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New double Wronskian solutions of the AKNS equation

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Abstract Soliton solutions, rational solutions, Matveev solutions, complexitons and interaction solutions of the AKNS equation are derived through a matrix method for constructing double Wronskian entries. The latter three solutions are novel. Moreover, rational solutions of the nonlinear Schrödinger equation are obtained by reduction.

Keywords: AKNS equation, the Wronskian technique, double Wronski determinant

MSC(2000): 35Q51, 35Q58

1 Introduction

In order to search for multi-soliton solutions to Lax integrable equations, various methods have been developed. Among them, the Wronskian technique has obvious advantages. Since each column of a Wronskian is the derivatives of the previous one, higher derivatives of it lead to the sums of determinants, but the length of the sum depends on the number of differentiations and not on the number n of solitons. So this type of solutions can be verified by the direct substitution into the soliton equation in the bilinear form[1].

The determination of Wronskian entries ϕ_i $(1 \leqslant i \leqslant n)$ is the key to constructing Wronskian solutions. In general, letting potential function $u = 0$ and spectral parameter $k = k_i$ in the Lax pair of the equation, one can get the corresponding solution of ϕ_i. Taking $\phi = (\phi_1, \phi_2, \ldots, \phi_n)^{\mathrm{T}}$, then ϕ satisfies a matrix equation with a diagonal coefficient matrix A.

In 1988, Sirianunpiboon et al.[2] generalized A to a triangular form, in order that the Wronskian can generate rational solutions and their interaction solutions with multi-solitons. About four years later, Matveev[3] introduced the generalized Wronskian to obtain another important kind of exact solutions called Positons for the KdV equation.

Recently, Ma et al.[4] considered the case where A is an arbitrary real matrix. Assuming that A has complex eigenvalues, they obtained complexitons for the KdV equation, including the positons which corresponds to positive eigenvalues of A.

In this paper, we would like to consider the second-order isospectral AKNS equation[5]

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = \begin{pmatrix} -q_{xx} + 2q^2 r \\ r_{xx} - 2qr^2 \end{pmatrix}. \tag{1.1}$$

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When $r=-q^*$, substituting $-it$ for t leads to the following nonlinear Schrödinger equation

$$iq_t+q_{xx}+2|q|^2q=0. \tag{1.2}$$

The Lax pair of (1.1) is

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}_x=\begin{pmatrix}-\eta & q\\ r & \eta\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}, \tag{1.3.1}$$

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}_t=\begin{pmatrix}-2\eta^2+qr & 2q\eta-q_x\\ 2r\eta+r_x & 2\eta^2-qr\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}. \tag{1.3.2}$$

Through the dependent variable transformation

$$q=\frac{g}{f},\quad r=\frac{h}{f}, \tag{1.4}$$

(1.1) is transformed into the bilinear form

$$ff_{xx}-f_x^2+gh=0, \tag{1.5.1}$$

$$g_tf-gf_t+g_{xx}f-2g_xf_x+gf_{xx}=0, \tag{1.5.2}$$

$$h_tf-hf_t-h_{xx}f+2h_xf_x-hf_{xx}=0. \tag{1.5.3}$$

Let us observe the matrix equations

$$\phi_x=-A\phi,\quad \psi_x=A\psi, \tag{1.6.1}$$

$$\phi_t=-2\phi_{xx},\quad \psi_t=2\psi_{xx}, \tag{1.6.2}$$

where $A=(a_{ij})$ is an $(n+m+2)\times(n+m+2)$ arbitrary real matrix independent of x and t,

$$\phi=(\phi_1,\phi_2,\dots,\phi_{n+m+2})^{\mathrm{T}},\quad \psi=(\psi_1,\psi_2,\dots,\psi_{n+m+2})^{\mathrm{T}}. \tag{1.7}$$

In this paper, we first prove that under the conditions (1.6), (1.5) has the following double Wronskian solution

$$f=W^{n+1,m+1}(\phi;\psi)=|\widehat{n};\widehat{m}|, \tag{1.8.1}$$

$$g=2W^{n+2,m}(\phi;\psi)=2|\widehat{n+1};\widehat{m-1}|, \tag{1.8.2}$$

$$h=-2W^{n,m+2}(\phi;\psi)=-2|\widehat{n-1};\widehat{m+1}|, \tag{1.8.3}$$

where

$$W^{j,l}(\phi;\psi)=|\phi,\partial_x\phi,\dots,\partial_x^{j-1}\phi;\psi,\partial_x\psi,\dots,\partial_x^{l-1}\psi|=|\widehat{j-1};\widehat{l-1}|. \tag{1.8.4}$$

Secondly, letting A be some special matrices, we obtain rational solutions, Matveev solutions, complexitons of (1.5). Finally, we illustrate how to produce more double Wronskian interaction solutions. The method for constructing solutions is a general one. It can be applied to other Lax integrable equations, such as the higher-order AKNS equation, the KdV equation, the Boussinesq equation, the KP equation and the DS equation.

The paper is organized as follows. In Section 2, we verify that the double Wronskian (1.8) solves (1.5). In Section 3, soliton solutions and rational solutions in double Wronskian form are obtained. In Section 4, Matveev solutions are provided. In Section 5, complexitons are constructed. Interaction solutions are given in Section 6.

2 Double Wronskian solutions of (1.5)

For convenience of proof, we first give the following lemmas.

Lemma 1.

$$|Q,a,b||Q,c,d|-|Q,a,c||Q,b,d|+|Q,a,d||Q,b,c|=0, \tag{2.1}$$

where Q is an $N\times(N-2)$ matrix, a,b,c and d represent N-dimensional column vectors.

Lemma 2.

$$\sum_{j=1}^{N}|\alpha_1,\ldots,\alpha_{j-1},\gamma\alpha_j,\alpha_{j+1},\ldots,\alpha_N|=\sum_{j=1}^{N}\gamma_j|\alpha_1,\ldots,\alpha_N|, \tag{2.2}$$

where α_j $(1\leqslant j\leqslant N)$ are N-dimensional column vectors and $\gamma\alpha_j$ denotes $(\gamma_1\alpha_{1j},\gamma_2\alpha_{2j},\ldots,\gamma_N\alpha_{Nj})^{\mathrm{T}}$.

