A New Class of Strong Orthogonal Arrays of Strength Three*

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DOI: 10.1007/s11424-023-3093-9
Received: 15 March 2023 / Revised: 29 May 2023

Abstract Strong orthogonal arrays (SOAs) were recently introduced and studied as a class of space-filling designs for computer experiments. To surely realize better space-filling properties, SOAs of strength three or higher are desirable. In addition, orthogonality is also an important property for designs of computer experiments, because it guarantees that the estimates of the main effects are uncorrelated. This paper first provides a systematic study on the construction of (nearly) orthogonal strength-three SOAs with better space-filling properties. The newly proposed strength-three SOAs enjoy almost the same space-filling properties of strength-four SOAs, and can accommodate much more columns than the latter. Moreover, they are (nearly) orthogonal and flexible in run sizes. The construction methods are straightforward to implement, and their theoretical supports are well established. In addition to the theoretical results, many designs are tabulated for practical needs.

Keywords Computer experiment, orthogonality, space-filling property, strong orthogonal array.

1 Introduction

Computer experiments call for space-filling designs[1, 2]. A space-filling design spreads its points in the design region uniformly, where the uniformity can be evaluated by some distance criteria, discrepancy criteria or stratification properties. A computer experiment always involves a large number of factors at an early stage while only a few of them are expected to be active[3–5], the so-called effect sparsity principle in [6] also applies to this situation. Therefore, designs with low-dimensional space-filling properties are preferred[7, 8]. There have been a fruitful literature

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*This work was supported by the National Natural Science Foundation of China under Grant Nos. 12131001 and 12226343, the MOE Project of Key Research Institute of Humanities and Social Sciences under Grant No. 22JJD110001, and the National Ten Thousand Talents Program of China.
◇ *This paper was recommended for publication by Editor HE Xu.*

on space-filling designs with good low-dimensional stratifications. Latin hypercube designs (LHDs), proposed by [9], are the most commonly used space-filling designs. An LHD with n runs possesses n equally spaced levels, which guarantees the maximum stratification in one dimension. Tang[10] constructed LHDs based on orthogonal arrays (OAs) for getting designs with attractive g-dimensional space-filling property for any g no more than t, where t is the strength of the corresponding OAs. Qian[11] introduced LHDs with nested structures, while [12] proposed LHDs with sliced structures, and this line of research has been further developed in [13] and [14].

As we all know, orthogonality is very important for designs of computer experiments, since it guarantees that the estimates of the main effects are uncorrelated. Approaches have been proposed in the literature for constructing orthogonal designs, see, e.g., [15–23] and the references therein.

He and Tang[24] introduced and studied strong orthogonal arrays (SOAs), where an SOA of strength t is more space-filling than the corresponding OA when projected onto any g dimensions for any g less than t. [25–28] studied orthogonal SOAs. It is worth noting that to surely realize better space-filling properties, SOAs of three or higher are desirable. He and Tang[29] theoretically solved the existence problem of strength-three SOAs. Recently, [30] constructed strength-three SOAs with better space-filling properties, that is, the strength-three SOAs enjoy some of the space-filling properties that only strength-four SOAs can offer. Whereas their resulting SOAs do not have the property of orthogonality. And their constructions are based on regular 2^{k-p} designs, which means that the run size must be the power of 2.

This paper aims at constructing a new class of strength-three SOAs that possess better orthogonality and space-filling properties. These strength-three SOAs enjoy almost the same space-filling properties of strength-four SOAs, and can accommodate much more columns than the latter. Moreover, they are (nearly) orthogonal and flexible in run sizes (for example, they do not necessarily need to be a power of 2). The construction methods are straightforward to implement, and their theoretical supports are well established. In addition to the theoretical results, many designs are tabulated for practical needs.

This paper is organized as follows. Section 2 introduces the definitions and notation used in this paper. Characterizations and constructions will be studied under various scenarios in Section 3. Concluding remarks are provided in Section 4. Four tables are listed in the Appendix.

2 Definitions and Notations

We use $D(n, s_1, \cdots, s_m)$ to denote a balanced design of n runs and m factors, with each of the s_i levels from $\{0, 1, \cdots, s_i - 1\}$ replicated equally often in the ith column. When all the s_i's are equal to s, the design becomes a symmetric balanced design $D(n, s^m)$. A $D(n, s_1, \cdots, s_m)$ becomes a mixed-level OA of strength t and $s_1, \cdots, s_m$ levels, denoted by $\text{OA}(n, m,\ s_1 \times \cdots \times s_m, t)$, if all possible level-combinations for any t columns occur with the same frequency. When all the s_i's are equal to s, the array is symmetric and denoted by $\text{OA}(n, m, s, t)$.

The correlation between two column vectors $a = (a_1, \cdots, a_n)^{\mathrm{T}}$ and $b = (b_1, \cdots, b_n)^{\mathrm{T}}$ is defined as

$$\rho(a,b) = \sum_{i=1}^{n}(a_i-\overline{a})(b_i-\overline{b}) \Bigg/ \left[\sum_{i=1}^{n}(a_i-\overline{a})^2\sum_{i=1}^{n}(b_i-\overline{b})^2\right]^{1/2},$$

where $\overline{a} = \sum_{i=1}^{n} a_i/n$ and $\overline{b} = \sum_{i=1}^{n} b_i/n$. Two column vectors are called orthogonal if the correlation between them is 0. The correlation matrix of a design D is denoted by $\rho(D) = (\rho(d_i, d_j))_{m\times m}$, where d_i and d_j are the ith and jth columns of D respectively, $1 \leq i,\ j \leq m$. A design is called orthogonal if any two columns of it are orthogonal.

For an array with n runs and m factors, we say it achieves a stratification on an $s_1 \times \cdots \times s_t$ grid in some t $(t \geq 2)$ dimensions if the corresponding t columns of it can be collapsed into an $\mathrm{OA}(n, t, s_1 \times \cdots \times s_t, t)$.

A design is called a regular design if any two factorial effects of it are either combinatorial-orthogonal to each other or fully aliased, where two vectors are called combinatorial-orthogonal if they form an OA of strength two.

An $n \times m$ matrix with entries from $\{0, 1, \cdots, s^t - 1\}$ is called an SOA of strength t, denoted by $\mathrm{SOA}(n, m, s^t, t)$, if any $n \times f$ submatrix, $1 \leq f \leq t$, can be collapsed into an $\mathrm{OA}(n, f, s^{\mu_1} \times \cdots \times s^{\mu_f}, f)$ for any positive integers $\mu_1, \cdots, \mu_f$ with $\mu_1 + \cdots + \mu_f = t$, where the s^t levels of a factor are collapsed into s^{μ_j} levels by $\lfloor x/s^{t-\mu_j} \rfloor$ for $x = 0, 1, \cdots, s^t - 1$, $1 \leq j \leq f$, therein $\lfloor z \rfloor$ represents the integer part of z. For an $\mathrm{SOA}(n, m, s^t, t)$, if it is orthogonal, we call it an orthogonal SOA of strength t, denoted by $\mathrm{OSOA}(n, m, s^t, t)$.

An $\mathrm{SOA}(n, m, s^4, 4)$ is more space-filling than an $\mathrm{SOA}(n, m, s^3, 3)$, as it also enjoys properties:

α: Stratifications on $s^2 \times s^2$ grids in all two dimensions;

β: Stratifications on $s \times s^3$ and $s^3 \times s$ grids in all two dimensions;

γ: Stratifications on $s \times s \times s^2$, $s \times s^2 \times s$ and $s^2 \times s \times s$ grids in all three dimensions.

Here the four-dimensional space-filling property is not of our concern due to the effect sparsity principle (cf. [6]). Note that if a design enjoys properties α and β, it has the same two-dimensional space-filling properties as the SOA of strength four. If a design enjoys property γ, it has the same three-dimensional space-filling properties as the SOA of strength four. The remainder of the paper is devoted to the construction of these two kinds of strength-three SOAs. In this paper, properties β and γ correspond to the two-dimensional ($s \times s^3$ and $s^3 \times s$) and three-dimensional ($s \times s \times s^2$, $s \times s^2 \times s$, and $s^2 \times s \times s$) stratification properties, respectively, while properties β and γ in [30] were defined differently, representing the three- and two-dimensional stratification properties, respectively.

