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Combined perturbation bounds: II. Polar decompositions

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Abstract In this paper, we study the perturbation bounds for the polar decomposition $A = QH$ where Q is unitary and H is Hermitian. The optimal (asymptotic) bounds obtained in previous works for the unitary factor, the Hermitian factor and singular values of A are $\sigma_r^2\|\Delta Q\|_F^2 \leqslant \|\Delta A\|_F^2$, $\frac{1}{2}\|\Delta H\|_F^2 \leqslant \|\Delta A\|_F^2$ and $\|\Delta\Sigma\|_F^2 \leqslant \|\Delta A\|_F^2$, respectively, where $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r, 0, \ldots, 0)$ is the singular value matrix of A and σ_r denotes the smallest nonzero singular value. Here we present some new combined (asymptotic) perturbation bounds $\sigma_r^2\|\Delta Q\|_F^2+\frac{1}{2}\|\Delta H\|_F^2 \leqslant \|\Delta A\|_F^2$ and $\sigma_r^2\|\Delta Q\|_F^2+\|\Delta\Sigma\|_F^2 \leqslant \|\Delta A\|_F^2$ which are optimal for each factor. Some corresponding absolute perturbation bounds are also given.

Keywords: polar decomposition, perturbation, singular value

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1 Introduction

Let $\mathcal{C}^{m\times n}$ be the set of $m\times n$ complex matrices and $\mathcal{C}_r^{m\times n}$ be the set of $m\times n$ complex matrices with rank r. Here we always assume that $m \geqslant n$. We denote by $\|\cdot\|_2$, $\|\cdot\|_F$ and $\|\cdot\|$ the spectral norm, the Frobenius norm and the general unitarily invariant norm, respectively. Let

$$A = U\begin{pmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{pmatrix}V^*$$

be the singular value decomposition (SVD) of A and

$$H = V_1\Sigma_1V_1^*, \qquad Q = U_1V_1^*,$$

where $U = (U_1, U_2) \in \mathcal{C}^{m\times m}$ and $V = (V_1, V_2) \in \mathcal{C}^{n\times n}$ are unitary, $U_1 \in \mathcal{C}_r^{m\times r}$, $V_1 \in \mathcal{C}_r^{n\times r}$, $\Sigma_1 = \mathrm{diag}(\sigma_1, \ldots, \sigma_r)$, $\sigma_i, i = 1, 2, \ldots, r$, define the singular values of A with $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_r > 0$ and the superscript $*$ denotes the conjugate transpose. The polar decomposition of the matrix A is defined by

$$A = QH, \tag{1.1}$$

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where Q is called the unitary polar factor of A, and H is called the Hermitian polar factor of A, which is denoted by $|A|$ sometimes. For $r < n$, it was proved that the polar decomposition (1.1) satisfying $R(Q^*) = R(H)$ is unique (e.g. see [1]). Some applications in the case $r = m = n$ can be found in [2]. In this paper we always assume that the polar decomposition is unique without further illustration.

Perturbation bounds for polar factors, Q and H, have been studied by many authors, e.g., see Sun and Chen[1], Barrlund[3], Bhatia[4], Bhatia and Mukherjea[5], Chatelin and Gratton[6], Li[7,8], Li and Sun[9, 10], Chen, Li and Sun[11] and Mathias[12]. Let $\widetilde{A}$ be the perturbed matrix and $\widetilde{A} = \widetilde{Q}\widetilde{H}$. Perturbation bounds for these two polar factors in these previous works can be given in an asymptotic sense by

$$\frac{1}{2}\|H - \widetilde{H}\|_F^2 \leqslant \|A - \widetilde{A}\|_F^2 \tag{1.2}$$

and

$$\sigma_r^2\|Q - \widetilde{Q}\|_F^2 \leqslant \|A - \widetilde{A}\|_F^2\,, \tag{1.3}$$

both of which are optimal in general. However, it is easy to find some examples to show that one of these two bounds is not sharp since the perturbation can arise only from one of these two factors. In this paper, we focus on the perturbation bounds in a combined form of unitary factors, Hermitian factors or singular values. We present several new perturbation bounds, in the asymptotic sense, which lead to

$$\sigma_r^2\|Q - \widetilde{Q}\|_F^2 + \frac{1}{2}\|H - \widetilde{H}\|_F^2 \leqslant \|A - \widetilde{A}\|_F^2 \tag{1.4}$$

and

$$\sigma_r^2\|Q - \widetilde{Q}\|_F^2 + \|\Sigma - \widetilde{\Sigma}\|_F^2 \leqslant \|A - \widetilde{A}\|_F^2\,. \tag{1.5}$$