Employing the Wronskian technique, we have

Theorem 1. *The* AKNS *equation* (1.5) *has double Wronskian solutions* (1.8), *where*

$$\phi_{j,x}=-k_j\phi_j,\quad \psi_{j,x}=k_j\psi_j, \tag{2.3.1}$$

$$\phi_{j,t}=-2\phi_{jxx},\quad \psi_{j,t}=2\psi_{j,xx}\quad (j=1,2,\ldots,m+n+2). \tag{2.3.2}$$

Proof. The derivatives of f can be easily computed

$$f_x=|\widehat{n-1},n+1;\widehat{m}|+|\widehat{n};\widehat{m-1},m+1|, \tag{2.4.1}$$

$$\begin{aligned}f_{xx}=&|\widehat{n-2},n,n+1;\widehat{m}|+|\widehat{n-1},n+2;\widehat{m}|+2|\widehat{n-1},n+1;\widehat{m-1},m+1|\\&+|\widehat{n};\widehat{m-2},m,m+1|+|\widehat{n};\widehat{m-1},m+2|.\end{aligned} \tag{2.4.2}$$

Noting

$$|\widehat{n};\widehat{m}|\Big(\sum_{j=1}^{n+m+2}k_j\Big)^2|\widehat{n};\widehat{m}|=\Big(\sum_{j=1}^{n+m+2}k_j|\widehat{n};\widehat{m}|\Big)^2, \tag{2.5.1}$$

from (2.3.1), we have

$$\begin{aligned}&|\widehat{n},\widehat{m}|(|\widehat{n-2},n,n+1;\widehat{m}|+|\widehat{n-1},n+2;\widehat{m}|-2|\widehat{n-1},n+1;\widehat{m-1},m+1|\\&\quad+|\widehat{n};\widehat{m-2},m,m+1|+|\widehat{n};\widehat{m-1},m+2|)=(|\widehat{n-1},n+1;\widehat{m}|-|\widehat{n};\widehat{m-1},m+1|)^2.\end{aligned} \tag{2.5.2}$$

Substituting (2.4) into the left-hand side of (1.5.1) and making use of (2.5), we get

$$\begin{aligned}&|\widehat{n};\widehat{m}||\widehat{n-1},n+1;\widehat{m-1},m+1|-|\widehat{n-1},n+1;\widehat{m}||\widehat{n};\widehat{m-1},m+1|\\&\quad-|\widehat{n+1};\widehat{m-1}||\widehat{n-1};\widehat{m+1}|.\end{aligned} \tag{2.6}$$

According to Lemma 1, (2.6) is equal to zero. So the proof of (1.5.1) is finished.

From (2.3.2), we obtain

$$g_t = 4(|\widehat{n-1}, n+1, n+2; \widehat{m-1}| - |\widehat{n}, n+3; \widehat{m-1}| - |\widehat{n+1}; \widehat{m-3}, m-1, m| + |\widehat{n+1}; \widehat{m-2}, m+1|). \tag{2.7}$$

Then,

$$(g_t + g_{xx})f = (6|\widehat{n-1}, n+1, n+2; \widehat{m-1}| - 2|\widehat{n}, n+3; \widehat{m-1}| + 4|\widehat{n}, n+2; \widehat{m-2}, m| - 2|\widehat{n+1}; \widehat{m-3}, m-1, m| + 6|\widehat{n+1}; \widehat{m-2}, m+1|)|\widehat{n}; \widehat{m}|. \tag{2.8.1}$$

Similarly,

$$g(-f_t + f_{xx}) = 2|\widehat{n+1}; \widehat{m-1}|(-|\widehat{n-2}, n, n+1; \widehat{m}| + 3|\widehat{n-1}, n+2; \widehat{m}| + 2|\widehat{n-1}, n+1; \widehat{m-1}, m+1| + 3|\widehat{n}; \widehat{m-2}, m, m+1| - |\widehat{n}; \widehat{m-1}, m+2|), \tag{2.8.2}$$

$$-2g_x f_x = -4(|\widehat{n}, n+2; \widehat{m-1}| + |\widehat{n+2}; \widehat{m-2}, m|)(|\widehat{n-1}, n+1; \widehat{m}| + |\widehat{n}; \widehat{m-1}, m+1|). \tag{2.8.3}$$

Utilizing the following identities which are similar to (2.5)

$$\begin{aligned}&|\widehat{n};\widehat{m}|(|\widehat{n-1}, n+1, n+2; \widehat{m-1}| + |\widehat{n}, n+3; \widehat{m-1}| - 2|\widehat{n}, n+2; \widehat{m-2}, m| \\ &\quad + |\widehat{n+1}; \widehat{m-3}, m-1, m| + |\widehat{n+1}; \widehat{m-2}, m+1|) \\ &\quad = (|\widehat{n-1}, n+1; \widehat{m}| - |\widehat{n}; \widehat{m-1}, m+1|)(|\widehat{n}, n+2; \widehat{m-1}| - |\widehat{n+1}; \widehat{m-2}, m|),\end{aligned} \tag{2.9.1}$$

$$\begin{aligned}&|\widehat{n+1}; \widehat{m-1}|(|\widehat{n-2}, n, n+1; \widehat{m}| + |\widehat{n-1}, n+2; \widehat{m}| - 2|\widehat{n-1}, n+1; \widehat{m-1}, m+1| \\ &\quad + |\widehat{n}; \widehat{m-2}, m, m+1| + |\widehat{n}; \widehat{m-1}, m+2|) \\ &\quad = (|\widehat{n}, n+2; \widehat{m-1}| - |\widehat{n+1}; \widehat{m-2}, m|)(|\widehat{n-1}, n+1; \widehat{m}| - |\widehat{n}; \widehat{m-1}, m+1|),\end{aligned} \tag{2.9.2}$$

The left-hand side of (1.5.2) is reduced as

$$\begin{aligned}&|\widehat{n};\widehat{m}||\widehat{n-1}, n+1, n+2; \widehat{m-1}| + |\widehat{n}; \widehat{m}||\widehat{n+1}; \widehat{m-2}, m+1| + |\widehat{n+1}; \widehat{m-1}||\widehat{n-1}, n+2; \widehat{m}| \\ &\quad + |\widehat{n+1}; \widehat{m-1}||\widehat{n}; \widehat{m-2}, m, m+1| - |\widehat{n}, n+2; \widehat{m-1}||\widehat{n-1}, n+1; \widehat{m}| \\ &\quad - |\widehat{n+1}; \widehat{m-2}, m||\widehat{n}; \widehat{m-1}, m+1|.\end{aligned} \tag{2.10}$$

Using Lemma 1, (2.10) is equal to zero. (1.5.3) can be verified similarly.

From (2.3), we deduce that

$$\phi_j = e^{-\xi_j} c_j, \quad \psi_j = e^{\xi_j} d_j, \quad \xi_j = 2k_j^2 t + k_j x, \tag{2.11}$$

where c_j and d_j $(j = 1, 2, \ldots, n+m+2)$ are arbitrary real constants.

