J Syst Sci Complex (2023) 36: 2515–2536

The Impact of General Correlation Under Multi-Period Mean-Variance Asset-Liability Portfolio Management*

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DOI: 10.1007/s11424-023-3019-6
Received: 18 January 2023 / Revised: 2 August 2023

Abstract This paper studies the multi-period mean-variance (MV) asset-liability portfolio management problem (MVAL), in which the portfolio is constructed by risky assets and liability. It is worth mentioning that the impact of general correlation is considered, i.e., the random returns of risky assets and the liability are not only statistically correlated to each other but also correlated to themselves in different time periods. Such a model with a general correlation structure extends the classical multi-period MVAL models with assumption of independent returns. The authors derive the explicit portfolio policy and the MV efficient frontier for this problem. Moreover, a numerical example is presented to illustrate the efficiency of the proposed solution scheme.

Keywords Asset-liability management, dynamic programming, mean-variance, multi-period portfolio, stochastic correlated returns.

1 Introduction

Since Markowitz[1] published his seminar work in 1952, the mean-variance (MV) portfolio selection model has become a standard tool of the modern investment analysis in both academic study and financial practice. This static MV portfolio selection model has been extended to dynamic one[2, 3], which is one of the most significant development in this area. The multi-period MV portfolio optimization models have been developed greatly in the past years. For example, the mean-field formulation was introduced in [4], the management fee was considered

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*This research was partially supported by the National Natural Science Foundation of China under Grant Nos. 72201067, 12201129, and 71973028, the Natural Science Foundation of Guangdong Province under Grant No. 2022A1515010839, the Project of Science and Technology of Guangzhou under Grant No. 202102020273, the Opening Project of Guangdong Province Key Laboratory of Computational Science at Sun Yat-sen University under Grant No. 2021004.
◇ *This paper was recommended for publication by Editor YAO Dacheng.*

in [5], the no-shorting constraint was investigated in [6, 7] and both no-shorting constraints and regime-switching were studied in [8].

On the other hand, some financial institutions, such as banks and insurance companies, also pay great attention to their liability. In fact, one can operate more soundly and lucratively when considering the effects of liability. The asset liability management (ALM) problem under MV framework was first investigated in a single-period setting[9]. It was then extended to multi-period setting[10] and also with different constraints, e.g., with an uncertain exiting time[11, 12], with probability constrains[13], with an uncontrolled cash flow[14] and with jump diffusion process[15]. It is noted that the returns of assets and liability in the above literatures are assumed to be statistical independent. However, as pointed out in [16], the correlation of the returns of the risky assets can not be neglected in portfolio management. In their work, they adopted the ARMA(1, 1) process to demonstrate the correlation characteristics of the risky assets returns. When investors take into account the fact that risky asset returns are both correlated to each other and serially correlated over different periods, they can achieve better performance, as discussed in Section 4. For the pure multi-period MV portfolio selection problem, some researchers considered the correlations of random returns of the risky assets in their models, e.g., [17–20]. As far as we know, however, there is few work investigating the general correlations of the random returns of the assets and liability for the MVAL model.

In this paper, we study a general model of the multi-period MVAL problem, in which all the assets in the portfolio are risky assets. Furthermore, the returns of these risky assets and the liability are not only correlated to each other but also serially correlated themselves in different time periods. Instead of assuming any particular stochastic processes of the random returns of the risky assets and liability, we adopt a general formulation to model the correlation, which enables us to derive the model-independent portfolio policy, i.e., the portfolio policy only depends on the first and second conditional moments of the random parameters. Once the first and second moments of the returns of the assets and liability are calibrated from market data, the optimal policy can be computed numerically. Our result is novel in the literature and it includes several previous ones as its special cases.

- Note that our model is a generalization of [21] in which a specific correlation structure of the returns is considered, i.e., regime switching. Our model can also be regarded as a generalization of [22] if it does not have an intertemporal constraint.

- If we do not consider the correlation between the returns of the risky assets and liabilities, our model will become simplified to the one presented in [10] and [23] without an intertemporal constraint.

- If we exclude the exogenous liabilities, our model is simplified to the pure multi-period mean-variance portfolio optimization problem with general correlations (see, e.g., [18–20]).

In addition, we can see from our result that the MVAL efficient frontiers generated from the model with all risky asset and the one with risk-free asset are all in shapes of hyperbola and are not tangent to each other. Such a phenomenon is consistent with the observation from pure continuous-time based MV portfolio selection model (see e.g., [24])*. The impacts of the correlation among the risky assets, the assets and the liability to the efficient frontier are also analyzed in the illustrative example.

The remaining of the paper is organized as follows. Section 2 presents the market setting and our model. The optimal portfolio strategies and efficient frontiers are derived in Section 3. Section 4 provides a numerical example to illustrate the derived theoretical results and presents the impacts of some parameters in the model. Finally, we conclude the paper in Section 5.

2 Formulation

Assume that the capital market consists of n risky assets and one liability, which can be traded in totally T periods. At each time period $t \in \{0, 1, \cdots, T-1\}$, the random return vector of these n risky assets and the random return of the liability are denoted by $\boldsymbol{e}_t = [e_t^1, \cdots, e_t^n]^{\mathrm{T}}$ and p_t, respectively. Note that, different from the independent assumption used in the literature [10, 12, 25], in this paper, we allow $\boldsymbol{e}_t$ and p_t to be statistically correlated to each other and also serially correlated in different periods†, such as Markov chain models (see, e.g., [26]) or the conventional time series models, which have important applications in financial decision making (see, e.g., [17, 22]). All the underlying uncertainties are modeled by a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the structure of the filtration satisfying $\mathcal{F}_t \subset \mathcal{F}_{t+1}, t = 0, 1, \cdots, T-1$ and $\mathcal{F}_T = \mathcal{F}$. The filtration $\mathcal{F}_t$ represents the information available at stage t‡. Under this model, at any time t, the return vector $\boldsymbol{e}_t$ and liability p_t are $\mathcal{F}_{t+1}$ measurable, i.e., the realization of $\boldsymbol{e}_t$ and p_t are known only at time $t+1$§. To simplify the notation, we use the $\mathbb{E}_t[\cdot]$ and $\mathrm{Var}_t[\cdot]$ to denote the conditional expectation $\mathbb{E}[\cdot|\mathcal{F}_t]$ and the conditional variance $\mathrm{Var}[\cdot|\mathcal{F}_t]$, respectively¶. The advantage of this model is that, there is no need to assume any particular stochastic process of $\boldsymbol{e}_t$ and p_t and all the results are expressed by the conditional moments of $\boldsymbol{e}_t$ and p_t for given filtration $\mathcal{F}_t$.

An investor joins the market at the beginning of period 0 with an initial wealth x_0 and initial liability l_0. He can adjust his portfolio at the beginning of each following $T-1$ consecutive periods. Let x_t and l_t be the wealth and liability of the investor at the beginning of period t respectively, then $x_t - l_t$ is the net wealth. We assume in this paper that the liability follows the dynamics, $l_{t+1} = p_t l_t$, $t = 0, \cdots, T-1$, which is exogenous, i.e., it is uncontrollable and cannot be affected by the investor's strategies. At period t, let $\boldsymbol{u}_t := (u_t^1, \cdots, u_t^n)^{\mathrm{T}}$ be the portfolio decision vector, where u_t^i is the amount invested in the i-th risky asset. We confine all admissible investment policies to be $\mathcal{F}_t$-adapted Markovian policy, i.e., $\boldsymbol{u}_t$ is only based on the information before time stage t.

*This phenomenon is observed in pure dynamic MV portfolio selection model and is named as the premium of dynamic trading in [24].

†The serial correlation indicates that $\boldsymbol{e}_t$ and p_t are dependent on the realization value of $\{\boldsymbol{e}_s, p_s\}_{s=0}^{t-1}$, for all $t = 0, 1, \cdots, T-1$.

‡Mathematically, $\mathcal{F}_t$ is the σ-algebra generated by the realization of $\{\boldsymbol{e}_0, \boldsymbol{e}_1, \cdots, \boldsymbol{e}_{t-1}, p_0, \cdots, p_{t-1}\}$.

§This type of model is very general since it does not require specifying the particular stochastic process that $\boldsymbol{e}_t$ and p_t follow.

¶At $t = 0$, $\mathbb{E}[\cdot|\mathcal{F}_0]$ and $\mathrm{Var}[\cdot|\mathcal{F}_0]$ are just the unconditional expectation $\mathbb{E}[\cdot]$ and variance $\mathrm{Var}[\cdot]$, respectively.

The investor adopts the multi-period MVAL model to guide his investment, which is to seek the best strategy, $\{\boldsymbol{u}_t^*\}|_{t=0}^{T-1}$, to solve the following problem

$$
\begin{aligned}
(\mathcal{P}):\quad \min_{\boldsymbol{u}_t}\quad & \omega\mathrm{Var}[x_T-l_T]-\mathbb{E}[x_T-l_T] \\
\text{s.t.}\quad & x_{t+1}=\boldsymbol{e}_t^{\mathrm{T}}\boldsymbol{u}_t && (1)\\
& x_t=\mathbf{1}^{\mathrm{T}}\boldsymbol{u}_t, && (2)\\
& l_{t+1}=p_tl_t,\quad t=0,\cdots,T-1, && (3)
\end{aligned}
$$

where x_0 and l_0 are known, $\omega>0$ is a weighting parameter and $\mathbf{1}$ is the vector whose elements are all 1. In this paper, we also consider the MVAL model including a risk-free asset. Let the return of the risk-free asset be r_t, $t=0,\cdots,T-1$, which is assumed to be deterministic. When the risk-free asset is included, we can represent allocation in the risk-free asset as $x_t-\mathbf{1}^{\mathrm{T}}\boldsymbol{u}_t$ for $t=0,\cdots,T-1$. Therefore, the wealth process can be written as

$$x_{t+1}=r_tx_t+\boldsymbol{d}_t^{\mathrm{T}}\boldsymbol{u}_t,\quad \text{for } t=0,\cdots,T-1, \tag{4}$$

where the excess return vector $\boldsymbol{d}_t\in\mathbb{R}^n$ is defined as, $\boldsymbol{d}_t\triangleq\left(d_t^1,d_t^2,\cdots,d_t^n\right)^{\mathrm{T}}=\boldsymbol{e}_t-r_t\mathbf{1}$. Then, the MVAL problem with n risky assets and a risk-free asset can be formulated as follows

$$
\begin{aligned}
(\mathcal{P}_f):\quad \min_{\boldsymbol{u}_t}\quad & \omega\mathrm{Var}[x_T-l_T]-\mathbb{E}[x_T-l_T] \\
\text{s.t.}\quad & \{x_t,l_t,\boldsymbol{u}_t\}\text{ satisfies dynamics (3) and (4)},\quad t=0,\cdots,T-1.
\end{aligned}
$$

Note that there is an alternative formulation for the problems $(\mathcal{P})$ and $(\mathcal{P}_f)$, i.e., maximizing the expected terminal net wealth for a given variance of the terminal net wealth. In fact, the two kinds of models are actually equivalent due to the convexity from the multi-objective optimization point of view. To guarantee the convexity of problem, we need to introduce the following assumption.

Assumption 2.1 For all time $t=0,1,\cdots,T-1$, the conditional covariance matrices satisfy $\mathrm{Cov}_t[\boldsymbol{e}_t]:=\mathbb{E}_t[\boldsymbol{e}_t\boldsymbol{e}_t^{\mathrm{T}}]-\mathbb{E}_t[\boldsymbol{e}_t]\mathbb{E}_t[\boldsymbol{e}_t^{\mathrm{T}}]\succ 0$, where the notation $A\succ 0$ means A is a positive definite matrix.

Remark 2.2 If there is no general correlation in our model, Assumption 2.1 reduces to the simplified assumption, i.e., "For all time $t=0,1,\cdots,T-1$, the covariance matrices of random return vector $\boldsymbol{e}_t$ satisfy $\mathrm{Cov}[\boldsymbol{e}_t]:=\mathbb{E}[\boldsymbol{e}_t\boldsymbol{e}_t^{\mathrm{T}}]-\mathbb{E}[\boldsymbol{e}_t]\mathbb{E}[\boldsymbol{e}_t^{\mathrm{T}}]\succ 0$, which further implies $\mathbb{E}[\boldsymbol{e}_t\boldsymbol{e}_t^{\mathrm{T}}]\succ 0$". This simplified assumption has been widely employed in numerous scholarly works as a foundation for addressing multi-period mean-variance portfolio selection problems without general correlation (see, e.g., [4, 27, 28]). This simplified assumption can exclude "redundant" securities (see, e.g., [29]) and ensure the convexity of the problem (see, e.g. [2, 27]).