By noting the Mirsky theorem (e.g. see [13] or [14])

$$\|\Sigma - \widetilde{\Sigma}\|_F^2 \leqslant \|A - \widetilde{A}\|_F^2\,, \tag{1.6}$$

the combined perturbation bounds in (1.4) and (1.5) are optimal for each factor.

2 Preliminaries

Let $A,\ \widetilde{A} \in \mathcal{C}_r^{m\times n}$, $m \geqslant n$ with the singular value decompositions

$$A = U\Sigma V^* \quad \text{and} \quad \widetilde{A} = \widetilde{U}\widetilde{\Sigma}\widetilde{V}^*, \tag{2.1}$$

where

$$\Sigma = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{C}_r^{m\times n} \quad \text{and} \quad \widetilde{\Sigma} = \begin{pmatrix} \widetilde{\Sigma}_1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{C}_r^{m\times n},$$

$\Sigma_1 = \mathrm{diag}(\sigma_1, \ldots, \sigma_r)$, $\widetilde{\Sigma}_1 = \mathrm{diag}(\widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_r)$, $\sigma_1 \geqslant \cdots \geqslant \sigma_r > 0$ and $\widetilde{\sigma}_1 \geqslant \cdots \geqslant \widetilde{\sigma}_r > 0$.

Let I_p be the $p \times p$ identity matrix and

$$I_{m,n}^{(p)} \equiv \begin{pmatrix} I_p & 0 \\ 0 & 0 \end{pmatrix}.$$

For simplicity we replace $I_{m,n}^{(p)}$ by $I^{(p)}$. Let $S = \widetilde{U}^* U$ and $T = \widetilde{V}^* V$ have the block form

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \in \mathcal{C}^{m\times m} \quad \text{and} \quad T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \in \mathcal{C}^{n\times n},$$

where both S_{11} and T_{11} are $r \times r$ matrices. Then S and T are the unitary matrices. Let

$$M = 2I - S_{11}^* T_{11} - T_{11}^* S_{11}, \qquad \widetilde{M} = 2I - T_{11} S_{11}^* - S_{11} T_{11}^*$$

and m_{ij} and $\widetilde{m}_{ij}$ denote the (i,j) entry of M and $\widetilde{M}$, respectively. We can see that M and $\widetilde{M}$ are Hermitian positive semidefinite.

Some basic properties are given below and the proofs can be found in [9,10].

$$\|Q - \widetilde{Q}\|_F^2 = \|SI^{(r)} - I^{(r)}T\|_F^2 = \mathrm{tr}(M) = \mathrm{tr}(\widetilde{M}), \quad \|A - \widetilde{A}\| = \|S\Sigma - \widetilde{\Sigma}T\|, \tag{2.2}$$

where $\mathrm{tr}(A)$ denotes the trace of A. Let

$$\Gamma = \Sigma - \sigma I^{(r)} \quad \text{and} \quad \widetilde{\Gamma} = \widetilde{\Sigma} - \sigma I^{(r)}.$$

It follows that

$$\|S\Sigma - \widetilde{\Sigma}T\|_F^2 = \sigma^2 \|Q - \widetilde{Q}\|_F^2 + \|S\Gamma - \widetilde{\Gamma}T\|_F^2 + 2\sigma \mathcal{R}e\, \mathrm{tr}[(SI^{(r)} - I^{(r)}T)(S\Gamma - \widetilde{\Gamma}T)^*], \tag{2.3}$$

where $\mathcal{R}e$ denotes the real part of a complex number.

