J Syst Sci Complex (2023) 36: 1560–1582

Exponential Stability of Impulsive Neutral Stochastic Functional Differential Equations with Markovian Switching*

XIAO Ke · LI Shuyong

DOI: 10.1007/s11424-023-1332-8
Received: 3 July 2021 / Revised: 20 April 2022

Abstract The aim of this paper is to the discussion of the exponential stability of a class of impulsive neutral stochastic functional differential equations with Markovian switching. Under the influence of impulsive disturbance, the solution for the system is discontinuous. By using the Razumikhin technique and stochastic analysis approaches, as well as combining the idea of mathematical induction and classification discussion, some sufficient conditions for the pth moment exponential stability and almost exponential stability of the systems are obtained. The stability conclusion is full time-delay. The results show that impulse, the point distance of impulse and Markovain switching affect the stability for the system. Finally, two examples are provided to illustrate the effectiveness of the results proposed.

Keywords Delay, exponential stability, impulsive, Markovian switching, neutral, Razumikhin technique, stochastic functional differential equations.

1 Introduction

Neutral functional differential equations (NFDEs) are usually used to simulate the systems that many dynamical systems not only depend on present and past states but also involve derivatives with delays (we can see [1–4]).

Meanwhile, some actual phenomenon has already been successfully modeled by the stochastic system such as chemical reactor control, population ecology, biological neural networks, applied economics, machine learning, as well as many branches in the field of science and industry, and the system also depends on the derivative of time-delay and the function itself. Such

XIAO Ke
School of Mathematics and Statistics, Sichuan University of Science & Engineering, Zigong 643002*, China; Department of Mathematics, Sichuan University, Chengdu* 610065*, China*. Email: xk_math@163.com.
LI Shuyong (Corresponding author)
School of Mathematics and Physics, Mianyang Teachers' College, Mianyang 621000, *China*.
Email: shuyongli@sicnu.edu.cn.
*This research was supported by the National Nature Science Foundation of China under Grant No. 11571245 and Young Crop Project of Sichuan University under Grant No. 2020SCUNL111.
◇ *This paper was recommended for publication by Editor TAN Ying.*

systems historically have been referred to as neutral stochastic functional differential equations (NSFDEs), which are more difficult to motivate than stochastic functional differential equations because they contain some special cases such as the latter. The investigation on stability is to help us understand the long-term behavior of the system, which is naturally concerned by researchers in various fields. In particular, owing to the emergence for the theory of aeroelasticity and the chemical engineering systems, [5] investigated neutral stochastic functional differential equations and discussed their stability. Furthermore, by using the Razumikhin technique, the pth moment general decay stability and general asymptotic stability of the solution for NSFDEs are discussed respectively in [6] and [7]. Further discussions on NSFDEs can be found in [8–14].

On the one hand, due to the unpredictable real environment, the structure and parameters of the stochastic systems may often encounter random abrupt changes, causing the system to switch between limited states. Continuous-time Markov chain is an efficient tool to model these abrupt changes gratifyingly. One of the important issue in researching Markov switching systems is the automatic control (such as [15, 16]). In addition, under the influence of the jump disturbance, the exponential stability analysis of neutral stochastic functional differential equations with Markovian switching (NSFDEs-MS) is more troublesome (we refer to [17–22]). For example, in [18], Zhou and Hu considered the exponential stability for a class of NSFDEs-MS by virtue of the Razumikhin technique and the stochastic analysis, which eliminated the influence of the discontinuity of the sample path on the stability. Impulses, on the other hand, may have significant effects on the systems' dynamics, either as a harmful or beneficial role on the stability, in which system's state changes suddenly at certain instants. Actually, the stability analysis and stabilization of the (stochastic) functional differential systems with impulsive effects have been considered in [23–28]. For instance, some results on the exponential stability of solutions to impulsive stochastic functional differential equations are giving in [26] and [28] separately. Considering the derivatives with functionals, some stability and stabilization problems for impulsive neutral stochastic functional differential equations (INSFDEs) have been discussed in [29–31]. It is worth mention that in [30], Benhardri, et al. discussed the existence and uniqueness as well as asymptotic stability for a kind of INSFDEs based on a contraction mapping principle and the analytic technique.

It should be noted that Markovian switching and stochastic perturbations often coexist on practical systems, which can be modeled by impulsive neutral stochastic functional differential equations with Markovian switching (INSFDEs-MS) (see [32, 33]). However, the stability analysis on the systems is more complex and still an open problem due to the discontinuity of the sample orbit which consider Markovian switching and impulsive effect simultaneously. As is known to all, little has been reported for stabilities of such systems. Recently, by using the Razumikhin technique and stochastic analysis theories, [32] derived some criteria for determining the exponential stability of the trivial solution for a kind of INSFDEs-MS, and Lassaad and Mohamed investigated the β-stability in qth moment for INSFDEs-MS in [33], which overcame difficulties of the forgoing discussion.