Remark 2.3 If there is general correlation among the returns of assets and liability, Assumption 2.1 is still commonly applied in the context of pure multi-period mean-variance portfolio optimization problem without the exogenous liability (see, e.g., [6, 19]). Similarly, Assumption 2.1 can also be applied to ensure the convexity of our problem, which further guarantees the existence and uniqueness of the optimal portfolio policy.

3 Optimal Portfolio Policy

3.1 The Solution for Problem ($\mathcal{P}$)

The primary problem ($\mathcal{P}$) is a nonseparable one in the sense of dynamic programming. To solve it, we consider the following auxiliary problem ($\overline{\mathcal{P}}(\lambda)$) for some given parameter λ

$$(\overline{\mathcal{P}}(\lambda)): \quad \min_{\boldsymbol{u}_t} \quad \omega\mathbb{E}[(x_T - l_T)^2] - \lambda\mathbb{E}[x_T - l_T]$$
$$\text{s.t.} \quad \{x_t, l_t, \boldsymbol{u}_t\} \text{ satisfies the dynamics (1)–(3).}$$

This embedding scheme was first introduced by Li and Ng[2]. Let the set of optimal policy for the problems ($\overline{\mathcal{P}}(\lambda)$) and ($\mathcal{P}$) be $\Pi(\overline{\mathcal{P}}(\lambda))$ and $\Pi(\mathcal{P})$, respectively. The following two lemmas characterize the relationship of the solutions between the problems ($\overline{\mathcal{P}}(\lambda)$) and ($\mathcal{P}$). On the basis of avoiding confusion, we use the notation $\boldsymbol{u}^*$ to denote $\{\boldsymbol{u}_t^*\}_{t=0}^{T-1}$. Denote $d(\boldsymbol{u}) = 1 + 2\omega\mathbb{E}[x_T - l_T]\big|_{\boldsymbol{u}}$.

Lemma 3.1 *For any $\boldsymbol{u}^* \in \Pi(\mathcal{P})$, we have $\boldsymbol{u}^* \in \Pi(\overline{\mathcal{P}}(d(\boldsymbol{u}^*)))$*[‖]*.*

Proof We will prove this lemma using the method of contradiction. If the solution $\boldsymbol{u}^*$ is not the optimal solution of $\Pi(\overline{\mathcal{P}}(d(\boldsymbol{u}^*)))$, we can find another solution $\boldsymbol{u}^\dagger$ such that the following inequality holds true:

$$\begin{aligned}&\Big\{\omega\mathbb{E}[(x_T - l_T)^2]\big|_{\boldsymbol{u}^\dagger} - \Big(1 + 2\omega\mathbb{E}[x_T - l_T]\big|_{\boldsymbol{u}^*}\Big)\mathbb{E}[(x_T - l_T)]\big|_{\boldsymbol{u}^\dagger}\Big\}\\ \le&\Big\{\omega\mathbb{E}[(x_T - l_T)^2]\big|_{\boldsymbol{u}^*} - \Big(1 + 2\omega\mathbb{E}[x_T - l_T]\big|_{\boldsymbol{u}^*}\Big)\mathbb{E}[(x_T - l_T)]\big|_{\boldsymbol{u}^*}\Big\}.\end{aligned} \tag{5}$$

Notice that ν^2 is a convex function with respect to ν, which implies that the following inequality holds true:

$$\begin{aligned}&\Big(\mathbb{E}[x_T - l_T]\big|_{\boldsymbol{u}^\dagger}\Big)^2 - \Big(\mathbb{E}[x_T - l_T]\big|_{\boldsymbol{u}^*}\Big)^2\\ \ge\ & 2\mathbb{E}[x_T - l_T]\big|_{\boldsymbol{u}^*} \times \Big(\mathbb{E}[x_T - l_T]\big|_{\boldsymbol{u}^\dagger} - \mathbb{E}[x_T - l_T]\big|_{\boldsymbol{u}^*}\Big).\end{aligned} \tag{6}$$

Combining the inequality (6) with (5) yields

$$\begin{aligned}&\Big\{\omega\Big(\mathbb{E}[(x_T - l_T)^2]\big|_{\boldsymbol{u}^*} - \Big(\mathbb{E}[x_T - l_T]\big|_{\boldsymbol{u}^*}\Big)^2\Big) - \mathbb{E}[x_T - l_T]\big|_{\boldsymbol{u}^*}\Big\}\\ \ge&\Big\{\omega\Big(\mathbb{E}[(x_T - l_T)^2]\big|_{\boldsymbol{u}^\dagger} - \Big(\mathbb{E}[x_T - l_T]\big|_{\boldsymbol{u}^\dagger}\Big)^2\Big) - \mathbb{E}[x_T - l_T]\big|_{\boldsymbol{u}^\dagger}\Big\},\end{aligned}$$

which contradicts the assumption that $\boldsymbol{u}^*$ is the optimal solution of the problem ($\mathcal{P}$). This completes the proof. ∎

The implication of Lemma 3.1 is that the solution set for the problem ($\mathcal{P}$) is a subset of the solution set for the problem ($\overline{\mathcal{P}}(\lambda)$). This allows us to transform the intractable primal problem ($\mathcal{P}$) into a tractable auxiliary problem ($\overline{\mathcal{P}}(\lambda)$) by utilizing a quadratic utility function.

The following lemma presents a necessary condition under which a solution of $\Pi(\overline{\mathcal{P}}(\lambda))$ represents an optimal multi-period portfolio policy of ($\mathcal{P}$).

‖This lemma is similar to Theorem 1 of Li and Ng[2].

Lemma 3.2 *If* $\boldsymbol{u}^* \in \Pi(\overline{\mathcal{P}}(\lambda^*))$, *a necessary condition for* $\boldsymbol{u}^* \in \Pi(\mathcal{P})$ *is*

$$\lambda^* = 1 + 2\omega\mathbb{E}[x_T - l_T]\big|_{\boldsymbol{u}^*}. \tag{7}$$

Proof Clearly, the solution set $(\overline{\mathcal{P}}(\lambda))$ can be parameterized by λ. That implies that each point within the set $\bigcup_\lambda \Pi(\overline{\mathcal{P}}(\lambda))$ can be expressed with respect to λ as $\Big\{\mathbb{E}\Big[\big(x_T-l_T\big)^2(\lambda)\Big], \mathbb{E}\Big[\big(x_T-l_T\big)(\lambda)\Big]\Big\}$. Therefore, considering that $\Pi(\mathcal{P}) \subseteq \Pi(\overline{\mathcal{P}}(\lambda))$, we can reformulate the problem $(\mathcal{P})$ into the following equivalent representation:

$$\begin{aligned}&\min_\lambda \ \omega\Big(\mathrm{Var}\Big[\big(x_T - l_T\big)(\lambda)\Big]\Big) - \Big(\mathbb{E}\Big[\big(x_T - l_T\big)(\lambda)\Big]\Big)\\ =&\min_\lambda \ \omega\mathbb{E}\Big[\big(x_T - l_T\big)^2(\lambda)\Big] - \Big(\omega\mathbb{E}^2\Big[\big(x_T - l_T\big)(\lambda)\Big] + \mathbb{E}\Big[\big(x_T - l_T\big)(\lambda)\Big]\Big).\end{aligned}$$

The first-order necessary optimality condition for the optimal λ^* is given as follows**:

$$\omega\frac{\partial\mathbb{E}\Big[\big(x_T - l_T\big)^2(\lambda^*)\Big]}{\partial\lambda} - \Big[1 + 2\omega\mathbb{E}[x_T - l_T]\big|_{\boldsymbol{u}^*}\Big]\frac{\partial\mathbb{E}\Big[\big(x_T - l_T\big)(\lambda^*)\Big]}{\partial\lambda} = 0. \tag{8}$$

On the other hand, when $\boldsymbol{u}^* \in \Pi(\overline{\mathcal{P}}(\lambda^*))$, the following result is derived from Reid and Citron[30]:

$$\omega\frac{\partial\mathbb{E}\Big[\big(x_T - l_T\big)^2(\lambda^*)\Big]}{\partial\lambda} - \lambda^*\frac{\partial\mathbb{E}\Big[\big(x_T - l_T\big)(\lambda^*)\Big]}{\partial\lambda} = 0.$$

From the above analysis, it is not hard to obtain that $\lambda^* = 1+2\omega\mathbb{E}[x_T-l_T]\big|_{\boldsymbol{u}^*}$, which completes the proof. ∎

The proof of Lemma 3.2 is similar to that of Theorem 2 in [2]. Before presenting the main results, we define the following random variables for $t = T-1, \cdots, 0$:

$$\begin{aligned}
Q_t &= \frac{1}{\mathbf{1}^{\mathrm{T}}\mathbb{E}_t^{-1}[Q_{t+1}\boldsymbol{e}_t\boldsymbol{e}_t^{\mathrm{T}}]\mathbf{1}},\\
R_t &= \mathbb{E}_t[R_{t+1}p_t^2] - \mathbb{E}_t[m_{t+1}\boldsymbol{e}_t^{\mathrm{T}}p_t]\mathbb{E}_t^{-1}[Q_{t+1}\boldsymbol{e}_t\boldsymbol{e}_t^{\mathrm{T}}]\mathbb{E}_t[m_{t+1}\boldsymbol{e}_t p_t]\\
&\quad + \frac{\big(\mathbf{1}^{\mathrm{T}}\mathbb{E}_t^{-1}[Q_{t+1}\boldsymbol{e}_t\boldsymbol{e}_t^{\mathrm{T}}]\mathbb{E}_t[m_{t+1}\boldsymbol{e}_t p_t]\big)^2}{\mathbf{1}^{\mathrm{T}}\mathbb{E}_t^{-1}[Q_{t+1}\boldsymbol{e}_t\boldsymbol{e}_t^{\mathrm{T}}]\mathbf{1}},\\
m_t &= \frac{\mathbf{1}^{\mathrm{T}}\mathbb{E}_t^{-1}[Q_{t+1}\boldsymbol{e}_t\boldsymbol{e}_t^{\mathrm{T}}]\mathbb{E}_t[m_{t+1}\boldsymbol{e}_t p_t]}{\mathbf{1}^{\mathrm{T}}\mathbb{E}_t^{-1}[Q_{t+1}\boldsymbol{e}_t\boldsymbol{e}_t^{\mathrm{T}}]\mathbf{1}},\\
q_t &= \frac{\mathbf{1}^{\mathrm{T}}\mathbb{E}_t^{-1}[Q_{t+1}\boldsymbol{e}_t\boldsymbol{e}_t^{\mathrm{T}}]\mathbb{E}_t[q_{t+1}\boldsymbol{e}_t]}{\mathbf{1}^{\mathrm{T}}\mathbb{E}_t^{-1}[Q_{t+1}\boldsymbol{e}_t\boldsymbol{e}_t^{\mathrm{T}}]\mathbf{1}},\\
h_t &= \mathbb{E}_t[h_{t+1}p_t] - \mathbb{E}_t[m_{t+1}\boldsymbol{e}_t^{\mathrm{T}}p_t]\mathbb{E}_t^{-1}[Q_{t+1}\boldsymbol{e}_t\boldsymbol{e}_t^{\mathrm{T}}]\mathbb{E}_t[q_{t+1}\boldsymbol{e}_t]
\end{aligned} \tag{9}$$

**This condition is a necessary condition since the λ that satisfies the condition (8) may not necessarily be the optimal value but could represent an extreme point.

$$+\frac{\mathbf{1}^{\mathrm{T}}\mathbb{E}_t^{-1}[Q_{t+1}\boldsymbol{e}_t\boldsymbol{e}_t^{\mathrm{T}}]\mathbb{E}_t[m_{t+1}\boldsymbol{e}_tp_t]\mathbf{1}^{\mathrm{T}}\mathbb{E}_t^{-1}[Q_{t+1}\boldsymbol{e}_t\boldsymbol{e}_t^{\mathrm{T}}]\mathbb{E}_t[q_{t+1}\boldsymbol{e}_t]}{\mathbf{1}^{\mathrm{T}}\mathbb{E}_t^{-1}[Q_{t+1}\boldsymbol{e}_t\boldsymbol{e}_t^{\mathrm{T}}]\mathbf{1}},$$

$$c_t=\mathbb{E}_t[c_{t+1}]-\frac{\big(\mathbf{1}^{\mathrm{T}}\mathbb{E}_t^{-1}[Q_{t+1}\boldsymbol{e}_t\boldsymbol{e}_t^{\mathrm{T}}]\mathbb{E}_t[q_{t+1}\boldsymbol{e}_t]\big)^2}{\mathbf{1}^{\mathrm{T}}\mathbb{E}_t^{-1}[Q_{t+1}\boldsymbol{e}_t\boldsymbol{e}_t^{\mathrm{T}}]\mathbf{1}}+\mathbb{E}_t[q_{t+1}\boldsymbol{e}_t^{\mathrm{T}}]\mathbb{E}_t^{-1}[Q_{t+1}\boldsymbol{e}_t\boldsymbol{e}_t^{\mathrm{T}}]\mathbb{E}_t[q_{t+1}\boldsymbol{e}_t],$$

with $Q_T=1$, $R_T=1$, $m_T=-1$, $q_T=1$, $h_T=-1$ and $c_T=0$. Note that this random variables can be characterized once the stochastic process of $\boldsymbol{e}_t$ and p_t is specified. The following lemma is useful in the proof of the main theorem.