3 Construction Results

In this section, we construct two families of strength-three SOAs. The first family are orthogonal and enjoy α and β, that is, they achieve the same two-dimensional space-filling properties as those of strength-four SOAs. The second family are nearly orthogonal and enjoy

γ, that is, they achieve the same three-dimensional space-filling properties as those of strength-four SOAs.

3.1 Orthogonal Strength-Three SOAs with Properties α and β

In this subsection, we present the construction of OSOAs of strength three that enjoy properties α and β.

Let E be an OA$(n_0, k, 4, 2)$, and F be the following OA$(4, 3, 2, 2)$ with entries from $GF(2)$

$$F = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}. \tag{1}$$

In each column of E, we replace the level $q-1$ by the qth row of F for $q = 1, 2, 3, 4$, then we can obtain a $D(n_0, 2^{3k})$, say G. It can be written as $G = (G_1, \cdots, G_k)$, where G_i is the ith group of three columns arising from replacing the levels in the ith column of E by the rows of F for $i = 1, \cdots, k$.

According to the expansive replacement method discussed in [31, Subsection 9.3], G is an OA$(n_0, 3k, 2, 2)$. Then we introduce the following proposition, which is crucial for later construction methods.

Proposition 3.1 *Any four columns, obtained by taking two columns from G_{i_1} and two columns from G_{i_2} with $i_1 \neq i_2$, must form an OA$(n_0, 4, 2, 4)$.*

Its proof is similar to that of Proposition 1 in [22]. According to Proposition 3.1, many four-column subarrays of G form OAs of strength four, while G is an OA of strength two.

Now we are ready to present the construction of OSOAs of strength three that enjoy properties α and β. The construction method is given in the following algorithm.

Algorithm 3.2

Input: An OA$(n_0, k, 4, 2)$, called E, and an OA$(4, 3, 2, 2)$, called F.

Output: An OSOA$(2n_0, k, 8, 3)$, called D_1.

Step 1 In each column of E, we replace the level $q-1$ by the qth row of F for $q = 1, 2, 3, 4$, then we can obtain $G = (G_1, \cdots, G_k)$, where G_i is the ith group of three columns arising from replacing the levels in the ith column of E by the rows of F for $i = 1, \cdots, k$.

Step 2 Create $H_j = (g_{1j}, \cdots, g_{kj})$ for $j = 1, 2, 3$, where g_{ij} represents the jth column of G_i for $i = 1, \cdots, k$.

Step 3 Let $A_1 = (H_1^{\mathrm{T}}, (H_1 \dot{+} 1)^{\mathrm{T}})^{\mathrm{T}}, B_1 = (H_2^{\mathrm{T}}, H_2^{\mathrm{T}})^{\mathrm{T}}$, $C_1 = (H_1^{\mathrm{T}}, H_1^{\mathrm{T}})^{\mathrm{T}}$, where $\dot{+}$ is the addition defined on $GF(2)$. Treat all entries as numbers and define $D_1 = 4A_1 + 2B_1 + C_1$.

For the resulting design D_1, we have the following theorem.

Theorem 3.3 *The design D_1 obtained in Algorithm* 3.2 *is an* OSOA$(2n_0, k, 8, 3)$ *with properties α and β.*

Proof For the convenience of the proof, let a_j, b_j, c_j and d_j be the ith columns of A_1, B_1, C_1 and D_1 respectively. Obviously, Theorem 3.3 holds if we can prove that the 8-level array D_1 can achieve (i) stratifications on 4×4, 2×8 and 8×2 grids in all two dimensions, (ii) stratifications on $2\times 2\times 2$ grids in all three dimensions, and (iii) D_1 is orthogonal.

Now we consider the three cases respectively.

(i) First, we check the two-dimensional stratification of D_1. According to Proposition 3.1, any four columns obtained by taking two columns from G_{l_1} and two columns from G_{l_2} with $l_1 \neq l_2$ must form an OA of strength four. Thus for $l_1 \neq l_2$, $(h_{1l_1}, h_{1l_2}, h_{2l_1}, h_{2l_2})$ is an OA of strength four, where h_{il} denotes the lth column of H_i for $i = 1, 2$ and $l = 1, \cdots, k$. As $A_1 = (H_1^{\rm T}, (H_1 \dot{+} 1)^{\rm T})^{\rm T}$, and $B = (H_2^{\rm T}, H_2^{\rm T})^{\rm T}$, it is easy to see that $(a_{l_1}, a_{l_2}, b_{l_1}, b_{l_2})$ is a strength-4 OA for any $l_1 \neq l_2$, indicating that D_1 can achieve stratifications on 4×4 grids in all two dimensions.

Recall that for $l_1 \neq l_2$, $(h_{1l_1}, h_{1l_2}, h_{2l_1}, h_{2l_2})$ is an OA of strength four, and thus $(h_{1l_1}, h_{1l_2}, h_{2l_1})$ is a strength-three OA. Now consider $(a_{l_1}, a_{l_2}, b_{l_1}, c_{l_1})$, it can be written as

$$\begin{pmatrix} h_{1l_1} & h_{1l_2} & h_{2l_1} & h_{1l_1} \\ h_{1l_1} \dot{+} 1 & h_{1l_2} \dot{+} 1 & h_{2l_1} & h_{1l_1} \end{pmatrix}.$$

As $(h_{1l_2}, h_{2l_1}, h_{1l_1})$ is a strength-three OA, $(h_{1l_2} \dot{+} 1, h_{2l_1}, h_{1l_1})$ forms a strength-three OA. And for any level combination $(\gamma, \alpha, \beta, \gamma)$ in $(h_{1l_1}, h_{1l_2}, h_{2l_1}, h_{1l_1})$, there is a corresponding level combination $(\gamma \dot{+} 1, \alpha, \beta, \gamma)$ in $(h_{1l_1} \dot{+} 1, h_{1l_2} \dot{+} 1, h_{2l_1}, h_{1l_1})$. Thus $(a_{l_1}, a_{l_2}, b_{l_1}, c_{l_1})$ is an OA of strength four for any $l_1 \neq l_2$. Therefore, D_1 can achieve stratifications on 2×8 and 8×2 grids in all two dimensions.

(ii) Next we consider the three-dimensional stratification. Let us check the strength of A_1 first. Note that H_1 can be obtained from A by replacing its four levels with the four elements of the first column of F, where F is an OA$(4, 3, 2, 2)$, then H_1 is an OA of strength two. According to Theorem 2.24 of [31], $A_1 = (H_1^{\rm T}, (H_1 \dot{+} 1)^{\rm T})^{\rm T}$ must be an OA of strength three. It is easy to see that after collapsing the factors of D_1 into 2 levels, D_1 becomes A_1. Thus D_1 achieves a stratification on a $2\times 2\times 2$ grid in any three dimensions.

(iii) We consider the orthogonality of D_1. Note that (A_1, B_1, C_1) is an array of strength two, then $(a_i, a_j), (b_i, b_j), (c_i, c_j), (a_i, b_j), (a_i, c_j)$ and (b_i, c_j) are all OAs of strength two with $i \neq j$. Without loss of generality, we assume the levels in A_1, B_1 and C_1 are centered. Correspondingly, we have $d_i^{\rm T} d_j = (4a_i + 2b_i + c_i)^{\rm T}(4a_j + 2b_j + c_j) = 16a_i^{\rm T} a_j + 8(a_i^{\rm T} b_j + b_i^{\rm T} a_j) + 4(a_i^{\rm T} c_j + b_i^{\rm T} b_j + c_i^{\rm T} a_j) + 2(b_i^{\rm T} c_j + c_i^{\rm T} b_j) + c_i^{\rm T} c_j = 0$, implying that any two distinct columns of D_1 are orthogonal. Thus D_1 is orthogonal. ∎

Now let us see an illustrative example.