Inspired by the works mentioned above and of [32, 34, 35] as well as [36], in this paper, we will investigate exponential stability for a class of INSFDEs-MS by means of the Razumikhin technique and combining the stochastic analysis theories as well as the inequality techniques. More precisely, we will improve the Razumikhin condition that the coefficients of the estimated upper bound for the diffusion operator of Lyapunov functionals can be a positive definite or negative definite function instead of negative value, and will establish some criteria to ensure the exponential stability for the trivial solution of the systems. The main contributions of this paper are summarized into two aspects:

1) Impulse stabilization has a wider application scope than before studies. Due to the unstable state of the stochastic dynamic of the system without impulses ($\mathrm{E}\mathcal{L}V \leq b(t)\mathrm{E}V$), some sufficiently strong impulses input are required to ensure the stability of the system ($q^{-1} < \rho_1 + \rho_2 < 1$), In [32], to promote the exponential stability of the solution for the system under the relatively satisfactory dynamic situation ($\mathrm{E}\mathcal{L}V \leq -\mu\mathrm{E}V$, $\mu > 0$), the system impulses input are considered ($|\gamma_k| < 1$ and $|\gamma_k|^p < \mathrm{e}^{\gamma\tau}q$).

2) The constraint of unstable impulses perturbation need not be so strict. When the system without impulsive input is in a relatively stable state ($\mathrm{E}\mathcal{L}V \leq -b(t)\mathrm{E}V$), it is not necessary to strictly restrict the impulse disturbance, i.e., $\rho_1+\rho_2 > 1$ and the condition (vi) in Theorem 3.6. In [32], the impulses disturbance must meet the requirement of $|\gamma_k| < 1$ before the stability conclusion can be established.

The contents of the paper are as follows. In Section 2, we will introduce some basic notations and definitions as well as present the mathematical model of INSFDEs-MS. In Section 3, we will give several new criteria on the pth moment and almost exponential stability of the systems. In Section 4, two illustrative examples are analyzed to show the usefulness and feasibility of results presented. Finally, a brief conclusion is given to end this work in Section 5.

2 Preliminaries

Notations Throughout this paper, unless otherwise specified, we shall use the follow notations. Let $|x|$ denote Euclidean norm of a vector $x \in \mathbb{R}^d$ and its transpose be denoted by A^{T}. $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq t_0}, P)$ stands for a complete probability space with filtration $\{\mathscr{F}_t\}_{t\geq t_0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathscr{F}_{t_0}$ contains all P-null sets). $w = w(t)(t \geq 0)$, $w(t) = (w_1(t), w_2(t), \cdots, w_m(t))^{\mathrm{T}}$ is a standard m-dimensional Brownian motion defined on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq t_0}, P)$. Let $\mathbb{R}_0 = [0, +\infty)$, and $\mathbb{N}$ denotes the set of positive integers. Let $PC([a, b], \mathbb{R}^d) = \{\varphi : [a, b] \to \mathbb{R}^d|\ \varphi(t^+) = \varphi(t)$ for all $t \in [a, b)$, $\varphi(t^-)$ exists and $\varphi(t^-) = \varphi(t)$ for all but at most a finite number of points $t \in (a, b]\}$ with the norm $\|\varphi\| = \sup_{a\leq t\leq b} |\varphi(\theta)|$, where $\varphi(t^+)$ and $\varphi(t^-)$ denote the right-hand and left-hand limits of function $\varphi(t)$ at t respectively. In specially, we define $PC \triangleq PC([-\tau, 0], \mathbb{R}^d)$ for all $\tau \geq 0$. $L^p_{\mathscr{F}_0}(\Omega, PC)$ denotes the family of all $\mathscr{F}_0$ measurable, PC-valued random variables $\varphi = \{\varphi(\theta) : -\tau \leq \theta \leq 0\}$ with $\|\varphi\|^p_0 = \sup_{-\tau\leq\theta\leq 0} \mathrm{E}|\varphi(\theta)|^p < \infty$, where E stands for the corresponding expectation operator with respect to the given probability measure P. Moreover, $\mathcal{M}_{d\times m}$ is the space of all real-valued $d \times m$ matrices with the norm $\|A\|_1 \triangleq (\sum_{i=1}^d \sum_{j=1}^m |A_{ij}A_{ji}|)^{1/2}$, where $A = (a_{ij})_{d\times m} \in \mathcal{M}_{d\times m}$. A_{ij} and A_{ji} are the cofactor of the elements a_{ij} and a_{ji} respectively.