Lemma 3.3 *For any $t=0,1,\cdots,T-1$, we have $\mathbb{E}_t[Q_{t+1}\boldsymbol{e}_t\boldsymbol{e}_t^{\mathrm{T}}]\succ 0$ and $0<c_t<1$.*

Please refer to Appendix A.1 for a more detailed proof of this lemma, which is similar to that of Lemma 3 in [18]. Next, we present the optimal portfolio policy of the problem $(\overline{\mathcal{P}}(\lambda))$.

Theorem 3.4 *The optimal portfolio policy of the problem $(\overline{\mathcal{P}}(\lambda))$ is*

$$\boldsymbol{u}_t^*=\mathbb{E}_t^{-1}[Q_{t+1}\boldsymbol{e}_t\boldsymbol{e}_t^{\mathrm{T}}]\bigg(\frac{\lambda}{2\omega}\Big(\mathbb{E}_t[q_{t+1}\boldsymbol{e}_t]-q_t\mathbf{1}\Big)+Q_tx_t\mathbf{1}+l_t\Big(m_t\mathbf{1}-\mathbb{E}_t[m_{t+1}p_t\boldsymbol{e}_t]\Big)\bigg),\tag{10}$$

for $t=0,\cdots,T-1$.

Proof We solve the problem $(\overline{\mathcal{P}}(\lambda))$ by approach of dynamic programming. At any time t, the value function is defined as

$$J_t(x_t,l_t):=\min_{\{\boldsymbol{u}_k\}_{k=t}^{T-1}}\mathbb{E}\Big[\omega(x_T-l_T)^2-\lambda(x_T-l_T)\big|\mathcal{F}_t\Big].$$

From Bellman's principle of optimality principle, we have the following recursion

$$J_t(x_t,l_t)=\min_{\boldsymbol{u}_t}\mathbb{E}\Big[J_{t+1}(x_{t+1},l_{t+1})\big|\mathcal{F}_t\Big],$$

where the terminal condition is $J_T(x_T,l_T)=\omega(x_T-l_T)^2-\lambda(x_T-l_T)$. Claim that the value function is in the following form

$$J_t(x_t,l_t)=\omega\big(Q_tx_t^2+R_tl_t^2+2m_tx_tl_t\big)-\lambda q_tx_t-\lambda h_tl_t-\frac{\lambda^2}{4\omega}c_t,\tag{11}$$

for $t=0,\cdots,T$. Obviously, when $t=T$, the claim (11) is true. Assume that the claim (11) is true at stage $t+1$. Now we check it is also right at stage t. Introducing a lagrange multiplier ϕ_t for the constraint (2) gives the lagrange function

$$\begin{aligned}L(\boldsymbol{u}_t,\phi_t)=&\mathbb{E}\bigg[\omega\big(Q_{t+1}x_{t+1}^2+R_{t+1}l_{t+1}^2+2m_{t+1}x_{t+1}l_{t+1}\big)\\&-\lambda\big(q_{t+1}x_{t+1}+h_{t+1}l_{t+1}\big)-\frac{\lambda^2}{4\omega}c_{t+1}\big|\mathcal{F}_t\bigg]+\lambda\phi_t(x_t-\mathbf{1}^{\mathrm{T}}\boldsymbol{u}_t)\\=&\omega\Big(\boldsymbol{u}_t^{\mathrm{T}}\mathbb{E}_t[Q_{t+1}\boldsymbol{e}_t\boldsymbol{e}_t^{\mathrm{T}}]\boldsymbol{u}_t+\mathbb{E}_t[R_{t+1}p_t^2]l_t^2+2\mathbb{E}_t[m_{t+1}\boldsymbol{e}_t^{\mathrm{T}}p_t]l_t\boldsymbol{u}_t\Big)\\&-\lambda\big(\mathbb{E}_t[q_{t+1}\boldsymbol{e}_t^{\mathrm{T}}]\boldsymbol{u}_t+\mathbb{E}_t[h_{t+1}p_t]l_t\big)-\frac{\lambda^2}{4\omega}\mathbb{E}_t[c_{t+1}]+\lambda\phi_t(x_t-\mathbf{1}^{\mathrm{T}}\boldsymbol{u}_t).\end{aligned}\tag{12}$$

Lemma 3.3 implies that $L(\boldsymbol{u}_t, \phi_t)$ defined in (12) is a strictly convex function with respect to $\boldsymbol{u}_t$. Thus, we have

$$\boldsymbol{u}_t^* = \arg\min_{\boldsymbol{u}_t} L(\boldsymbol{u}_t, \phi_t) = -\mathbb{E}_t^{-1}[Q_{t+1}\boldsymbol{e}_t\boldsymbol{e}_t^{\mathrm{T}}]\Big(\mathbb{E}_t[m_{t+1}\boldsymbol{e}_t p_t]l_t - \frac{\lambda}{2\omega}\big(\mathbb{E}_t[q_{t+1}\boldsymbol{e}_t] + \phi_t\mathbf{1}\big)\Big). \quad (13)$$

Substituting $\boldsymbol{u}_t^*$ to (12) yields the optimal lagrange multiplier

$$\phi_t^* = \frac{\mathbf{1}^{\mathrm{T}}\mathbb{E}_t^{-1}[Q_{t+1}\boldsymbol{e}_t\boldsymbol{e}_t^{\mathrm{T}}]\Big(\frac{2\omega}{\lambda}\mathbb{E}_t[m_{t+1}\boldsymbol{e}_t p_t]l_t - \mathbb{E}_t[q_{t+1}\boldsymbol{e}_t]\Big) + \frac{2\omega}{\lambda}x_t}{\mathbf{1}^{\mathrm{T}}\mathbb{E}_t^{-1}[Q_{t+1}\boldsymbol{e}_t\boldsymbol{e}_t^{\mathrm{T}}]\mathbf{1}}.$$

Then the result (10) as well as the expression (11) of the value function $J_t(x_t, l_t)$ can be derived, where parameters Q_t, R_t, m_t, q_t, h_t and c_t are defined in (9). ▮

Theorem 3.5 *Under the optimal policy $\boldsymbol{u}_t^*$ of $(\overline{\mathcal{P}}(\lambda))$, the following expressions hold on for the terminal net wealth $x_T - l_T$:*

$$\mathbb{E}_t[x_T^* - l_T^*] = q_t x_t + h_t l_t + \frac{\lambda}{2\omega}c_t, \quad (14)$$

$$\mathbb{E}_t[(x_T^* - l_T^*)^2] = Q_t x_t^2 + R_t l_t^2 + 2m_t x_t l_t + \frac{\lambda^2}{4\omega^2}c_t, \quad (15)$$

$$\begin{aligned}\mathrm{Var}_t[x_T^* - l_T^*] = {} & (Q_t - q_t^2)x_t^2 + (R_t - h_t^2)l_t^2 + 2(m_t - q_t h_t)x_t l_t \\ & - \frac{\lambda}{\omega}(q_t x_t c_t + h_t l_t c_t) - \frac{\lambda^2}{4\omega^2}(c_t^2 - c_t). \end{aligned} \quad (16)$$

We use induction method to prove the formulations (14) and (15). The equation (16) is then derived easily by them. The proof is simple but long. Please see Appendix A.2.

Theorem 3.6 *The policy* (10) *solves the primary problem* $(\mathcal{P})$ *with* λ *being*

$$\lambda^* = \frac{1 + 2\omega(q_0 x_0 + h_0 l_0)}{1 - c_0}, \quad (17)$$

and the optimal mean-variance pair of the terminal net wealth $x_T - l_T$ are given by

$$\mathbb{E}[x_T^* - l_T^*] = \frac{q_0 x_0 + h_0 l_0}{1 - c_0} + \frac{c_0}{2\omega(1 - c_0)}, \quad (18)$$

$$\mathrm{Var}[x_T^* - l_T^*] = \Big(Q_0 - \frac{q_0^2}{1 - c_0}\Big)x_0^2 + \Big(R_0 - \frac{h_0^2}{1 - c_0}\Big)l_0^2 + 2\Big(m_0 - \frac{q_0 h_0}{1 - c_0}\Big)x_0 l_0 + \frac{c_0}{4\omega^2(1 - c_0)}. \quad (19)$$

Moreover, the MV efficient frontier of net wealth is

$$\begin{aligned}\Big(\mathbb{E}[x_T^* - l_T^*] - \frac{q_0 x_0 + h_0 l_0}{1 - c_0}\Big)^2 = {} & \frac{c_0}{1 - c_0}\Big(\mathrm{Var}[x_T^* - l_T^*] - \Big(Q_0 - \frac{q_0^2}{1 - c_0}\Big)x_0^2 \\ & - \Big(R_0 - \frac{h_0^2}{1 - c_0}\Big)l_0^2 - 2\Big(m_0 - \frac{q_0 h_0}{1 - c_0}\Big)x_0 l_0\Big). \end{aligned} \quad (20)$$

Proof At $t = 0$, we have

$$\lambda^* = 1 + 2\omega(q_0 x_0 + h_0 l_0) + \lambda^* c_0$$

by substituting (14) into (7), which further gives the expression (17). Then the rest results of the theorem can be easily derived by Theorem 3.5. Note that the efficient frontier (20) is well defined due to $0 < c_0 < 1$ in Lemma 3.3. ∎

Theorem 3.6 suggests the correlation information of realized random returns before time t has a significant impact on the value of the stochastic parameters Q_t, R_t, m_t, q_t, h_t, c_t, which further affects $\boldsymbol{u}_t^*$. This implies that, even if the initial boundary conditions are the same, the portfolio strategy and MV efficient frontier can still be different due to the different correlation structures of random returns.

Generally speaking, once the stochastic processes of $\{\boldsymbol{e}_t\}|_{t=0}^{T-1}$ and $\{p_t\}|_{t=0}^{T-1}$ are specified, we can employ some numerical methods for the systems of (9) to compute $\boldsymbol{u}_t^*$. Note that if the sequences $\{\boldsymbol{e}_t\}|_{t=0}^{T-1}$ and $\{p_t\}|_{t=0}^{T-1}$ are statistically independent over time, all the conditional expectations in (9) would become the unconditional expectations, resulting in deterministic parameters. However, if $\{\boldsymbol{e}_t\}|_{t=0}^{T-1}$ and $\{p_t\}|_{t=0}^{T-1}$ exhibit serial correlation over time, all the conditional expectations $\mathbb{E}_t[\cdot]$ become dependent on the filtration $\mathcal{F}_t$. In other words, all parameters Q_t, R_t, m_t, q_t, h_t, c_t become random variables. In such cases, numerical methods are typically necessary to discretize the sample space and solve the systems of (9) for each sample path.

