \\ Q(y, S, T) \ = \exp\{-\gamma c(y, S)\}, \end{cases} \tag{1.1}$$

where

$$c(y, S) = \begin{cases} (1+\lambda)yS & \text{if } \ y < 0, \\ (1-\mu)yS & \text{if } \ y \geqslant 0, \end{cases} \tag{1.2}$$

and $\sigma > 0,\ \alpha > r \geqslant 0,\ 0 \leqslant \lambda < 1,\ 0 \leqslant \mu < 1$ and $\gamma > 0$ are constants.

In Appendix we shall present the financial and stochastic background of this problem.

(1.1) is a backward PDE problem, we transform it to a familiar forward parabolic problem,

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so letting $\tau = T - t$ we have

$$\begin{cases} \max\Big\{-(\partial_y Q + \gamma(1+\lambda)SQe^{r\tau}),\ \partial_y Q + \gamma(1-\mu)SQe^{r\tau}, \\ \qquad \partial_\tau Q - \dfrac{\sigma^2}{2}S^2\partial_{SS}Q - \alpha S\partial_S Q\Big\} = 0, \quad y \in \mathbb{R},\ S > 0,\ 0 < \tau \leqslant T, \\ Q(y,S,0) = \exp\{-\gamma c(y,S)\}. \end{cases} \tag{1.3}$$

In the next section we transform problem (1.3) into a one dimensional parabolic problem (2.4) with gradient constraints (see [2–4]). And then transform it into a usual two obstacles problem (2.7) with respect to a new unknown function (see [5]), this is the key idea of this paper. In [5] the authors studied an investment problem with transaction costs and log or power utility functions. There if utility function is exponential, then the problem is similar to problem (2.4).

In Section 3 we will prove the existence and uniqueness of $W_p^{2,1}$ solution to the parabolic variational inequality (2.7). The main work is in the Section 4 where we prove that two free boundaries are monotonic and C^∞-smooth. As a general rule, the monotonicity of the solution $V(z,\tau)$ in (2.7) with respect to τ is very important for proving smoothness of the free boundary[6]. For the free boundary $\partial\{V = 1-\mu\}$ we need $\partial_\tau V \geqslant 0$ because $1-\mu$ is the lower obstacle. On the other hand for the free boundary $\partial\{V = 1+\lambda\}$ we need $\partial_\tau V \leqslant 0$ because $1+\lambda$ is the upper obstacle. In fact in our case $\partial_\tau V \geqslant 0$ (see (4.3)). In this way the proof of smoothness of the free boundary $\partial\{V = 1+\lambda\}$ is very difficult. We construct function ψ in the step 3 in the proof of Theorem 4.4 for proving that $V(z,\tau)$ possesses local cone property (4.27). Applying this property we know that the free boundary is Lipshitz continuous. Moreover we will show the start points of two free boundaries. In Section 5 we construct the solution of problems (2.4) and (1.3).

2 Transformations

Transformation 1. In order to transform the constraints of first order differential inequalities into the gradient constraints, we let $U = \ln Q$ ($Q > 0$ can be inferred from the reality or from the result in the paper), then U satisfies

$$\begin{cases} \max\Big\{-(\partial_y U + \gamma(1+\lambda)Se^{r\tau}), \quad \partial_y U + \gamma(1-\mu)Se^{r\tau}, \\ \qquad \partial_\tau U - \dfrac{\sigma^2}{2}S^2\partial_{SS}U - \alpha S\partial_S U - \dfrac{\sigma^2}{2}(S\partial_S U)^2\Big\} = 0, \quad y \in \mathbb{R},\ S > 0,\ 0 < \tau \leqslant T, \\ U(y,S,0) = -\gamma c(y,S). \end{cases} \tag{2.1}$$

Transformation 2. Problem (2.1) is in two-dimensional case. We modify it into one dimensional problem by letting

$$z = e^{r\tau}yS, \qquad V^*(z,\tau) = U(y,S,\tau). \tag{2.2}$$

Then

$$\partial_y U(y,S,\tau) = e^{r\tau}S\partial_z V^*(z,\tau), \qquad \partial_\tau U(y,S,\tau) = \partial_\tau V^*(z,\tau) + rz\partial_z V^*(z,\tau),$$
$$\partial_S U(y,S,\tau) = e^{r\tau}y\partial_z V^*(z,\tau), \qquad \partial_{SS}U(y,S,\tau) = e^{2r\tau}y^2\partial_{zz}V^*(z,\tau).$$

Thus

$$\begin{aligned}
&\partial_y U(y,S,\tau)+\gamma(1+\lambda)Se^{r\tau}=e^{r\tau}S[\partial_z V^*(z,\tau)+\gamma(1+\lambda)],\\
&\partial_y U(y,S,\tau)+\gamma(1-\mu)Se^{r\tau}=e^{r\tau}S[\partial_z V^*(z,\tau)+\gamma(1-\mu)],\\
&\partial_\tau U-\frac{\sigma^2}{2}S^2\partial_{SS}U-\alpha S\partial_S U-\frac{\sigma^2}{2}(S\partial_S U)^2=\partial_\tau V^*-LV^*,
\end{aligned}$$

where

$$LV^*=\frac{\sigma^2}{2}z^2\partial_{zz}V^*+(\alpha-r)z\partial_z V^*+\frac{\sigma^2}{2}(z\partial_z V^*)^2. \tag{2.3}$$

Therefore, $V^*(z,\tau)$ satisfies

$$\begin{cases}
\max\{-(\partial_z V^*+\gamma(1+\lambda)),\ \partial_z V^*+\gamma(1-\mu),\ \partial_\tau V^*-LV^*\}=0,\ z\in\mathbb{R}^1,\ 0<\tau\leqslant T,\\
V^*(z,0)\ =\begin{cases}-\gamma(1+\lambda)z, & \text{if } z<0,\\ -\gamma(1-\mu)z, & \text{if } z\geqslant 0.\end{cases}
\end{cases} \tag{2.4}$$

Remark. In (2.2) if we define $z=yS$, then problem (2.1) is transformed into one dimensional case as well, but its free boundaries should not enjoy the monotonicitiy in Section 4.

Transformation 3. In singular stochastic control literature, the spatial C^2 regularity of the value function has been called the principle of smooth fit (see [7]). It has been instrumental in the analysis of one constraint upon a spatial derivative of unknown function in one dimensional case (see [8]) and in multidimensional case (see [9]). This paper develops this idea to the case of two gradient constraints in one dimensional case.

Formally, taking the derivative with respect to z in (2.3),

$$\begin{aligned}
\frac{\partial}{\partial z}(LV^*)=&\frac{\sigma^2}{2}z^2\partial_{zz}(\partial_z V^*)+(\alpha-r+\sigma^2)z\partial_z(\partial_z V^*)+(\alpha-r)(\partial_z V^*)\\
&+\sigma^2 z(\partial_z V^*)(z\partial_z(\partial_z V^*)+(\partial_z V^*)),
\end{aligned} \tag{2.5}$$

hence if we denote

$$V(z,\tau)=-\frac{1}{\gamma}\,\partial_z V^*(z,\tau) \tag{2.6}$$

just in mind, and postulate that $V(z,\tau)$ satisfies

$$\begin{cases}
\partial_\tau V-\mathcal{L}_z V=0, & \text{if } 1-\mu<V<1+\lambda;\\
\partial_\tau V-\mathcal{L}_z V\leqslant 0, & \text{if } \ V=1+\lambda;\\
\partial_\tau V-\mathcal{L}_z V\geqslant 0, & \text{if } \ V=1-\mu;\\
V(z,0)\ =\begin{cases}1+\lambda, & \text{if } z<0,\\ 1-\mu, & \text{if } z\geqslant 0,\end{cases}
\end{cases} \tag{2.7}$$

where, by (2.5) and (2.6),

$$\mathcal{L}_z V=\frac{\sigma^2}{2}z^2\partial_{zz}V+(\sigma^2+\alpha-r)z\partial_z V+(\alpha-r)V-\gamma\sigma^2 zV(z\partial_z V+V). \tag{2.8}$$

Once we have the solution $V(z,\tau)$ of (2.7) we will apply (2.6) to get the solution to problem (2.4) in Section 5.

3 The existence and uniqueness of $W^{2,1}_{p,\,\mathrm{loc}}$ solution of problem (2.7)

The operator $\mathcal{L}_z$ is degenerate at $z=0$. Thanks to the Fichera Theorem in [10], we can study the problem in the domains $\{z>0\}$ and $\{z<0\}$ independently.

Moreover we find that $V(z,\tau)=1+\lambda$ is the solution of problem (2.7) in the domain $\{z<0\}$. In the following we only consider the problem (2.7) in the domain domain $\{z>0\}$. To do this, let $z=e^x$, $v(x,\tau)=V(z,\tau)$, then $v(x,\tau)$ satisfies

$$\begin{cases} \partial_\tau v-\mathcal{L}_x v=0, & \text{if } 1-\mu<v<1+\lambda, \quad x\in\mathbb{R},\ 0<\tau\leqslant T,\\ \partial_\tau v-\mathcal{L}_x v\geqslant 0, & \text{if } \quad v=1-\mu, \quad x\in\mathbb{R},\ 0<\tau\leqslant T,\\ \partial_\tau v-\mathcal{L}_x v\leqslant 0, & \text{if } \quad v=1+\lambda, \quad x\in\mathbb{R},\ 0<\tau\leqslant T,\\ v(x,0)=1-\mu, & x\in\mathbb{R}, \end{cases} \tag{3.1}$$

where

$$\mathcal{L}_x v=\frac{\sigma^2}{2}\partial_{xx}v+\left(\alpha-r+\frac{\sigma^2}{2}\right)\partial_x v+(\alpha-r)v-\gamma\sigma^2e^x v(\partial_x v+v). \tag{3.2}$$

Define $\Omega_T=\mathbb{R}\times(0,T)$, $\Omega_T^R=(-R,R)\times(0,T)$.

