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Transitivity, mixing and chaos for a class of set-valued mappings

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Abstract Consider the continuous map $f : X \to X$ and the continuous map $\bar{f}$ of $\mathcal{K}(X)$ into itself induced by f, where X is a metric space and $\mathcal{K}(X)$ the space of all non-empty compact subsets of X endowed with the Hausdorff metric. According to the questions whether the chaoticity of f implies the chaoticity of $\bar{f}$ posed by Román-Flores and when the chaoticity of f implies the chaoticity of $\bar{f}$ posed by Fedeli, we investigate the relations between f and $\bar{f}$ in the related dynamical properties such as transitivity, weakly mixing and mixing, etc. And by using the obtained results, we give the satisfied answers to Román-Flores's question and Fedeli's question.

Keywords: set-valued mapping, transitivity, weak mixing, mixing, chaos.

Throughout this paper we always suppose that f is a continuous map of the metric space (X, d) into itself, $\bar{f}$ the set-valued mapping induced by f, i.e. the natural extension of f to $\mathcal{K}(X)$, where $\mathcal{K}(X)$ is the space of all non-empty compact subsets of X endowed with the Hausdorff metric induced by d. As is known to all, the main task to investigate the system (X, f) is to clear how the points of X move. Nevertheless, in many fields or problems such as biological species, demography, numerical simulation and attractors, etc, it is not enough to know only how the points of X move[1,2], one has to know how the subsets of X move. So it is necessary to study the set-valued dynamical system $(\mathcal{K}(X), \bar{f})$ associated to the system (X, f).

The aim of this paper is to investigate the relationships between the transitivity, mixing and chaoticity of f and those of $\bar{f}$. The meaning that a map is chaotic is that it posesses three properties in the definition of Devaney: (i) transitivity, (ii) periodic density, (iii) sensitive dependence(on initial conditions)[3].

To determine if a map is chaotic, it is sufficient to consider whether it posesses the transitivity and the periodic density, since the properties (i) and (ii) in the definition of Devaney imply (iii) for the case that f is infinite (refs. [4,5] or see ref. [6]). One can find from refs. [7–9] that the transitivity and the periodic density imply not only the sensitive dependence but also grimmer even surprising chaotic behaviours.

Román-Flores[10] posed the following questions:

(Q1) Whether does the chaoticity of f imply the chaoticity of $\bar{f}$?

To resolve this question, he showed by the irrational rotation of the unit circle that the transitivity of f does not need to imply the transitivity of $\bar{f}$. Since the irrational rotation of the circle is not chaotic (without periodic point), we easily see that (Q1) has not been resolved. Later, Fedeli[11] asked:

(Q2) When does the chaoticity of f imply the chaoticity of $\bar{f}$?

The response of Fedeli himself is : the periodic density of f and the transitivity of $\bar{f}$ imply the chaoticity of $\bar{f}$. Obviously, this cannot make one satisfied also, because we can not deduce, from the chaoticity of f, whether $\bar{f}$ is transitive.

The study of this paper is exactly to resolve (Q1) and (Q2). The concreteness is organized as follows. Section 1 contains some necessary preliminaries with basic notions and results. In Section 2 we investigate in a deep going way the relationships between f and $\bar{f}$ in the related dynamical properties, and as an application, (Q2) will obtain a precise response. In Section 3, by considering the interval maps we prove that there exists a mapping f such that it is chaotic but $\bar{f}$ is not so, and by this we give (Q1) a negative answer. Finally, in Section 4 we give several examples, which contain not only the applications of the results obtained in Sections 2 and 3, but also a proof of the conclusion that the periodic density of $\bar{f}$ does not need to imply the periodic density of f. After this we pose a question waiting for response.

1 Preliminaries

f is (topologically) transitive if for any non-empty open sets U and V, there is a positive integer n such that $f^n(U) \cap V \neq \phi$. f is (topologically) weakly mixing if the Cartesian product $f \times f$ of two f's is transitive, that is for any non-empty open sets U_1, U_2, V_1 and V_2, there is a positive integer n such that $f^n(U_i) \cap V_i \neq \phi, i = 1, 2$. The Cartesian product of m f's is simply denoted as f_m. f is (topologically) mixing if for any non-empty open sets U and V, there is a positive integer N such that for all $n \geqslant N$, $f^n(U) \cap V \neq \phi$.

It is clear that a mixing map is weakly mixing and a weakly mixing map is transitive.