We then turn to consider the special case that the problem without the general correlation structure. Assume that all the random returns $\boldsymbol{e}_t$ and p_t are uncorrelated, i.e., $\boldsymbol{e}_t$ and p_t are statistically independent and serially independent over time. Under this case, the conditional expectation $\mathbb{E}_t[\cdot]$ reduces to unconditional expectation $\mathbb{E}[\cdot]$. This implies that,

$$
\begin{aligned}
&\mathbb{E}_t[Q_{t+1}\boldsymbol{e}_t\boldsymbol{e}_t^{\mathrm{T}}] = Q_{t+1}\mathbb{E}[\boldsymbol{e}_t\boldsymbol{e}_t^{\mathrm{T}}], \quad \mathbb{E}_t[q_{t+1}\boldsymbol{e}_t] = q_{t+1}\mathbb{E}[\boldsymbol{e}_t], \quad \mathbb{E}_t[R_{t+1}p_t^2] = R_{t+1}\mathbb{E}[p_t^2],\\
&\mathbb{E}_t[m_{t+1}\boldsymbol{e}_t^{\mathrm{T}}p_t] = m_{t+1}\mathbb{E}[p_t]\mathbb{E}[\boldsymbol{e}_t^{\mathrm{T}}], \quad \mathbb{E}_t[h_{t+1}p_t] = h_{t+1}\mathbb{E}[p_t],
\end{aligned}
$$

for all $t = 0, 1, \cdots, T-1$. Furthermore, the stochastic processes defined in (9) become deterministic for $t = 0, 1, \cdots, T-1$,

$$
\begin{aligned}
&Q_t = \prod_{k=t}^{T-1}\widehat{Q}_k, \quad m_t = -Q_t\Big(\prod_{k=t}^{T-1}\widehat{m}_k\Big), \quad q_t = Q_t\Big(\prod_{k=t}^{T-1}\widehat{q}_k\Big),\\
&R_t = R_{t+1}\mathbb{E}[p_t^2] - \frac{m_{t+1}^2}{Q_{t+1}}\mathbb{E}[\boldsymbol{e}_t^{\mathrm{T}}]\mathbb{E}^{-1}[\boldsymbol{e}_t\boldsymbol{e}_t^{\mathrm{T}}]\mathbb{E}[\boldsymbol{e}_t]\mathbb{E}[p_t^2] + \frac{m_t^2}{Q_t},\\
&h_t = h_{t+1}\mathbb{E}[p_t] - \frac{m_{t+1}q_{t+1}}{Q_{t+1}}\mathbb{E}[\boldsymbol{e}_t^{\mathrm{T}}]\mathbb{E}^{-1}[\boldsymbol{e}_t\boldsymbol{e}_t^{\mathrm{T}}]\mathbb{E}[\boldsymbol{e}_t]\mathbb{E}[p_t] + \frac{m_tq_t}{Q_t},\\
&c_t = c_{t+1} + \frac{q_{t+1}^2}{Q_{t+1}}\mathbb{E}[\boldsymbol{e}_t^{\mathrm{T}}]\mathbb{E}^{-1}[\boldsymbol{e}_t\boldsymbol{e}_t^{\mathrm{T}}]\mathbb{E}[\boldsymbol{e}_t] - \frac{q_t^2}{Q_t},
\end{aligned}
\tag{21}
$$

with $\widehat{Q}_t \triangleq 1/\mathbf{1}^{\mathrm{T}}\mathbb{E}^{-1}[\boldsymbol{e}_t\boldsymbol{e}_t^{\mathrm{T}}]\mathbf{1}$, $\widehat{m}_t \triangleq \mathbf{1}^{\mathrm{T}}\mathbb{E}^{-1}[\boldsymbol{e}_t\boldsymbol{e}_t^{\mathrm{T}}]\mathbb{E}[\boldsymbol{e}_t]\mathbb{E}[p_t]$, $\widehat{q}_t \triangleq \mathbf{1}^{\mathrm{T}}\mathbb{E}^{-1}[\boldsymbol{e}_t\boldsymbol{e}_t^{\mathrm{T}}]\mathbb{E}[\boldsymbol{e}_t]$. By combining (10), (17), (20), and (21), we can easily obtain the optimal investment policies and MV efficient frontier for the problem $(\mathcal{P})$ with independent random returns.

3.2 The Solution for Problem $(\mathcal{P}_f)$

To solve the problem $(\mathcal{P}_f)$, in which the risk-free asset is included, we define the following random variables ρ_t, η_t, and ζ_t as follows:

$$\rho_t = \mathbb{E}_t[\rho_{t+1}] - \mathbb{E}_t[\rho_{t+1}\boldsymbol{d}_t^{\mathrm{T}}]\mathbb{E}_t^{-1}[\rho_{t+1}\boldsymbol{d}_t\boldsymbol{d}_t^{\mathrm{T}}]\mathbb{E}_t[\rho_{t+1}\boldsymbol{d}_t], \tag{22}$$

$$\eta_t = \mathbb{E}_t[p_t\eta_{t+1}] - \mathbb{E}_t[\rho_{t+1}\boldsymbol{d}_t^{\mathrm{T}}]\mathbb{E}_t^{-1}[\rho_{t+1}\boldsymbol{d}_t\boldsymbol{d}_t^{\mathrm{T}}]\mathbb{E}_t[p_t\eta_{t+1}\boldsymbol{d}_t], \tag{23}$$

$$\zeta_t = \mathbb{E}_t[p_t^2\zeta_{t+1}] - \mathbb{E}_t[p_t\eta_{t+1}\boldsymbol{d}_t^{\mathrm{T}}]\mathbb{E}_t^{-1}[\rho_{t+1}\boldsymbol{d}_t\boldsymbol{d}_t^{\mathrm{T}}]\mathbb{E}_t[p_t\eta_{t+1}\boldsymbol{d}_t], \tag{24}$$

for $t = T-1, \cdots, 0$ with $\rho_T = 1$, $\eta_T = -1$, $\zeta_T = 1$. Define further the discount factor as $\gamma_t := \prod_{k=t}^{T-1} r_k$ with $\gamma_T = 1$. Then the process ρ_t has the following property.

Lemma 3.7 *For all $t = T-1, \cdots, 0$, it has $0 < \rho_t < 1$ almost surely.*

The proof of Lemma 3.7 is omitted here since Lemma 3.7 is just a special case of Lemma 1 in [17] with $s = T$.

Theorem 3.8 *The optimal portfolio policy for problem $(\mathcal{P}_f)$ is*

$$\boldsymbol{u}_t^* = -\mathbb{E}_t[\rho_{t+1}\boldsymbol{d}_t\boldsymbol{d}_t^{\mathrm{T}}]^{-1}\left[\left(r_tx_t - \prod_{k=0}^{t} r_kx_0 - \frac{1+2\omega\eta_0 l_0}{2\omega\rho_0\gamma_{t+1}}\right)\mathbb{E}_t[\rho_{t+1}\boldsymbol{d}_t] + \frac{1}{\gamma_{t+1}}\mathbb{E}_t[p_t\eta_{t+1}\boldsymbol{d}_t]\right], \tag{25}$$

with the MV efficient frontier of the net wealth being

$$\left(\mathbb{E}[x_T^* - l_T^*] - \gamma_0x_0 - \frac{\eta_0 l_0}{\rho_0}\right)^2 = \frac{1-\rho_0}{\rho_0}\left(\mathrm{Var}[x_T^* - l_T^*] - \left(\zeta_0 - \frac{\eta_0^2}{\rho_0}\right)l_0^2\right). \tag{26}$$

The similar result as the equation (20) has been reported by Zhang and Li[20], where the dynamic programming approach is adopted. However, as we have claimed, the problem $(\mathcal{P}_f)$ is a special case of the problem $(\mathcal{P})$ if we regard one risky asset to be risk-free. Let $\widehat{e}_t \in \mathbb{R}^{n+1}$ be the augmented return vector, i.e., $\widehat{e}_t = \left(r_t, e_t^{\mathrm{T}}\right)^{\mathrm{T}}$, for $t = 0, \cdots, T-1$. Due to the special structure of $\widehat{e}_t$, we can consider a problem $(\mathcal{P})$ whose portfolio decision vector is dimension $n+1$, where the first one is for the risk-free asset and the remained n elements are for the risky assets. Then, the random variables Q_t, R_t, m_t, q_t, h_t and c_t defined in (9) can be simplified as

$$Q_t = \gamma_t^2\rho_t, \quad R_t = \zeta_t, \quad m_t = \gamma_t\eta_t, \quad q_t = \gamma_t\rho_t, \quad h_t = \eta_t, \quad c_t = 1 - \rho_t, \tag{27}$$

where ρ_t, η_t, and ζ_t are defined in (22), (23), and (24). We can easily verify the equation (27) hold by using induction method, and details are shown in Appendix A.3. Thus, the optimal portfolio policy (25) can be achieved by rewriting the portfolio policy given in Theorem 3.6 and using the random variables ρ_t, η_t and ζ_t, $t = 0, \cdots, T-1$. Note that the efficient frontier (26) is well defined due to Lemma 3.7.

When all random returns $\boldsymbol{e}_t$ and p_t are uncorrelated, we can rewrite (22), (23), and (24) as

$$\begin{aligned}
\rho_t &= \rho_{t+1}\Big(1 - \mathbb{E}[\boldsymbol{d}_t^{\mathrm{T}}]\mathbb{E}^{-1}[\boldsymbol{d}_t\boldsymbol{d}_t^{\mathrm{T}}]\mathbb{E}[\boldsymbol{d}_t]\Big),\\
\eta_t &= \eta_{t+1}\Big(\mathbb{E}[p_t] - \mathbb{E}[\boldsymbol{d}_t^{\mathrm{T}}]\mathbb{E}^{-1}[\boldsymbol{d}_t\boldsymbol{d}_t^{\mathrm{T}}]\mathbb{E}[\boldsymbol{d}_t]\mathbb{E}[p_t]\Big),\\
\zeta_t &= \zeta_{t+1}\mathbb{E}[p_t^2] - \frac{\eta_{t+1}^2}{\rho_{t+1}}\mathbb{E}[p_t^2]\mathbb{E}[\boldsymbol{d}_t^{\mathrm{T}}]\mathbb{E}^{-1}[\boldsymbol{d}_t\boldsymbol{d}_t^{\mathrm{T}}]\mathbb{E}[\boldsymbol{d}_t].
\end{aligned} \tag{28}$$

Note that ρ_t, η_t and ζ_t are all deterministic variables for $t = 0, 1, \cdots, T-1$. The following corollary sheds light on the optimal investment policy and MV efficient frontier in the absence of the correlation structure among random returns.

Corollary 3.9 *Under the assumption of independent random returns, the optimal portfolio strategy becomes*

$$\boldsymbol{u}_t^* = -\mathbb{E}[\boldsymbol{d}_t\boldsymbol{d}_t^{\mathrm{T}}]^{-1}\left[\left(r_t x_t - \prod_{k=0}^{t} r_k x_0 - \frac{1+2\omega\eta_0 l_0}{2\omega\rho_0\gamma_{t+1}}\right)\mathbb{E}[\boldsymbol{d}_t] + \frac{\eta_{t+1}}{\rho_{t+1}\gamma_{t+1}}\mathbb{E}[p_t]\mathbb{E}[\boldsymbol{d}_t]\right], \tag{29}$$

where $t = 0, 1, \cdots, T-1$. *In addition, the MV efficient frontier can be expressed as* (26) *when* ρ_t, η_t, *and* ζ_t *follow the deterministic processes* (28).

The results given in Corollary 3.9 are consistent with those presented by Leippold, et al.[10].