Taking $c_j = d_j = 1$, the double Wronskian solution of (1.5) is obtained as follows:

$$f = |e^{-\xi_j}, \partial_x e^{-\xi_j}, \ldots, \partial_x^n e^{-\xi_j}; e^{\xi_j}, \partial_x e^{\xi_j}, \ldots, \partial_x^m e^{\xi_j}|, \tag{2.12.1}$$

$$g = 2|e^{-\xi_j}, \partial_x e^{-\xi_j}, \ldots, \partial_x^{n+1} e^{-\xi_j}; e^{\xi_j}, \partial_x e^{\xi_j}, \ldots, \partial_x^{m-1} e^{\xi_j}|, \tag{2.12.2}$$

$$h = -2|e^{-\xi_j}, \partial_x e^{-\xi_j}, \ldots, \partial_x^{n-1} e^{-\xi_j}; e^{\xi_j}, \partial_x e^{\xi_j}, \ldots, \partial_x^{m+1} e^{\xi_j}|. \tag{2.12.3}$$

Letting $n = m = 0$ gives

$$f = e^{\xi_2-\xi_1} - e^{\xi_1-\xi_2}, \quad g = 2(k_1-k_2)e^{-\xi_1-\xi_2}, \quad h = 2(k_1-k_2)e^{\xi_1+\xi_2}. \tag{2.13}$$

The corresponding one-soliton solution of (1.1) is

$$q = (k_1-k_2)\frac{e^{-\xi_1-\xi_2}}{\sinh(\xi_2-\xi_1)}, \quad r = (k_1-k_2)\frac{e^{\xi_1+\xi_2}}{\sinh(\xi_2-\xi_1)}. \tag{2.14}$$

Choosing $n = 1, m = 0$ yields

$$f = (k_1-k_2)e^{-\xi_1-\xi_2+\xi_3} + (k_3-k_1)e^{-\xi_1+\xi_2-\xi_3} + (k_2-k_3)e^{\xi_1-\xi_2-\xi_3}, \tag{2.15.1}$$

$$g = 2(k_1-k_2)(k_2-k_3)(k_1-k_3)e^{-\xi_1-\xi_2-\xi_3}, \tag{2.15.2}$$

$$h = -2[(k_3-k_2)e^{-\xi_1+\xi_2+\xi_3} + (k_2-k_1)e^{\xi_1+\xi_2-\xi_3} + (k_1-k_3)e^{\xi_1-\xi_2+\xi_3}], \tag{2.15.3}$$

so we have

$$q = 2\frac{(k_1-k_2)(k_2-k_3)(k_1-k_3)e^{-\xi_1-\xi_2-\xi_3}}{(k_1-k_2)e^{-\xi_1-\xi_2+\xi_3} + (k_3-k_1)e^{-\xi_1+\xi_2-\xi_3} + (k_2-k_3)e^{\xi_1-\xi_2-\xi_3}}, \tag{2.16.1}$$

$$r = 2\frac{(k_2-k_3)e^{-\xi_1+\xi_2+\xi_3} + (k_1-k_2)e^{\xi_1+\xi_2-\xi_3} + (k_3-k_1)e^{\xi_1-\xi_2+\xi_3}}{(k_1-k_2)e^{-\xi_1-\xi_2+\xi_3} + (k_3-k_1)e^{-\xi_1+\xi_2-\xi_3} + (k_2-k_3)e^{\xi_1-\xi_2-\xi_3}}. \tag{2.16.2}$$

Similarly, when $n = 0, m = 1$, we gain

$$q = -2\frac{(k_1-k_2)e^{-\xi_1-\xi_2+\xi_3} + (k_3-k_1)e^{-\xi_1+\xi_2-\xi_3} + (k_2-k_3)e^{\xi_1-\xi_2-\xi_3}}{(k_1-k_2)e^{\xi_1+\xi_2-\xi_3} + (k_3-k_1)e^{\xi_1-\xi_2+\xi_3} + (k_2-k_3)e^{-\xi_1+\xi_2+\xi_3}}, \tag{2.17.1}$$

$$r = 2\frac{(k_1-k_2)(k_2-k_3)(k_1-k_3)e^{\xi_1+\xi_2+\xi_3}}{(k_1-k_2)e^{\xi_1+\xi_2-\xi_3} + (k_3-k_1)e^{\xi_1-\xi_2+\xi_3} + (k_2-k_3)e^{-\xi_1+\xi_2+\xi_3}}. \tag{2.17.2}$$

In order to prove that (1.8) solves (1.5) under the conditions (1.6), we give the following lemma.

Lemma 3. *Assume that $P = (p_{ij})$ is an $l \times l$ operator matrix and its entries p_{ij} are differential operators. $B = (b_{ij})$ is an $l \times l$ function matrix with column vector set b_i and row vector set b'_j ($i = 1, 2, \ldots, l; j = 1, 2, \ldots, l$), then*

$$\sum_{i=1}^{l} |b_1, \ldots, p_i b_i, \ldots, b_l| = \sum_{j=1}^{l} \begin{vmatrix} b'_1 \\ \vdots \\ p'_j b'_j \\ \vdots \\ b'_l \end{vmatrix}, \tag{2.18}$$

where $p_i b_i = (p_{1i}b_{1i}, p_{2i}b_{2i}, \ldots, p_{li}b_{li})^{\mathrm{T}}, \quad p'_j b'_j = (p_{j1}b_{j1}, p_{j2}b_{j2}, \ldots, p_{jl}b_{jl})$[6].

In fact, we only need to verify that identities (2.5.2) and (2.9) hold, where ϕ and ψ enjoy the conditions (1.6).

(1) If $\operatorname{tr} A \neq 0$, setting

$$p_{ij} = \begin{cases} -\partial_x & 1 \leqslant i \leqslant n+m+2\ ; 1 \leqslant j \leqslant n+1; \\ \partial_x & 1 \leqslant i \leqslant n+m+2\ ; n+2 \leqslant j \leqslant n+m+2, \end{cases}$$

from Lemma 3, we can get

$$\sum_{j=1}^{n+m+2}\begin{vmatrix} \phi_1 & \cdots & \partial_x^n\phi_1 & \psi_1 & \partial_x\psi_1 & \cdots & \partial_x^m\psi_1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\partial_x\phi_j & \cdots & -\partial_x(\partial_x^n\phi_j) & \partial_x\psi_j & \partial_x(\partial_x\psi_j) & \cdots & \partial_x(\partial_x^m\psi_j) \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi_{n+m+2} & \cdots & \partial_x^n\phi_{n+m+2} & \psi_{n+m+2} & \partial_x\psi_{n+m+2} & \cdots & \partial_x^m\psi_{n+m+2} \end{vmatrix}$$
$$= -|\widehat{n-1}, n+1; \widehat{m}| + |\widehat{n}; \widehat{m-1}, m+1|. \tag{2.19.1}$$

Making use of (1.6.1), the left-hand side of (2.19.1) is equal to

$$\sum_{j=1}^{n+m+2}\begin{vmatrix} \phi_1 & \cdots & \partial_x^n\phi_1 & \psi_1 & \partial_x\psi_1 & \cdots & \partial_x^m\psi_1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \sum_{l=1}^{n+m+2} a_{jl}\phi_l & \cdots & \sum_{l=1}^{n+m+2} a_{jl}\partial_x^n\phi_l & \sum_{l=1}^{n+m+2} a_{jl}\psi_l & \sum_{l=1}^{n+m+2} a_{jl}\partial_x\psi_l & \cdots & \sum_{l=1}^{n+m+2} a_{jl}\partial_x^m\psi_l \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi_{n+m+2} & \cdots & \partial_x^n\phi_{n+m+2} & \psi_{n+m+2} & \partial_x\psi_{n+m+2} & \cdots & \partial_x^m\psi_{n+m+2} \end{vmatrix}$$
$$= \sum_{j=1}^{n+m+2} a_j j|\widehat{n}; \widehat{m}|, \tag{2.19.2}$$

so that

$$\mathrm{tr}A|\widehat{n}; \widehat{m}| = -|\widehat{n-1}, n+1; \widehat{m}| + |\widehat{n}; \widehat{m-1}, m+1|. \tag{2.20}$$