Example 3.4 Giving an OA$(48, 13, 4, 2)$, denoted by $E = (e_1, \cdots, e_{13})$, as shown in Table A.1, we obtain an OSOA$(96, 13, 8, 3)$ with properties α and β as follows. For $i = 1, \cdots, 13$, replacing the four levels of e_i by the four rows of F in (1) respectively, we obtain 13 sets of three columns, denoted as $G_1, \cdots, G_{13}$. Taking one column at a time from these 13 groups and putting them together, we obtain 3 sets of 13 columns, denoted as H_1, H_2 and H_3. Let

$A_1 = (H_1^{\mathrm{T}}, (H_1 \dot{+} 1)^{\mathrm{T}})^{\mathrm{T}}$, $B_1 = (H_2^{\mathrm{T}}, H_2^{\mathrm{T}})^{\mathrm{T}}$, $C_1 = (H_1^{\mathrm{T}}, H_1^{\mathrm{T}})^{\mathrm{T}}$. Treat all entries as numbers, then we can get an OSOA$(96, 13, 8, 3)$ by taking $D_1 = 4A_1 + 2B_1 + C_1$, which is shown in the left part of Table A.2. We can check that, as an OSOA$(96, 13, 8, 3)$, D_1 is orthogonal and achieves stratifications on 2×4 and 4×2 grids in all two dimensions, and stratifications on $2 \times 2 \times 2$ grids in all three dimensions. In addition, by checking all the two-tuples, we can find that D_1 achieves stratifications on 4×4, 2×8 and 8×2 grids in all two dimensions. In summary, D_1 is an OSOA of strength three, and enjoys the same two-dimensional space-filling properties as the SOAs of strength four.

From Theorem 3.3, many OSOAs of strength three with properties α and β can be constructed. The OAs needed in Algorithm 3.2 are available in the library of OAs maintained by Dr. Sloane N J A (http://neilsloane.com/oadir/index.html) and [31]. Table 1 summarizes some generated designs (Column 2) as well as the OSOAs of strength three and four in [25] (Columns 5 and 6). It is clear that the resulting OSOAs of strength three from Algorithm 3.2 enjoy the same orthogonality and two-dimensional space-filling property as the OSOAs of strength four, and can accommodate much more columns than the latter ones. Thus they are more suitable choices for computer experiments.

Remark 3.5 The underlying arrays in Algorithm 3.2 are not unique, the setting of $A_2 = (H_2^{\mathrm{T}}, (H_2 \dot{+} 1)^{\mathrm{T}})^{\mathrm{T}}, B_2 = (H_3^{\mathrm{T}}, (H_3 \dot{+} 1)^{\mathrm{T}})^{\mathrm{T}}$ and $C_2 = (H_2^{\mathrm{T}}, H_2^{\mathrm{T}})^{\mathrm{T}}$ results in another OSOA$(2n_0, k, 8, 3)$ with properties α and β, denoted as D_2, these arrays are related through $D_2 = 4A_2 + 2B_2 + C_2$. Furthermore, these two OSOAs can be combined to obtain a grouped design $D = (D_1, D_2)$ with the properties stated in Theorem 3.6.

In order to describe the properties of the design more conveniently, we introduce the notations π_o, π_α and π_β, where π_o presents the proportion of two-tuples that achieve orthogonality, π_α represents the proportion of two-tuples that achieve stratifications on 4×4 grids, π_β denotes the proportion of two-tuples that achieve stratifications on 2×8 and 8×2 grids.

Theorem 3.6 *$D = (D_1, D_2)$ is an* SOA$(2n_0, 2k, 8, 3)$ *with properties α, β and orthogonality in a proportion $\pi_\alpha = \pi_\beta = \pi_o = 2(k-1)/(2k-1)$. Therein any two-tuple $(d_{i_1 j_1}, d_{i_2 j_2})$ achieves α, β and orthogonality except for the case $i_1 \neq i_2$ and $j_1 = j_2$, and the correlation between any two non-orthogonal columns is* $2/21 = 0.0952$*, where d_{ij} represents the jth column of D_i for $i = 1, 2$ and $j = 1, \cdots, k$.*

Proof For the convenience of the proof, define $A = (A_1, A_2), B = (B_1, B_2)$ and $C = (C_1, C_2)$, then D can be written as

$$D = 4A + 2B + C = (D_1, D_2) = (d_{11}, \cdots, d_{1k}, d_{21}, \cdots, d_{2k}),$$

where $d_{ij} = 4a_{ij} + 2b_{ij} + c_{ij}$ and $d_{ij}, a_{ij}, b_{ij}, c_{ij}$ are the jth columns of D_i, A_i, B_i and C_i respectively for $i = 1, 2,\ j = 1, \cdots, k$.

In fact, we can prove that D_2 is an OSOA$(2n_0, k, 8, 3)$ with properties α and β by similar arguments as in the proof of Theorem 3.3. Then Theorem 3.6 holds if we can show that (i) A is an OA of strength three (to ensure that D is an SOA of strength three); (ii) any two-tuple

Table 1 Some OSOAs and SOAs of strength three from Theorems 3.3 and 3.6 and related designs

E	D_1	D: SOA$(2n_0, 2k, 8, 3)$[2]		OSOA$(n, m, 8, 3)$[3]	OSOA$(n, m, 16, 4)$[4]	SOA$(n, m, 8, 3)$[5]
OA$(n_0, k, 4, 2)$	OSOA$(2n_0, k, 8, 3)$[1]	Design	$\pi_o/\pi_\alpha/\pi_\beta$(%)			
OA$(16, 5, 4, 2)$	OSOA$(32, 5, 8, 3)$	SOA$(32, 10, 8, 3)$	88.89	OSOA$(32, 8, 8, 3)$	OSOA$(32, 2, 16, 4)$	SOA$(32, 7, 8, 3)$
OA$(32, 9, 4, 2)$	OSOA$(64, 9, 8, 3)$	SOA$(64, 18, 8, 3)$	94.12	OSOA$(64, 16, 8, 3)$	OSOA$(64, 4, 16, 4)$	SOA$(64, 15, 8, 3)$
OA$(48, 13, 4, 2)$	OSOA$(96, 13, 8, 3)$	SOA$(96, 26, 8, 3)$	96.00	OSOA$(96, 24, 8, 3)$	OSOA$(96, 2, 16, 4)$	-
OA$(64, 21, 4, 2)$	OSOA$(128, 21, 8, 3)$	SOA$(128, 42, 8, 3)$	97.56	OSOA$(128, 32, 8, 3)$	OSOA$(128, 4, 16, 4)$	SOA$(128, 31, 8, 3)$
OA$(96, 23, 4, 2)$	OSOA$(192, 23, 8, 3)$	SOA$(192, 46, 8, 3)$	97.78	OSOA$(192, 48, 8, 3)$	OSOA$(192, 4, 16, 4)$	-
OA$(128, 41, 4, 2)$	OSOA$(256, 41, 8, 3)$	SOA$(256, 82, 8, 3)$	98.77	OSOA$(256, 64, 8, 3)$	OSOA$(256, 6, 16, 4)$	SOA$(256, 63, 8, 3)$
OA$(160, 17, 4, 2)$	OSOA$(320, 17, 8, 3)$	SOA$(320, 34, 8, 3)$	96.97	OSOA$(320, 80, 8, 3)$	OSOA$(320, 4, 16, 4)$	-
OA$(256, 85, 4, 2)$	OSOA$(512, 85, 8, 3)$	SOA$(512, 170, 8, 3)$	99.41	OSOA$(512, 128, 8, 3)$	OSOA$(512, 10, 16, 4)$	SOA$(512, 127, 8, 3)$
OA$(512, 169, 4, 2)$	OSOA$(1024, 169, 8, 3)$	SOA$(1024, 338, 8, 3)$	99.70	OSOA$(1024, 256, 8, 3)$	OSOA$(1024, 14, 16, 4)$	SOA$(1024, 255, 8, 3)$
OA$(1024, 341, 4, 2)$	OSOA$(2048, 341, 8, 3)$	SOA$(2048, 682, 8, 3)$	99.85	OSOA$(2048, 512, 8, 3)$	OSOA$(2048, 16, 16, 4)$	SOA$(2048, 511, 8, 3)$
OA$(2048, 681, 4, 2)$	OSOA$(4096, 681, 8, 3)$	SOA$(4096, 1362, 8, 3)$	99.93	OSOA$(4096, 1024, 8, 3)$	OSOA$(4096, 30, 16, 4)$	SOA$(4096, 1023, 8, 3)$

[1] OSOA of strength three from Theorem 3.3 with properties α and β;

[2] SOA$(2n_0, 2k, 8, 3)$ from Theorem 3.6 with properties α, β and orthogonality in corresponding proportions;

[3] OSOA$(n, m, 2^t, t)$ in [25] for $t = 3$;

[4] OSOA$(n, m, 2^t, t)$ in [25] for $t = 4$;

[5] SOA$(n, m, 8, 3)$ with properties α, β and γ in [30];

Symbol-indicates that the corresponding design is not available.