3.3 Extension to a Market Model with No-Shorting Constraints

Inspired by certain markets with no-shorting constraints, we are also interested in the multi-period MVAL problem with shorting prohibition and general correlation. That is to say, we consider the following feasible portfolio set

$$\mathcal{U}_t := \left\{\boldsymbol{u}_t \in \mathbb{R}^n \middle| \boldsymbol{u}_t \geq \boldsymbol{0}_{n\times 1}\right\}, \quad t = 0, \cdots, T-1, \tag{30}$$

which gives rise to the following optimization problem

$$\begin{aligned}(\mathcal{P}'):\quad & \min_{\boldsymbol{u}_t} \quad \omega\mathrm{Var}[x_T - l_T] - \mathbb{E}[x_T - l_T] \\ & \text{s.t.} \quad \{x_t, l_t, \boldsymbol{u}_t\} \text{ satisfies the dynamics (1), (2), (3) and (30)}, \quad t = 0, \cdots, T-1\end{aligned}$$

and

$$\begin{aligned}(\mathcal{P}'_f):\quad & \min_{\boldsymbol{u}_t} \quad \omega\mathrm{Var}[x_T - l_T] - \mathbb{E}[x_T - l_T] \\ & \text{s.t.} \quad \{x_t, l_t, \boldsymbol{u}_t\} \text{ satisfies the dynamics (3), (4) and (30)}, \quad t = 0, \cdots, T-1.\end{aligned}$$

It should be noted that $(\mathcal{P})$ and $(\mathcal{P}_f)$ are unconstrained optimization problems, whereas the problems $(\mathcal{P}')$ and $(\mathcal{P}'_f)$ involve constraints. This fundamental distinction requires us to approach these models in different ways.

Wu, et al.[6] proposed an interesting state separation theorem and successfully derive the explicit solution for the constrained scalar-state stochastic linear-quadratic control problem. However, it is quite difficult to obtain the analytic optimal portfolio policy for our constrained MVAL problem since there are two state variables x_t and l_t in the model (see, e.g., [6]). To overcome this difficulty, we need to introduce an alternative method to identify the optimal portfolio policy for our constrained MVAL problem. In order to achieve this objective, we may have to turn to follow the approach of "Progressive Hedging Algorithm", which was developed by [31] and adopted by [32]. Roughly speaking, Dynamic programming (DP) is an algorithm that is based on time decomposition, while the progressive hedging algorithm (PHA) proposed by [31] is a scenario decomposition-based scheme. PHA was developed specifically to solve multistage stochastic decision-making problems with finite number of scenarios. Undoubtedly, this is a subject that deserves further investigation. In our future research, we intend to delve more deeply into this issue.

4 An Illustrative Example

We consider a simple example with $n = 2$ and $T = 4$ to illustrate the solution procedure developed in previous sections and show how the investment performance is affected by the correlation of stocks and liability. Assume that the investor invests in a market where short selling is permitted, and let $x_0 = 2$, $l_0 = 1$, $\omega = 2$. Suppose $\boldsymbol{e}_t$ and p_t follow a joined AR(1) process as follows

$$(\boldsymbol{e}_{t+1}^{\mathrm{T}}, p_{t+1})^{\mathrm{T}} = \Gamma + \Psi(\boldsymbol{e}_t^{\mathrm{T}}, p_t)^{\mathrm{T}} + \varepsilon_t, \tag{31}$$

with $\Psi = \Phi \triangleq \begin{pmatrix} \alpha A & \gamma C \\ \gamma C^{\mathrm{T}} & 0.02 \end{pmatrix}$ and initial value $\boldsymbol{e}_{-1} = (1.07, 1.05)^{\mathrm{T}}$, $p_{-1} = 1.1$. Other parameters associated with the AR(1) process are $A = \begin{pmatrix} 1.8 & -0.6 \\ -0.6 & 3.6 \end{pmatrix} \times 10^{-3}$, $C = (1.6, -1.2)^{\mathrm{T}} \times 10^{-2}$, $\Gamma = (1.05, 1.05, 1.07)^{\mathrm{T}}$, and $\varepsilon_t \in \mathbb{R}^3$ is a white noise vector. Note that, the parameter α controls the correlation of $\boldsymbol{e}_t$ between different time periods and the parameter γ controls the correlation between $\boldsymbol{e}_t$ and p_t. To simplify the discussion, we choose ε_t normal distributed with variance 0.09 and randomly generate 8 values (8 nodes) in interval $(-0.5, 0.5)$ at time $t = 0$. The tree structure then will have 4096 branches in $T = 4$. By using Theorems 3.4 and 3.6, we can express all the random variables, Q_t, R_t, m_t, q_t, h_t and c_t and compute the portfolio policy for each node of the scenario tree explicitly. From the equation (20), we can plot the MV efficient frontier of the net wealth $x_T - l_T$, which is affected by different parameters α, γ, the investment horizon T and the exist of risk-free asset[††].

4.1 The Impact of α

We present the MV efficient frontiers with different α when $\gamma = 0$ and $\gamma \neq 0$ in Figure 1. It can be seen that for fixed γ, the efficient frontier generated from larger α dominates the one generated from smaller α whenever $\gamma = 0$ or $\gamma \neq 0$. That means the performance of dynamic MVAL policy is enhanced if the correlation of the returns between different stage is strong, i.e., the statistics has more predict power. Furthermore, the differences of each α when there is correlation between $\boldsymbol{e}_t$ and p_t are bigger than that of no correlation.

4.2 The Impact of γ

One of the contribution of our paper is that the correlation between $\boldsymbol{e}_t$ and p_t is considered. We plot the MV efficient frontiers with different γ by fixing $\alpha = 1$ in Figure 2. It can be seen that the efficient frontier of $\gamma = 0$ is worst. And the larger the $|\gamma|$, the better the efficient frontier. In addition, for the same $|\gamma|$, the efficient frontiers when $\boldsymbol{e}_t$ and p_t are negative correlated are better than that of positive correlated, which coincides with the real market.

[††]Based on the equations (10) and (20), it can be observed that the parameter ω solely influences the portfolio strategy and has no influence on the efficient frontier.

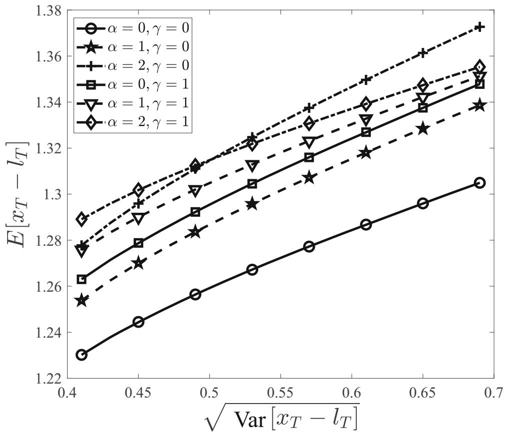

Figure 1 The efficient frontiers of different α for the problem $(\mathcal{P})$

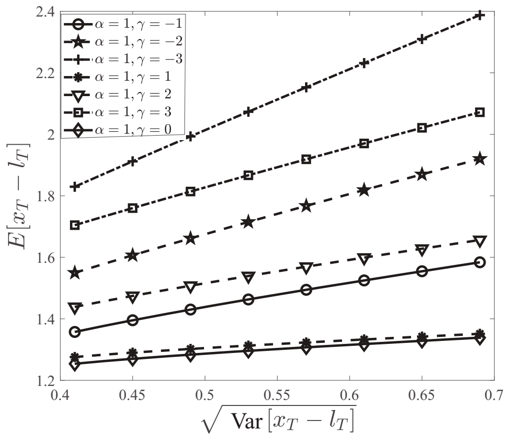

Figure 2 The efficient frontiers of different γ for the problem $(\mathcal{P})$

4.3 The Impact of Risk-Free Asset

Figure 3 plots the efficient frontiers when one risk-free asset is included, in which the risk-free rate is 1%. We can observe that for the same parameters α and γ, to gain the same expection for the problem $(\mathcal{P})$ as $(\mathcal{P}_f)$ takes higher risk for the price. Furthermore, the gap of the efficient frontiers generated from the problems $(\mathcal{P}_f)$ and $(\mathcal{P})$, respectively, is enhanced when α is increased. Furthermore, we investigate the influences of the risk-free rate and investment horizon on the problem $(\mathcal{P}_f)$. Figure 4 plots the efficient frontiers with different r_f and T for the problem $(\mathcal{P}_f)$ when $\alpha=\gamma=1$. From Figure 4, we can easily see that the longer investment horizon outperforms its corresponding shorter horizon. On the other hand, the movement of the efficient frontier is opposite to that of the risk-free rate. This further implies that our investment strategy can yield superior performance in a market environment with lower risk-free returns.

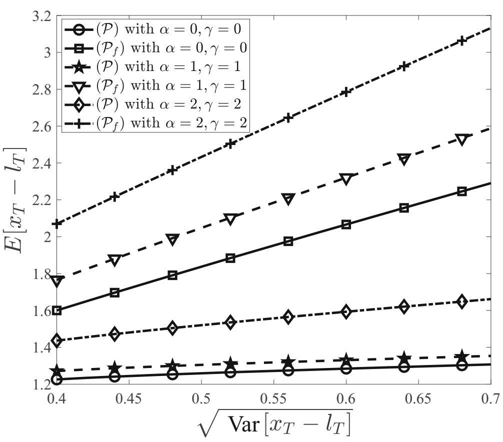

Figure 3 The efficient frontiers for the problems $(\mathcal{P})$ and $(\mathcal{P}_f)$

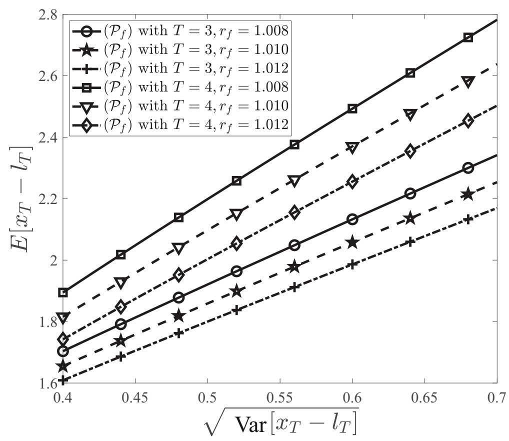

Figure 4 The efficient frontiers of different r_f and T for the problem $(\mathcal{P}_f)$

4.4 The Impact of Correlation and Investment Horizon

Let us begin by examining the effect of correlation. For simplicity, we will focus on the different processes given by (31) with $\Psi = \psi\Phi$, and suppose that $\alpha = 1$ and $\gamma = 2$ without loss of generality. Note that, the parameter ψ measures the degree of correlation between risky assets and liability. Actually, the AR(1) processes with $\psi = 0$ would become simple processes which suggest that all random returns are statistically independent,

$$(\boldsymbol{e}_{t+1}^{\mathrm{T}}, p_{t+1})^{\mathrm{T}} = \Gamma + \varepsilon_t. \tag{32}$$

Figure 5 illustrates the efficient frontiers with differing values of ψ. Notably, the efficient frontiers obtained for various correlation degrees outperform those generated by random returns that are statistically independent. Moreover, the larger the $|\psi|$, the better the efficient frontier. It's worth noting that negative correlation can lead to better investment performance compared to positive correlation.

Clearly, the efficient frontier would be influenced by the investment horizon T as it impacts the values of parameters Q_t, R_t, m_t, q_t, h_t, and c_t. Figure 6 illustrates the efficient frontiers for different investment horizons T, under both correlated and statistically independent scenarios. The findings suggest that as the investment horizon increases, the $(\mathcal{P})$ model's performance improves. In addition, we also see that investors can achieve considerable investment performance within a relatively short investment horizon when the market displays asset return correlation.

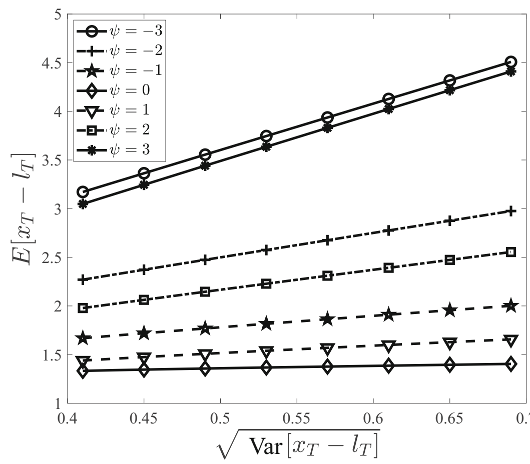

Figure 5 The efficient frontiers of different ψ for the problem $(\mathcal{P})$

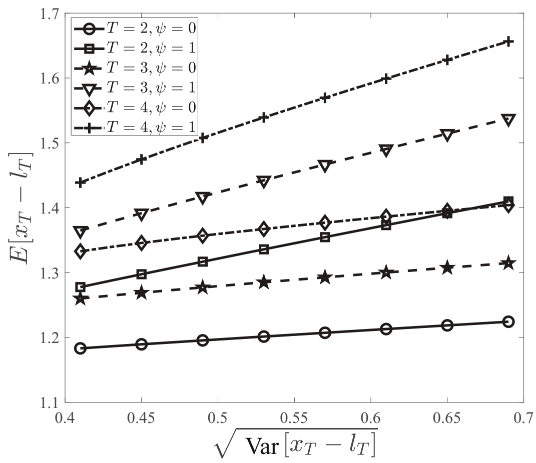

Figure 6 The efficient frontiers of different T for the problem $(\mathcal{P})$

5 Conclusion

In this paper, we study the multi-period MVAL problem in which returns of the assets and liability have general correlations. Without assuming any particular stochastic process of the random returns, we derive the portfolio policy, MV efficient frontier and the impact of general correlation for this problem. Although the explicit solution is developed for our general MVAL model, the practical implementation of this model is far from trivial. Calibrating our model in real-life ALM practice by using the real market data for some particular stochastic process of the random returns could be an interesting future research topic.