Recall that $\mathcal{K}(X)$ is the family of all non-empty compact subsets of X. Let H denote the Hausdorff metric on $\mathcal{K}(X)$ which is defined as: for any $A, B \in \mathcal{K}(X)$,

$$H(A, B) = \max\{\mathrm{dist}(A, B), \mathrm{dist}(B, A)\},$$

where $\mathrm{dist}(A, B) = \sup_{a \in A} \inf_{b \in B} d(a, b)$. It is worth noting that the topology induced by the Hausdorff metric H on $\mathcal{K}(X)$ is exactly the Vietoris topology on $\mathcal{K}(X)$, and a base of the Vietoris topology consists of all sets of the form

$$\mathcal{B}(U_1,\cdots,U_n)=\{F\in\mathcal{K}(X)|F\subset\bigcup_{i=1}^{n}U_i, F\cap U_i\neq\phi, i=1,\cdots,n\},$$

where $U_1,\cdots,U_n$ are non-empty open sets of X (see p. 8 in ref. [12], where, however, "base" was wrongly written as "subset").

The extension $\bar{f}$ of f to $\mathcal{K}(X)$ is defined as: for any $A\in\mathcal{K}(X)$, $\bar{f}(A)=\{f(a)|a\in A\}$. It is not difficult to prove that f continuous implies $\bar{f}$ continuous. We use $B(A,\epsilon)$ to denote the set $\{F\in\mathcal{K}(X)|H(A,F)<\epsilon\}$.

The following Lemmas 1.1 and 1.2 are obvious.

Lemma 1.1. f is transitive if and only if for any non-empty open sets U and V, there is a positive integer n such that $f^{-n}(U)\cap V\neq\phi$.

Lemma 1.2. f is mixing if and only if for any $m>0$, f_m is mixing.

Lemma 1.3. The following are equivalent:

(i) f is weakly mixing;

(ii) for any $m\geqslant 2$, f_m is transitive;

(iii) for any non-empty open sets U and V, there is an $n>0$ such that

$$f^n(U)\cap V\neq\phi \quad\text{and}\quad f^n(V)\cap V\neq\phi.$$

Proof. (i) $\Rightarrow$ (ii). Let f be weakly mixing. By the definition, f_2 is transitive. Assume that for some $k\geqslant 2$, the transitivity of f_k has been shown. Let $U_1,U_2,\cdots,U_k,U_{k+1}$, $V_1,V_2,\cdots,V_k,V_{k+1}$ be any $2(k+1)$ non-empty open sets of X. Since f_2 is transitive, by Lemma 1.1, there is $l>0$ such that

$$U=U_k\cap f^{-l}(U_{k+1}),\quad V=V_k\cap f^{-l}(V_{k+1})$$

are both non-empty open sets. Also, since f_k is transitive, there exists $n>0$ such that for each $i=1,2,\cdots k-1$, $f^n(U_i)\cap V_i\neq\phi$ and $f^n(U)\cap V\neq\phi$. Noting that

$$f^n(U_k)\cap V_k\supset f^n(U)\cap V\neq\phi$$

and

$$f^n(U_{k+1})\cap V_{k+1}\supset f^n(f^l(U))\cap f^l(V)\supset f^l(f^n(U)\cap V)\neq\phi,$$

we see that f_{k+1} is transitive. By induction, what we want to prove follows.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i). Let U_1,U_2,V_1,V_2 be non-empty open sets of X. First since we easily see that f is transitive, it follows from Lemma 1.1 that for some $n_1>0$, $A=V_1\cap f^{-n_1}(V_2)\neq\phi$, and for some $n_2>0$, $B=f^{-n_2}(A)\cap f^{-n_1}(U_2)\neq\phi$. Secondly, for the non-empty open sets B and U_1, there is $n_3>0$ such that

$f^{n_3}(B) \cap B \neq \phi$ and $f^{n_3}(U_1) \cap B \neq \phi$. Finally, putting $n = n_2 + n_3$, we have

$$\begin{aligned} f^n(U_1) \cap V_1 &= f^{n_2+n_3}(U_1) \cap V_1 \\ &\supset f^{n_2+n_3}(U_1) \cap f^{n_2} f^{-n_2}(A) \\ &\supset f^{n_2}(f^{n_3}(U_1) \cap f^{-n_2}(A)) \\ &\supset f^{n_2}(f^{n_3}(U_1) \cap B) \neq \phi. \end{aligned}$$

Since

$$\begin{aligned} f^{-n_1} f^{-n_2}(f^{n_2+n_3}(U_2) \cap V_2) &\supset f^{-n_1} f^{n_3}(U_2) \cap f^{-(n_1+n_2)}(V_2) \\ &\supset f^{n_3} f^{-n_1}(U_2) \cap f^{-(n_1+n_2)}(V_2) \\ &\supset f^{n_3}(B) \cap B \neq \phi, \end{aligned}$$

we also have $f^n(U_2) \cap V_2 \neq \phi$. So f is weakly mixing.