First consider the problem in bounded domain,

$$\begin{cases} \partial_\tau v_n-\mathcal{L}_x v_n=0, & \text{if } 1-\mu<v_n<1+\lambda \text{ and } (x,\tau)\in\Omega_T^n;\\ \partial_\tau v_n-\mathcal{L}_x v_n\geqslant 0, & \text{if } v_n=1-\mu \text{ and } (x,\tau)\in\Omega_T^n;\\ \partial_\tau v_n-\mathcal{L}_x v_n\leqslant 0, & \text{if } v_n=1+\lambda \text{ and } (x,\tau)\in\Omega_T^n;\\ \partial_x v_n(x,\tau)=0, & x=\pm n\ ,\ 0\leqslant\tau\leqslant T;\\ v_n(x,0)=1-\mu, & -n\leqslant x\leqslant n. \end{cases} \tag{3.3}$$

Lemma 3.1 *For any fixed* $n\in\mathbb{Z}^+$, *there exists a unique solution* $v_n\in C(\overline{\Omega}_T^n)\cap W_p^{2.1}(\Omega_T^n)$ *to problem* (3.3), *where* $1<p<+\infty$. *Moreover,*

$$\partial_x v_n\leqslant 0 \ \ in\ \Omega_T^n;\qquad \partial_\tau v_n\geqslant 0\ \ a.e.\ in\ \Omega_T^n. \tag{3.4}$$

Proof. As usual we define a penalty function $\beta_\varepsilon(t)$ which satisfies

$$\begin{aligned} &\beta_\varepsilon(t)\in C^2(-\infty,+\infty),\ \ \beta_\varepsilon(t)\leqslant 0,\\ &\beta_\varepsilon(0)=-C_0,\quad C_0=\max\{\gamma\sigma^2(1-\mu)^2e^n,(\alpha-r)(1+\lambda)\},\\ &\beta'_\varepsilon(t)\geqslant 0,\quad \beta''_\varepsilon(t)\leqslant 0, \end{aligned} \tag{3.5}$$

and moreover,

$$\lim_{\varepsilon\to 0^+}\beta_\varepsilon(t)=\begin{cases} 0, & t>0,\\ -\infty, & t<0. \end{cases}$$

Following the idea in [11], construct an approximate problem

$$\begin{cases} \partial_\tau v_{\varepsilon,\,n} - \mathcal{L}_x v_{\varepsilon,\,n} + \beta_\varepsilon(v_{\varepsilon,\,n} - (1-\mu)) - \beta_\varepsilon(-v_{\varepsilon,\,n} + (1+\lambda)) = 0 \quad \text{in } \Omega_T^n, \\ \partial_x v_{\varepsilon,\,n}(x,\tau) = 0, \qquad x = \pm n, \ \ 0 \leqslant \tau \leqslant T, \\ v_{\varepsilon,\,n}(x,0) = 1-\mu, \quad -n \leqslant x \leqslant n. \end{cases} \tag{3.6}$$

Applying the Leray-Schauder fixed point theorem, it is not difficult to get the existence of the solution. The proof of uniqueness is a standard way as well, so we omit the details.

It is deduced that, by $\varepsilon \to 0^+$, $v_{\varepsilon,n} \rightharpoonup v_n$ in $W_p^{2,1}(\Omega_T^n)$ weakly, where v_n is the solution of problem (3.3). It is noted that $C_0 \geqslant (\alpha - r)(1+\lambda)$ is important to the conclusion.

Differentiate (3.6) with respect to τ, and denote $w = \partial_\tau v_{\varepsilon,n}$, then

$$\begin{cases} \partial_\tau w - \dfrac{\sigma^2}{2}\partial_{xx} w - \left(\alpha - r + \dfrac{\sigma^2}{2}\right)\partial_x w - (\alpha - r)w + \gamma\sigma^2 e^x[v_{\varepsilon,n}\partial_x w + (\partial_x v_{\varepsilon,n} + 2v_{\varepsilon,n})w] \\ \qquad +\beta'_\varepsilon(\cdot)w + \beta'_\varepsilon(\cdot\cdot)w = 0 \qquad \text{in } \Omega_T^n, \\ \partial_x w(x,\tau) = 0, \quad x = \pm n, \ \ 0 \leqslant \tau \leqslant T, \\ w(x,0) = (\alpha - r)(1-\mu) - [\gamma\sigma^2(1-\mu)^2 e^x + \beta_\varepsilon(0)] \geqslant 0 \quad (\text{by } (3.5)). \end{cases} \tag{3.7}$$

Applying the minimum principle, we have $\partial_\tau v_{\varepsilon,n} \geqslant 0$.

On the other hand, differentiate (3.6) with respect to x, and denote $W = \partial_x v_{\varepsilon,n}$, then

$$\begin{cases} \partial_\tau W - \dfrac{\sigma^2}{2}\partial_{xx} W - \left(\alpha - r + \dfrac{\sigma^2}{2}\right)\partial_x W - (\alpha - r)W + \gamma\sigma^2 e^x[v_{\varepsilon,n}\partial_x W + 3v_{\varepsilon,n}W + W^2] \\ \qquad +\beta'_\varepsilon(\cdot)W + \beta'_\varepsilon(\cdot\cdot)W = -\gamma\sigma^2 e^x v_{\varepsilon,n}^2 \leqslant 0 \quad \text{in } \Omega_T^n, \\ W(x,\tau) = 0 \quad \text{on } \partial_p\Omega_T^n. \end{cases} \tag{3.8}$$

Applying the maximum principle, we have $\partial_x v_{\varepsilon,n} \leqslant 0$.

At last we prove uniqueness. Suppose v_1 and v_2 are two $W_p^{2.1}$ solutions to problem (3.3), we denote

$$\mathcal{N} = \{(x,\tau) : v_1(x,\tau) < v_2(x,\tau), \ |x| < n, \ 0 < \tau \leqslant T\}.$$

Suppose it is not empty, then if $(x,\tau) \in \mathcal{N}$,

$$\begin{aligned} v_1(x,\tau) < 1+\lambda, &\quad \partial_\tau v_1 - \mathcal{L}_x v_1 \geqslant 0; \\ v_2(x,\tau) > 1-\mu, &\quad \partial_\tau v_2 - \mathcal{L}_x v_2 \leqslant 0. \end{aligned}$$

Denote $v^* = v_2 - v_1$, then v^* satisfies

$$\begin{cases} \partial_\tau v^* - \dfrac{\sigma^2}{2}\partial_{xx} v^* - \left(\alpha - r + \dfrac{\sigma^2}{2}\right)\partial_x v^* - (\alpha - r)v^* \\ \qquad +\gamma\sigma^2 e^x[v_2\partial_x v^* + (v_1 + v_2 + \partial_x v_1)v^*] \leqslant 0, \qquad (x,\tau) \in \mathcal{N}, \\ v^*(x,0) = 0 \qquad \text{on } \partial_p\mathcal{N} \cap \{|x| < n\}, \\ \partial_x v^*(x,0) = 0 \quad \text{on } \partial_p\mathcal{N} \cap \{|x| = n\}, \end{cases}$$

where $\partial_p\mathcal{N}$ is parabolic boundary of the domain $\mathcal{N}$. Applying the maximum principle, we have $v^* \leqslant 0$ in $\mathcal{N}$, which contradicts the definition of $\mathcal{N}$.

We complete the proof of Lemma 3.1.

Theorem 3.2. *There exists a unique solution* $v \in C(\overline{\Omega}_T) \cap W_p^{2,1}(\Omega_T^R)$ *to problem* (3.1), *where* $\forall R > 0,\ 1 < p < +\infty$. *And*

$$\partial_x v \leqslant 0 \ \text{ in } \Omega_T; \qquad \partial_\tau v \geqslant 0 \ \text{ a.e. in } \Omega_T. \tag{3.9}$$

Moreover for any fixed $X \in \mathbb{R}$, $v \in C^{\alpha,\alpha/2}(\overline{(-\infty, X) \times (0,T)}), 0 < \alpha < 1$, *and*

$$|v|_{C^{\alpha,\alpha/2}(\overline{(-\infty,X)\times(0,T)})} \leqslant C_X, \tag{3.10}$$

where C_X *is a positive constant depending on* X.

Proof. Applying

$$(\partial_\tau - \mathcal{L}_x)(1+\lambda) = -(\alpha - r)(1+\lambda) + \gamma\sigma^2 e^x (1+\lambda)^2;$$
$$(\partial_\tau - \mathcal{L}_x)(1-\mu) = -(\alpha - r)(1-\mu) + \gamma\sigma^2 e^x (1-\mu)^2,$$

recalling $v_n \in W_p^{2,1}(\Omega_T^n)$, we rewrite problem (3.3) as

$$\begin{cases} \partial_\tau v_n - \mathcal{L}_x v_n = f(x,\tau) \quad \text{in } \Omega_T^n, \\ \partial_x v_n(x,\tau) = 0, \quad x = \pm n\ ,\ 0 \leqslant \tau \leqslant T, \\ v_n(x,0) = 1-\mu, \quad -n \leqslant x \leqslant n, \end{cases} \tag{3.11}$$

where

$$\begin{aligned} f(x,\tau) = {} & I_{\{v_n = 1-\mu\}}[-(\alpha - r)(1-\mu) + \gamma\sigma^2 e^x (1-\mu)^2] \\ & + I_{\{v_n = 1+\lambda\}}[-(\alpha - r)(1+\lambda) + \gamma\sigma^2 e^x (1+\lambda)^2], \ \text{a.e. in } \Omega_T^n, \end{aligned} \tag{3.12}$$

and I_A denotes the indicator function of the set A.