$(d_{i_1j_1}, d_{i_2j_2})$, with $i_1 \neq i_2$ and $j_1 = j_2$, achieves stratifications on 2×4 and 4×2 grids, and the correlation between these two columns is 2/21=0.0952; and (iii) any two-tuple $(d_{i_1j_1}, d_{i_2j_2})$, with $i_1 \neq i_2$ and $j_1 \neq j_2$, achieves stratifications on 4×4, 2×8 and 8×2 grids, and orthogonality.

(i) First we check the strength of A, where

$$A = \begin{pmatrix} H_1 & H_2 \\ \vdots & \vdots \\ H_1 + 1 & H_2 + 1 \end{pmatrix}.$$

Due to the structure of G, it is an OA of strength two, then as a sub-matrix of G, (H_1, H_2) must be an OA of strength two. In this way, A is an OA of strength three (cf. Theorem 2.24 of [31]).

Now we consider the stratification and orthogonality in two dimensions.

(ii) For any two-tuple (d_{1j_1}, d_{2j_2}) with $j_1 = j_2 = j$, the six columns $(a_{1j}, a_{2j}, b_{1j}, b_{2j}, c_{1j}, c_{2j})$ corresponds to the same G_j, and we have $b_{1j} = c_{2j}$ or $c_{1j} = b_{2j}$. Therein $(a_{1j}, a_{2j}, b_{1j}, b_{2j})$ forms an OA of strength three, indicating that (d_{1j}, d_{2j}) achieves the stratifications on 2×4 and 4×2 grids. As $b_{1j} = c_{2j}$ or $c_{1j} = b_{2j}$, so (d_{1j}, d_{2j}) cannot achieve the stratifications on 4×4, 2×8 and 8×2 grid. The correlation between columns $d_{1j_1} = 4a_{1j_1} + 2b_{1j_1} + c_{1j_1}$ and $d_{2j_1} = 4a_{2j_1} + 2b_{2j_1} + c_{2j_1}$ is $2/(4^2 + 2^2 + 1) = 2/21 = 0.0952$.

(iii) For any two-tuple (d_{1j_1}, d_{2j_2}) with $j_1 \neq j_2$, the six columns $(a_{1j_1}, a_{2j_2}, b_{1j_1}, b_{2j_2}, c_{1j_1}, c_{2j_2})$ corresponds to G_{j_1} and G_{j_2} and is an OA of strength two. Therein $(a_{1j_1}, a_{2j_2}, b_{1j_1}, b_{2j_2})$ is an OA of strength four, indicating that (d_{1j_1}, d_{2j_2}) achieves a stratification on a 4×4 grid. $(a_{1j_1}, a_{2j_2}, b_{1j_1}, c_{1j_1})$, where $(a_{1j_1}, b_{1j_1}, c_{1j_1})$ corresponds to G_{j_1} and a_{2j_2} corresponds to G_{j_2}, is also an OA of strength four, indicating that (d_{1j_1}, d_{2j_2}) achieves stratifications on 2×8 and 8×2 grids. Furthermore, as a sub-matrix of G, $(a_{1j_1}, a_{2j_2}, b_{1j_1}, b_{2j_2}, c_{1j_1}, c_{2j_2})$ is an OA of strength two, which means D is orthogonal. ∎

From Theorems 3.3 and 3.6, we can see that D has twice as many columns as D_1, with $\pi_o = \pi_\alpha = \pi_\beta = 2(k-1)/(2k-1)$. In other words, we can sacrifice just a little proportion of properties of D_1 in exchange for another k columns. It is also worth noting that there is a close relationship between these SOAs and the OSOAs of strength four, that is, the SOAs of strength three from Theorem 3.6 can be regarded as nearly OSOAs of strength four, where the proportions π_α and π_β measure the degree of proximity in terms of two-dimensional space-filling properties, and the proportion π_o measures the degree of proximity in terms of orthogonality.

Let us see an example.

Example 3.7 (Example 3.4 continued) Let H_1, H_2, H_3 and D_1 be the same matrices as in Example 3.4. As shown in Example 3.4, D_1 is an OSOA$(96, 13, 8, 3)$ with properties α and β. Now we obtain another OSOA$(96, 13, 8, 3)$ with properties α and β, say D_2, by taking $D_2 = 4A_2 + 2B_2 + C_2$ with $A_2 = (H_2^{\mathrm{T}}, (H_2 \dotplus 1)^{\mathrm{T}})^{\mathrm{T}}$, $B_2 = (H_3^{\mathrm{T}}, (H_3 \dotplus 1)^{\mathrm{T}})^{\mathrm{T}}$, and $C_2 = (H_2^{\mathrm{T}}, H_2^{\mathrm{T}})^{\mathrm{T}}$. Define $D = (D_1, D_2)$, then we obtain an array with 26 columns, as shown in Table A.2. We can check that D is an SOA$(96, 26, 8, 3)$, that is, D achieves stratifications on 2×4 and 4×2 grids in all two dimensions, and stratifications on $2 \times 2 \times 2$ grids in all three dimensions. In addition, we can check that its two-tuple $(d_{i_1j_1}, d_{i_2j_2})$ achieves stratifications on

4×4, 2×8, 8×2 grids and orthogonality except for the case $i_1\neq i_2$ and $j_1=j_2$. Furthermore, the correlation between any two-tuple $(d_{i_1j_1}, d_{i_2j_2})$ with $i_1\neq i_2$ and $j_1=j_2$ is 0.0952, implying that even if these two columns are not orthogonal, their correlation is really acceptable. Thus we have $\pi_o=\pi_\alpha=\pi_\beta=96.00\%$. In summary, D enjoys properties α,β and orthogonality to a large extent, thus can be seen as a nearly OSOA of strength four.

Table 1 summarizes some SOAs of strength three from Theorem 3.6 (Columns 3–4), the OSOAs of strengths three and four in [25] (Columns 5–6) and the third family of strength-three SOAs in [30] (Column 7). As we have discussed, the resulting SOAs of strength three from Theorem 3.6 can be regarded as nearly OSOAs of strength four, where the proportions π_α and π_β measure the degree of proximity in terms of two-dimensional space-filling properties, and proportion π_o measures the degree of proximity in terms of orthogonality. From Table 1, we can see that the values of these proportions are very close to 1, which means that these SOAs of strength three have almost the same desirable orthogonality as well as two-dimensional space-filling properties as the OSOAs of strength four. Besides, they can accommodate much more columns than the latter ones.

Compared with the strength-three OSOAs in [25], our strength-three SOAs have better two-dimensional space-filling properties and can accommodate more columns, while guaranteeing almost the same orthogonality. Compared with the strength-four OSOAs in [25], our strength-three SOAs have almost the same desirable orthogonality as well as two-dimensional space-filling properties, and they can accommodate much more columns than the latter ones. Compared with the third family of strength-three SOAs in [30], which enjoy properties α,β and γ, the resulting strength-three SOAs from Theorem 3.6 enjoy better orthogonality property and can accommodate more columns when the third family of strength-three SOAs are available. Besides, the resulting SOAs are particularly useful when the run size n is a multiple of 32 but not a power of 2, where the third family of strength-three SOAs are not available. That is, the resulting SOAs can fill the gap between the run sizes of the available third family of strength-three SOAs. For example, we can construct strength-three SOAs with attractive two-dimensional space-filling properties of 96 and 160 runs while such third family of strength-three SOAs are not available. All these desirable properties ensure the resulting SOAs to be competitive designs for computer experiments.

3.2 Nearly Orthogonal Strength-Three SOAs with Property γ

Now, we present the construction of the nearly orthogonal SOAs of strength three that enjoy property γ, that is, the same three-dimensional space-filling properties as the strength-four SOAs. The construction method is given in the following algorithm.

Algorithm 3.8
Input: An OA$(n_0,p,2,3)$, called U.
Output: An SOA$(2n_0,p,8,3)$, called $\widetilde{D}$.
Step 1 Let U be an OA$(n_0,p,2,3)$ with entries from $GF(2)$.
Step 2 Create $X=(U^{\rm T},(U\dot{+}1)^{\rm T})^{\rm T}$, $Y=(U^{*{\rm T}},U^{*{\rm T}})^{\rm T}$ and $Z=(U^{\rm T},U^{\rm T})^{\rm T}$, where $\dot{+}$ is

the addition defined on $GF(2)$, and $*$ is a right circular shift of the columns of a design, that is $D^* = (d_m, d_1, \cdots, d_{m-1})$ for any design $D = (d_1, d_2, \cdots, d_m)$.