2 Transitivity, mixing and a response to Fedeli's question

Theorem 2.1. The following are equivalent.

(i) f is weakly mixing.

(ii) $\bar{f}$ is weakly mixing.

(iii) $\bar{f}$ is transitive.

Proof. (i) $\Rightarrow$ (ii). Let f be weakly mixing, and let $\mathcal{U}, \mathcal{V}$ be non-empty open sets of $\mathcal{K}(X)$. There exist non-empty open sets $U_1, U_2, \cdots, U_s$, $V_1, V_2, \cdots, V_t$ of X, such that $\mathcal{B}_1 = \mathcal{B}(U_1, \cdots, U_s) \subset \mathcal{U}$, $\mathcal{B}_2 = \mathcal{B}(V_1, \cdots, V_t) \subset \mathcal{V}$. Let $m = \max\{s, t\}$, and put

$$\bar{U}_i = \begin{cases} U_i, & 1 \leqslant i \leqslant s, \\ U_s, & s \leqslant i \leqslant m, \\ V_i, & m+1 \leqslant i \leqslant m+t, \\ V_t, & m+t \leqslant i \leqslant 2m; \end{cases}$$

$$\bar{V}_i = \begin{cases} V_i, & 1 \leqslant i \leqslant t \quad \text{or} \quad m+1 \leqslant i \leqslant m+t, \\ V_t, & t \leqslant i \leqslant m \quad \text{or} \quad m+t \leqslant i \leqslant 2m. \end{cases}$$

Evidently, $\bar{U}_i, \bar{V}_i$ are both non-empty open sets of X. Since f is weakly mixing, by Lemma 1.3, f_{2m} is transitive. So for some $n > 0$, $f^n(\bar{U}_i) \cap \bar{V}_i \neq \phi$ for $1 \leqslant i \leqslant 2m$. Thus for each $i = 1, 2, \cdots, 2m$, we can choose $x_i \in \bar{U}_i$, $y_i \in \bar{V}_i$ such that $f^n(x_i) = y_i$. Put

$$E_1 = \{x_1, x_2, \cdots, x_m\}, \quad F_1 = \{y_1, y_2, \cdots, y_m\},$$

$$E_2 = \{x_{m+1}, x_{m+2}, \cdots, x_{2m}\}, \quad F_2 = \{y_{m+1}, y_{m+2}, \cdots, y_{2m}\}.$$

We have $\overline{f}^n(E_i) = F_i$, $i = 1, 2$. Since we easily see that $E_1 \in \mathcal{B}_1$, and $E_2, F_1, F_2 \in \mathcal{B}_2$, it follows that $\overline{f}^n(\mathcal{U}) \cap \mathcal{V} \supset \overline{f}^n(\mathcal{B}_1) \cap \mathcal{B}_2 \neq \phi$ and $\overline{f}^n(\mathcal{V}) \cap \mathcal{V} \supset \overline{f}^n(\mathcal{B}_2) \cap \mathcal{B}_2 \neq \phi$. By Lemma 1.3, $\bar{f}$ is weakly mixing.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i). Let $\bar{f}$ be transitive, U, V the non-empty open sets of X. Let $\mathcal{U} = \mathcal{B}(U, V)$, $\mathcal{V} = \mathcal{B}(V)$. Then $\mathcal{U}, \mathcal{V}$ are non-empty open sets of $\mathcal{K}(X)$. Thus for some $n > 0$, $\overline{f}^n(\mathcal{U}) \cap \mathcal{V} \neq \phi$, i.e. there exist $K \in \mathcal{U}$, $F \in \mathcal{V}$ such that $\overline{f}^n(K) = F$. Choose arbitrarily $x \in K \cap U$, $y \in K \cap V$, we have $f^n(x) \in V$, $f^n(y) \in V$. This shows $f^n(U) \cap V \neq \phi$ and $f^n(V) \cap V \neq \phi$. Hence f is weakly mixing.

By Theorem 2.1 we have immediately

Corollary 2.1. If f is mixing or weakly mixing then $\bar{f}$ is transitive.

For if f is periodically dense then so is $\bar{f}$[11], we also have

Corollary 2.2. When f is chaotic and weakly mixing, $\bar{f}$ is chaotic; Conversely, if $\bar{f}$ is chaotic then f must be weakly mixing.