It is obvious that $|f(x,\tau)| \leqslant Ce^R$ if $-R \leqslant x \leqslant R$, where C is independent of R and n. For any fixed $R > 0$, we choose $n > R$, then we have the following $W_p^{2,1}$ uniform estimate in the domain $\overline{\Omega}_T^R$:

$$\|v_n\|_{W_p^{2,1}(\Omega_T^R)} \leqslant C(\|v_n\|_{L^\infty(\Omega_T^R)} + (1-\mu) + \|f(x,\tau)\|_{L^\infty(\Omega_T^R)}) \leqslant C,$$

where C depends on R, but is independent of n. Let $n \to \infty$, then we have, possibly a subsequence,

$$v_n \rightharpoonup v_R \ \text{ in } W_p^{2,1}(\Omega_T^R) \quad \text{and} \quad v_n \to v_R \ \text{ in } C(\overline{\Omega}_T^R) \quad \text{as} \quad n \to +\infty.$$

Define $v = v_R$ if $x \in [-R, R]$, it is clear that v is reasonably defined and v is the solution of problem (3.1).

(3.9) is a consequences of (3.4). Now we prove (3.10). Note that

$$\begin{cases} \partial_\tau v - \mathcal{L}_x v = f(x,\tau) \quad \text{in } \Omega_T, \\ v(x,0) = 1-\mu, \quad x \in \mathbb{R}. \end{cases} \tag{3.13}$$

We can see that, from (3.12), $f(x,\tau)$ is bounded on $(-\infty, X)\times(0,T)$, the bound of which depends on X. Hence (3.10) follows from the standard C^α theory of parabolic equation (see [12]).

The proof of the uniqueness is the same as in Lemma 3.1.

We complete the proof of Theorem 3.2.

4 Characterizations of the free boundaries

In this section, we mainly consider problem (2.7). Denote

$$\mathbf{BR} = \{(z,\tau)|V(z,\tau)=1+\lambda\} \quad \text{(buy region)},$$
$$\mathbf{NR} = \{(z,\tau)|1-\mu<V(z,\tau)<1+\lambda\} \quad \text{(no transaction region)},$$
$$\mathbf{SR} = \{(z,\tau)|V(z,\tau)=1-\mu\} \quad \text{(sell region) (Figure 1)}.$$

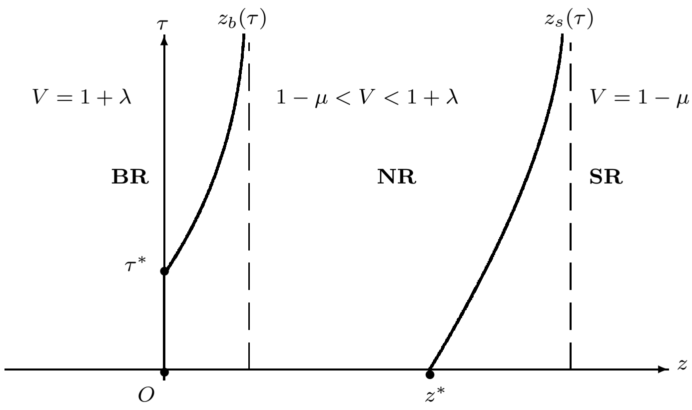

Figure 1

Note that, from (2.8),

$$(\partial_\tau V-\mathcal{L}_z)(1+\lambda) = -\mathcal{L}_z(1+\lambda) = -(1+\lambda)[(\alpha-r)-\gamma\sigma^2 z(1+\lambda)] \leqslant 0, \qquad \text{in } \mathbf{BR},$$
$$(\partial_\tau V-\mathcal{L}_z)(1-\mu) = -\mathcal{L}_z(1-\mu) = -(1-\mu)[(\alpha-r)-\gamma\sigma^2 z(1-\mu)] \geqslant 0, \qquad \text{in } \mathbf{SR},$$

hence

$$\mathbf{BR} \subset \left\{z \leqslant \frac{\alpha-r}{\gamma\sigma^2(1+\lambda)}\right\}, \tag{4.1}$$
$$\mathbf{SR} \subset \left\{z \geqslant \frac{\alpha-r}{\gamma\sigma^2(1-\mu)}\right\}. \tag{4.2}$$

On the other hand it is deduced by (3.9) and (3.10) that,

$$\partial_\tau V \geqslant 0, \quad \partial_z V = e^{-x}\partial_x V \leqslant 0, \tag{4.3}$$
$$|V(z,\tau)|_{C_\tau^{\alpha/2}[0,T]} \leqslant C_Z, \qquad 0<z\leqslant Z, \tag{4.4}$$

where constant C_Z depends on Z. Applying the second inequality in (4.3) we can define the free boundaries

$$z_b(\tau) = \sup\{z|\,V(z,\tau)=1+\lambda\}, \qquad 0<\tau\leqslant T,$$
$$z_s(\tau) = \inf\{z|\,V(z,\tau)=1-\mu\}, \qquad 0<\tau\leqslant T.$$

It is obvious that $z_s(\tau)$ and $z_b(\tau)$ are increasing by (4.3).

Lemma 4.1. *There exists a positive constant M_s such that*

$$0 \leqslant z_b(\tau) \leqslant \frac{\alpha - r}{\gamma\sigma^2(1+\lambda)}, \tag{4.5}$$

$$M_s \geqslant z_s(\tau) \geqslant \frac{\alpha - r}{\gamma\sigma^2(1-\mu)}, \tag{4.6}$$

where M_s is independent of T.

Proof. Since $V = 1+\lambda$ for every $z < 0$, $z_b(\tau) \geqslant 0$. The second parts of (4.5) and (4.6) are the consequences of (4.1) and (4.2).

Next we prove that $z_s(\tau) \leqslant M_s$, where M_s is independent of T. Firstly, we pay attention to the stationary problem of (2.7):

$$\begin{cases} -\mathcal{L}_z W(z) = 0, & \text{if } 1-\mu < W(z) < 1+\lambda \quad \text{and} \quad z \in \mathbb{R}^+, \\ -\mathcal{L}_z W(z) \leqslant 0, & \text{if} \quad W(z) = 1+\lambda \quad \text{and} \quad z \in \mathbb{R}^+, \\ -\mathcal{L}_z W(z) \geqslant 0, & \text{if} \quad W(z) = 1-\mu \quad \text{and} \quad z \in \mathbb{R}^+. \end{cases} \tag{4.7}$$

By the Fichera Theorem in [10], we consider the problem without the boundary value at $z = 0$.

Applying the similar method in Section 3, we can prove the existence and uniqueness of the solution to problem (4.7), $W \in C^1(\mathbb{R}^+) \cap W_p^2(1/R, R)$, where $\forall R > 0$, $1 < p < +\infty$. Moreover, $W'(z) \leqslant 0$. So we can define $\mathbf{SR}^* = \{z \in \mathbb{R}^+ |\, W(z) = 1-\mu\}$.

If we denote $W^*(z,\tau) = W(z)$, then $W^*(z,\tau)$ satisfies

$$\begin{cases} \partial_\tau W^* - \mathcal{L}_z W^* = 0, & \text{if } 1-\mu < W^* < 1+\lambda \quad \text{and} \quad (z,\tau) \in \mathbb{R}^+ \times (0,T], \\ \partial_\tau W^* - \mathcal{L}_z W^* \geqslant 0, & \text{if} \quad W^* = 1-\mu \quad \text{and} \quad (z,\tau) \in \mathbb{R}^+ \times (0,T], \\ \partial_\tau W^* - \mathcal{L}_z W^* \leqslant 0, & \text{if} \quad W^* = 1+\lambda \quad \text{and} \quad (z,\tau) \in \mathbb{R}^+ \times (0,T], \\ W^*(z,0) = W(z) \geqslant 1-\mu = V(z,0), \quad z \in \mathbb{R}^+. \end{cases}$$

Applying the comparison principle, we have $V(z,\tau) \leqslant W^*(z,\tau) = W(z)$. If we can prove that there exists a positive constant M_s such that

$$\mathbf{SR}^* \supset [\,M_s, +\infty), \tag{4.8}$$

then we obtain $[\,M_s, +\infty) \times [\,0,T\,] \subset \mathbf{SR}^* \times [\,0,T\,] \subset \mathbf{SR}$, it means $z_s(\tau) \leqslant M_s$.

We next prove (4.8). It is deduced by (4.7) that if $1-\mu < W(z) < 1+\lambda$,

$$\frac{d}{dz}\left[\frac{\sigma^2}{2} z(zW)' + \left(\alpha - \frac{\sigma^2}{2} - r\right)(zW) - \frac{\gamma\sigma^2}{2}(zW)^2\right] = 0.$$

Then

$$z(zW)' + \left(\frac{2(\alpha - r)}{\sigma^2} - 1\right)(zW) - \gamma(zW)^2 = \overline{C},$$

where $\overline{C}$ is a unknown constant. Denote

$$\widehat{W} = zW + \frac{1}{2\gamma}\left[1 - \frac{2(\alpha - r)}{\sigma^2}\right],$$

then $z\widehat{W}' - \gamma\widehat{W}^2 = C$, where $C = \overline{C} - \frac{1}{4\gamma^2}[1 - \frac{2(\alpha-r)}{\sigma^2}]^2$. There are only the following three possibilities:

(1) If $C < 0$, $z\widehat{W}' - \gamma\widehat{W}^2 = -C_1^2$ ($C_1 = \sqrt{-C} > 0$), solving the equation we have

$$\widehat{W} = \frac{C_1}{\sqrt{\gamma}}\left(\frac{2}{1 - C_2 z^{2\sqrt{\gamma}C_1}} - 1\right), \qquad W = \frac{1}{z}\left(\widehat{W} + \frac{\alpha - r}{\gamma\sigma^2} - \frac{1}{2\gamma}\right).$$

As $z \to +\infty$, $W \to 0$, which contradicts $1 - \mu \leqslant W \leqslant 1 + \lambda$. So there exists an $M_s > 0$ such that $\mathbf{SR}^* \supset [\,M_s, +\infty)$.