Step 3 Treat all entries as numbers and define $\widetilde{D} = 4X + 2Y + Z$.

For the resulting design, we have the following theorem.

Theorem 3.9 *The design $\widetilde{D}$ obtained in Algorithm 3.8 is an $\mathrm{SOA}(2n_0, p, 8, 3)$ with property γ, while it achieves property β and orthogonality in a proportion $\pi_\beta = \pi_o = (p-3)/(p-1)$. Therein any two columns are orthogonal and achieve stratifications on 2×8 and 8×2 grids if and only if they are not adjacent, and the correlation between any two adjacent columns is $2/21 = 0.0952$, where the first and last columns are regarded as two adjacent columns.*

Proof By the similar proof of Theorem 1 and Remark 1 in [32], we can easily verify that $\widetilde{D}$ is an SOA of strength three. Now we only need to prove (i) property γ holds for $\widetilde{D}$; (ii) $\widetilde{D}$ enjoys property β and orthogonality in a proportion $\pi = (p-3)/(p-1)$; and (iii) the correlation between any two adjacent columns is 2/21=0.0952.

For the convenience of the proof, let u_i, x_i, y_i, z_i and d_i be the ith columns of U, X, Y, Z and $\widetilde{D}$ respectively.

(i) First we consider the three-dimensional stratification of $\widetilde{D}$, where $\widetilde{D} = 4X + 2Y + Z$ with

$$X = \begin{pmatrix} u_1 & u_2 & \cdots & u_p \\ u_1 \dot{+} 1 & u_2 \dot{+} 1 & \cdots & u_p \dot{+} 1 \end{pmatrix},$$

$$Y = \begin{pmatrix} u_p & u_1 & \cdots & u_{p-1} \\ u_p & u_1 & \cdots & u_{p-1} \end{pmatrix}, \text{ and } Z = \begin{pmatrix} u_1 & u_2 & \cdots & u_p \\ u_1 & u_2 & \cdots & u_p \end{pmatrix}.$$

To show that $\widetilde{D}$ enjoys property γ, we only need to prove that (x_i, x_j, x_k, y_k) is an OA of strength four for all $i \neq j, i \neq k$ and $j \neq k$, which are in the following two cases.

(a) Case 1: $k-1 = i$ or $k-1 = j$. Without loss of generality, we suppose $k-1 = i$, then (x_i, x_j, x_k, y_k) can be written as

$$\begin{pmatrix} u_i & u_j & u_k & u_i \\ u_i \dot{+} 1 & u_j \dot{+} 1 & u_k \dot{+} 1 & u_i \end{pmatrix}.$$

Note that (u_i, u_j, u_k) is an OA of strength three, then $(u_i \dot{+} 1, u_j \dot{+} 1, u_k \dot{+} 1)$ is also an OA of strength three. And for any level combination $(\alpha, \beta, \gamma, \alpha)$ in (u_i, u_j, u_k, u_i), there is a corresponding level combination $(\alpha, \beta, \gamma, \alpha \dot{+} 1)$ in $(u_i \dot{+} 1, u_j \dot{+} 1, u_k \dot{+} 1, u_i)$. Thus (x_i, x_j, x_k, y_k) is an OA of strength four.

(b) Case 2: $k-1 = l$ with $l \neq i$ and $l \neq j$, then (x_i, x_j, x_k, y_k) can be written as

$$\begin{pmatrix} u_i & u_j & u_k & u_l \\ u_i \dot{+} 1 & u_j \dot{+} 1 & u_k \dot{+} 1 & u_l \end{pmatrix}.$$

Therein (u_i, u_j, u_k) and $(u_i \dot{+} 1, u_j \dot{+} 1, u_k \dot{+} 1)$ are OAs of strength three. And for any level combination $(\alpha, \beta, \gamma, \eta_1)$ in (u_i, u_j, u_k, u_l), there is a corresponding level combination $(\alpha, \beta, \gamma, \eta_2)$ in $(u_i \dot{+} 1, u_j \dot{+} 1, u_k \dot{+} 1, u_l)$, where η_1 and η_2 are distinct with each other and they are a permutation on $\{0, 1\}$. Thus (x_i, x_j, x_k, y_k) is an OA of strength four.

(ii) We consider the two-dimensional stratification of $\widetilde{D}$. For any two-tuple (d_i, d_j), it achieves property β if and only if (x_i, x_j, y_j, z_j) and (x_i, y_i, z_i, x_j) are OAs of strength four. Note that (x_i, x_j, y_j, z_j) and (x_i, y_i, z_i, x_j) can be written as

$$\begin{pmatrix} u_i & u_j & u_{j-1} & u_j \\ u_i \dot{+} 1 & u_j \dot{+} 1 & u_{j-1} & u_j \end{pmatrix} \text{ and } \begin{pmatrix} u_i & u_{i-1} & u_i & u_j \\ u_i \dot{+} 1 & u_{i-1} & u_i & u_j \dot{+} 1 \end{pmatrix}$$

respectively. Obviously, (x_i, x_j, y_j, z_j) is an OA of strength four if and only if $j - 1 \neq i$, while (x_i, y_i, z_i, x_j) is an OA of strength four if and only if $i - 1 \neq j$. That is, (d_i, d_j) can achieve stratifications on 2×8 and 8×2 grids if and only if $i - 1 \neq j$ and $j - 1 \neq i$. Therefore, any two columns of $\widetilde{D}$ can achieve stratifications on 2×8 and 8×2 grids if and only if they are not adjacent.

Next we consider the orthogonality of $\widetilde{D}$. According to its structure, (d_i, d_j) is orthogonal if and only if the following array is an OA of strength two:

$$\begin{pmatrix} u_i & u_{i-1} & u_i & u_j & u_{j-1} & u_j \\ u_i \dot{+} 1 & u_{i-1} & u_i & u_j \dot{+} 1 & u_{j-1} & u_j \end{pmatrix}, \tag{2}$$

which is equivalent to that $i - 1 \neq j$ and $j - 1 \neq i$. In this way, for any two columns of $\widetilde{D}$, they are not orthogonal to each other if and only if they are adjacent. Thus, $\widetilde{D}$ enjoy property β and orthogonality in a proportion $\pi = (p - 3)/(p - 1)$.

(iii) For two adjacent columns d_i and d_j of $\widetilde{D}$, we have $i - 1 = j$ or $j - 1 = i$. Then according to (2), the correlation between these two columns is $2/(4^2 + 2^2 + 1) = 2/21 = 0.0952$. ∎

The following is an illustrative example.

Example 3.10 We now construct an SOA$(48, 12, 8, 3)$ with property γ. Let $U = (u_1, \cdots, u_{12})$ be an OA$(24, 12, 2, 3)$ with entries from $GF(2)$, as shown in Table A.3. Define

$$X = \begin{pmatrix} u_1 & u_2 & \cdots & u_{12} \\ u_1 \dot{+} 1 & u_2 \dot{+} 1 & \cdots & u_{12} \dot{+} 1 \end{pmatrix},$$

$$Y = \begin{pmatrix} u_{12} & u_1 & \cdots & u_{11} \\ u_{12} & u_1 & \cdots & u_{11} \end{pmatrix}, \text{ and } Z = \begin{pmatrix} u_1 & u_2 & \cdots & u_{12} \\ u_1 & u_2 & \cdots & u_{12} \end{pmatrix}.$$

Treat all entries as numbers then we can obtain an SOA$(48, 12, 8, 3)$ by taking $\widetilde{D} = 4X + 2Y + Z$, which is shown in Table A.4. We can check that any three columns of $\widetilde{D}$ achieve stratifications on $2 \times 2 \times 4$, $2 \times 4 \times 2$ and $4 \times 2 \times 2$ grids, i.e., $\widetilde{D}$ enjoys property γ. In addition, any two columns are orthogonal and achieve stratifications on 2×8 and 8×2 grids if and only if they are not adjacent. Thus we have $\pi_\beta = \pi_o = 81.82\%$. Furthermore, we can check that the correlation between any two adjacent columns is 0.0952, implying that even if any two adjacent columns are not orthogonal, the correlation between them is really acceptable.