The equation (2.3) has been studied by several authors. Li[7] proved that $\mathcal{R}e\, \mathrm{tr}[(SI^{(r)} - I^{(r)}T)(S\Gamma - \widetilde{\Gamma}T)^*]$ is non-negative when $\sigma \leqslant \min\{\sigma_r, \widetilde{\sigma}_r\}$, which results in the perturbation bound

$$\|Q - \widetilde{Q}\|_F \leqslant \frac{1}{\min\{\sigma_r, \widetilde{\sigma}_r\}} \|A - \widetilde{A}\|_F .$$

It was studied more precisely in our recent work[9], where we proved that for $A, \widetilde{A} \in \mathcal{C}_r^{m\times n}$,

$$\begin{aligned} \|A - \widetilde{A}\|_F^2 \geqslant{}& \sigma^2 \|Q - \widetilde{Q}\|_F^2 + \|S\Gamma - \widetilde{\Gamma}T\|_F^2 + \sigma(\sigma_{r-1} + \widetilde{\sigma}_{r-1} - 2\sigma)\mathrm{tr}(M) \\ & -\sigma(\sigma_{r-1} - \sigma_r) m_{rr} - \sigma(\widetilde{\sigma}_{r-1} - \widetilde{\sigma}_r)\widetilde{m}_{rr} . \end{aligned} \tag{2.4}$$

Lemma 2.1[4,Theorem VII.5.7]**.** *Let B and $\widetilde{B}$ be any two matrices. Then*

$$\||B| - |\widetilde{B}|\|_F^2 + \||B^*| - |\widetilde{B}^*|\|_F^2 \leqslant 2\|B - \widetilde{B}\|_F^2, \tag{2.5}$$

where by $|\ast|$ we denote the Hermitian positive semidefinite factor of $\ast$.

Clearly, $|A| = H$ and $|\widetilde{A}| = \widetilde{H}$. Let $H' = U\Sigma^* U^*$ and $\widetilde{H}' = \widetilde{U}\widetilde{\Sigma}^* \widetilde{U}^*$. Then H' and $\widetilde{H}'$ are the Hermitian factors of A^* and $\widetilde{A}^*$, respectively, i.e. $|A^*| = H'$ and $|\widetilde{A}^*| = \widetilde{H}'$.

Lemma 2.2. *If $\sigma = \min\{\sigma_r, \widetilde{\sigma}_r\}$,*

$$\|S\Gamma - \widetilde{\Gamma}T\|_F^2 \geqslant \frac{1}{2}\|H - \widetilde{H}\|_F^2 + \frac{1}{2}\|H' - \widetilde{H}'\|_F^2 \tag{2.6}$$

and

$$\|S\Gamma - \widetilde{\Gamma}T\|_F^2 \geqslant \frac{1}{2}\|H - \widetilde{H}\|_F^2 + \frac{1}{2}\|\Sigma - \widetilde{\Sigma}\|_F^2. \tag{2.7}$$

Proof. For simplicity, we assume that $\sigma_r \leqslant \widetilde{\sigma}_r$. Let $\sigma = \sigma_r$. We obtain

$$\|S\Gamma - \widetilde{\Gamma}T\|_F = \|U\Gamma V^* - \widetilde{U}\widetilde{\Gamma}\widetilde{V}^*\|_F = \|B - \widetilde{B}\|_F, \tag{2.8}$$

where $B = U\Gamma V^*$ and $\widetilde{B} = \widetilde{U}\widetilde{\Gamma}\widetilde{V}^*$. Since $\sigma - \sigma_r \leqslant 0$ and $\sigma - \widetilde{\sigma}_r \leqslant 0$, $B = U\Gamma V^*$ and $\widetilde{B} = \widetilde{U}\widetilde{\Gamma}\widetilde{V}^*$ are the SVDs of B and $\widetilde{B}$ and

$$\begin{aligned}\||B| - |\widetilde{B}|\|_F &= \|V\Gamma V^* - \widetilde{V}\widetilde{\Gamma}\widetilde{V}^*\|_F = \|\widetilde{V}^*V\Gamma - \widetilde{\Gamma}\widetilde{V}^*V\|_F \\ &= \|\widetilde{V}^*V\Sigma - \widetilde{\Sigma}\widetilde{V}^*V\|_F = \|H - \widetilde{H}\|_F.\end{aligned}$$

Similarly, we have

$$\||B^*| - |\widetilde{B}^*|\|_F = \|H' - \widetilde{H}'\|_F. \tag{2.9}$$

(2.6) follows immediately from Lemma 2.1.