(2) If $C > 0$, $z\widehat{W}' - \gamma\widehat{W}^2 = C_1^2$ ($C_1 = \sqrt{C} > 0$), solving the equation we obtain

$$\widehat{W} = \frac{C_1}{\sqrt{\gamma}}\tan(C_1\sqrt{\gamma}\ln z + C_2), \qquad W = \frac{1}{z}\left(\widehat{W} + \frac{\alpha - r}{\gamma\sigma^2} - \frac{1}{2\gamma}\right).$$

So $\liminf_{z\to+\infty} W = 0$, which contradicts $1-\mu \leqslant W \leqslant 1+\lambda$. So, we have the same conclusion.

(3) If $C = 0$, $z\widehat{W}' - \gamma\widehat{W}^2 = 0$, it implies that

$$\widehat{W} = \frac{-1}{C_2 + \gamma\ln z}, \qquad W = \frac{1}{z}\left(\widehat{W} + \frac{\alpha - r}{\gamma\sigma^2} - \frac{1}{2\gamma}\right).$$

As $z \to +\infty$, $W \to 0$. So, whatever, there exists an $M_s > 0$ such that (4.8) holds.

We complete the proof of Lemma 4.1.

Lemma 4.2. *There exist $z_0 > 0$, $\tau_0 > 0$, such that*

$$(0, z_0) \times (0, \tau_0) \subset \mathbf{NR}, \tag{4.9}$$

and

$$\textit{all partial derivatives of } V(z,\tau) \textit{ are bounded on } (0, z_0) \times (0, \tau_0). \tag{4.10}$$

Proof. Since $V(z, 0) = 1 - \mu$, applying (4.4) we know that for any fixed $Z > 0$, there exists a $\tau_0 > 0$ such that

$$V(z,\tau) < 1 + \lambda. \qquad (z,\tau) \in (0, Z) \times (0, \tau_0). \tag{4.11}$$

On the other hand, from (4.2), we have that for any $0 < z_0 < \frac{\alpha - r}{\gamma\sigma^2(1-\mu)}$,

$$V(z,\tau) > 1 - \mu, \qquad (z,\tau) \in (0, z_0) \times (0, T]. \tag{4.12}$$

Combining (4.11) and (4.12), we obtain (4.9). Thus

$$\begin{cases} \partial_\tau V - \mathcal{L}_z V = 0 \quad \text{in } (0, z_0) \times (0, \tau_0), \\ V(z_0, \tau) \in C^\infty[0, \tau_0], \\ V(z, 0) = 1 - \mu, \quad 0 < z < z_0. \end{cases} \tag{4.13}$$

Let $x = \ln z$, $x_0 = \ln z_0$, $v(x,\tau) = V(z,\tau)$. Recalling (3.2) we have

$$\begin{cases} \partial_\tau v - \dfrac{\sigma^2}{2}\partial_{xx} v - \left(\alpha - r + \dfrac{\sigma^2}{2}\right)\partial_x v - (\alpha - r)v \\ \qquad + \gamma\sigma^2 e^x v(\partial_x v + v) = 0, \qquad (x,\tau) \in (-\infty, x_0) \times (0, \tau_0]; \\ v(x_0, \tau) \in C^\infty[0, \tau_0], \\ v(x, 0) = 1 - \mu, \quad x \in (-\infty, x_0). \end{cases} \tag{4.14}$$

Employing (3.10) and Schauder theory of parabolic equation (see [10]), we have

$$\|v\|_{C^{2+\alpha,\,1+\alpha/2}(\overline{(-\infty,x_0)\times(0,\tau_0)})} \leqslant C_{x_0},$$

where C_{x_0} depends on x_0. Using a bootstrap argument we obtain that all partial derivatives of $v(x,\tau)$ are bounded on $(-\infty,x_0)\times(0,\tau_0)$.

In order to prove that $\partial_z V(z,\tau) = e^{-x}\partial_x v(x,\tau)$ is bounded, we take the derivative with respect to x in (4.14) to have

$$\partial_\tau(\partial_x v) - \frac{\sigma^2}{2}\partial_{xx}(\partial_x v) - \left(\alpha - r + \frac{\sigma^2}{2}\right)\partial_x(\partial_x v) - (\alpha - r)(\partial_x v)$$
$$+\ \gamma\sigma^2 e^x[(\partial_x v + v)^2 + v(\partial_{xx}v + \partial_x v)] = 0.$$

Denote $W = e^{-x}\partial_x v$, then

$$\begin{cases} \partial_\tau W - \dfrac{\sigma^2}{2}\partial_{xx}W - \left(\alpha - r + \dfrac{3}{2}\sigma^2\right)\partial_x W - (2\alpha - 2r + \sigma^2)W \\ \qquad = -\gamma\sigma^2[(\partial_x v + v)^2 + v(\partial_{xx}v + \partial_x v)], \qquad (x,\tau)\in(-\infty,x_0)\times(0,\tau_0]; \\ W(x_0,\tau)\in C^\infty[0,\tau_0]; \\ W(x,0) = 0, \quad x\in(-\infty,x_0). \end{cases}$$

Since the right-hand side of the equation is bounded, $\partial_z V = e^{-x}\partial_x v$ is bounded.

In an analogous way we can prove that $\partial_{zz}V = \partial_z W = e^{-2x}(\partial_{xx}v - \partial_x v)$ is bounded. Moreover all partial derivatives are bounded on $(0,z_0)\times(0,\tau_0)$ by the bootstrap argument.

Theorem 4.3. *$z_b(\tau)\in C[0,T]$ and is strictly increasing on $[\tau^*,T]$, $z_b(\tau) = 0$, $0\leqslant\tau\leqslant\tau^*$, $\lim_{z\to 0^+} V(z,\tau) = V_0(\tau)$, where*

$$\tau^* = \frac{1}{\alpha - r}\ \ln\frac{1+\lambda}{1-\mu}, \tag{4.15}$$

$$V_0(\tau) = \begin{cases} (1-\mu)e^{(\alpha-r)\tau}, & 0\leqslant\tau\leqslant\tau^*; \\ 1+\lambda, & \tau > \tau^*. \end{cases} \tag{4.16}$$

Proof. In the first we prove $z_b(\tau)\in C[0,T]$. Otherwise, there exists a domain $(z_1,z_2)\times(0,\tau_1)$ $(0\leqslant z_1 < z_2 \leqslant \frac{\alpha-r}{\gamma\sigma^2(1+\lambda)})$, in which the following equation holds:

$$\begin{cases} \partial_\tau V - \mathcal{L}_z V = 0, & (z,\tau)\ \in\ (z_1,z_2)\times(0,\tau_1); \\ V(z,\tau_1) = 1+\lambda, & z_1\leqslant z\leqslant z_2. \end{cases}$$

Then $W = \partial_z V$ satisfies the following equation in the domain $(z_1,z_2)\times(0,\tau_1)$:

$$\begin{cases} \partial_\tau W - \dfrac{\sigma^2}{2}z^2\partial_{zz}W - (2\sigma^2 + \alpha - r)z\partial_z W - [\sigma^2 + 2(\alpha - r)]W + \gamma\sigma^2 z^2 V\partial_z W \\ \qquad +\gamma\sigma^2 z(z\partial_z V + 4V)W = -\gamma\sigma^2 V^2 \leqslant 0; \\ W(z,\tau_1) = 0, \quad z_1\leqslant z\leqslant z_2. \end{cases}$$

Since $W = \partial_z V \leqslant 0$, W achieves its non-negative maximum on $\tau = \tau_1$. By the maximum principle, we deduce $\partial_z V = W \equiv 0$ in the domain $(z_1,z_2)\times(0,\tau_1)$. It is obviously impossible.

Secondly, by (4.10), we see that there exists a $V_0(\tau)\in C[0,T]$ such that

$$\lim_{z\to0^+}V(z,\tau)=V_0(\tau),\qquad \lim_{z\to0^+}\partial_\tau V(z,\tau)=V_0'(\tau).$$

Applying (4.10) and letting $z\to0^+$ in (4.13) deduce that

$$\begin{cases} V_0'(\tau)-(\alpha-r)V_0(\tau)=0, & 0<\tau<\tau_0,\\ V_0(0)=1-\mu. \end{cases}$$

It implies that $V_0(\tau)=(1-\mu)e^{(\alpha-r)\tau}$, $0<\tau<\tau_0$. Let $V_0(\tau^*)=1+\lambda$, we have that

$$\tau^*=\frac{1}{\alpha-r}\ln\frac{1+\lambda}{1-\mu}.$$

Now we prove that $z_b(\tau)$ is strictly increasing on $[\tau^*,T]$. Otherwise, there exists a domain $(z_1,z_2)\times(\tau_1,\tau_2)$ $(0\leqslant z_1<z_2\leqslant\frac{\alpha-r}{\gamma\sigma^2(1+\lambda)},\ \tau^*\leqslant\tau_1<\tau_2\leqslant T)$, in which the following equation holds:

$$\begin{cases} \partial_\tau V-\mathcal{L}_zV=0, & (z,\tau)\in(z_1,z_2)\times(\tau_1,\tau_2);\\ V(z_1,\tau)=1+\lambda, & \tau_1<\tau<\tau_2. \end{cases}$$

Then $W=\partial_\tau v$ satisfies the following equation in the domain $(z_1,z_2)\times(\tau_1,\tau_2)$:

$$\begin{cases} \partial_\tau W-\dfrac{\sigma^2}{2}z^2\partial_{zz}W-(\sigma^2+\alpha-r)z\partial_zW-(\alpha-r)W\\ \qquad+\gamma\sigma^2z[zV\partial_zW+(z\partial_zV+2V)W]=0;\\ W(z_1,\tau)=0, \quad \tau_1<\tau<\tau_2. \end{cases}$$

Since $W=\partial_\tau V\geqslant0$, W achieves non-positive minimum on $z=z_1$; employing the maximum principle we have $\partial_zW(z_1,\tau)<0$. On the other hand, we can infer $\partial_zV(z_1,\tau)=0$ by $\partial_zV\in C((-\infty,\infty)\times[\tau^*,T])$. So $\partial_zW(z_1,\tau)=\partial_{z\tau}(z_1,\tau)=0$ and we get a contradiction.