Remark 2.1. Corollary 2.2 is to say that only if f is weakly mixing, the conclusion, that f chaotic implies $\bar{f}$ chaotic, holds, which, obviously, gives a precise answer to the question posed by Fedeli.

Theorem 2.2. $\bar{f}$ is mixing if and only if so is f.

Proof. The necessity is obvious. The sufficiency will use Lemma 1.2, the proof is similar to (i) $\Rightarrow$ (ii) in Theorem 2.1, so omitted.

Remark 2.2. When X is any topological space ($\mathcal{K}(X)$ is a topological space endowed with Vietoris topology, respectively), Lemmas 1.1–1.3 follow, so we see from the proofs that Theorems 2.1 and 2.2 are also true.

Remark 2.3. Refs. [13,14] describe the complex dynamical behaviours for mixing maps and weakly mixing maps, the above Theorems 2.1 and 2.2 have added new contents for the known results.

3 Interval mappings and a response to Rom*á*n-Flores's question

In the sequel, I will denote the unit closed interval [0, 1].

Lemma 3.1[15]**.** If $f : I \to I$ is a transitive continuous map, then one of the following conditions holds:

(i) f is mixing.

(ii) There is a fixed point $e \in (0, 1)$ such that

$$f([0, e]) = [e, 1] \quad \text{and} \quad f([e, 1]) = [0, e]. \tag{3.1}$$

Lemma 3.2. Let $f : I \to I$ be the tent map, that is $f(x) = |1 - 2x|$. Then f is mixing.

Proof. As is well known, f is transitive[3]. Noting that $e = 1/3$ is the only fixed point of f and $f([e, 1]) = I \neq [0, e]$, it follows from Lemma 3.1 that f is mixing.

Proposition 3.1. If $f : I \to I$ is continuous, then the following are equivalent.

(i) f is mixing.

(ii) f is weakly mixing.

(iii) $\bar{f}$ is chaotic.

(iv) $\bar{f}$ is transitive.

(v) $\bar{f}$ is mixing.

(vi) $\bar{f}$ is weakly mixing.

Proof. To prove the result, it is sufficient from Theorems 2.1 and 2.2 to prove (i)$\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii).

(i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (iii). Since f weakly mixing $\Rightarrow f$ transitive $\Rightarrow f$ chaotic[16], it follows from Corollary 2.2 that $\bar{f}$ is chaotic.

(iii) $\Rightarrow$ (i). Let $\bar{f}$ be chaotic. By Corollary 2.2, f is weakly mixing, further, f is transitive. If f is not mixing, then by Lemma 3.1, there exists $e \in (0,1)$ such that (3.1) holds. Set $\varepsilon = \min\{e/2, (1-e)/2\}$. Consider the open subsets of $\mathcal{K}(X)$ $B(I,\varepsilon)$ and $B(\{1\},\varepsilon)$. If $K \in B(\{1\},\varepsilon)$, then $K \subset [e,1]$. So for any $n > 0$, $f^n(K)$ is contained in either $[0,e]$ or $[e,1]$. Whatever happens we have $H(\overline{f}^n(K), I) \geqslant \varepsilon$. Thus $\overline{f}^n(B(\{1\},\varepsilon)) \cap B(I,\varepsilon) = \phi$. Moreover $\bar{f}$ is not transitive , which contradicts the chaoticity of $\bar{f}$.

Remark 3.1. The equivalence of (i) and (ii) in Proposition 3.1 is a known result in ref. [13], but the others are new.

Proposition 3.2. There is a continuous map $f : I \to I$ such that f is chaotic but $\bar{f}$ is not chaotic.

Proof. Let $f : I \to I$ be defined as: for any $x \in I$,

$$f(x) = \begin{cases} 2x + 1/2, & 0 \leqslant x \leqslant 1/4, \\ 3/2 - 2x, & 1/4 \leqslant x \leqslant 1/2, \\ 1 - x, & 1/2 \leqslant x \leqslant 1. \end{cases}$$

Observe easily that f is continuous, $1/2$ is a fixed point of f and $f([0,1/2]) = [1/2,1]$, $f([1/2,1]) = [0,1/2]$. Since it is not hard to check that the restrictions $f^2|_{[0,1/2]}$ and $f^2|_{[1/2,1]}$ are both topologically conjugate to the tent map of I into itself, it follows from Lemma 3.2 that they are both mixing. By Proposition 4.3 in ref. [13], f is transitive, so f is chaotic[16]. Evidently, f is not mixing , it follows from Proposition 3.1 that $\bar{f}$ is not chaotic.