Since $|B^*| = U\Gamma^*U^*$ and $|\widetilde{B}^*| = \widetilde{U}\widetilde{\Gamma}^*\widetilde{U}^*$, by the Mirsky theorem (1.6) we have

$$\||B^*| - |\widetilde{B}^*|\|_F^2 = \|U\Gamma^*U^* - \widetilde{U}\widetilde{\Gamma}^*\widetilde{U}^*\|_F^2 \geqslant \|\Gamma - \widetilde{\Gamma}\|_F^2 = \|\Sigma - \widetilde{\Sigma}\|_F^2.$$

(2.7) is obtained from Lemma 2.1 and (2.8).

3 Combined perturbation bounds

Taking $\sigma = \sigma_r = \min\{\sigma_r, \widetilde{\sigma}_r\}$ in (2.4) gives

$$\|A - \widetilde{A}\|_F^2 \geqslant \sigma_r\widetilde{\sigma}_r\|Q - \widetilde{Q}\|_F^2 + \|S\Gamma - \widetilde{\Gamma}T\|_F^2,$$

where we have used (2.2) and noted the fact that $m_{rr} \leqslant \|Q - \widetilde{Q}\|_F^2, \widetilde{m}_{rr} \leqslant \|Q - \widetilde{Q}\|_F^2$. We have the following theorem by Lemma 2.2.

Theorem 3.1. *Let A, $\widetilde{A} \in \mathcal{C}_r^{m\times n}$ have the singular value decompositions in* (2.1). *Then*

$$\sigma_r\widetilde{\sigma}_r\|Q - \widetilde{Q}\|_F^2 + \frac{1}{2}\|H - \widetilde{H}\|_F^2 + \frac{1}{2}\|H' - \widetilde{H}'\|_F^2 \leqslant \|A - \widetilde{A}\|_F^2. \tag{3.1}$$

By the Mirsky theorem (1.6),

$$\|\Sigma - \widetilde{\Sigma}\|_F^2 \leqslant \|H - \widetilde{H}\|_F^2, \quad \|\Sigma - \widetilde{\Sigma}\|_F^2 \leqslant \|H' - \widetilde{H}'\|_F^2,$$

and hence

$$\|\Sigma - \widetilde{\Sigma}\|_F^2 \leqslant \frac{1}{2}\|H - \widetilde{H}\|_F^2 + \frac{1}{2}\|H' - \widetilde{H}'\|_F^2.$$

Therefore, we have the following corollaries.

Corollary 3.1. *Let A, $\widetilde{A} \in \mathcal{C}_r^{m\times n}$ have the singular value decompositions in* (2.1). *Then*

$$\sigma_r\widetilde{\sigma}_r\|Q - \widetilde{Q}\|_F^2 + \|\Sigma - \widetilde{\Sigma}\|_F^2 \leqslant \|A - \widetilde{A}\|_F^2. \tag{3.2}$$

Corollary 3.2. *Under the assumption above, the asymptotic bounds* (1.4) *and* (1.5) *hold.*

It is obvious that the combined bounds in (3.1) and (3.2) are sharper than those previous bounds. The following example shows that the equalities in (3.1) and (3.2) may hold.

Example 3.1. Let A be a 2×2 matrix and $A = U\Sigma V^*$ be its singular value decomposition with

$$\Sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \ \sigma_1 > \sigma_2.$$

Let $\widetilde{A} = U\widetilde{\Sigma}(V(I+D))^*$ with

$$\widetilde{\Sigma} = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \widetilde{\sigma}_2 \end{pmatrix}, \quad \sigma_1 \geqslant \widetilde{\sigma}_2 > 0 \text{ and } \widetilde{\sigma}_2 \neq \sigma_2,$$

and

$$D = \begin{pmatrix} 0 & 0 \\ 0 & -a + i\sqrt{2a - a^2} \end{pmatrix}, \quad 0 < a < 2.$$

Then $\widetilde{U} = U$, $\widetilde{V} = V(I+D)$. A straightforward calculation gives

$$\begin{aligned} &\|Q - \widetilde{Q}\|_F = \|UD^*V^*\|_F = \|D^*\|_F = \sqrt{2a}\,, \\ &\|H - \widetilde{H}\|_F = \|H' - \widetilde{H}'\|_F = |\widetilde{\sigma}_2 - \sigma_2|, \\ &\|A - \widetilde{A}\|_F = \|\Sigma - \widetilde{\Sigma}(I+D)^*\|_F = \sqrt{(\widetilde{\sigma}_2 - \sigma_2)^2 + 2\widetilde{\sigma}_2\sigma_2 a}\,. \end{aligned}$$

Thus the equalities in (3.1) and (3.2) hold.