Thus we complete the proof of Theorem 4.3.

Theorem 4.4. $z_b(\tau)\in C^\infty(\tau^*,T]$.

Proof. The proof is divided into six steps.

Step 1. Prove that

$$\partial_xv+v\geqslant2K_1e^{-K_2\tau}\qquad x\in\mathbb{R},\ \ 0\leqslant\tau\leqslant T,\tag{4.17}$$

where $K_1=(1-\mu)/2$, $K_2=2\gamma\sigma^2e^M(1+\lambda)$, in which $M=\ln M_s+2$ (M_s was defined in Lemma 4.1, see (4.6)).

In fact, if we denote $\mathbf{BR}_x$, $\mathbf{NR}_x$, $\mathbf{SR}_x$ are the counterparts of $\mathbf{BR}\cap\{z>0\}$, $\mathbf{NR}\cap\{z>0\}$, $\mathbf{SR}\cap\{z>0\}$ by the transformation $x=\ln z$ respectively, then $\partial_xv+v=1+\lambda$ in $\mathbf{BR}_x$, $\partial_xv+v=1-\mu$ in $\mathbf{SR}_x$. Letting $W=\partial_xv+v$, from (3.2) we have

$$\begin{cases} \partial_\tau W-\dfrac{\sigma^2}{2}\partial_{xx}W-\left(\alpha-r+\dfrac{\sigma^2}{2}\right)\partial_xW-(\alpha-r)W\\ \qquad+\gamma\sigma^2e^x[v\partial_xW+(2v+\partial_xv)W]=0 \quad \text{in } \mathbf{NR}_x;\\ W\geqslant1-\mu \quad \text{on } \partial_p\mathbf{NR}_x. \end{cases}\tag{4.18}$$

If we denote $w = 2K_1e^{-K_2\tau}$, since $\mathbf{NR}_x \subset (-\infty, M\,]$ and $\alpha - r \geqslant 0$, $\partial_x v \leqslant 0$, $v \leqslant 1+\lambda$, we have

$$\begin{aligned}&\partial_\tau w - \frac{\sigma^2}{2}\partial_{xx}w - \left(\alpha - r + \frac{\sigma^2}{2}\right)\partial_x w - (\alpha-r)w + \gamma\sigma^2 e^x[v\partial_x w + (2v+\partial_x v)w]\\&= -K_2 w - (\alpha-r)w + \gamma\sigma^2e^x(2v+\partial_x v)w\\&\leqslant w(-K_2 + 2\gamma\sigma^2e^M(1+\lambda)) = 0 \quad \text{in } \mathbf{NR}_x,\end{aligned}$$

so w is a sub-solution of (4.18). Hence $\partial_x v + v = W \geqslant w = 2K_1e^{-K_2\tau}$. We complete the proof of (4.17).

In what follows we come back to (3.6). Since $\mathbf{SR}_x \supset [\,M, \infty)$, we can rewrite (3.6) as

$$\begin{cases}\partial_\tau v_{\varepsilon,n} - \mathcal{L}_x v_{\varepsilon,n} + \beta_\varepsilon(v_{\varepsilon,n} - (1-\mu)) - \beta_\varepsilon(-v_{\varepsilon,n} + (1+\lambda)) = 0 \quad \text{in } (-n, M)\times(0,T\,];\\ \partial_x v_{\varepsilon,n}(x,\tau) = 0, \quad x=-n,\ x=M,\ 0\leqslant\tau\leqslant T;\\ v_{\varepsilon,n}(x,0) = 1-\mu, \quad -n\leqslant x\leqslant M,\end{cases} \tag{4.19}$$

where, recalling (3.5), $\beta_\varepsilon(0) = -C_0$. Now we define $C_0 = \max\{\gamma\sigma^2(1-\mu)^2e^M, (\alpha-r)(1+\lambda)\}$.

From (4.17), we know that if ε is small enough, then

$$\partial_x v_{\varepsilon,n} + v_{\varepsilon,n} \geqslant K_1e^{-K_2\tau}, \quad -n\leqslant x\leqslant M,\ 0\leqslant\tau\leqslant T. \tag{4.20}$$

Step 2. Prove that

$$\partial_\tau v_{\varepsilon,n} \leqslant K_3e^{(\alpha-r)\tau}, \quad -n\leqslant x\leqslant M,\ 0\leqslant\tau\leqslant T, \tag{4.21}$$

where $K_3 = \gamma\sigma^2e^M + 2(\alpha-r)(1+\lambda)$. Indeed, if we denote $w = \partial_\tau v_{\varepsilon,n}$, then from (3.7), we have

$$\begin{cases}\mathcal{T}w = 0 \quad \text{in} \quad (-n,M)\times(0,T\,];\\ \partial_x w(x,\tau) = 0, \quad x=-n,\ x=M,\ 0\leqslant\tau\leqslant T;\\ w(x,0) = (\alpha-r)(1-\mu) - [\gamma\sigma^2(1-\mu)^2e^x + \beta_\varepsilon(0)] \leqslant (\alpha-r)(1-\mu) - \beta_\varepsilon(0),\end{cases} \tag{4.22}$$

where

$$\begin{aligned}\mathcal{T}w = {}&\partial_\tau w - \frac{\sigma^2}{2}\partial_{xx}w - \left(\alpha - r + \frac{\sigma^2}{2}\right)\partial_x w - (\alpha-r)w\\&+ \gamma\sigma^2e^x[v_{\varepsilon,n}\partial_x w + (\partial_x v_{\varepsilon,n} + 2v_{\varepsilon,n})w] + \beta'_\varepsilon(\cdot)w + \beta'_\varepsilon(\cdot\cdot)w.\end{aligned}$$

From $\partial_x v_{\varepsilon,n} + 2v_{\varepsilon,n} \geqslant 0$ and $\beta'_\varepsilon \geqslant 0$, we have

$$\mathcal{T}(K_3e^{(\alpha-r)\tau}) \geqslant K_3e^{(\alpha-r)\tau}((\alpha-r)-(\alpha-r)) = 0 \qquad \text{in } (-n,M)\times(0,T\,],$$

recalling $K_3 = \gamma\sigma^2e^M + 2(\alpha-r)(1+\lambda)$, then $K_3e^{(\alpha-r)\tau} \geqslant w$ while $\tau = 0$. Hence $K_3e^{(\alpha-r)\tau}$ is supersolution to (4.22), thus we get (4.21).

Step 3. Construct function $\psi(\xi,\tau)$ which will be used in step 4. For any fixed x_0, we define

$$\psi(\xi,\tau) = e^{(K_4+\alpha-r)\tau}(e^{|\xi|} - 1 - |\,\xi|), \qquad \xi = x - x_0,$$

where

$$K_4 = \sigma^2 + 2K_5 + 2; \quad K_5 = \alpha - r + \frac{\sigma^2}{2} + \gamma\sigma^2e^M(1+\lambda),$$

then

$$\partial_x\psi(\xi,\tau)=\begin{cases} e^{(K_4+\alpha-r)\tau}(e^{|\xi|}-1), & \xi\geqslant 0,\\ -e^{(K_4+\alpha-r)\tau}(e^{|\xi|}-1), & \xi<0,\end{cases}$$

$$\partial_{xx}\psi(\xi,\tau)=e^{(K_4+\alpha-r)\tau}e^{|\xi|},$$

and from $\psi\geqslant 0,\ \partial_x v_{\varepsilon,n}+2v_{\varepsilon,n}\geqslant 0,\ \beta'_\varepsilon\geqslant 0,\ \alpha\geqslant r,\ x\leqslant M,\ 1-\mu\leqslant v_{\varepsilon,n}\leqslant 1+\lambda$, we obtain

$$\begin{aligned}\mathcal{T}\psi&\geqslant[\partial_\tau\psi-(\alpha-r)\psi]-\frac{1}{2}\sigma^2\partial_{xx}\psi-\left[\left(\alpha-r+\frac{\sigma^2}{2}\right)-\gamma\sigma^2e^x v_{\varepsilon,n}\right]\partial_x\psi\\ &\geqslant e^{(K_4+\alpha-r)\tau}\left[K_4(e^{|\xi|}-1-|\xi|)-\frac{\sigma^2}{2}e^{|\xi|}-K_5(e^{|\xi|}-1)\right].\end{aligned}$$

Since $\frac{1}{2}e^{|\xi|}-1-|\xi|\geqslant\frac{1}{2}e^3-4>0$ while $|\xi|\geqslant 3$, then

$$\mathcal{T}\psi\geqslant\begin{cases} e^{(K_4+\alpha-r)\tau}\left(\dfrac{K_4}{2}e^{|\xi|}-\dfrac{\sigma^2}{2}e^{|\xi|}-K_5e^{|\xi|}\right)\geqslant\dfrac{K_4-\sigma^2-2K_5}{2}e^{|\xi|}\geqslant e^3, & |\xi|\geqslant 3,\\ -K_6e^{(K_4+\alpha-r)\tau}, & |\xi|<3,\end{cases}\tag{4.23}$$

where K_6 is a positive constant depending on K_4 and K_5, but independent of ε, n, ξ.