From (2.20) we derive further

$$(\mathrm{tr}\,A)^2|\widehat{n}; \widehat{m}| = |\widehat{n-2}, n, n+1; \widehat{m}| + |\widehat{n-1}, n+2; \widehat{m}| - 2|\widehat{n-1}, n+1; \widehat{m-1}, m+1|$$
$$+ |\widehat{n}; \widehat{m-2}, m, m+1| + |\widehat{n}; \widehat{m-1}, m+2|, \tag{2.21.1}$$

$$(\mathrm{tr}\,A)^2|\widehat{n+1}; \widehat{m-1}| = |\widehat{n-1}, n+1, n+2; \widehat{m-1}| + |\widehat{n}, n+3; \widehat{m-1}| - 2|\widehat{n}, n+2; \widehat{m-2}, m|$$
$$+ |\widehat{n+1}; \widehat{m-3}, m-1, m| + |\widehat{n+1}; \widehat{m-2}, m+1|, \tag{2.21.2}$$

It is obvious that (2.5.2) holds, so does (2.9).

(2) If $\mathrm{tr}\,A = 0$, we can consider this as a limit case where $\mathrm{tr}\,A$ tends to zero. Then (2.20) and (2.21) become

$$|\widehat{n-1}, n+1; \widehat{m}| = |\widehat{n}; \widehat{m-1}, m+1|, \tag{2.22}$$

$$|\widehat{n-1}, n+2; \widehat{m}| - 2|\widehat{n-1}, n+1; \widehat{m-1}, m+1| + |\widehat{n}; \widehat{m-2}, m, m+1|$$
$$= -|\widehat{n-2}, n, n+1; \widehat{m}| - |\widehat{n}; \widehat{m-1}, m+2|, \tag{2.23.1}$$

$$|\widehat{n}, n+3; \widehat{m-1}| - 2|\widehat{n}, n+2; \widehat{m-2}, m| + |\widehat{n+1}; \widehat{m-3}, m-1, m|$$
$$= -|\widehat{n-1}, n+1, n+2; \widehat{m-1}| - |\widehat{n+1}; \widehat{m-2}, m+1|. \tag{2.23.2}$$

Using (2.22) and (2.23), (1.8) still solves (1.5).

From (1.6), we get the general solution

$$\phi = e^{-2A^2t-Ax}C, \quad \psi = e^{2A^2t+Ax}D, \tag{2.24}$$

where $C = (c_1, c_2, \ldots, c_{n+m+2})^{\mathrm{T}}$ and $D = (d_1, d_2, \ldots, d_{n+m+2})^{\mathrm{T}}$ are real constant vectors. Thus, we have the following

Theorem 2. *$A = (a_{ij})$ is an $(n+m+2) \times (n+m+2)$ arbitrary real matrix independent of x and t. (1.5) has double Wronskian solution (1.8), where ϕ, ψ are constructed by (2.24). The corresponding solution of (1.1) can be expressed as*

$$q = 2\frac{W^{n+2,m}(\phi;\psi)}{W^{n+1,m+1}(\phi;\psi)}, \quad r = -2\frac{W^{n,m+2}(\phi;\psi)}{W^{n+1,m+1}(\phi;\psi)}. \tag{2.25}$$

3 Soliton solutions and rational solutions

Expanding (2.24) leads to

$$\phi = e^{-2A^2t}e^{-Ax}C = \sum_{s=0}^{\infty}\left[\sum_{l=0}^{[\frac{s}{2}]}\frac{(-1)^{s-l}2^l}{l!(s-2l)!}t^l x^{s-2l}\right]A^sC, \tag{3.1.1}$$

$$\psi = e^{2A^2t}e^{Ax}D = \sum_{s=0}^{\infty}\left[\sum_{l=0}^{[\frac{s}{2}]}\frac{2^l}{l!(s-2l)!}t^l x^{s-2l}\right]A^sD. \tag{3.1.2}$$

If

$$A = \begin{pmatrix} k_1 & & & 0 \\ & k_2 & & \\ & & \ddots & \\ 0 & & & k_{n+m+2} \end{pmatrix}, \quad k_i \neq k_j \ (i \neq j), \tag{3.2}$$

we can obtain soliton solutions of (1.5), where

$$\phi_j = c_j e^{-2k_j^2t-k_jx}, \quad \psi_j = d_j e^{2k_j^2t+k_jx} \quad (j = 1, 2, \ldots, n+m+2). \tag{3.3}$$

If

$$A = \begin{pmatrix} 0 & & & 0 \\ 1 & 0 & & \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{pmatrix}_{(n+m+2)\times(n+m+2)}, \tag{3.4}$$

it is obvious to know that $A^{n+m+2} = 0$. Thus (3.1) can be truncated as

$$\phi = \sum_{s=0}^{n+m+1}\left[\sum_{l=0}^{[\frac{s}{2}]}\frac{(-1)^{s-l}2^l}{l!(s-2l)!}t^l x^{s-2l}\right]A^sC, \tag{3.5.1}$$

$$\psi = \sum_{s=0}^{n+m+1}\left[\sum_{l=0}^{[\frac{s}{2}]}\frac{2^l}{l!(s-2l)!}t^l x^{s-2l}\right]A^sD. \tag{3.5.2}$$

The components of ϕ and ψ are

$$\phi_j = c_j - c_{j-1}x + c_{j-2}\left(-2t + \frac{x^2}{2}\right) + \cdots + c_1 \sum_{l=0}^{[\frac{j-1}{2}]} \frac{(-1)^{j-l-1}2^l}{l!(j-1-2l)!} t^l x^{j-1-2l}, \quad (3.6.1)$$

$$\psi_j = d_j + d_{j-1}x + d_{j-2}\left(2t + \frac{x^2}{2}\right) + \cdots + d_1 \sum_{l=0}^{[\frac{j-1}{2}]} \frac{2^l}{l!(j-1-2l)!} t^l x^{j-1-2l}$$
$$(j = 1, 2, \ldots, n+m+2). \quad (3.6.2)$$

In (3.6), taking $c_1 = d_1 = 1, c_k = d_k = 0$ $(k = 2, 3, \ldots, n+m+2)$, then (3.6) becomes

$$\phi_j = \sum_{l=0}^{[\frac{j-1}{2}]} \frac{(-1)^{j-l-1}2^l}{l!(j-1-2l)!} t^l x^{j-1-2l}, \quad \psi_j = \sum_{l=0}^{[\frac{j-1}{2}]} \frac{2^l}{l!(j-1-2l)!} t^l x^{j-1-2l}. \quad (3.7)$$