Remark 3.1. The asymptotic perturbation bounds in (1.3) and (1.4) are optimal since they are optimal for each factor. The corresponding absolute bounds seem more complicated. The best absolute perturbation bound obtained until now for the unitary factor is

$$\left(\frac{\sigma_r + \widetilde{\sigma}_r}{2}\right)^2 \|Q - \widetilde{Q}\|_F^2 \leqslant \|A - \widetilde{A}\|_F^2, \tag{3.3}$$

which was proved by Li[8] for the case $r = m = n$ and by Li and Sun[9] for the general cases. It was shown in [8, 10] that the bound in (3.3) is optimal for any perturbation. The above example also shows that the simple combination

$$\left(\frac{\sigma_r + \widetilde{\sigma}_r}{2}\right)^2 \|Q - \widetilde{Q}\|_F^2 + \frac{1}{2}\|H - \widetilde{H}\|_F^2 \leqslant \|A - \widetilde{A}\|_F^2$$

does not hold. To obtain more precisely absolute combined bounds, we need to do restriction on the perturbation.

Theorem 3.2. *Let A, $\widetilde{A} \in \mathcal{C}_r^{m\times n}$ have the singular value decompositions in* (2.1). *If*

$$\|A - \widetilde{A}\|_F \leqslant \frac{\sigma_r + \widetilde{\sigma}_r}{2|\sigma_r - \widetilde{\sigma}_r|}\|\Sigma - \widetilde{\Sigma}\|_F, \tag{3.4}$$

then

$$\frac{1}{2}(\sigma_r^2 + \widetilde{\sigma}_r^2)\|Q - \widetilde{Q}\|_F^2 + \frac{1}{2}\|H - \widetilde{H}\|_F^2 \leqslant \|A - \widetilde{A}\|_F^2. \tag{3.5}$$

Proof. Without loss of generality we assume that $\sigma_r \leqslant \widetilde{\sigma}_r$. By (3.3), the assumption (3.4) implies that $\|Q - \widetilde{Q}\|_F \leqslant \frac{1}{|\sigma_r - \widetilde{\sigma}_r|}\|\Sigma - \widetilde{\Sigma}\|_F$ and by taking $\sigma = \sigma_r$ in (2.4) and Lemma 2.2 we have

$$\begin{aligned} \|A - \widetilde{A}\|_F^2 \geqslant\; & [\sigma_r^2 + \sigma_r(\sigma_{r-1} + \widetilde{\sigma}_{r-1} - 2\sigma_r) - \sigma_r(\sigma_{r-1} - \sigma_r) - \sigma_r(\widetilde{\sigma}_{r-1} - \widetilde{\sigma}_r)]\|Q - \widetilde{Q}\|_F^2 \\ & + \frac{1}{2}\|H - \widetilde{H}\|_F^2 + \frac{1}{2}\|\Sigma - \widetilde{\Sigma}\|_F^2 \\ \geqslant\; & \sigma_r\widetilde{\sigma}_r\|Q - \widetilde{Q}\|_F^2 + \frac{1}{2}(\sigma_r - \widetilde{\sigma}_r)^2\|Q - \widetilde{Q}\|_F^2 + \frac{1}{2}\|H - \widetilde{H}\|_F^2 \\ \geqslant\; & \frac{1}{2}(\sigma_r^2 + \widetilde{\sigma}_r^2)\|Q - \widetilde{Q}\|_F^2 + \frac{1}{2}\|H - \widetilde{H}\|_F^2. \end{aligned}$$

This completes the proof of the theorem.

Under some assumption of the perturbation being small, a further improvement on (3.3) was given in [10],

$$\frac{1}{2}(\sigma_r^2+\widetilde{\sigma}_r^2)\|Q-\widetilde{Q}\|_F^2 \leqslant \|A-\widetilde{A}\|_F^2.$$

Our bound in (3.5) is a combination based on the optimal bound (1.2) and the bound in the last equation.