Step 4. Prove that for any $(x,\tau)\in\mathbf{BR}_x\cup\mathbf{NR}_x$

$$\tau\partial_\tau v(x,\tau)\leqslant -Ce^{-x}\partial_x v(x,\tau),\tag{4.24}$$

where $C>0$ is independent of x, τ.

In fact, for any $(x_0,\tau_0)\in(-\infty,M-2)\times[0,T]$, we choose $n>\max\{-x_0+2,\,2\}$. Define

$$\phi(x,\tau)=\tau\partial_\tau v_{\varepsilon,n}(x,\tau)+Ce^{-x_0}\partial_x v_{\varepsilon,n}(x,\tau)-K_7\psi(\xi,\tau),$$

where $C>0$ is determined later and $K_7=(1+T)K_3e^{(\alpha-r)T}$. If we denote $w=\partial_\tau v_{\varepsilon,n},\ W=\partial_x v_{\varepsilon,n}$, then from (4.22), (4.21) and (4.19), we have

$$\begin{cases}\mathcal{T}w=0 & \text{in}\quad(-n,M)\times(0,T],\\ w(x,\tau)\leqslant K_3e^{(\alpha-r)\tau} & \text{on}\quad\partial_p\left((-n,M)\times(0,T]\right);\end{cases}\tag{4.25}$$

$$\begin{cases}\mathcal{T}W=-\gamma\sigma^2e^xv_{\varepsilon,n}(v_{\varepsilon,n}+\partial_xv_{\varepsilon,n}) & \text{in}\quad(-n,M)\times(0,T],\\ W(x,\tau)=0 & \text{on}\quad\partial_p\left((-n,M)\times(0,T]\right).\end{cases}\tag{4.26}$$

From (4.25), (4.26), (4.20), (4.21) and $v_{\varepsilon,n}\geqslant 1-\mu$, we have

$$\begin{aligned}\mathcal{T}\phi&=w-Ce^{-x_0}\gamma\sigma^2e^xv_{\varepsilon,n}(v_{\varepsilon,n}+\partial_xv_{\varepsilon,n})-K_7\mathcal{T}\psi(\xi,\tau)\\ &\leqslant K_3e^{(\alpha-r)T}-Ce^{-x_0}\gamma\sigma^2e^x(1-\mu)K_1e^{-K_2T}-K_7\mathcal{T}\psi(\xi,\tau).\end{aligned}$$

Applying (4.23), we see that if $|x-x_0|=|\xi|\geqslant 3$, $\mathcal{T}\phi\leqslant K_3e^{(\alpha-r)T}-K_7e^3\leqslant 0$; and if $|x-x_0|=|\xi|<3$, we can choose C large enough, which is independent of ε, n, x_0, τ_0, such that

$$\mathcal{T}\phi\leqslant K_3e^{(\alpha-r)T}+K_7K_6e^{(K_4+\alpha-r)T}-Ce^{-x_0}\gamma\sigma^2(1-\mu)K_1e^{x_0-3-K_2T}\leqslant 0.$$

Moreover, $\phi(x,0) \leqslant 0$ by $\partial_x v_{\varepsilon,n}(x,0) = 0$, $\psi(\xi,0) \geqslant 0$. If $x = -n \leqslant x_0 - 2$, then $|\xi| = |x - x_0| \geqslant 2$, combining the boundary conditions in (4.25) and (4.26), we have

$$\begin{aligned}\phi(-n,\tau) &\leqslant \tau\partial_\tau v_{\varepsilon,n}(-n,\tau) - K_7 e^{(K_4+\alpha-r)\tau}(e^{|\xi|} - |\xi| - 1)\\ &\leqslant TK_3 e^{(\alpha-r)T} - K_7 \leqslant 0.\end{aligned}$$

If $x = M$, we can get $\phi(M,\tau) \leqslant 0$ by the same method. So we get $\phi(x,\tau) \leqslant 0$ for any $x \in (-n, M)$ by the maximum principle. Particularly

$$\phi(x_0,\tau_0) = \tau_0\partial_\tau v_{\varepsilon,n}(x_0,\tau_0) + Ce^{-x_0}\partial_x v_{\varepsilon,n}(x_0,\tau_0) \leqslant 0,$$

for any $(x_0,\tau_0) \in (-n+2, M-2) \times [0,T]$. Since C is independent of ε, n, x_0, τ_0, let $\varepsilon \to 0^+$, $n \to \infty$, we know that for any $(x,\tau) \in (-\infty, \ln M_s) \times [0,T] \supset \mathbf{BR}_x \cup \mathbf{NR}_x$, (4.24) holds.

Step 5. Considering $\partial_\tau v = \partial_x v = 0$ in $\mathbf{SR}_x \supset (\ln M_s, +\infty) \times [0,T]$, then we have for any $(x,\tau) \in (-\infty,+\infty) \times [0,T]$, $\tau\,\partial_\tau v(x,\tau) \leqslant -Ce^{-x}\partial_x v(x,\tau)$. Going back to the (z,τ) coordinate system we obtain

$$\tau\,\partial_\tau V(z,\tau) \leqslant -C\partial_z V(z,\tau), \quad 0 \leqslant z < +\infty, \quad 0 \leqslant \tau \leqslant T.$$

Since $\partial_\tau V \geqslant 0$, we have an important cone property

$$\pm\tau\,\partial_\tau V(z,\tau) \leqslant -C\partial_z V(z,\tau), \quad 0 \leqslant z < +\infty, \quad 0 \leqslant \tau \leqslant T. \tag{4.27}$$

From this property we see that $V(z,\tau)$ is monotonic decreasing in the directions $(C, \pm\tau)$, therefore $z_b(\tau) \in C^{0,1}(\tau^*, T]$ and $\partial_\tau V$ is continuous across the free-boundary $z_b(\tau)$ while $\tau > \tau^*$.

Step 6. Moreover, we can get $z_b(\tau) \in C^\infty(\tau^*, T]$ by the bootstrap argument.

Theorem 4.5. *$z_s(\tau) \in C[0,T] \cap C^\infty(0,T]$ and is strictly increasing with $z_s(0) = z^*$, where*

$$z^* = \frac{\alpha - r}{\gamma\sigma^2(1-\mu)}.$$

Proof. First we prove $z^* = \frac{\alpha-r}{\gamma\sigma^2(1-\mu)}$. From (4.2) we know that $z^* \geqslant \frac{\alpha-r}{\gamma\sigma^2(1-\mu)}$. Now we prove that z^* cannot be greater than $\frac{\alpha-r}{\gamma\sigma^2(1-\mu)}$. Otherwise, there exists a domain $(\frac{\alpha-r}{\gamma\sigma^2(1-\mu)}, z_2) \times (0,T)$, such that

$$\begin{cases} \partial_\tau V - \mathcal{L}_z V = 0, & (z,\tau) \in \left(\dfrac{\alpha-r}{\gamma\sigma^2(1-\mu)}, z_2\right) \times (0,T) \\ V(z,0) = 1-\mu, & \dfrac{\alpha-r}{\gamma\sigma^2(1-\mu)} \leqslant z \leqslant z_2. \end{cases}$$

Thus

$$\partial_\tau V(z,0) = (\alpha - r)(1-\mu) - \gamma\sigma^2 z(1-\mu)^2 < 0, \quad \text{for } \frac{\alpha-r}{\gamma\sigma^2(1-\mu)} < z \leqslant z_2.$$

It contradicts $\partial_\tau V \geqslant 0$. Applying the same method, we can prove $z_s(\tau) \in C[0,T]$.

We deduce that $z_s(\tau)$ is strictly increasing applying a same method as in the proof of $z_b(\tau)$ being strictly increasing.

Since $\partial_\tau V \geqslant 0$ and $1-\mu$ is lower obstacle, it is not difficult to prove $z_s(\tau) \in C^{0,1}(0,T]$ by a method developed by Friedman in [13]. Moreover $z_s(\tau) \in C^\infty(0,T]$ by the bootstrap argument.

On the other hand we can prove $z_s(\tau) \in C^{0,1}(0,T]$ by the same method as in the proof of Theorem 4.4 as well.

5 The solutions of problems (2.4) and (1.3)

According to (2.6), there should be two functions $A(\tau)$ and $B(\tau)$ such that

$$V^*(z,\tau) = \begin{cases} A(\tau) - \gamma(1+\lambda)z, & (z,\tau) \in \mathbf{BR}, \\ B(\tau) - \gamma(1-\mu)z, & (z,\tau) \in \mathbf{SR}. \end{cases}$$

Now we define $A(\tau)$ and $B(\tau)$.