Conflict of Interest

The authors declare no conflict of interest.

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Appendix

A.1 The Proof of Lemma 3.3

Before giving the details, we introduce the following lemma.

Lemma A.1 *Given* $a > 0$, $\boldsymbol{h}, \boldsymbol{v} \in \mathbb{R}^n$, *and* $\boldsymbol{H} \in \S^n_{++}$ *is invertible. If* $\boldsymbol{H} - \boldsymbol{h}\boldsymbol{h}^{\mathrm{T}} \succ 0$ *and* $1 - \boldsymbol{v}^{\mathrm{T}}\boldsymbol{H}^{-1}\boldsymbol{h} \neq 0$, *then we have*

$$\boldsymbol{h}^{\mathrm{T}}\boldsymbol{H}^{-1}\boldsymbol{h} = 1 - \frac{1}{\boldsymbol{h}^{\mathrm{T}}(\boldsymbol{H} - \boldsymbol{h}\boldsymbol{h}^{\mathrm{T}})^{-1}\boldsymbol{h}}, \tag{33}$$

$$(\boldsymbol{H} - \boldsymbol{h}\boldsymbol{v}^{\mathrm{T}})^{-1} = \boldsymbol{H}^{-1} + \frac{\boldsymbol{H}^{-1}\boldsymbol{h}\boldsymbol{v}^{\mathrm{T}}\boldsymbol{H}^{-1}}{1 - \boldsymbol{v}^{\mathrm{T}}\boldsymbol{H}^{-1}\boldsymbol{h}}, \tag{34}$$

and

$$\begin{bmatrix} a^2 & a\boldsymbol{h}^{\mathrm{T}} \\ a\boldsymbol{h} & \boldsymbol{H} \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -\frac{1}{a}\boldsymbol{h}^{\mathrm{T}} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \frac{1}{a^2} & \boldsymbol{0} \\ \boldsymbol{0} & (\boldsymbol{H} - \boldsymbol{h}\boldsymbol{h}^{\mathrm{T}})^{-1} \end{bmatrix} \begin{bmatrix} 1 & \boldsymbol{0} \\ -\frac{1}{a}\boldsymbol{h} & \boldsymbol{I} \end{bmatrix}, \tag{35}$$

where $\boldsymbol{I}$ *and* $\boldsymbol{0}$ *denote the identity matrix and the zero matrix, respectively.*

Proof The proof of lemma A.1 can be found in [33]. ∎

Now we focus on proving Lemma 3.3. Note that, Assumption 2.1 also indicates that $\mathbb{E}_t[\boldsymbol{e}_t\boldsymbol{e}_t^{\mathrm{T}}]$ are positive definite for all $t = 0, 1, \cdots, T-1$. Thus, it has $\mathbb{E}_{T-1}[Q_T\boldsymbol{e}_{T-1}\boldsymbol{e}_{T-1}^{\mathrm{T}}] \succ 0$ that holds true since $Q_T = 1$, which further implies $Q_{T-1} > 0$ (see the definition of Q_{T-1} given in (9)). Now we assume that $Q_{t+1} > 0$ holds. At time t, we have $\mathbb{E}_t[Q_{t+1}\boldsymbol{e}_t\boldsymbol{e}_t^{\mathrm{T}}] \succ 0$, which yields $Q_t > 0$. From the above analysis, it is not hard to obtain $\mathbb{E}_t[Q_{t+1}\boldsymbol{e}_t\boldsymbol{e}_t^{\mathrm{T}}] \succ 0$ for $t = 0, 1, \cdots, T-1$.

Then we need to prove that $0 < c_t < 1$ holds for all $t = 0, 1, \cdots, T-1$. To simplify the notations, we introduce two new symbols, namely, $\boldsymbol{G}_t \triangleq \mathbb{E}_t[Q_{t+1}\boldsymbol{e}_t\boldsymbol{e}_t^{\mathrm{T}}]$ and $\boldsymbol{W}_t \triangleq \mathbb{E}_t[q_{t+1}\boldsymbol{e}_t]$, which allow us to rewrite Q_t, q_t, c_t as follows

$$Q_t = \frac{1}{\boldsymbol{1}^{\mathrm{T}}\boldsymbol{G}_t^{-1}\boldsymbol{1}}, \quad q_t = \frac{\boldsymbol{1}^{\mathrm{T}}\boldsymbol{G}_t^{-1}\boldsymbol{W}_t}{\boldsymbol{1}^{\mathrm{T}}\boldsymbol{G}_t^{-1}\boldsymbol{1}}, \quad c_t = \mathbb{E}_t[c_{t+1}] - \frac{q_t^2}{Q_t} + \boldsymbol{W}_t^{\mathrm{T}}\boldsymbol{G}_t^{-1}\boldsymbol{W}_t, \tag{36}$$

for $t = 0, 1, \cdots, T-1$. As $\boldsymbol{G}_t \succ 0$ for $t = 0, 1, \cdots, T-1$, it follows that $\boldsymbol{g}^{\mathrm{T}}\boldsymbol{G}_t^{-1}\boldsymbol{g} > 0$ for any non-zero vector $\boldsymbol{g} \in \mathbb{R}^n$. Therefore, the following inequality holds

$$\begin{aligned}
&(\boldsymbol{W}_t^{\mathrm{T}}\boldsymbol{G}_t^{-1}\boldsymbol{W}_t)(\boldsymbol{1}^{\mathrm{T}}\boldsymbol{G}_t^{-1}\boldsymbol{1})\left[\boldsymbol{W}_t^{\mathrm{T}}\boldsymbol{G}_t^{-1}\boldsymbol{W}_t - \frac{q_t^2}{Q_t}\right] \\
=&\boldsymbol{W}_t^{\mathrm{T}}\boldsymbol{G}_t^{-1}\boldsymbol{W}_t\left[(\boldsymbol{W}_t^{\mathrm{T}}\boldsymbol{G}_t^{-1}\boldsymbol{W}_t)(\boldsymbol{1}^{\mathrm{T}}\boldsymbol{G}_t^{-1}\boldsymbol{1}) - (\boldsymbol{1}^{\mathrm{T}}\boldsymbol{G}_t^{-1}\boldsymbol{W}_t)^2\right] \\
=&\left[(\boldsymbol{1}^{\mathrm{T}}\boldsymbol{G}_t^{-1}\boldsymbol{W}_t)\boldsymbol{W}_t - (\boldsymbol{W}_t^{\mathrm{T}}\boldsymbol{G}_t^{-1}\boldsymbol{W}_t)\boldsymbol{1}\right]^{\mathrm{T}}\boldsymbol{G}_t^{-1}\left[(\boldsymbol{1}^{\mathrm{T}}\boldsymbol{G}_t^{-1}\boldsymbol{W}_t)\boldsymbol{W}_t - (\boldsymbol{W}_t^{\mathrm{T}}\boldsymbol{G}_t^{-1}\boldsymbol{W}_t)\boldsymbol{1}\right] > 0,
\end{aligned}$$

where $t = 0, 1, \cdots, T-1$. Thus, we can obtain the following inequality since $\boldsymbol{W}_t^{\mathrm{T}}\boldsymbol{G}_t^{-1}\boldsymbol{W}_t > 0$ and $\boldsymbol{1}^{\mathrm{T}}\boldsymbol{G}_t^{-1}\boldsymbol{1} > 0$,

$$\boldsymbol{W}_t^{\mathrm{T}}\boldsymbol{G}_t^{-1}\boldsymbol{W}_t - \frac{q_t^2}{Q_t} > 0, \quad t = 0, 1, \cdots, T-1. \tag{37}$$

By using the results (36) and (37), we can easily obtain $c_{T-1} > 0$ since $c_T = 0$. On the other hand, the result (33) implies the following result holds true

$$\boldsymbol{W}_{T-1}^{\mathrm{T}}\boldsymbol{G}_{T-1}^{-1}\boldsymbol{W}_{T-1} = 1 - \frac{1}{\boldsymbol{W}_{T-1}^{\mathrm{T}}(\boldsymbol{G}_{T-1} - \boldsymbol{W}_{T-1}\boldsymbol{W}_{T-1}^{\mathrm{T}})^{-1}\boldsymbol{W}_{T-1}}.$$

Note that $\boldsymbol{G}_{T-1} - \boldsymbol{W}_{T-1}\boldsymbol{W}_{T-1}^{\mathrm{T}} = \mathbb{E}_{T-1}[\boldsymbol{e}_{T-1}\boldsymbol{e}_{T-1}^{\mathrm{T}}] - \mathbb{E}_{T-1}[\boldsymbol{e}_{T-1}]\mathbb{E}_{T-1}[\boldsymbol{e}_{T-1}^{\mathrm{T}}] \succ 0$. Thus, it has

$$1 - c_{T-1} = 1 + \frac{q_{T-1}^2}{Q_{T-1}} - \left[1 - \frac{1}{\boldsymbol{W}_{T-1}^{\mathrm{T}}(\boldsymbol{G}_{T-1} - \boldsymbol{W}_{T-1}\boldsymbol{W}_{T-1}^{\mathrm{T}})^{-1}\boldsymbol{W}_{T-1}}\right] > \frac{q_{T-1}^2}{Q_{T-1}} > 0,$$

which further implies $c_{T-1} < 1$.

Now, we suppose that the inequality $1 > 1 - c_{t+1} > \frac{q_{t+1}^2}{Q_{t+1}} > 0$ holds at time $t+1$, which also means that $0 < c_{t+1} < 1$. Clearly, $c_t > 0$ holds due to (36) and (37), which implies $1 - c_t < 1$. Next, we need to prove that $1 - c_t > \frac{q_t^2}{Q_t}$. Note that $Q_{t+1} > 0$ and $Y(z) = Q_{t+1}z^2 + 2q_{t+1}z$ is a quadratic function, we can derive

$$Q_{t+1}(\boldsymbol{g}^{\mathrm{T}}\boldsymbol{e}_t)(\boldsymbol{g}^{\mathrm{T}}\boldsymbol{e}_t) + 2q_{t+1}(\boldsymbol{g}^{\mathrm{T}}\boldsymbol{e}_t) + (1 - c_{t+1}) \geq 1 - c_{t+1} - \frac{q_{t+1}^2}{Q_{t+1}} > 0,$$

for any $\boldsymbol{g} \in \mathbb{R}^n$. Taking conditional expectation with respect to $\mathcal{F}_t$ yields

$$\boldsymbol{g}^{\mathrm{T}}\boldsymbol{G}_t\boldsymbol{g} + 2\boldsymbol{g}^{\mathrm{T}}\boldsymbol{W}_t + 1 - \mathbb{E}_t[c_{t+1}] > 0, \quad \forall\, \boldsymbol{g} \in \mathbb{R}^n,$$

which further means that

$$\begin{bmatrix} \boldsymbol{G}_t & \boldsymbol{W}_t \\ \boldsymbol{W}_t^{\mathrm{T}} & 1 - \mathbb{E}_t[c_{t+1}] \end{bmatrix} \succ \boldsymbol{0}. \tag{38}$$