Now we consider the case $A,\ \widetilde{A}\in\mathcal{C}_n^{n\times n}$. In this case, the condition (3.4) can be weakened. For simplicity we always assume that $\sigma_n\leqslant\widetilde{\sigma}_n$.

Theorem 3.3. *Let $A,\ \widetilde{A}\in\mathcal{C}_n^{n\times n}$ have the singular value decompositions in* (2.1). *If*

$$\|A-\widetilde{A}\|_2\leqslant\frac{\sigma_n+\widetilde{\sigma}_n}{2}, \tag{3.6}$$

then

$$\min\left\{\sigma_n(\sigma_{n-1}+\widetilde{\sigma}_{n-1}-\sigma_n),\ \frac{1}{2}(\sigma_n^2+\widetilde{\sigma}_n^2)\right\}\|Q-\widetilde{Q}\|_F^2+\frac{1}{2}\|H-\widetilde{H}\|_F^2\leqslant\|A-\widetilde{A}\|_F^2. \tag{3.7}$$

Proof. We take $\sigma=\sigma_n$ in (2.4). By Lemma 2.2,

$$\begin{aligned}\|A-\widetilde{A}\|_F^2\geqslant{}&\sigma_n^2\|Q-\widetilde{Q}\|_F^2+\frac{1}{2}\|H-\widetilde{H}\|_F^2+\frac{1}{2}\|\Sigma-\widetilde{\Sigma}\|_F^2\\&+\sigma_n(\sigma_{n-1}+\widetilde{\sigma}_{n-1}-2\sigma_n)\mathrm{tr}(M)\\&-\sigma_n(\sigma_{n-1}-\sigma_n)m_{nn}-\sigma_n(\widetilde{\sigma}_{n-1}-\widetilde{\sigma}_n)\widetilde{m}_{nn}.\end{aligned}$$

Noting the fact[9] that

$$m_{nn}\leqslant\|Q-\widetilde{Q}\|_2^2,\qquad \widetilde{m}_{nn}\leqslant\|Q-\widetilde{Q}\|_2^2,$$

and by (2.2), we obtain

$$\begin{aligned}\|A-\widetilde{A}\|_F^2\geqslant{}&\sigma_n(\sigma_{n-1}+\widetilde{\sigma}_{n-1}-\sigma_n)\|Q-\widetilde{Q}\|_F^2+\frac{1}{2}(\sigma_n-\widetilde{\sigma}_n)^2\\&-\sigma_n(\sigma_{n-1}-\sigma_n+\widetilde{\sigma}_{n-1}-\widetilde{\sigma}_n)\|Q-\widetilde{Q}\|_2^2+\frac{1}{2}\|H-\widetilde{H}\|_F^2.\end{aligned}$$

By [8] and the assumption (3.6), we have $\|Q-\widetilde{Q}\|_2\leqslant\frac{2}{\sigma_n+\widetilde{\sigma}_n}\|A-\widetilde{A}\|_2\leqslant1$. It follows that

$$\begin{aligned}\|A-\widetilde{A}\|_F^2\geqslant{}&\sigma_n(\sigma_{n-1}+\widetilde{\sigma}_{n-1}-\sigma_n)\|Q-\widetilde{Q}\|_F^2\\&-\left[\sigma_n(\sigma_{n-1}+\widetilde{\sigma}_{n-1})-\frac{3}{2}\sigma_n^2-\frac{1}{2}\widetilde{\sigma}_n^2\right]\|Q-\widetilde{Q}\|_2^2+\frac{1}{2}\|H-\widetilde{H}\|_F^2.\end{aligned} \tag{3.8}$$

If $\sigma_n(\sigma_{n-1}+\widetilde{\sigma}_{n-1})-\frac{3}{2}\sigma_n^2-\frac{1}{2}\widetilde{\sigma}_n^2\geqslant0$, (3.8) becomes

$$\|A-\widetilde{A}\|_F^2\geqslant\frac{1}{2}(\sigma_n^2+\widetilde{\sigma}_n^2)\|Q-\widetilde{Q}\|_F^2+\frac{1}{2}\|H-\widetilde{H}\|_F^2.$$

If $\sigma_n(\sigma_{n-1}+\widetilde{\sigma}_{n-1})-\frac{3}{2}\sigma_n^2-\frac{1}{2}\widetilde{\sigma}_n^2<0$, we obtain

$$\|A-\widetilde{A}\|_F^2\geqslant\sigma_n(\sigma_{n-1}+\widetilde{\sigma}_{n-1}-\sigma_n)\|Q-\widetilde{Q}\|_F^2+\frac{1}{2}\|H-\widetilde{H}\|_F^2.$$

The bound (3.7) follows from the last two equations and the proof is complete.