Applying the idea of $\partial_\tau V^* - LV^* = 0$ on $z = z_b(\tau)$, recalling (2.3) we have

$$A'(\tau) = -\gamma(\alpha - r)(1+\lambda)z_b(\tau) + \gamma^2 \frac{\sigma^2}{2}(1+\lambda)^2 z_b^2(\tau).$$

On the other hand, from initial condition in (2.4), we see $A(0) = 0$, hence we define

$$A(\tau) = \gamma(1+\lambda)\int_0^\tau \left[\frac{\sigma^2}{2}\gamma(1+\lambda)z_b^2(t) - (\alpha - r)z_b(t)\right]dt. \tag{5.1}$$

Then, in view of (2.6), define

$$V^*(z,\tau) = A(\tau) - \gamma\int_0^z V(\xi,\tau)d\xi. \tag{5.2}$$

Lemma 5.1. *$V^*(z,\tau)$, $\partial_\tau V^*(z,\tau)$, $z\partial_z V^*$ and $z^2\partial_{zz}V^* \in C(\mathbb{R}\times[0,T])$, moreover*

$$V^*(z,\tau) = \begin{cases} A(\tau) - \gamma(1+\lambda)z, & z \leqslant z_b(\tau), \\ B(\tau) - \gamma(1-\mu)z, & z \geqslant z_s(\tau), \end{cases} \tag{5.3}$$

where $A(\tau)$ is defined in (5.1) *and*

$$B(\tau) = A(\tau) - \gamma\int_0^{z_s(\tau)} V(\xi,\tau)d\xi + \gamma(1-\mu)z_s(\tau). \tag{5.4}$$

Proof. First we prove (5.3). If $z \leqslant z_b(\tau)$, from (5.2) we obtain

$$V^*(z,\tau) = A(\tau) - \gamma\int_0^z (1+\lambda)d\xi = A(\tau) - \gamma(1+\lambda)z,$$

and if $z \geqslant z_s(\tau)$,

$$\begin{aligned} V^*(z,\tau) &= A(\tau) - \gamma\int_0^{z_s(\tau)} V(\xi,\tau)d\xi - \gamma\int_{z_s(\tau)}^z (1-\mu)d\xi \\ &= A(\tau) - \gamma\int_0^{z_s(\tau)} V(\xi,\tau)d\xi + \gamma(1-\mu)z_s(\tau) - \gamma(1-\mu)z \\ &= B(\tau) - \gamma(1-\mu)z, \qquad \text{(by (5.4))}. \end{aligned}$$

Now we analyze the smoothness of $V^*(z,\tau)$. Since $z_b(\tau) \in C[0,T]$, $A(\tau) \in C^1[0,T]$ by (5.1); moreover, $V \in L^\infty(\mathbb{R}\times[0,T])$ and is continuous with respect to τ, therefore $V^*(z,\tau) \in C(\mathbb{R}\times[0,T])$ by (5.2).

Next we prove $\partial_\tau V^*(z,\tau) \in C(\mathbb{R}\times[0,T])$. In fact

$$\partial_\tau V^*(z,\tau) = A'(\tau) - \gamma \int_0^z \partial_\tau V(\xi,\tau)\, d\xi. \tag{5.5}$$

It is clear that $\partial_\tau V^*$ is continuous across $z=0$ by (5.5). On the other hand (5.5) can be rewritten as

$$\partial_\tau V^*(z,\tau) = A'(\tau) - \gamma \int_{z_b(\tau)}^z \partial_\tau V(\xi,\tau)\, d\xi. \tag{5.6}$$

And

$$\partial_\tau V^*(z,\tau) = A'(\tau) = -\gamma(\alpha-r)(1+\lambda)z_b(\tau) + \gamma^2\frac{\sigma^2}{2}(1+\lambda)^2 z_b^2(\tau), \qquad z \leqslant z_b(\tau). \tag{5.7}$$

And if $z_b(\tau) \leqslant z \leqslant z_s(\tau)$,

$$\begin{aligned}
&\int_{z_b(\tau)}^z \partial_\tau V(\xi,\tau)\, d\xi \\
&\quad= \int_{z_b(\tau)}^z \mathcal{L}_z V(\xi,\tau) d\xi \\
&\quad= \int_{z_b(\tau)}^z \frac{\partial}{\partial \xi}\left[\frac{\sigma^2}{2}\xi^2\partial_\xi V(\xi,\tau) + (\alpha-r)\xi V(\xi,\tau) - \gamma\frac{\sigma^2}{2}(\xi V(\xi,\tau))^2\right] d\xi \qquad \text{(by (2.8))} \\
&\quad= \left[\frac{\sigma^2}{2}\xi^2\partial_\xi V(\xi,\tau) + (\alpha-r)\xi V(\xi,\tau) - \gamma\frac{\sigma^2}{2}(\xi V(\xi,\tau))^2\right]_{\xi=z_b(\tau)}^{\xi=z}.
\end{aligned} \tag{5.8}$$

Substituting (5.8) into (5.6) we have

$$\begin{aligned}
&\partial_\tau V^*(z,\tau) \\
&\quad= A'(\tau) - \gamma\left[\frac{\sigma^2}{2}\xi^2\partial_\xi V(\xi,\tau) + (\alpha-r)\xi V(\xi,\tau) - \gamma\frac{\sigma^2}{2}(\xi V(\xi,\tau))^2\right]_{\xi=z_b(\tau)}^{\xi=z} \\
&\quad= -\gamma\left[\frac{\sigma^2}{2}z^2\partial_z V(z,\tau) + (\alpha-r)zV(z,\tau) - \gamma\frac{\sigma^2}{2}(zV(z,\tau))^2\right] \qquad \text{(by (5.7))},
\end{aligned} \tag{5.9}$$

if $z_b(\tau) \leqslant z \leqslant z_s(\tau)$, (5.9) shows that $\partial_\tau V^*(z,\tau) \in C([z_b(\tau), z_s(\tau)]\times[0,T])$. (5.7) and (5.9) imply that $\partial_\tau V^*$ is continuous across $z = z_b(\tau)$.

Moreover, by (5.9),

$$\lim_{z\to z_s^-(\tau)} \partial_\tau V^*(z,\tau) = -\gamma(\alpha-r)(1-\mu)z_s(\tau) + \gamma^2\frac{\sigma^2}{2}(1-\mu)^2 z_s^2(\tau). \tag{5.10}$$

On the other hand if $z \geqslant z_s(\tau)$,

$$\partial_\tau V^*(z,\tau) = B'(\tau) = A'(\tau) - \gamma\int_0^{z_s(\tau)} \partial_\tau V(\xi,\tau)\, d\xi = A'(\tau) - \gamma\int_{z_b(\tau)}^{z_s(\tau)} \partial_\tau V(\xi,\tau)\, d\xi.$$

Substituting $z = z_s(\tau)$ in (5.8) and combining (5.7), we see that

$$B'(\tau) = -\gamma(\alpha-r)(1-\mu)z_s(\tau) + \gamma^2\frac{\sigma^2}{2}(1-\mu)^2 z_s^2(\tau). \tag{5.11}$$

(5.10) and (5.11) show that $\partial_\tau V^*$ is continuous across $z = z_s(\tau)$.

Finally notice that $z\partial_z V^*(z,\tau) = -\gamma z V(z,\tau)$ is continuous on $\mathbb{R}\times[0,T]$, and $\partial_z[zV(z,\tau)]$ is bounded on $\mathbb{R}\times[0,T]$, moreover

$$z^2\partial_{zz}V^*(z,\tau) = -\gamma z^2\partial_z V(z,\tau) = -\gamma z\{\partial_z[zV(z,\tau)] - V(z,\tau)\}$$

is continuous on $\mathbb{R}\times[0,T]$.

We complete the proof of Lemma 5.1.

Theorem 5.2. *$V^*(z,\tau)$, defined by* (5.2), *is the solution of the problem* (2.4), *i.e.*

$$\partial_\tau V^* - LV^* \leqslant 0 \quad in \quad \mathbb{R}\times(0,T); \tag{5.12}$$

$$-\gamma(1+\lambda) \leqslant \partial_z V^* \leqslant -\gamma(1-\mu) \quad in \quad \mathbb{R}\times(0,T); \tag{5.13}$$

$$\partial_\tau V^* - LV^* = 0, \quad if \quad -\gamma(1+\lambda) < \partial_z V^* < -\gamma(1-\mu); \tag{5.14}$$

$$V^*(z,0) = \begin{cases} -\gamma(1+\lambda)z, & if \quad z<0, \\ -\gamma(1-\mu)z, & if \quad z\geqslant 0. \end{cases} \tag{5.15}$$

Proof. Since $\partial_z V^* = -\gamma V$ and $1-\mu \leqslant V \leqslant 1+\lambda$, we obtain (5.13). More precisely,

$$\begin{cases} \partial_z V^* = -\gamma(1+\lambda), & \text{if} \quad z\leqslant z_b(\tau); \\ -\gamma(1+\lambda) < \partial_z V^* < -\gamma(1-\mu), & \text{if} \quad z_b(\tau)<z<z_s(\tau); \\ \partial_z V^* = -\gamma(1-\mu), & \text{if} \quad z \geqslant z_s(\tau). \end{cases}$$

From the definition (5.2) and the initial value of V, we get (5.15).

In the following we prove (5.14). Substituting $V(z,\tau) = -\frac{1}{\gamma}\,\partial_z V^*(z,\tau)$ into the right-hand side of (5.9), combining (2.3), we obtain

$$\partial_\tau V^* - LV^* = 0, \qquad \text{if} \quad z_b(\tau)\leqslant z\leqslant z_s(\tau). \tag{5.16}$$

Finally we establish (5.12). Notice that, from Lemma 5.1, $\partial_\tau V^* - LV^*$ is continuous on $\mathbb{R}\times[0,T]$. From (2.5)–(2.8) we know that

$$\frac{\partial}{\partial z}(\partial_\tau V^* - LV^*) = -\gamma(\partial_\tau V - \mathcal{L}_z V)\begin{cases} \geqslant 0, & \text{if} \quad z\leqslant z_b(\tau); \\ =0, & \text{if} \quad z_b(\tau)<z<z_s(\tau); \\ \leqslant 0, & \text{if} \quad z\geqslant z_s(\tau). \end{cases}$$

Combining (5.16) we complete the proofs of (5.12) and Theorem 5.2.