Thus, we can calculate several rational solutions of (1.1).

$$q = r = -\frac{1}{x}, \quad (3.8)$$

$$q = \frac{1}{2t + x^2}, \quad r = -2\frac{2t - x^2}{2t + x^2}, \quad (3.9.1)$$

$$q = 2\frac{2t + x^2}{2t - x^2}, \quad r = \frac{1}{2t - x^2}, \quad (3.9.2)$$

$$q = -\frac{3}{2}\ \frac{1}{6tx + x^3}, \quad r = -2\frac{12t^2 + x^4}{6tx + x^3}, \quad (3.10.1)$$

$$q = 2\frac{6tx + x^3}{12t^2 + x^4}, \quad r = -2\frac{6tx - x^3}{12t^2 + x^4}, \quad (3.10.2)$$

$$q = 2\frac{12t^2 + x^4}{6tx - x^3}, \quad r = \frac{3}{2}\ \frac{1}{6tx - x^3}. \quad (3.10.3)$$

From (3.7), substituting $-it$ for t, it is easy to see that

$$\psi_j^* = (-1)^{j-1}\phi_j, \quad (j = 1, 2, \ldots, n+m+2), \quad (3.11)$$

so we deduce that

$$f^* = W^{(n+1,n+1)}(\phi^*; \psi^*) = (-1)^{n+1}W^{(n+1,n+1)}(\psi; \phi) = f, \quad (3.12.1)$$

$$g^* = 2W^{(n+2,n)}(\phi^*; \psi^*) = 2(-1)^{n+1}W^{(n,n+2)}(\psi; \phi) = -h, \quad (3.12.2)$$

or $r = -q^*$. Thus, we can get rational solutions to (1.2).

4 Matveev solutions

Let A be a Jordan matrix

$$A = \begin{pmatrix} J(k_1) & & & 0 \\ & J(k_2) & & \\ & & \ddots & \\ 0 & & & J(k_s) \end{pmatrix}_{(n+m+2)\times(n+m+2)}. \quad (4.1)$$

Without loss of generality, we observe the following Jordan block (dropping the subscript of k)

$$J(k)=\begin{pmatrix} k & & & 0\\ 1 & k & & \\ & \ddots & \ddots & \\ 0 & & 1 & k\end{pmatrix}_{l_i\times l_i}=kI_{l_i}+E_{l_i},\qquad E_{l_i}=\begin{pmatrix} 0 & & & 0\\ 1 & 0 & & \\ & \ddots & \ddots & \\ 0 & & 1 & 0\end{pmatrix}_{l_i\times l_i}, \tag{4.2}$$

where I_{l_i} denotes an $l_i\times l_i$ unite matrix. We have

$$J(k)^s=(kI_{l_i}+E_{l_i})^s=\left(I_{l_i}+E_{l_i}\partial_k+\frac{1}{2!}E_{l_i}^2\partial_k^2+\cdots+\frac{1}{j!}E_{l_i}^j\partial_k^j+\cdots+\frac{1}{s!}E_{l_i}^s\partial_k^s\right)k^s, \tag{4.3.1}$$

i.e.,

$$J(k)^s=T_kk^s,\qquad T_k=\begin{pmatrix} 1 & & & & & 0\\ \partial_k & 1 & & & & \\ \frac{1}{2}\partial_k^2 & \partial_k & 1 & & & \\ \frac{1}{6}\partial_k^3 & \frac{1}{2}\partial_k^2 & \partial_k & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \\ \frac{1}{(l_i-1)!}\partial_k^{l_i-1} & \cdots & \frac{1}{6}\partial_k^3 & \frac{1}{2}\partial_k^2 & \partial_k & 1\end{pmatrix}. \tag{4.3.2}$$

In [7], the similar result about the KdV equation was derived. Substituting (4.2) into (3.1), we get

$$\phi(k)=T(k)e^{-2k^2t-kx}C,\qquad \psi(k)=T(k)e^{2k^2t+kx}D. \tag{4.4}$$

The components of $\phi(k)$ and $\psi(k)$ are

$$\phi_j(k)=\left(c_1\frac{1}{(j-1)!}\partial_k^{j-1}+\cdots+c_{j-1}\partial_k+c_j\right)e^{-2k^2t-kx}, \tag{4.5.1}$$

$$\psi_j(k)=\left(d_1\frac{1}{(j-1)!}\partial_k^{j-1}+\cdots+d_{j-1}\partial_k+d_j\right)e^{2k^2t+kx}\ (j=1,2,\ldots,l_i). \tag{4.5.2}$$

Specially, taking $c_1=d_1=1$ and $c_j=0$, $d_j=0$ $(j=2,3\ldots,l_i)$, (4.5) becomes

$$\phi_j(k)=\frac{1}{(j-1)!}\partial_k^{j-1}e^{-2k^2t-kx},\quad \psi_j(k)=\frac{1}{(j-1)!}\partial_k^{j-1}e^{2k^2t+kx}. \tag{4.6}$$

Thus Matveev solutions of (1.1) are obtained, where

$$\phi=(\phi_1(k_1),\ldots,\phi_{l_1}(k_1);\phi_1(k_2)\ldots,\phi_{l_2}(k_2);\ldots,\phi_1(k_s),\ldots,\phi_{l_s}(k_s))^{\mathrm{T}}, \tag{4.7.1}$$

$$\psi=(\psi_1(k_1),\ldots,\psi_{l_1}(k_1);\psi_1(k_2)\ldots,\psi_{l_2}(k_2);\ldots,\psi_1(k_s),\ldots,\psi_{l_s}(k_s))^{\mathrm{T}} \tag{4.7.2}$$

$$(l_1+l_2+\cdots+l_s=n+m+2).$$

In (4.7), taking

$$\phi=(\phi_1(k),\phi_2(k))^{\mathrm{T}},\quad \psi=(\psi_1(k),\psi_2(k))^{\mathrm{T}}, \tag{4.8}$$

where $\phi_j(k)$ and $\psi_j(k)$ are generated from (4.6), we can compute the Matveev solution of (1.1),

$$q=-\frac{1}{4kt+x}e^{-2(2k^2t+kx)},\quad r=-\frac{1}{4kt+x}e^{2(2k^2t+kx)}. \tag{4.9}$$

Similarly, choosing

$$\phi = (\phi_1(k), \phi_2(k), \phi_3(k))^{\mathrm{T}}, \quad \psi = (\psi_1(k), \psi_2(k), \psi_3(k))^{\mathrm{T}} \tag{4.10}$$

and $(n, m) = (1, 0)$, we have

$$q = \frac{1}{2t + (4kt + x)^2}\ e^{-2(2k^2t+kx)}, \qquad r = \frac{-2t + (4kt + x)^2}{2t + (4kt + x)^2}\ e^{2(2k^2t+kx)}. \tag{4.11}$$