Since $1 - \mathbb{E}_t[c_{t+1}] > 0$, we can apply Schur's Complement Theorem (see, e.g., [33]) to (38) and gives rise to $\boldsymbol{G}_t - \frac{\boldsymbol{W}_t\boldsymbol{W}_t^{\mathrm{T}}}{1-\mathbb{E}_t[c_{t+1}]} \succ 0$. This also implies

$$\boldsymbol{G}_t\big(1 - \mathbb{E}_t[c_{t+1}]\big) - \boldsymbol{W}_t\boldsymbol{W}_t^{\mathrm{T}} \succ 0.$$

Note that $\boldsymbol{G}_t(1 - \mathbb{E}_t[c_{t+1}]) \succ 0$ and using (33), we have

$$\begin{aligned}
1 - c_t &= 1 - \mathbb{E}_t[c_{t+1}] - \boldsymbol{W}_t^{\mathrm{T}}\boldsymbol{G}_t^{-1}\boldsymbol{W}_t + \frac{q_t^2}{Q_t} \\
&= \Big(1 - \mathbb{E}_t[c_{t+1}]\Big)\Big[1 - \boldsymbol{W}_t^{\mathrm{T}}\Big(\boldsymbol{G}_t\big(1 - \mathbb{E}_t[c_{t+1}]\big)\Big)^{-1}\boldsymbol{W}_t\Big] + \frac{q_t^2}{Q_t} \\
&= \frac{1 - \mathbb{E}_t[c_{t+1}]}{\boldsymbol{W}_t^{\mathrm{T}}\Big[\boldsymbol{G}_t\big(1 - \mathbb{E}_t[c_{t+1}]\big) - \boldsymbol{W}_t\boldsymbol{W}_t^{\mathrm{T}}\Big]^{-1}\boldsymbol{W}_t} + \frac{q_t^2}{Q_t} \\
&> \frac{q_t^2}{Q_t} \\
&> 0,
\end{aligned}$$

which further implies $c_t < 1$, and thus the proof is completed. ∎

A.2 The Proof of Theorem 3.5

Proof We use induction method to prove the formulations (14) and (15). Obviously, (14) and (15) are true at time T. Suppose that (14) is true at time $t+1$, then we have

$$\mathbb{E}_t[x_T^* - l_T^*] = \mathbb{E}_t\Big[\mathbb{E}_{t+1}[x_T^* - l_T^*]\Big] = \mathbb{E}_t[q_{t+1}\boldsymbol{e}_t^{\mathrm{T}}]\boldsymbol{u}_t + \mathbb{E}_t[h_{t+1}p_t]l_t + \frac{\lambda}{2\omega}\mathbb{E}_t[c_{t+1}].$$

Substituting (10) into the above equation and by (9) we can derive

$$\begin{aligned}
&\mathbb{E}_t[x_T^* - l_T^*]\\
=&\mathbb{E}_t[q_{t+1}\boldsymbol{e}_t^{\mathrm{T}}]\mathbb{E}_t^{-1}[Q_{t+1}\boldsymbol{e}_t\boldsymbol{e}_t^{\mathrm{T}}]\bigg(\frac{\lambda}{2\omega}\Big(\mathbb{E}_t[q_{t+1}\boldsymbol{e}_t] - q_t\mathbf{1}\Big) + Q_t x_t\mathbf{1} + l_t\Big(m_t\mathbf{1} - \mathbb{E}_t[m_{t+1}p_t\boldsymbol{e}_t]\Big)\bigg)\\
&+\mathbb{E}_t[h_{t+1}p_t]l_t + \frac{\lambda}{2\omega}\mathbb{E}_t[c_{t+1}]\\
=&q_t x_t + h_t l_t + \frac{\lambda}{2\omega}c_t.
\end{aligned}$$

Similarly, for (15), we assume it holds true at time $t+1$, then we have

$$\begin{aligned}
&\mathbb{E}_t[(x_T^* - l_T^*)^2]\\
=&\mathbb{E}_t\Big[\mathbb{E}_{t+1}[(x_T^* - l_T^*)^2]\Big]\\
=&\mathbb{E}_t\Big[Q_{t+1}x_{t+1}^2 + R_{t+1}l_{t+1}^2 + 2m_{t+1}x_{t+1}l_{t+1} + \frac{\lambda^2}{4\omega^2}c_{t+1}\Big]\\
=&\boldsymbol{u}_t^{\mathrm{T}}\mathbb{E}_t[Q_{t+1}\boldsymbol{e}_t\boldsymbol{e}_t^{\mathrm{T}}]\boldsymbol{u}_t + \mathbb{E}_t[R_{t+1}p_t^2]l_t^2 + 2\mathbb{E}_t[m_{t+1}p_t\boldsymbol{e}_t^{\mathrm{T}}]\boldsymbol{u}_t l_t + \frac{\lambda^2}{4\omega^2}\mathbb{E}_t[c_{t+1}].
\end{aligned}$$

By using (9) and (10) we can derive

$$\begin{aligned}
&\mathbb{E}_t[(x_T^* - l_T^*)^2]\\
=&\bigg(\frac{\lambda}{2\omega}\Big(\mathbb{E}_t[q_{t+1}\boldsymbol{e}_t] - q_t\mathbf{1}\Big) + Q_t x_t\mathbf{1} + l_t\Big(m_t\mathbf{1} - \mathbb{E}_t[m_{t+1}p_t\boldsymbol{e}_t]\Big)\bigg)^{\mathrm{T}}\mathbb{E}_t^{-1}[Q_{t+1}\boldsymbol{e}_t\boldsymbol{e}_t^{\mathrm{T}}]\\
&\times\bigg(\frac{\lambda}{2\omega}\Big(\mathbb{E}_t[q_{t+1}\boldsymbol{e}_t] - q_t\mathbf{1}\Big) + Q_t x_t\mathbf{1} + l_t\Big(m_t\mathbf{1} - \mathbb{E}_t[m_{t+1}p_t\boldsymbol{e}_t]\Big)\bigg) + \mathbb{E}_t[R_{t+1}p_t^2]l_t^2\\
&+2\mathbb{E}_t[m_{t+1}p_t\boldsymbol{e}_t^{\mathrm{T}}]\mathbb{E}_t^{-1}[Q_{t+1}\boldsymbol{e}_t\boldsymbol{e}_t^{\mathrm{T}}]\bigg(\frac{\lambda}{2\omega}\Big(\mathbb{E}_t[q_{t+1}\boldsymbol{e}_t] - q_t\mathbf{1}\Big) + Q_t x_t\mathbf{1}\\
&+l_t\Big(m_t\mathbf{1} - \mathbb{E}_t[m_{t+1}p_t\boldsymbol{e}_t]\Big)\bigg)l_t + \frac{\lambda^2}{4\omega^2}\mathbb{E}_t[c_{t+1}]\\
=&\frac{1}{\mathbf{1}^{\mathrm{T}}\mathbb{E}_t^{-1}[Q_{t+1}\boldsymbol{e}_t\boldsymbol{e}_t^{\mathrm{T}}]\mathbf{1}}x_t^2 + \bigg(\mathbb{E}_t[R_{t+1}p_t^2] - \mathbb{E}_t[m_{t+1}\boldsymbol{e}_t^{\mathrm{T}}p_t]\mathbb{E}_t^{-1}[Q_{t+1}\boldsymbol{e}_t\boldsymbol{e}_t^{\mathrm{T}}]\mathbb{E}_t[m_{t+1}\boldsymbol{e}_t p_t]\\
&+\frac{\big(\mathbf{1}^{\mathrm{T}}\mathbb{E}_t^{-1}[Q_{t+1}\boldsymbol{e}_t\boldsymbol{e}_t^{\mathrm{T}}]\mathbb{E}_t[m_{t+1}\boldsymbol{e}_t p_t]\big)^2}{\mathbf{1}^{\mathrm{T}}\mathbb{E}_t^{-1}[Q_{t+1}\boldsymbol{e}_t\boldsymbol{e}_t^{\mathrm{T}}]\mathbf{1}}\bigg)l_t^2 + 2\frac{\mathbf{1}^{\mathrm{T}}\mathbb{E}_t^{-1}[Q_{t+1}\boldsymbol{e}_t\boldsymbol{e}_t^{\mathrm{T}}]\mathbb{E}_t[m_{t+1}\boldsymbol{e}_t p_t]}{\mathbf{1}^{\mathrm{T}}\mathbb{E}_t^{-1}[Q_{t+1}\boldsymbol{e}_t\boldsymbol{e}_t^{\mathrm{T}}]\mathbf{1}}x_t l_t\\
&+\frac{\lambda^2}{4\omega^2}\bigg(\mathbb{E}_t[c_{t+1}] - \frac{\big(\mathbf{1}^{\mathrm{T}}\mathbb{E}_t^{-1}[Q_{t+1}\boldsymbol{e}_t\boldsymbol{e}_t^{\mathrm{T}}]\mathbb{E}_t[q_{t+1}\boldsymbol{e}_t]\big)^2}{\mathbf{1}^{\mathrm{T}}\mathbb{E}_t^{-1}[Q_{t+1}\boldsymbol{e}_t\boldsymbol{e}_t^{\mathrm{T}}]\mathbf{1}} + \mathbb{E}_t[q_{t+1}\boldsymbol{e}_t^{\mathrm{T}}]\mathbb{E}_t^{-1}[Q_{t+1}\boldsymbol{e}_t\boldsymbol{e}_t^{\mathrm{T}}]\mathbb{E}_t[q_{t+1}\boldsymbol{e}_t]\bigg)\\
=&Q_t x_t^2 + R_t l_t^2 + 2m_t x_t l_t + \frac{\lambda^2}{4\omega^2}c_t.
\end{aligned}$$

The equation (16) can be derived easily by substituting (14) and (15) into

$$\mathrm{Var}_t[x_T^* - l_T^*] = \mathbb{E}_t[(x_T^* - l_T^*)^2] - \big(\mathbb{E}_t[x_T^* - l_T^*]\big)^2.$$

The proof is completed. ∎

A.3 The Proof of Verifying Equation (27)

To simplify the expression, define

$$\begin{aligned}
B_t &= \mathbb{E}_t[\rho_{t+1}\boldsymbol{d}_t^{\mathrm{T}}]\mathbb{E}_t^{-1}[\rho_{t+1}\boldsymbol{d}_t\boldsymbol{d}_t^{\mathrm{T}}]\mathbb{E}_t[\rho_{t+1}\boldsymbol{d}_t],\\
\widehat{B}_t &= \mathbb{E}_t[\rho_{t+1}\boldsymbol{d}_t^{\mathrm{T}}]\mathbb{E}_t^{-1}[\rho_{t+1}\boldsymbol{d}_t\boldsymbol{d}_t^{\mathrm{T}}]\mathbb{E}_t[p_t\eta_{t+1}\boldsymbol{d}_t],\\
\widetilde{B}_t &= \mathbb{E}_t[p_t\eta_{t+1}\boldsymbol{d}_t^{\mathrm{T}}]\mathbb{E}_t^{-1}[\rho_{t+1}\boldsymbol{d}_t\boldsymbol{d}_t^{\mathrm{T}}]\mathbb{E}_t[p_t\eta_{t+1}\boldsymbol{d}_t],
\end{aligned}$$

then

$$\rho_t = \mathbb{E}_t[\rho_{t+1}] - B_t, \quad \eta_t = \mathbb{E}_t[p_t\eta_{t+1}] - \widehat{B}_t, \quad \zeta_t = \mathbb{E}_t[p_t^2\zeta_{t+1}] - \widetilde{B}_t.$$