In the following we construct the solution of problem (1.3). Recalling transformations 1 and 2 in Section 2, we define

$$Q(y,S,\tau) = \exp\{V^*(e^{r\tau}yS,\tau)\}, \tag{5.17}$$

$$y_b(S,\tau) = \frac{1}{e^{r\tau}S}\,z_b(\tau), \tag{5.18}$$

$$y_s(S,\tau) = \frac{1}{e^{r\tau}S}\,z_s(\tau), \tag{5.19}$$

then

$$\partial_y Q = e^{V^*}\partial_z V^* e^{r\tau} S, \quad \partial_S Q = e^{V^*}\partial_z V^* e^{r\tau} y,$$
$$\partial_{SS} Q = e^{V^*} e^{2r\tau} y^2[\partial_{zz} V^* + (\partial_z V^*)^2], \quad \partial_\tau Q = e^{V^*}(\partial_\tau V^* + re^{r\tau} yS\partial_z V^*).$$

It can be seen that, by Lemma 5.1, $\partial_y Q$ is bounded in bound domain and $S\partial_S Q$, $S^2\partial_{SS} Q$, $\partial_\tau Q$ are continuous. Thus we obtain

Theorem 5.3. *$Q(y, S, \tau)$ is the solution to the problem* (1.3). *And*

$$\mathbf{BR} = \{(y, S, \tau) | y \leqslant y_b(S, \tau)\},$$
$$\mathbf{NR} = \{(y, S, \tau) | y_b(S, \tau) < y < y_s(S, \tau)\},$$
$$\mathbf{SR} = \{(y, S, \tau) | y \geqslant y_s(S, \tau)\}.$$

Moreover

(1) *For a fixed τ, $0 \leqslant \tau \leqslant \tau^*$, the free boundaries $y = y_b(S, \tau) = 0$ and $y = y_s(S, \tau)$ is one branch of hyperbola in the first quadrant* (*Figure* 2).

(2) *For a fixed τ, $\tau^* < \tau \leqslant T$, two free boundaries are the branches of hyperbola in the first quadrant* (*Figure* 3).

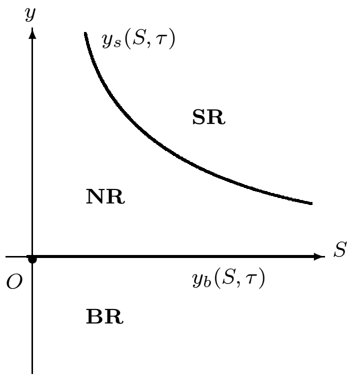

Figure 2 *τ-section for $0 \leqslant \tau \leqslant \tau^*$*

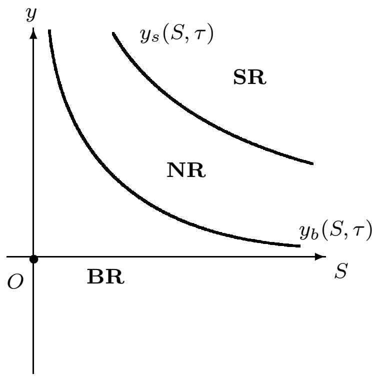

Figure 3 *τ-section for $\tau^* < \tau \leqslant T$*

Appendix: formulation of the model

In Black-Scholes model[14], the price of a European call option without divided and transaction costs satisfies a linear PDE

$$\begin{cases} \partial_t V + \dfrac{\sigma^2}{2} S^2 \partial_{SS} V + rS\partial_S V - rV = 0, & (S, t) \in (0, +\infty) \times [0, T), \\ V(S, T) = (S - E)^+, & S \in [0, +\infty), \end{cases} \tag{6.1}$$

where V is the price of the European call option, S is the price of the underlying asset, r is riskless interest, σ is volatility.

The following derivation is from [1], where the price of the option is the difference between the two value functions, which are, respectively, the solutions of two variational inequalities arising from two stochastic control problems.

The utility maximization is critical in the model. So, in the first, we research how to make utility maximization in the market. It is now assumed that investors must pay transaction costs, which are proportional to the amount transferred from the stock to the bank. The market model similar to that of Davis and Norman in [15].

We consider a time interval $[0, T]$ and a market, which consists of a stock whose price $S(t)$ is assumed to be stochastic processes on a given probability space $(\Omega, \mathscr{F}, P)$, their natural filtration is $\mathscr{F}_t = \sigma\{S(u) : 0 \leqslant u \leqslant t\}$. The cash value of a number of shares $y(t)$ of the stock is

$$c(y(t), S(t)) = \begin{cases} (1+\lambda)y(t)S(t), & \text{if } y(t) < 0, \\ (1-\mu)y(t)S(t), & \text{if } y(t) \geqslant 0, \end{cases} \tag{6.2}$$

where λ and μ are the fraction of the traded amount in stock, which the investor pays in transaction costs when buying or selling stock, respectively. The market model equations are

$$dB(t) = rB(t) - (1+\lambda)S(t)dL(t) + (1-\mu)S(t)dM(t); \tag{6.3}$$

$$dy(t) = dL(t) - dM(t); \tag{6.4}$$

$$dS(t) = S(t)(\alpha dt + \sigma^2 dR(t)), \tag{6.5}$$

where $L(t)$ and $M(t)$ are the cumulative number of shares bought or sold, respectively, over $[0, T]$ by an investor, $R(t)$ is a P- Brownian motion that represents the single source of uncertainty, and $r,\ \alpha,\ \sigma$ are non-random constants, representing riskless interest, risky interest and volatility, respectively. It is obvious that $\alpha > r$.

Let $\mathscr{T}_{(s,\,S)}(B, y)$ denote the set of admissible trading strategies available to an investor who starts at time s with an amount B in cash and y shares of the stock at the price of S. We assume $\mathscr{T}_{(s,\,S)}(B, y)$ consists of all the two-dimensional, right-continuous, measurable processes $(B^\pi(t), y^\pi(t))$, where π is an element in $\mathscr{T}_{(s,\,S)}(B, y)$, $B^\pi(t),\ y^\pi(t)$ are the solution of equations (6.3)–(6.5), corresponding to some pair of right-continuous, measurable, $\mathscr{F}_t$-adapted, processes $(L(t), M(t))$, such that $(B^\pi(t), y^\pi(t), S(t)) \in \mathscr{E}_K,\ \forall\, t \in [s, T]$, where K is constant, which may depend on the policy π and $\mathscr{E}_K = \{(B, y, S) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+ : B + c(y, S) > -K\}$. By convention, $L(0-) = M(0-) = 0$.

We define the following value function:

$$V_\Phi(s, B, y, S) = \sup_{\pi \in \mathscr{T}_{(s,\,S)}(B,\,y)} E\{\mathcal{U}(\Phi)\},$$

where E denote expectation, and $\mathcal{U} = 1 - \exp(-\gamma x)$ is exponential utility function, γ is index of risk aversion, independent of the investor's wealth. Φ is the wealth function at terminal time T, $V_\Phi(B)$ is maximum of the utility expectation with the initial endowment B.

Define $\Phi_1(T, B(T), y(T), S(T)) = B(T) + c(y(T), S(T)),\ V_1(s, B, y, S) = V_{\Phi_1}(s, B, y, S)$, which is the wealth of an investor without option at terminal time T.

Define $\Phi_2(T, B(T), y(T), S(T)) = B(T) + I_{(S(T) \leqslant E)} c(y(T), S(T)) + I_{(S(T) > E)}[c(y(T) - 1, S(T)) + E]$, $V_2(s, B, y, S) = V_{\Phi_2}(s, B, y, S)$, which is the wealth of an investor at terminal time T, who write an European call option.

Define $B_1(t, S) = \inf\{B^* : V_1(t, B^*, 0, S) \geqslant 0\},\ B_2(t, S) = \inf\{B^* : V_2(t, B^*, 0, S) \geqslant 0\}$.

Then $p(t, S) = B_2(t, S) - B_1(t, S)$ is the price of the European call option, where $B_1 \leqslant 0$, since clearly $V_1(t, 0, 0, S) \geqslant 0$. Think of $-B_1$ as the "entry fee" that the writer is prepared to pay to get into the market, and p is difference between going to the market to hedge the option and going into the market strictly on his own account.

By the knowledge of stochastic control, if we let $Q_j(t,y,S) = 1 - V_j(t,0,y,S)$ $(j = 1,2)$, Q_j satisfy the following two variational inequality problems, see [1, page 478, (4.27)–(4.29)].

$$\min\Big\{\partial_y Q_j + \gamma(1+\lambda)SQ_j e^{r(T-t)},\ -(\partial_y Q_j + \gamma(1-\mu)SQ_j e^{r(T-t)}),$$
$$\partial_t Q_j + \frac{\sigma^2}{2}S^2\partial_{SS}Q_j + \alpha S\partial_S Q_j\Big\} = 0, \qquad S > 0,\ \ y \in \mathbb{R},\ \ 0 < t \leqslant T,\ \ j = 1,2, \tag{6.6}$$

$$Q_1(T,y,S) = \exp\{-\gamma c(y,S)\}, \tag{6.7}$$

$$Q_2(T,y,S) = \exp\{-\gamma(I_{(S(T)\leqslant E)}c(y,S) + I_{(S(T)>E)}[c(y-1,S)+E])\}. \tag{6.8}$$

Moreover

$$V_j(t,B,y,S) = 1 - \exp\{-B\gamma e^{r(T-t)}\}Q_j(t,y,S)$$

and the price of the European call option is

$$p(t,S) = \gamma^{-1}e^{-r(T-t)}(\ln Q_2(t,0,S) - \ln Q_1(t,0,S)).$$

In this paper, we only consider the variational inequality with respect to Q_1, (6.6) with $j = 1$ and (6.7). Problem (6.6) with $j = 2$ and (6.8) will be considered in the future which cannot be simplified to one dimensional case.

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