Remark 3.2. The conditions (3.4) and (3.6) cannot be omitted. For example, let A and $\widetilde{A}$ be defined as in Example 3.1 with $a > \frac{1}{2}$ and $\sigma_1\sigma_2 > \widetilde{\sigma}_2^2$. Then

$$\|A - \widetilde{A}\|_F = \|A - \widetilde{A}\|_2 > \sqrt{(\widetilde{\sigma}_2 - \sigma_2)^2 + \widetilde{\sigma}_2\sigma_2} \geqslant \frac{\widetilde{\sigma}_2 + \sigma_2}{2}.$$

We can see that $\sigma_2(\sigma_1 + \widetilde{\sigma}_1 - \sigma_2) > \frac{1}{2}(\sigma_2^2 + \widetilde{\sigma}_2^2)$ and

$$\frac{1}{2}(\sigma_2^2 + \widetilde{\sigma}_2^2)\|Q - \widetilde{Q}\|_F^2 + \frac{1}{2}\|H - \widetilde{H}\|_F^2 > \|A - \widetilde{A}\|_F^2.$$

Corollary 3.3. *Under the assumptions in Theorem* 3.3 *and the assumption*

$$\widetilde{\sigma}_n^2 \leqslant \sigma_n\widetilde{\sigma}_{n-1}, \tag{3.9}$$

the combined perturbation bound (3.5) *holds.*

4 Concluding remarks

We have presented some new perturbation bounds for the polar decomposition $A = QH$, which are based on a combination of perturbation bounds for the unitary factor, the Hermitian factor and singular values. Our bounds are optimal since they are optimal for each factor. In [15] the authors obtained some optimal combination of the perturbation bounds for the spectral decomposition and the singular value decomposition. However, such an optimal combination of the perturbation bounds cannot be extended to any other matrix decompositions. The following example shows that it is not true for QR decomposition of a matrix. For $A, \widetilde{A} \in \mathcal{C}_n^{n\times n}$, the classical perturbation bounds for Q and R are given by (e.g. see [16])

$$\frac{\sigma_n^2}{2}\|\widetilde{Q} - Q\|_F^2 \leqslant \|\widetilde{A} - A\|_F^2$$

and

$$\frac{1}{\kappa_R^2(A)}\|\widetilde{R} - R\|_F^2 \leqslant \|\widetilde{A} - A\|_F^2\,,$$

respectively.

Example 4.1. Let $Q = I$, $R = I$, $A = QR = I$ and $\widetilde{A} = \widetilde{Q}\widetilde{R}$ with

$$\widetilde{Q} = \begin{pmatrix} 1-\delta & -\sqrt{2\delta - \delta^2} \\ \sqrt{2\delta - \delta^2} & 1-\delta \end{pmatrix}, \quad \widetilde{R} = \begin{pmatrix} 1 & \sqrt{2\delta - \delta^2} \\ 0 & 1 \end{pmatrix}.$$

Then

$$\|\widetilde{A} - A\|_F^2 = \left\|\begin{pmatrix} -\delta & -\delta\sqrt{2\delta - \delta^2} \\ \sqrt{2\delta - \delta^2} & \delta - \delta^2 \end{pmatrix}\right\|_F^2 = 2\delta + \delta^2,$$

$$\|\widetilde{Q} - Q\|_F^2 = 4\delta, \quad \|\widetilde{R} - R\|_F^2 = 2\delta - \delta^2, \quad \kappa_R(A) = 1\,.$$

The combined perturbation bound

$$\frac{\sigma_2^2}{2}\|\widetilde{Q} - Q\|_F^2 + c(A)\|\widetilde{R} - R\|_F^2 \leqslant \|\widetilde{A} - A\|_F^2$$

holds only if $c(A) \equiv 0$.

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