From Theorem 3.9, we can obtain many nearly orthogonal SOAs of strength three with property γ. The OAs needed in Algorithm 3.8 are available in the library of OAs maintained by Dr. N. J. A. Sloane (http://neilsloane.com/oadir/index.html) and [31]. Table 2 summarizes some resulting designs and related designs. We can see that most values of π_β (π_o) are very close to 1, indicating that the resulting designs enjoy orthogonality and property β to a large extent. Shi and Tang[30] constructed strength-three SOAs with property γ using regular designs. Compared with their strength-three SOAs, the resulting SOAs in Theorem 3.9 have a better orthogonality and two-dimensional stratification property with the same numbers of columns when n is a power of 2. Besides, our strength-three SOAs are particularly useful when the run size n is a multiple of 16 but not a power of 2, which cannot be constructed by their method. That is, we can fill the gap between the run sizes of their available strength-three SOAs. For example, we can construct strength-three SOAs of 48 and 80 runs while such SOAs are not available in the literature. All these desirable properties ensure the strength-three SOAs to be competitive designs for computer experiments.

Table 2 Some SOAs of strength three from Theorem 3.9 and related designs

C: OA$(n_0, p, 2, 3)$	$\widetilde{D}$: SOA$(2n_0, p, 8, 3)^\dagger$		SOA$(n, m, 8, 3)^\ddagger$
	Design	$\pi_\beta = \pi_o(\%)$	
OA(16, 8, 2, 3)	SOA(32, 8, 8, 3)	71.43	SOA(32, 8, 8, 3)
OA(24, 12, 2, 3)	SOA(48, 12, 8, 3)	81.82	-
OA(32, 16, 2, 3)	SOA(64, 16, 8, 3)	86.67	SOA(64, 16, 8, 3)
OA(40, 20, 2, 3)	SOA(80, 20, 8, 3)	89.47	-
OA(48, 24, 2, 3)	SOA(96, 24, 8, 3)	91.30	-
OA(56, 28, 2, 3)	SOA(112, 28, 8, 3)	92.59	-
OA(64, 32, 2, 3)	SOA(128, 32, 8, 3)	93.55	SOA(128, 32, 8, 3)
OA(72, 36, 2, 3)	SOA(144, 36, 8, 3)	94.29	-
OA(80, 40, 2, 3)	SOA(160, 40, 8, 3)	94.87	-
OA(88, 44, 2, 3)	SOA(176, 44, 8, 3)	95.35	-
OA(96, 48, 2, 3)	SOA(192, 48, 8, 3)	95.74	-
OA(104, 52, 2, 3)	SOA(208, 52, 8, 3)	96.08	-
OA(112, 56, 2, 3)	SOA(224, 56, 8, 3)	96.36	-
OA(120, 60, 2, 3)	SOA(240, 60, 8, 3)	96.61	-
OA(128, 64, 2, 3)	SOA(256, 64, 8, 3)	96.83	SOA(256, 64, 8, 3)

† SOA$(2n_0, p, 8, 3)$ from Theorem 3.9 that enjoys property γ, as well as property β and orthogonality in a proportion $\pi_\beta = \pi_o = (p-3)/(p-1)$;

‡ SOA$(n, m, 8, 3)$ with property γ in [30] (Theorem 4), where n is a power of 2 and $m = n/4$; Symbol-indicates that the corresponding design is not available.

4 Concluding Remarks

For designs of computer experiments, space-filling property and orthogonality are two important properties. As a class of space-filling designs, SOAs are getting more and more attention as they have better space-filling properties than ordinary OAs. To surely realize better space-filling properties, SOAs of strength three or higher are desirable. This paper develops methods for constructing (nearly) orthogonal SOAs of strength three with better space-filling properties. Based on OAs which can be either regular or nonregular, the proposed methods are general and simple. The newly generated strength-three SOAs enjoy almost the same space-filling properties of strength-four SOAs, and can accommodate much more columns than the latter. Moreover, they are (nearly) orthogonal and flexible in run sizes. All these desirable properties make them competitive designs for computer experiments. Examples are provided throughout to illustrate the methods, and we tabulate many resulting designs for practical needs.

Conflict of Interest

The authors declare no conflict of interest.

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Appendix: Four tables in Examples 3.4, 3.7 and 3.10

Table A.1 The OA(48, 13, 4, 2) $E = (e_1, \cdots, e_{13})$ in Example 3.4

e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	3	3	3	2	2	2	0
0	0	0	3	3	3	2	2	2	1	1	1	0
0	3	1	2	1	3	1	2	0	3	0	2	1
0	3	1	3	2	1	0	1	2	2	3	0	1
0	3	1	1	3	2	2	0	1	0	2	3	1
0	1	2	3	0	2	1	0	3	1	3	2	2
0	1	2	2	3	0	3	1	0	2	1	3	2
0	1	2	0	2	3	0	3	1	3	2	1	2
0	2	3	1	2	0	1	3	2	1	0	3	3
0	2	3	0	1	2	2	1	3	3	1	0	3
0	2	3	2	0	1	3	2	1	0	3	1	3
1	1	1	1	1	1	1	1	1	1	1	1	0
1	1	1	0	0	0	2	2	2	3	3	3	0
1	1	1	2	2	2	3	3	3	0	0	0	0
1	2	0	3	0	2	0	3	1	2	1	3	1
1	2	0	2	3	0	1	0	3	3	2	1	1
1	2	0	0	2	3	3	1	0	1	3	2	1
1	0	3	2	1	3	0	1	2	0	2	3	2
1	0	3	3	2	1	2	0	1	3	0	2	2
1	0	3	1	3	2	1	2	0	2	3	0	2
1	3	2	0	3	1	0	2	3	0	1	2	3
1	3	2	1	0	3	3	0	2	2	0	1	3
1	3	2	3	1	0	2	3	0	1	2	0	3
2	2	2	2	2	2	2	2	2	2	2	2	0
2	2	2	3	3	3	1	1	1	0	0	0	0
2	2	2	1	1	1	0	0	0	3	3	3	0
2	1	3	0	3	1	3	0	2	1	2	0	1
2	1	3	1	0	3	2	3	0	0	1	2	1
2	1	3	3	1	0	0	2	3	2	0	1	1
2	3	0	1	2	0	3	2	1	3	1	0	2
2	3	0	0	1	2	1	3	2	0	3	1	2
2	3	0	2	0	1	2	1	3	1	0	3	2
2	0	1	3	0	2	3	1	0	3	2	1	3
2	0	1	2	3	0	0	3	1	1	3	2	3
2	0	1	0	2	3	1	0	3	2	1	3	3
3	3	3	3	3	3	3	3	3	3	3	3	0
3	3	3	2	2	2	0	0	0	1	1	1	0
3	3	3	0	0	0	1	1	1	2	2	2	0
3	0	2	1	2	0	2	1	3	0	3	1	1
3	0	2	0	1	2	3	2	1	1	0	3	1
3	0	2	2	0	1	1	3	2	3	1	0	1
3	2	1	0	3	1	2	3	0	2	0	1	2
3	2	1	1	0	3	0	2	3	1	2	0	2
3	2	1	3	1	0	3	0	2	0	1	2	2
3	1	0	2	1	3	2	0	1	2	3	0	3
3	1	0	3	2	1	1	2	0	0	2	3	3
3	1	0	1	3	2	0	1	2	3	0	2	3