When $(n, m) = (0, 1)$, we get

$$q = \frac{2t + (4kt + x)^2}{2t - (4kt + x)^2}\ e^{-2(2k^2t+kx)}, \qquad r = \frac{1}{2t - (4kt + x)^2}\ e^{2(2k^2t+kx)}. \tag{4.12}$$

Assume that

$$\phi = (\phi_1(k_1), \phi_2(k_1), \phi_1(k_2))^{\mathrm{T}}, \qquad \psi = (\psi_1(k_1), \psi_2(k_1), \psi_1(k_2))^{\mathrm{T}}, \tag{4.13}$$

setting $(n, m) = (1, 0)$ gives

$$q = -2\frac{(k_1 - k_2)^2}{[1 + 2(k_2 - k_1)(4k_1t + x)]e^{2\xi_1} - e^{2\xi_2}}, \tag{4.14.1}$$

$$r = -2\frac{[1 + 2(k_2 - k_1)(4k_1t + x)]e^{2\xi_2} - e^{2\xi_1}}{1 + 2(k_2 - k_1)(4k_1t + x) - e^{2\xi_2-2\xi_1}}. \tag{4.14.2}$$

Similarly, taking $(n, m) = (0, 1)$ yields

$$q = 2\frac{[1 + 2(k_2 - k_1)(4k_1t + x)]e^{-2\xi_2} - e^{-2\xi_1}}{1 + 2(k_2 - k_1)(4k_1t + x) - e^{2\xi_1-2\xi_2}}, \tag{4.15.1}$$

$$r = -2\frac{(k_1 - k_2)^2}{[1 + 2(k_2 - k_1)(4k_1t + x)]e^{-2\xi_1} - e^{-2\xi_2}}, \tag{4.15.2}$$

where $\xi_i = 2k_i^2t + k_ix (i = 1, 2)$.

From (4.9), (2.14), (4.11), (4.12) and (2.16), (2.17), the Matveev solutions are the limit solutions when k_2 tends to $k_1 = k$, k_2, k_3 tends to $k_1 = k$. General conclusions are also correct.

5 Complexitons

We would like to consider that A is a real Jordan matrix as follows:

$$A = \begin{pmatrix} J_1 & & & 0 \\ & J_2 & & \\ & & \ddots & \\ 0 & & & J_h \end{pmatrix}, \tag{5.1.1}$$

where

$$J_i = \begin{pmatrix} A_i & & & 0 \\ I_2 & A_i & & \\ & \ddots & \ddots & \\ 0 & & I_2 & A_i \end{pmatrix}, \quad A_i = \begin{pmatrix} \alpha_i & -\beta_i \\ \beta_i & \alpha_i \end{pmatrix}, \tag{5.1.2}$$

and α_i, β_i $(i = 1, 2, \ldots, h)$ are real constants. Then, from (3.1), complexitons can be obtained.

In order to prove that, we first observe the simplest case when

$$A = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} = \alpha I_2 + \beta\sigma_2, \qquad \sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{5.2}$$

Substituting (5.2) into (3.1.1) gives rise to

$$\phi = e^{-[2(\alpha^2-\beta^2)t+\alpha x]I_2} \cdot e^{-(4\alpha\beta t+\beta x)\sigma_2} C. \tag{5.3}$$

Expanding the above ϕ and taking advantage of $\sigma_2^2 = -I_2$, we have

$$\phi = e^{-2(\alpha^2-\beta^2)t-\alpha x}[\cos(4\alpha\beta t + \beta x)I_2 - \sin(4\alpha\beta t + \beta x)\sigma_2]C. \tag{5.4.1}$$

Similarly,

$$\psi = e^{2(\alpha^2-\beta^2)t+\alpha x}[\cos(4\alpha\beta t + \beta x)I_2 + \sin(4\alpha\beta t + \beta x)\sigma_2]D. \tag{5.4.2}$$

Further, we consider the matrix A as a Jordan block J_i

$$A = J_i = A' + E', \tag{5.5.1}$$

$$A' = I_{l_i} \otimes A_i = \begin{pmatrix} A_i & & & 0 \\ & A_i & & \\ & & \ddots & \\ 0 & & & A_i \end{pmatrix}, \tag{5.5.2}$$

$$E' = E_{l_i} \otimes I_i = \begin{pmatrix} 0 & & & 0 \\ I_2 & 0 & & \\ & \ddots & \ddots & \\ 0 & & I_2 & 0 \end{pmatrix}_{2l_i\times 2l_i}, \tag{5.5.3}$$

where the symbol $\otimes$ denotes tensor product of matrices. Noting that $A'E' = E'A'$, we get

$$A^s = (A' + E')^s = \left(I_{2l_i} + E'\partial_{\alpha_i} + \cdots + \frac{1}{j!}E'^j\partial_{\alpha_i}^j + \cdots + \frac{1}{s!}E'^s\partial_{\alpha_i}^s\right)A'^s. \tag{5.6}$$

Employing the following formula

$$\partial_{\alpha_i}A_i^p = \partial_{\alpha_i}(\alpha_i I_2 + \beta_i\sigma_2)^p = p(\alpha_i I_2 + \beta_i\sigma_2)^{p-1} \quad (p = 1, 2, 3, \ldots), \tag{5.7}$$

(5.6) can be written as

$$A^s = \begin{pmatrix} I_2 & & & & 0 \\ I_2\partial_{\alpha_i} & I_2 & & & \\ \frac{1}{2}I_2\partial_{\alpha_i}^2 & I_2\partial_{\alpha_i} & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & \\ \frac{1}{(l_i-1)!}I_2\partial_{\alpha_i}^{l_i-1} & \ldots & \frac{1}{2}I_2\partial_{\alpha_i}^2 & I_2\partial_{\alpha_i} & I_2 \end{pmatrix} A'^s = T(\partial_{\alpha_i})A'^s. \tag{5.8}$$

Substituting (5.8) into (3.1) yields

$$\phi_j(\alpha_i) = T(\partial_{\alpha_i})e^{-2A'^2t-A'x}C = T(\partial_{\alpha_i})(I_{l_i} \otimes e^{-2A_i^2t-A_ix})C, \tag{5.9.1}$$

$$\psi_j(\alpha_i) = T(\partial_{\alpha_i})e^{2A'^2t+A'x}D = T(\partial_{\alpha_i})(I_{l_i} \otimes e^{2A_i^2t+A_ix})D, \tag{5.9.2}$$

or

$$\phi_j(\alpha_i) = \frac{1}{(j-1)!}\partial_{\alpha_i}^{j-1}e^{-2A_i^2t-A_ix}c_1 + \cdots + \partial_{\alpha_i}e^{-2A_i^2t-A_ix}c_{j-1} + e^{-2A_i^2t-A_ix}c_j, \tag{5.10.1}$$

$$\psi_j(\alpha_i) = \frac{1}{(j-1)!}\partial_{\alpha_i}^{j-1}e^{2A_i^2t+A_ix}d_1 + \cdots + \partial_{\alpha_i}e^{2A_i^2t+A_ix}d_{j-1} + e^{2A_i^2t+A_ix}d_j, \tag{5.10.2}$$