Remark 3.2. Proposition 3.2 has given a negative answer to the question posed by Román-Flores.

4 Examples and a question

Example 4.1. Let f be the tent map. By Lemma 3.2 and Proposition 3.1,

$\bar{f}$ is chaotic. This is a simple proof of the same result in ref. [10] and ref. [11].

Example 4.2. Let λ be an irrational number, $T : S^1 \to S^1$ is defined by $T(e^{ix}) = e^{i(x+2\pi\lambda)}$, i.e. T is an irrational rotation of unit circle on complex plane. As is well known, T is transitive[17]. Let (p, q) denote the open arc from p counterclockwise to q. Consider the open arcs $U = (1, e^{i\pi/4})$ and $V = (-1, e^{i5\pi/4})$ on S^1. Since T is isometric, it is not hard to see that for any $n > 0$, $T^n(U) \cap V \neq \phi$ and $T^n(V) \cap V \neq \phi$ cannot be both true. So T is not weakly mixing. Furthermore, by Theorem 2.1, $\bar{T}$ is not transitive. This gives a simple proof to the same result in ref. [10].

Example 4.3. Let $\Sigma = \{s = (s_0, s_1, s_2, \cdots) | s_j = 0 \text{ or } 1\}$ be the symbolic space with two symbols, where the metric of any two points $s = (s_0, s_1, s_2, \cdots)$ and $t = (t_0, t_1, t_2, \cdots)$ in Σ is defined by

$$d(s, t) = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i}.$$

It is easy to show that $d(s, t) \leqslant 1/2^n$ if and only if for each $i = 0, 1, \cdots, n$, $s_i = t_i$. Let $\sigma : \Sigma \to \Sigma$ be the shift. σ is continuous and chaotic[3]. For any non-empty open set U, $\sigma^n(U) = \Sigma$ for some $n > 0$, we then easily see that σ is mixing, in particular, weakly mixing. Thus by Corollary 2.2 $\bar{\sigma}$ is chaotic.

Example 4.4. With Σ as in Example 4.3. For any points $s = (s_0, s_1, s_2, \cdots)$ and $t = (t_0, t_1, t_2, \cdots)$ in Σ,

$$s + t = (u_0, u_1, u_2, \cdots)$$

is defined as follows: if $s_0 + t_0 < 2$ then $u_0 = s_0 + t_0$, if $s_0 + t_0 \geqslant 2$ then $u_0 = s_0 + t_0 - 2$ and carries 1 to the next position. The terms $u_1, u_2, \cdots$ are successively determined in the same fashion. With this definition of addition, Σ is a compact topological group, denoted by Σ_+.

Let $\tau : \Sigma_+ \to \Sigma_+$ be defined by $\tau(s) = s + \mathbf{1}$, where $s \in \Sigma_+$, $\mathbf{1} = (1, 0, 0, \cdots,)$. τ is continuous and has no periodic points[17]. Let $\mathcal{B}(U_1, \cdots, U_m)$ be any member in the base of $\mathcal{K}(\Sigma_+)$. For each U_i, there is a member V_i in the base of Σ_+, that is a column of the form

$$[s_0, \cdots, s_k] = \{t \in \Sigma_+ | t_i = s_i, 0 \leqslant i \leqslant k\},$$

such that $V_i \subset U_i$. Set $V = V_1 \cup V_2 \cup \cdots \cup V_m$. Every V_i is a compact subset of Σ_+, so we easily see that $V \in \mathcal{K}(\Sigma_+) \cap \mathcal{B}(U_1, \cdots, U_n)$. Also, since for each V_i, there is an $n_i > 0$ such that $\tau^{n_i}(V_i) = V_i$, it follows that if put $n = n_1 n_2 \cdots n_m$ then $\tau^n(V) = V$. Hence V is a periodic point of $\overline{\tau}$. And then $\bar{\tau}$ is periodically dense.

Remark 4.1. Example 4.4 shows that the periodic density of $\bar{f}$ does not imply periodic density of f, this is exactly why Corollary 2.2 does not contain the version that $\bar{f}$ chaotic implies f chaotic. Nevertheless, observing that $\bar{\tau}$ in Example 4.4 is not chaotic (since it is not transitive), we still have the following question.

(Q3) Whether does the chaoticity of $\bar{f}$ imply the chaoticity of f?

Remark 4.2. Still, there exist many considerable questions besides (Q3). For example, what relations are there between topological entropy of $\bar{f}$ and that of f? which systems, besides $\bar{f}$, have the equivalence between transitivity and weakly mixing ? etc. We hope that these can provoke the interest of the same trade.

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