Table A.2 The SOA(96, 26, 8, 3) $D = (D_1, D_2)$ in Examples 3.4 and 3.7

$D_1 = (d_{11}, \cdots, d_{113})$													$D_2 = (d_{21}, \cdots, d_{213})$												
d_{11}	d_{12}	d_{13}	d_{14}	d_{15}	d_{16}	d_{17}	d_{18}	d_{19}	d_{110}	d_{111}	d_{112}	d_{113}	d_{21}	d_{22}	d_{23}	d_{24}	d_{25}	d_{26}	d_{27}	d_{28}	d_{29}	d_{210}	d_{211}	d_{212}	d_{213}
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	2	2	2	7	7	7	5	5	5	0	0	0	0	7	7	7	5	5	5	2	2	2	0
0	0	0	7	7	7	5	5	5	2	2	2	0	0	0	0	5	5	5	2	2	2	7	7	7	0
0	7	2	5	2	7	2	5	0	7	0	5	2	0	5	7	2	7	5	7	2	0	5	0	2	7
0	7	2	7	5	2	0	2	5	5	7	0	2	0	5	7	5	2	7	0	7	2	2	5	0	7
0	7	2	2	7	5	5	0	2	0	5	7	2	0	5	7	7	5	2	2	0	7	0	2	5	7
0	2	5	7	0	5	2	0	7	2	7	5	5	0	7	2	5	0	2	7	0	5	7	5	2	2
0	2	5	5	7	0	7	2	0	5	2	7	5	0	7	2	2	5	0	5	7	0	2	7	5	2
0	2	5	0	5	7	0	7	2	7	5	2	5	0	7	2	0	2	5	0	5	7	5	2	7	2
0	5	7	2	5	0	2	7	5	2	0	7	7	0	2	5	7	2	0	7	5	2	7	0	5	5
0	5	7	0	2	5	5	2	7	7	2	0	7	0	2	5	0	7	2	2	7	5	5	7	0	5
0	5	7	5	0	2	7	5	2	0	7	2	7	0	2	5	2	0	7	5	2	7	0	5	7	5
2	2	2	2	2	2	2	2	2	2	2	2	0	7	7	7	7	7	7	7	7	7	7	7	7	0
2	2	2	0	0	0	5	5	5	7	7	7	0	7	7	7	0	0	0	2	2	2	5	5	5	0
2	2	2	5	5	5	7	7	7	0	0	0	0	7	7	7	2	2	2	5	5	5	0	0	0	0
2	5	0	7	0	5	0	7	2	5	2	7	2	7	2	0	5	0	2	0	5	7	2	7	5	7
2	5	0	5	7	0	2	0	7	7	5	2	2	7	2	0	2	5	0	7	0	5	5	2	7	7
2	5	0	0	5	7	7	2	0	2	7	5	2	7	2	0	0	2	5	5	7	0	7	5	2	7
2	0	7	5	2	7	0	2	5	0	5	7	5	7	0	5	2	7	5	0	7	2	0	2	5	2
2	0	7	7	5	2	5	0	2	7	0	5	5	7	0	5	5	2	7	2	0	7	5	0	2	2
2	0	7	2	7	5	2	5	0	5	7	0	5	7	0	5	7	5	2	7	2	0	2	5	0	2
2	7	5	0	7	2	0	5	7	0	2	5	7	7	5	2	0	5	7	0	2	5	0	7	2	5
2	7	5	2	0	7	7	0	5	5	0	2	7	7	5	2	7	0	5	5	0	2	2	0	7	5
2	7	5	7	2	0	5	7	0	2	5	0	7	7	5	2	5	7	0	2	5	0	7	2	0	5
5	5	5	5	5	5	5	5	5	5	5	5	0	2	2	2	2	2	2	2	2	2	2	2	2	0
5	5	5	7	7	7	2	2	2	0	0	0	0	2	2	2	5	5	5	7	7	7	0	0	0	0
5	5	5	2	2	2	0	0	0	7	7	7	0	2	2	2	7	7	7	0	0	0	5	5	5	0
5	2	7	0	7	2	7	0	5	2	5	0	2	2	7	5	0	5	7	5	0	2	7	2	0	7
5	2	7	2	0	7	5	7	0	0	2	5	2	2	7	5	7	0	5	2	5	0	0	7	2	7
5	2	7	7	2	0	0	5	7	5	0	2	2	2	7	5	5	7	0	0	2	5	2	0	7	7
5	7	0	2	5	0	7	5	2	7	2	0	5	2	5	0	7	2	0	5	2	7	5	7	0	2
5	7	0	0	2	5	2	7	5	0	7	2	5	2	5	0	0	7	2	7	5	2	0	5	7	2
5	7	0	5	0	2	5	2	7	2	0	7	5	2	5	0	2	0	7	2	7	5	7	0	5	2
5	0	2	7	0	5	7	2	0	7	5	2	7	2	0	7	5	0	2	5	7	0	5	2	7	5
5	0	2	5	7	0	0	7	2	2	7	5	7	2	0	7	2	5	0	0	5	7	7	5	2	5
5	0	2	0	5	7	2	0	7	5	2	7	7	2	0	7	0	2	5	7	0	5	2	7	5	5
7	7	7	7	7	7	7	7	7	7	7	7	0	5	5	5	5	5	5	5	5	5	5	5	5	0
7	7	7	5	5	5	0	0	0	2	2	2	0	5	5	5	2	2	2	0	0	0	7	7	7	0
7	7	7	0	0	0	2	2	2	5	5	5	0	5	5	5	0	0	0	7	7	7	2	2	2	0
7	0	5	2	5	0	5	2	7	0	7	2	2	5	0	2	7	2	0	2	7	5	0	5	7	7
7	0	5	0	2	5	7	5	2	2	0	7	2	5	0	2	0	7	2	5	2	7	7	0	5	7
7	0	5	5	0	2	2	7	5	7	2	0	2	5	0	2	2	0	7	7	5	2	5	7	0	7
7	5	2	0	7	2	5	7	0	5	0	2	5	5	2	7	0	5	7	2	5	0	2	0	7	2
7	5	2	2	0	7	0	5	7	2	5	0	5	5	2	7	7	0	5	0	2	5	7	2	0	2
7	5	2	7	2	0	7	0	5	0	2	5	5	5	2	7	5	7	0	5	0	2	0	7	2	2
7	2	0	5	2	7	5	0	2	5	7	0	7	5	7	0	2	7	5	2	0	7	2	5	0	5
7	2	0	7	5	2	2	5	0	0	5	7	7	5	7	0	5	2	7	7	2	0	0	2	5	5
7	2	0	2	7	5	0	2	5	7	0	5	7	5	7	0	7	5	2	0	7	2	5	0	2	5
4	4	4	4	4	4	4	4	4	4	4	4	4	6	6	6	6	6	6	6	6	6	6	6	6	6
4	4	4	6	6	6	3	3	3	1	1	1	4	6	6	6	1	1	1	3	3	3	4	4	4	6
4	4	4	3	3	3	1	1	1	6	6	6	4	6	6	6	3	3	3	4	4	4	1	1	1	6
4	3	6	1	6	3	6	1	4	3	4	1	6	6	3	1	4	1	3	1	4	6	3	6	4	1
4	3	6	3	1	6	4	6	1	1	3	4	6	6	3	1	3	4	1	6	1	4	4	3	6	1
4	3	6	6	3	1	1	4	6	4	1	3	6	6	3	1	1	3	4	4	6	1	6	4	3	1
4	6	1	3	4	1	6	4	3	6	3	1	1	6	1	4	3	6	4	1	6	3	1	3	4	4
4	6	1	1	3	4	3	6	4	1	6	3	1	6	1	4	4	3	6	3	1	6	4	1	3	4
4	6	1	4	1	3	4	3	6	3	1	6	1	6	1	4	6	4	3	6	3	1	3	4	1	4
4	1	3	6	1	4	6	3	1	6	4	3	3	6	4	3	1	4	6	1	3	4	1	6	3	3
4	1	3	4	6	1	1	6	3	3	6	4	3	6	4	3	6	1	4	4	1	3	3	1	6	3
4	1	3	1	4	6	3	1	6	4	3	6	3	6	4	3	4	6	1	3	4	1	6	3	1	3

Table A.2 (Continued) The SOA$(96, 26, 8, 3)$ $D = (D_1, D_2)$ in Examples 3.4 and 3.7