where

$$\phi_j(\alpha_i) = (\phi_{j1}(\alpha_i), \phi_{j2}(\alpha_i))^{\mathrm{T}}, \quad \phi(\alpha_i) = (\phi_1(\alpha_i)^{\mathrm{T}}, \phi_2(\alpha_i)^{\mathrm{T}}, \ldots, \phi_{l_i}(\alpha_i)^{\mathrm{T}})^{\mathrm{T}}, \tag{5.11.1}$$

$$c_j = (c_{j1}, c_{j2})^{\mathrm{T}}, \quad C = (c_1^{\mathrm{T}}, c_2^{\mathrm{T}}, \ldots, c_{l_i}^{\mathrm{T}})^{\mathrm{T}}; \tag{5.11.2}$$

$$\psi_j(\alpha_i) = (\psi_{j1}(\alpha_i), \psi_{j2}(\alpha_i))^{\mathrm{T}}, \quad \psi(\alpha_i) = (\psi_1(\alpha_i)^{\mathrm{T}}, \psi_2(\alpha_i)^{\mathrm{T}}, \ldots, \psi_{l_i}(\alpha_i)^{\mathrm{T}})^{\mathrm{T}}, \tag{5.11.3}$$

$$d_j = (d_{j1}, d_{j2})^{\mathrm{T}}, \quad D = (d_1^{\mathrm{T}}, d_2^{\mathrm{T}}, \ldots, d_{l_i}^{\mathrm{T}})^{\mathrm{T}}. \tag{5.11.4}$$

According to (5.4), (5.10) can be expressed as the following explicit form:

$$\phi_j(\alpha_i) = \begin{pmatrix} \phi_{j1}(\alpha_i) \\ \phi_{j2}(\alpha_i) \end{pmatrix} = \sum_{s=1}^{j} \frac{1}{(j-s)!}\partial_{\alpha_i}^{j-s}\Bigg[e^{-2(\alpha_i^2-\beta_i^2)t-\alpha_ix} \cdot \begin{pmatrix} c_{s1}\cos(4\alpha_i\beta_it+\beta_ix) + c_{s2}\sin(4\alpha_i\beta_it+\beta_ix) \\ -c_{s1}\sin(4\alpha_i\beta_it+\beta_ix) + c_{s2}\cos(4\alpha_i\beta_it+\beta_ix) \end{pmatrix}\Bigg], \tag{5.12.1}$$

$$\psi_j(\alpha_i) = \begin{pmatrix} \psi_{j1}(\alpha_i) \\ \psi_{j2}(\alpha_i) \end{pmatrix} = \sum_{s=1}^{j} \frac{1}{(j-s)!}\partial_{\alpha_i}^{j-s}\Bigg[e^{2(\alpha_i^2-\beta_i^2)t+\alpha_ix} \cdot \begin{pmatrix} d_{s1}\cos(4\alpha_i\beta_it+\beta_ix) - d_{s2}\sin(4\alpha_i\beta_it+\beta_ix) \\ d_{s1}\sin(4\alpha_i\beta_it+\beta_ix) + d_{s2}\cos(4\alpha_i\beta_it+\beta_ix) \end{pmatrix}\Bigg]. \tag{5.12.2}$$

Thus, double Wronskian (1.8) is the complexiton of (1.5), where

$$\phi = (\phi_1(\alpha_1)^{\mathrm{T}}, \ldots, \phi_{l_1}(\alpha_1)^{\mathrm{T}}; \phi_1(\alpha_2)^{\mathrm{T}}, \ldots; \phi_{l_2}(\alpha_2)^{\mathrm{T}}; \ldots; \phi_1(\alpha_h)^{\mathrm{T}}, \ldots, \phi_{l_h}(\alpha_h)^{\mathrm{T}})^{\mathrm{T}}, \tag{5.13.1}$$

$$\psi = (\psi_1(\alpha_1)^{\mathrm{T}}, \ldots, \psi_{l_1}(\alpha_1)^{\mathrm{T}}; \psi_1(\alpha_2)^{\mathrm{T}}, \ldots; \psi_{l_2}(\alpha_2)^{\mathrm{T}}; \ldots; \psi_1(\alpha_h)^{\mathrm{T}}, \ldots, \psi_{l_h}(\alpha_h)^{\mathrm{T}})^{\mathrm{T}} \quad (l_1 + l_2 + \cdots + l_h = n + m + 2). \tag{5.13.2}$$

On the other hand, for $\partial_{\alpha_i}A_i^n = -\sigma_2\partial_{\beta_i}A_i^n$, the partial derivative with respect to α_i can be replaced by the partial derivative with respect to β_i in (5.10) and (5.12). In [8], the similar formula of Complexitons of KdV equation was presented.

For example, taking $n = m = 0$, $\xi = 2(\alpha^2 - \beta^2)t + \alpha x$, $\eta = 4\alpha\beta t + \beta x$ (dropping the subscript) and

$$\phi = (e^{-\xi}\cos\eta, -e^{-\xi}\sin\eta)^{\mathrm{T}}, \quad \psi = (e^{\xi}\cos\eta, e^{\xi}\sin\eta)^{\mathrm{T}}, \tag{5.14}$$

then,

$$q = -\beta \frac{e^{-2(\alpha^2-\beta^2)t-\alpha x}}{\sin 2(4\alpha\beta t+\beta x)}, \quad r = \beta \frac{e^{2(\alpha^2-\beta^2)t+\alpha x}}{\sin 2(4\alpha\beta t+\beta x)}. \tag{5.15}$$

When $\beta = \alpha$, (5.15) is the periodic solution about t, and

$$q = -\alpha \frac{e^{-\alpha x}}{\sin 2\alpha(4\alpha t+x)}, \quad r = \alpha \frac{e^{\alpha x}}{\sin 2\alpha(4\alpha t+x)}. \tag{5.16}$$

6 Interaction solutions

In order to obtain more exact solutions to (1.1) in double Wronskian form, we assume that A is composed of (3.4) (A_r), (4.1) (A_m) and (5.1) (A_c). i.e.

$$A = \begin{pmatrix} A_r & & 0 \\ & A_m & \\ 0 & & A_c \end{pmatrix}. \tag{6.1}$$

Then, according to (1.6), we have

$$\phi = (\phi_r^{\mathrm{T}}, \phi_m^{\mathrm{T}}, \phi_c^{\mathrm{T}})^{\mathrm{T}}, \quad \psi = (\psi_r^{\mathrm{T}}, \psi_m^{\mathrm{T}}, \psi_c^{\mathrm{T}})^{\mathrm{T}}, \tag{6.2}$$

where

$$\phi_{p,x} = -A_p\phi_p\ , \quad \psi_{p,x} = A_p\psi_p, \tag{6.3.1}$$

$$\phi_{p,t} = -2\phi_{p,xx}, \quad \psi_{p,t} = 2\psi_{p,xx} \quad (p = r, m, c). \tag{6.3.2}$$

It is obvious that (1.8) constructed by (6.2) still solves (1.5), and this type of solutions is called interaction solutions.