Now we can verify the equation (27) is true. Obviously, when $t = T$, the claim (27) holds. Assume that the equation (27) is true at stage $t+1$. Then, using the equation (35), we can easily obtain

$$\begin{aligned}
\mathbb{E}_t^{-1}[Q_{t+1}\widehat{e}_t\widehat{e}_t^{\mathrm{T}}] &= \frac{1}{\gamma_{t+1}^2}\begin{bmatrix} r_t^2\widehat{\alpha}_t & r_t\boldsymbol{b}_t^{\mathrm{T}} \\ r_t\boldsymbol{b}_t & \boldsymbol{H}_t \end{bmatrix}^{-1}\\
&= \frac{1}{\gamma_{t+1}^2}\begin{bmatrix} 1 & -\frac{1}{r_t\widehat{\alpha}_t}\boldsymbol{b}_t^{\mathrm{T}} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix}\begin{bmatrix} \frac{1}{r_t^2\widehat{\alpha}_t} & \boldsymbol{0} \\ \boldsymbol{0} & \widehat{\alpha}_t\big(\widehat{\alpha}_t\boldsymbol{H}_t - \boldsymbol{b}_t\boldsymbol{b}_t^{\mathrm{T}}\big)^{-1} \end{bmatrix}\begin{bmatrix} 1 & \boldsymbol{0} \\ -\frac{1}{r_t\widehat{\alpha}_t}\boldsymbol{b}_t & \boldsymbol{I} \end{bmatrix}
\end{aligned}$$

and

$$(1\ \ \boldsymbol{1}^{\mathrm{T}})\mathbb{E}_t^{-1}[Q_{t+1}\boldsymbol{e}_t\boldsymbol{e}_t^{\mathrm{T}}](1\ \ \boldsymbol{1}^{\mathrm{T}})^{\mathrm{T}} = \frac{1}{\gamma_t^2\widehat{\alpha}_t} + \frac{1}{\gamma_t^2\widehat{\alpha}_t}(\boldsymbol{b}_t - r_t\widehat{\alpha}_t\boldsymbol{1})^{\mathrm{T}}(\widehat{\alpha}_t\boldsymbol{H}_t - \boldsymbol{b}_t\boldsymbol{b}_t^{\mathrm{T}})^{-1}(\boldsymbol{b}_t - r_t\widehat{\alpha}_t\boldsymbol{1}),$$

where $\widehat{\alpha}_t = \mathbb{E}_t[\rho_{t+1}]$, $\boldsymbol{b}_t = \mathbb{E}_t[\rho_{t+1}\boldsymbol{e}_t]$, and $\boldsymbol{H}_t = \mathbb{E}_t[\rho_{t+1}\boldsymbol{e}_t\boldsymbol{e}_t^{\mathrm{T}}]$. Note that

$$\mathbb{E}_t[\rho_{t+1}\boldsymbol{d}_t] = \boldsymbol{b}_t - r_t\widehat{\alpha}_t\boldsymbol{1}, \quad \widehat{\alpha}_t\boldsymbol{H}_t - \boldsymbol{b}_t\boldsymbol{b}_t^{\mathrm{T}} = \widehat{\alpha}_t\mathbb{E}_t[\rho_{t+1}\boldsymbol{d}_t\boldsymbol{d}_t^{\mathrm{T}}] - \mathbb{E}_t[\rho_{t+1}\boldsymbol{d}_t]\mathbb{E}_t[\rho_{t+1}\boldsymbol{d}_t^{\mathrm{T}}].$$

And according to (9) and (34), we have

$$\begin{aligned}
Q_t &= 1/\Big((1\ \ \boldsymbol{1}^{\mathrm{T}})\mathbb{E}_t^{-1}[Q_{t+1}\widehat{e}_t\widehat{e}_t^{\mathrm{T}}](1\ \ \boldsymbol{1}^{\mathrm{T}})^{\mathrm{T}}\Big)\\
&= \frac{\gamma_t^2\widehat{\alpha}_t}{1 + \mathbb{E}_t[\rho_{t+1}\boldsymbol{d}_t^{\mathrm{T}}]\big(\widehat{\alpha}_t\mathbb{E}_t[\rho_{t+1}\boldsymbol{d}_t\boldsymbol{d}_t^{\mathrm{T}}] - \mathbb{E}_t[\rho_{t+1}\boldsymbol{d}_t]\mathbb{E}_t[\rho_{t+1}\boldsymbol{d}_t^{\mathrm{T}}]\big)^{-1}\mathbb{E}_t[\rho_{t+1}\boldsymbol{d}_t]}\\
&= \gamma_t^2(\widehat{\alpha}_t - B_t)\\
&= \gamma_t^2\rho_t.
\end{aligned}$$

Using (35) again, we have

$$(1 \quad \mathbf{1}^{\mathrm{T}})\mathbb{E}_t^{-1}[Q_{t+1}\widehat{e}_t\widehat{e}_t^{\mathrm{T}}]\mathbb{E}_t[q_{t+1}\widehat{e}_t] = (1 \quad \mathbf{1}^{\mathrm{T}})\mathbb{E}_t^{-1}[Q_{t+1}\widehat{e}_t\widehat{e}_t^{\mathrm{T}}](\widehat{\alpha}_t\gamma_t \quad \gamma_{t+1}\boldsymbol{b}_t^{\mathrm{T}})^{\mathrm{T}} = \frac{1}{\gamma_t},$$

$$\begin{aligned}(1 \quad \mathbf{1}^{\mathrm{T}})\mathbb{E}_t^{-1}[Q_{t+1}\widehat{e}_t\widehat{e}_t^{\mathrm{T}}]\mathbb{E}_t[m_{t+1}p_t\widehat{e}_t] &= (1 \quad \mathbf{1}^{\mathrm{T}})\mathbb{E}_t^{-1}[Q_{t+1}\widehat{e}_t\widehat{e}_t^{\mathrm{T}}](\beta_t\gamma_t \quad \gamma_{t+1}\boldsymbol{D}_t^{\mathrm{T}})^{\mathrm{T}} \\ &= \frac{\beta_t}{\gamma_t\widehat{\alpha}_t} + \frac{\beta_t}{\gamma_t\widehat{\alpha}_t}(\boldsymbol{b}_t - r_t\widehat{\alpha}_t\mathbf{1})^{\mathrm{T}}(\widehat{\alpha}_t\boldsymbol{H}_t - \boldsymbol{b}_t\boldsymbol{b}_t^{\mathrm{T}})^{-1}\left(\boldsymbol{b}_t - \frac{\widehat{\alpha}_t}{\beta_t}\boldsymbol{D}_t\right),\end{aligned}$$

$$\mathbb{E}_t[m_{t+1}p_t\widehat{e}_t^{\mathrm{T}}]\mathbb{E}_t^{-1}[Q_{t+1}\widehat{e}_t\widehat{e}_t^{\mathrm{T}}]\mathbb{E}_t[q_{t+1}\widehat{e}_t] = (\beta_t\gamma_t \quad \gamma_{t+1}\boldsymbol{D}_t^{\mathrm{T}})\mathbb{E}_t^{-1}[Q_{t+1}\widehat{e}_t\widehat{e}_t^{\mathrm{T}}](\widehat{\alpha}_t\gamma_t \quad \gamma_{t+1}\boldsymbol{b}_t^{\mathrm{T}})^{\mathrm{T}} = \beta_t,$$

$$\begin{aligned}&\mathbb{E}_t[m_{t+1}p_t\widehat{e}_t^{\mathrm{T}}]\mathbb{E}_t^{-1}[Q_{t+1}\widehat{e}_t\widehat{e}_t^{\mathrm{T}}]\mathbb{E}_t[m_{t+1}p_t\widehat{e}_t] = (\beta_t\gamma_t \quad \gamma_{t+1}\boldsymbol{D}_t^{\mathrm{T}})\mathbb{E}_t^{-1}[Q_{t+1}\widehat{e}_t\widehat{e}_t^{\mathrm{T}}](\beta_t\gamma_t \quad \gamma_{t+1}\boldsymbol{D}_t^{\mathrm{T}})^{\mathrm{T}} \\ =&\frac{\beta_t^2}{\widehat{\alpha}_t} + \frac{\beta_t^2}{\widehat{\alpha}_t}\left(\frac{B_t}{\widehat{\alpha}_t - B_t}\right) - 2\beta_t\left(\frac{\widehat{B}_t}{\widehat{\alpha}_t - B_t}\right) + \frac{\widehat{\alpha}_t\widetilde{B}_t - \widetilde{B}_tB_t + \widehat{B}_t^2}{\widehat{\alpha}_t - B_t} \\ =&\frac{\eta_t^2}{\rho_t} + \widetilde{B}_t,\end{aligned}$$

$$\mathbb{E}_t[q_{t+1}\widehat{e}_t^{\mathrm{T}}]\mathbb{E}_t^{-1}[Q_{t+1}\widehat{e}_t\widehat{e}_t^{\mathrm{T}}]\mathbb{E}_t[q_{t+1}\widehat{e}_t] = (\widehat{\alpha}_t\gamma_t \quad \gamma_{t+1}\boldsymbol{b}_t^{\mathrm{T}})\mathbb{E}_t^{-1}[Q_{t+1}\widehat{e}_t\widehat{e}_t^{\mathrm{T}}](\widehat{\alpha}_t\gamma_t \quad \gamma_{t+1}\boldsymbol{b}_t^{\mathrm{T}})^{\mathrm{T}} = \widehat{\alpha}_t,$$

with $\beta_t = \mathbb{E}_t[\eta_{t+1}p_t]$, $\boldsymbol{D}_t = \mathbb{E}_t[\eta_{t+1}p_t\boldsymbol{e}_t]$. Note that

$$\boldsymbol{b}_t - \frac{\widehat{\alpha}_t}{\beta_t}\boldsymbol{D}_t = \big(\mathbb{E}_t[\rho_{t+1}\boldsymbol{e}_t] - \widehat{\alpha}_tr_t\mathbf{1}\big) - \widehat{\alpha}_t\left(\frac{\boldsymbol{D}_t - \beta_tr_t\mathbf{1}}{\beta_t}\right) = \mathbb{E}_t[\rho_{t+1}\boldsymbol{d}_t] - \frac{\widehat{\alpha}_t}{\beta_t}\mathbb{E}_t[\eta_{t+1}p_t\boldsymbol{d}_t].$$

And combining (9) with (34), we have

$$\begin{aligned}q_t &= \frac{(1 \quad \mathbf{1}^{\mathrm{T}})\mathbb{E}_t^{-1}[Q_{t+1}\widehat{e}_t\widehat{e}_t^{\mathrm{T}}]\mathbb{E}_t[q_{t+1}\widehat{e}_t]}{(1 \quad \mathbf{1}^{\mathrm{T}})\mathbb{E}_t^{-1}[Q_{t+1}\widehat{e}_t\widehat{e}_t^{\mathrm{T}}](1 \quad \mathbf{1}^{\mathrm{T}})^{\mathrm{T}}} = \gamma_t\rho_t, \\ m_t &= \frac{(1 \quad \mathbf{1}^{\mathrm{T}})\mathbb{E}_t^{-1}[Q_{t+1}\widehat{e}_t\widehat{e}_t^{\mathrm{T}}]\mathbb{E}_t[m_{t+1}p_t\widehat{e}_t]}{(1 \quad \mathbf{1}^{\mathrm{T}})\mathbb{E}_t^{-1}[Q_{t+1}\widehat{e}_t\widehat{e}_t^{\mathrm{T}}](1 \quad \mathbf{1}^{\mathrm{T}})^{\mathrm{T}}} \\ &= \gamma_t^2\rho_t\left(\frac{\beta_t}{\gamma_t\widehat{\alpha}_t} + \frac{\beta_t}{\gamma_t\widehat{\alpha}_t}\left(\frac{\widehat{\alpha}_t}{\widehat{\alpha}_t - B_t} - 1\right) - \frac{1}{\gamma_t}\left(\frac{\widehat{B}_t}{\widehat{\alpha}_t - B_t}\right)\right) = \gamma_t\eta_t, \\ h_t &= \mathbb{E}[\eta_{t+1}p_t] - \beta_t + \frac{m_tq_t}{Q_t} = \mathbb{E}[\eta_{t+1}p_t] - \beta_t + \gamma_t\eta_t \times \frac{\gamma_t\rho_t}{\gamma_t^2\rho_t} = \eta_t, \\ R_t &= \mathbb{E}_t[\zeta_{t+1}p_t^2] - \left(\frac{\eta_t^2}{\rho_t} + \widetilde{B}_t\right) + \frac{m_t^2}{Q_t} = \mathbb{E}_t[\zeta_{t+1}p_t^2] - \left(\frac{\eta_t^2}{\rho_t} + \widetilde{B}_t\right) + \frac{\eta_t^2}{\rho_t} = \zeta_t, \\ c_t &= \mathbb{E}_t[1 - \rho_{t+1}] - \frac{q_t^2}{Q_t} + \widehat{\alpha}_t = 1 - \mathbb{E}_t[\rho_{t+1}] - \rho_t + \widehat{\alpha}_t = 1 - \rho_t.\end{aligned}$$

Then the proof is completed. ∎