$D_1 = (d_{11}, \cdots, d_{113})$													$D_2 = (d_{21}, \cdots, d_{213})$												
d_{11}	d_{12}	d_{13}	d_{14}	d_{15}	d_{16}	d_{17}	d_{18}	d_{19}	d_{110}	d_{111}	d_{112}	d_{113}	d_{21}	d_{22}	d_{23}	d_{24}	d_{25}	d_{26}	d_{27}	d_{28}	d_{29}	d_{210}	d_{211}	d_{212}	d_{213}
6	6	6	6	6	6	6	6	6	6	6	6	4	1	1	1	1	1	1	1	1	1	1	1	1	6
6	6	6	4	4	4	1	1	1	3	3	3	4	1	1	1	6	6	6	4	4	4	3	3	3	6
6	6	6	1	1	1	3	3	3	4	4	4	4	1	1	1	4	4	4	3	3	3	6	6	6	6
6	1	4	3	4	1	4	3	6	1	6	3	6	1	4	6	3	6	4	6	3	1	4	1	3	1
6	1	4	1	3	4	6	4	3	3	1	6	6	1	4	6	4	3	6	1	6	3	3	4	1	1
6	1	4	4	1	3	3	6	4	6	3	1	6	1	4	6	6	4	3	3	1	6	1	3	4	1
6	4	3	1	6	3	4	6	1	4	1	3	1	1	6	3	4	1	3	6	1	4	6	4	3	4
6	4	3	3	1	6	1	4	6	3	4	1	1	1	6	3	3	4	1	4	6	1	3	6	4	4
6	4	3	6	3	1	6	1	4	1	3	4	1	1	6	3	1	3	4	1	4	6	4	3	6	4
6	3	1	4	3	6	4	1	3	4	6	1	3	1	3	4	6	3	1	6	4	3	6	1	4	3
6	3	1	6	4	3	3	4	1	1	4	6	3	1	3	4	1	6	3	3	6	4	4	6	1	3
6	3	1	3	6	4	1	3	4	6	1	4	3	1	3	4	3	1	6	4	3	6	1	4	6	3
1	1	1	1	1	1	1	1	1	1	1	1	4	4	4	4	4	4	4	4	4	4	4	4	4	6
1	1	1	3	3	3	6	6	6	4	4	4	4	4	4	4	3	3	3	1	1	1	6	6	6	6
1	1	1	6	6	6	4	4	4	3	3	3	4	4	4	4	1	1	1	6	6	6	3	3	3	6
1	6	3	4	3	6	3	4	1	6	1	4	6	4	1	3	6	3	1	3	6	4	1	4	6	1
1	6	3	6	4	3	1	3	4	4	6	1	6	4	1	3	1	6	3	4	3	6	6	1	4	1
1	6	3	3	6	4	4	1	3	1	4	6	6	4	1	3	3	1	6	6	4	3	4	6	1	1
1	3	4	6	1	4	3	1	6	3	6	4	1	4	3	6	1	4	6	3	4	1	3	1	6	4
1	3	4	4	6	1	6	3	1	4	3	6	1	4	3	6	6	1	4	1	3	4	6	3	1	4
1	3	4	1	4	6	1	6	3	6	4	3	1	4	3	6	4	6	1	4	1	3	1	6	3	4
1	4	6	3	4	1	3	6	4	3	1	6	3	4	6	1	3	6	4	3	1	6	3	4	1	3
1	4	6	1	3	4	4	3	6	6	3	1	3	4	6	1	4	3	6	6	3	1	1	3	4	3
1	4	6	4	1	3	6	4	3	1	6	3	3	4	6	1	6	4	3	1	6	3	4	1	3	3
3	3	3	3	3	3	3	3	3	3	3	3	4	3	3	3	3	3	3	3	3	3	3	3	3	6
3	3	3	1	1	1	4	4	4	6	6	6	4	3	3	3	4	4	4	6	6	6	1	1	1	6
3	3	3	4	4	4	6	6	6	1	1	1	4	3	3	3	6	6	6	1	1	1	4	4	4	6
3	4	1	6	1	4	1	6	3	4	3	6	6	3	6	4	1	4	6	4	1	3	6	3	1	1
3	4	1	4	6	1	3	1	6	6	4	3	6	3	6	4	6	1	4	3	4	1	1	6	3	1
3	4	1	1	4	6	6	3	1	3	6	4	6	3	6	4	4	6	1	1	3	4	3	1	6	1
3	1	6	4	3	6	1	3	4	1	4	6	1	3	4	1	6	3	1	4	3	6	4	6	1	4
3	1	6	6	4	3	4	1	3	6	1	4	1	3	4	1	1	6	3	6	4	3	1	4	6	4
3	1	6	3	6	4	3	4	1	4	6	1	1	3	4	1	3	1	6	3	6	4	6	1	4	4
3	6	4	1	6	3	1	4	6	1	3	4	3	3	1	6	4	1	3	4	6	1	4	3	6	3
3	6	4	3	1	6	6	1	4	4	1	3	3	3	1	6	3	4	1	1	4	6	6	4	3	3
3	6	4	6	3	1	4	6	1	3	4	1	3	3	1	6	1	3	4	6	1	4	3	6	4	3

Table A.3 The OA$(24, 12, 2, 3)$ $U = (u_1, \cdots, u_{12})$ in Example 3.10

u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}	u_{11}	u_{12}
0	1	1	1	1	1	1	1	1	1	1	1
0	0	1	0	1	1	1	0	0	0	1	0
0	0	0	1	0	1	1	1	0	0	0	1
0	1	0	0	1	0	1	1	1	0	0	0
0	0	1	0	0	1	0	1	1	1	0	0
0	0	0	1	0	0	1	0	1	1	1	0
0	0	0	0	1	0	0	1	0	1	1	1
0	1	0	0	0	1	0	0	1	0	1	1
0	1	1	0	0	0	1	0	0	1	0	1
0	1	1	1	0	0	0	1	0	0	1	0
0	0	1	1	1	0	0	0	1	0	0	1
0	1	0	1	1	1	0	0	0	1	0	0
1	0	0	0	0	0	0	0	0	0	0	0
1	1	0	1	0	0	0	1	1	1	0	1
1	1	1	0	1	0	0	0	1	1	1	0
1	0	1	1	0	1	0	0	0	1	1	1
1	1	0	1	1	0	1	0	0	0	1	1
1	1	1	0	1	1	0	1	0	0	0	1
1	1	1	1	0	1	1	0	1	0	0	0
1	0	1	1	1	0	1	1	0	1	0	0
1	0	0	1	1	1	0	1	1	0	1	0
1	0	0	0	1	1	1	0	1	1	0	1
1	1	0	0	0	1	1	1	0	1	1	0
1	0	1	0	0	0	1	1	1	0	1	1

Table A.4 The SOA(48, 12, 8, 3) $\widetilde{D} = (d_1, \cdots, d_{12})$ with property γ in Example 3.10

d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9	d_{10}	d_{11}	d_{12}
2	5	7	7	7	7	7	7	7	7	7	7
0	0	5	2	5	7	7	2	0	0	5	2
2	0	0	5	2	5	7	7	2	0	0	5
0	5	2	0	5	2	5	7	7	2	0	0
0	0	5	2	0	5	2	5	7	7	2	0
0	0	0	5	2	0	5	2	5	7	7	2
2	0	0	0	5	2	0	5	2	5	7	7
2	5	2	0	0	5	2	0	5	2	5	7
2	5	7	2	0	0	5	2	0	5	2	5
0	5	7	7	2	0	0	5	2	0	5	2
2	0	5	7	7	2	0	0	5	2	0	5
0	5	2	5	7	7	2	0	0	5	2	0
5	2	0	0	0	0	0	0	0	0	0	0
7	7	2	5	2	0	0	5	7	7	2	5
5	7	7	2	5	2	0	0	5	7	7	2
7	2	5	7	2	5	2	0	0	5	7	7
7	7	2	5	7	2	5	2	0	0	5	7
7	7	7	2	5	7	2	5	2	0	0	5
5	7	7	7	2	5	7	2	5	2	0	0
5	2	5	7	7	2	5	7	2	5	2	0
5	2	0	5	7	7	2	5	7	2	5	2
7	2	0	0	5	7	7	2	5	7	2	5
5	7	2	0	0	5	7	7	2	5	7	2
7	2	5	2	0	0	5	7	7	2	5	7
6	1	3	3	3	3	3	3	3	3	3	3
4	4	1	6	1	3	3	6	4	4	1	6
6	4	4	1	6	1	3	3	6	4	4	1
4	1	6	4	1	6	1	3	3	6	4	4
4	4	1	6	4	1	6	1	3	3	6	4
4	4	4	1	6	4	1	6	1	3	3	6
6	4	4	4	1	6	4	1	6	1	3	3
6	1	6	4	4	1	6	4	1	6	1	3
6	1	3	6	4	4	1	6	4	1	6	1
4	1	3	3	6	4	4	1	6	4	1	6
6	4	1	3	3	6	4	4	1	6	4	1
4	1	6	1	3	3	6	4	4	1	6	4
1	6	4	4	4	4	4	4	4	4	4	4
3	3	6	1	6	4	4	1	3	3	6	1
1	3	3	6	1	6	4	4	1	3	3	6
3	6	1	3	6	1	6	4	4	1	3	3
3	3	6	1	3	6	1	6	4	4	1	3
3	3	3	6	1	3	6	1	6	4	4	1
1	3	3	3	6	1	3	6	1	6	4	4
1	6	1	3	3	6	1	3	6	1	6	4
1	6	4	1	3	3	6	1	3	6	1	6
3	6	4	4	1	3	3	6	1	3	6	1
1	3	6	4	4	1	3	3	6	1	3	6
3	6	1	6	4	4	1	3	3	6	1	3