Taking

$$\phi = (\phi_{1r}, \phi_{2r}, \phi_m(k))^{\mathrm{T}}, \quad \psi = (\psi_{1r}, \psi_{2r}, \psi_m(k))^{\mathrm{T}}, \tag{6.4}$$

and $(n, m) = (1, 0)$, it is easy to get

$$q = \frac{2k^2}{e^{2(2k^2t+kx)} - 2kx - 1}, \tag{6.5.1}$$

$$r = \frac{2(2kx-1)e^{2(2k^2t+kx)} + 2}{e^{2(2k^2t+kx)} - 2kx - 1}. \tag{6.5.2}$$

When $(n, m) = (0, 1)$, we can obtain

$$q = \frac{2(2kx+1)e^{-2(2k^2t+kx)} - 2}{e^{-2(2k^2t+kx)} + 2kx - 1}, \tag{6.6.1}$$

$$r = -\frac{2k^2}{e^{-2(2k^2t+kx)} + 2kx - 1}. \tag{6.6.2}$$

Letting

$$\phi = (\phi_{1r}, \phi_{2r}, \phi_{1m}(k), \phi_{2m}(k))^{\mathrm{T}}, \quad \psi = (\psi_{1r}, \psi_{2r}, \psi_{1m}(k), \psi_{2m}(k))^{\mathrm{T}}$$

and $(n,m)=(2,0)$, $(1,1)$, we have

$$q=\frac{k^3}{(4k^2t+kx-1)e^{2(2k^2t+kx)}+kx+1}, \tag{6.7.1}$$

$$r=\frac{e^{4(2k^2t+kx)}+[4k^2x(4kt+x)-2]e^{2(2k^2t+kx)}+1}{k[(4k^2t+kx-1)e^{2(2k^2t+kx)}+kx+1]}; \tag{6.7.2}$$

$$q=4k\frac{(kx+1)e^{-2(2k^2t+kx)}+4k^2t+kx-1}{e^{2(2k^2t+kx)}+e^{-2(2k^2t+kx)}+4k^2x(4kt+x)-2}, \tag{6.8.1}$$

$$r=4k\frac{(kx-1)e^{2(2k^2t+kx)}+4k^2t+kx+1}{e^{2(2k^2t+kx)}+e^{-2(2k^2t+kx)}+4k^2x(4kt+x)-2}, \tag{6.8.2}$$

respectively.

When $(n,m)=(0,2)$, we compute

$$q=\frac{e^{-4(2k^2t+kx)}+[4k^2x(4kt+x)-2]e^{-2(2k^2t+kx)}+1}{k[(4k^2t+kx+1)e^{-2(2k^2t+kx)}+kx-1]}, \tag{6.9.1}$$

$$r=-\frac{k^3}{(4k^2t+kx+1)e^{-2(2k^2t+kx)}+kx-1}. \tag{6.9.2}$$

Obviously, (6.5), (6.7), (6.8) and (6.9) are interaction solutions between rational and Matveev solutions.

Choosing

$$\phi=(\phi_{1r},\phi_{2r};\phi_{1c},\phi_{2c})^{\mathrm{T}},\quad \phi_{1c}=e^{-\xi}\cos\eta,\quad \phi_{2c}=-e^{-\xi}\sin\eta, \tag{6.10.1}$$

$$\psi=(\psi_{1r},\psi_{2r};\psi_{1c},\psi_{2c})^{\mathrm{T}},\quad \psi_{1c}=e^{\xi}\cos\eta,\quad \psi_{2c}=e^{\xi}\sin\eta, \tag{6.10.2}$$

and $(n,m)=(2,0)$, we have

$$q=\frac{2\beta(\alpha^2+\beta^2)^2e^{-2\xi}}{-2\beta[(\alpha^2+\beta^2)x+\alpha]e^{-2\xi}-(\alpha^2-\beta^2)\sin 2\eta+2\alpha\beta\cos 2\eta}, \tag{6.11.1}$$

$$r=\frac{4\beta(\cos 2\eta-\cosh 2\xi)+4(\alpha^2+\beta^2)x\sin 2\eta}{2\beta[(\alpha^2+\beta^2)x+\alpha]e^{-2\xi}+(\alpha^2-\beta^2)\sin 2\eta-2\alpha\beta\cos 2\eta}. \tag{6.11.2}$$

where $\xi=2(\alpha^2-\beta^2)t+\alpha x,\ \eta=4\alpha\beta t+\beta x$.

While $(n,m)=(1,1)$, we derive that

$$q=\frac{-2\beta[(\alpha^2+\beta^2)x+\alpha]e^{-2\xi}-2(\alpha^2-\beta^2)\sin 2\eta+2\alpha\beta\cos 2\eta}{\beta(\cos 2\eta-\cosh 2\xi)+(\alpha^2+\beta^2)x\sin 2\eta}, \tag{6.12.1}$$

$$r=-\frac{2\beta[(\alpha^2+\beta^2)x-\alpha]e^{2\xi}+2(\alpha^2-\beta^2)\sin 2\eta+2\alpha\beta\cos 2\eta}{\beta(\cos 2\eta-\cosh 2\xi)+(\alpha^2+\beta^2)x\sin 2\eta}. \tag{6.12.2}$$

Similarly, when $(n,m)=(0,2)$, the solution of (1.1) is

$$q=\frac{4\beta(\cos 2\eta-\cosh 2\xi)+4(\alpha^2+\beta^2)x\sin 2\eta}{2\beta[(\alpha^2+\beta^2)x-\alpha]e^{2\xi}-(\alpha^2-\beta^2)\sin 2\eta-2\alpha\beta\cos 2\eta}, \tag{6.13.1}$$

$$r=-\frac{2\beta(\alpha^2+\beta^2)e^{2\xi}}{2\beta[(\alpha^2+\beta^2)x-\alpha]e^{2\xi}-(\alpha^2-\beta^2)\sin 2\eta-2\alpha\beta\cos 2\eta}. \tag{6.13.2}$$

(6.11), (6.12) and (6.13) are interaction solutions of (1.1) between rational solutions and complexitons.

7 Conclusions

(1) Taking the second-order AKNS equation as an example, we provide a matrix equation satisfied by double Wronskian entries. By searching for the general solutions of ϕ and ψ and expanding them as the series of A, we can obtain rational solutions, Matveev solutions, complexitons and interaction solutions of the AKNS equation. Moreover, rational solutions of the nonlinear Schrödinger equation are constructed by reducing. According to our knowledge, these solutions are novel and have not been reported in literature.

(2) The method for constructing double Wronskian entries can be applied to other Lax integrable equation, such as higher-order AKNS equation, mKdV equation, Toda lattice and so on. This method is simple and can derive general formula of column vectors ϕ and ψ.

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