Let $\{r(t), t \geq 0\}$ be a right-continuous Markov chain on the probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq t_0}, P)$ taking values in a finite state space $S = \{1, 2, \cdots, \mathbb{N}\}$ with generator $Q = \{q_{ij}\}_{\mathbb{N}\times\mathbb{N}}$ (also called the state transition matrix of $r(t)$) given by

$$P\{r(t+\triangle t) = j | r(t) = i\} = \begin{cases} q_{ij}\triangle t + o(\triangle t), & \text{if } i \neq j, \\ 1 + q_{ii}\triangle t + o(\triangle t), & \text{if } i = j, \end{cases}$$

where $\triangle t > 0$ and $\lim_{\triangle t\to 0} \frac{o(\triangle t)}{\triangle t} = 0$. Here, $q_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while $q_{ii} = -\Sigma_{j\neq i} q_{ij}$. We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $w(\cdot)$. It is well known (see [37]) that almost every sample path of $r(t)$ is a constant except for a finite number of simple jumps in any finite subinterval of $[t_0, +\infty)$, and almost every sample path of $r(t)$ is right continuous. In other words, there is a sequence of stopping times that $0 = \tau_0 < \tau_1 < \cdots < \tau_k < \cdots \to \infty$ and $r(t)$ is constant almost everywhere on every interval $[\tau_k, \tau_{k+1})$, i.e.,

$$r(t) = r(\tau_k), \quad \forall t \in [\tau_k, \tau_{k+1}), \quad k \in \mathbb{N}.$$

The topic of our analysis is the following d-dimensional INSFDEs-MS:

$$\begin{cases} d[x(t) - D(t, x_t, r(t))] \\ = f(t, x_t, r(t))dt + g(t, x_t, r(t))dw(t), \quad t \geq t_0, \quad t \neq t_k, \quad k \in \mathbb{N}, \\ \triangle x(t_k) = I_k\big(t_k^-, x_{t_k^-}, r(t_k^-)\big), \qquad k \in \mathbb{N}, \\ x_{t_0} \triangleq \xi = \{\xi(\theta) : -\tau \leq \theta \leq 0\} \in L^p_{\mathscr{F}_{t_0}}(\Omega, PC), \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^d$, $x_t \triangleq x_t(\theta) = \{x(t+\theta) : -\tau \leq \theta \leq 0\}$ is regard as a PC-valued stochastic process. Both $D : [t_0, +\infty) \times PC \times S \to \mathbb{R}^d$, $f : [t_0, +\infty) \times PC \times S \to \mathbb{R}^d$ and $g : [t_0, +\infty) \times PC \times S \to \mathcal{M}_{d\times m}$ are continuous functionals. $I_k\big(t_k^-, x_{t_k^-}, r(t_k^-)\big) : [t_0, +\infty) \times PC \times S \to \mathbb{R}^d$ represents the impulsive perturbation of x at t_k. The fixed moments of impulsive times t_k satisfy $0 \leq t_0 < t_1 < \cdots < t_k < \cdots$ and $t_k \to \infty$ (as $k \to \infty$). $\triangle x(t_k) = x(t_k) - x(t_k^-)$, where $x(t_k^-) = \lim_{h\to 0^-} x(t_k + h)$. Let $\widetilde{x}(t) \triangleq x(t) - D(t, x_t, r(t))$.

Without loss of generality, we assume that the functionals D, f, g and I_k $(k \in N)$ satisfy necessary assumptions (such as the conditions H1 and H2 in [32]) so that, for all $t \geq t_0$ and the initial function ξ, System (1) has a unique global solution $x(t;\xi) \in PC([t_0-\tau, +\infty], \mathbb{R}^d)$ (record briefly $x(t)$). In addition, we assume that $D(t, 0, r(t)) \equiv 0$, $f(t, 0, r(t)) \equiv 0$, $g(t, 0, r(t)) \equiv 0$ for all $t \geq t_0$, and $I_k(t_k^-, 0, r(t_k^-)) \equiv 0$ for all $t \geq t_0$, $k \in \mathbb{N}$, then System (1) admits always a trivial solution $x(t) \equiv 0$.

Let $C^{1,2}([t_0 - \tau, +\infty) \times \mathbb{R}^d \times S)$ be the family of all nonnegative functions $V(t, x, i)$ from $[t_0 - \tau, +\infty) \times \mathbb{R}^d \times S$ to $\mathbb{R}_0$ which is continuously once differentiable in t and twice in x. For each $V \in C^{1,2}([t_0 - \tau, +\infty) \times \mathbb{R}^d \times S)$, define an operator $\mathcal{L}V : [t_0 - \tau, +\infty) \times \mathbb{R}^d \times S \longrightarrow \mathbb{R}$

associated with System (1) by

$$\mathcal{L}V(t,\varphi,i) = V_t(t,\widetilde{\varphi}(0),i) + V_x(t,\widetilde{\varphi}(0),i)f(t,\varphi,i) + \frac{1}{2}\mathrm{trace}[g_i^{\mathrm{T}}(t)V_{xx}(t,\widetilde{\varphi}(0),i)g_i(t)] + \sum_{j=1}^{\mathbb{N}} q_{ij}V(t,\widetilde{\varphi}(0),j), \tag{2}$$

where

$$g_i(t) = g(t,\varphi,i), \quad V_x(t,\widetilde{\varphi}(0),i) = \left(\frac{\partial V(t,x,i)}{\partial x_1}, \cdots, \frac{\partial V(t,x,i)}{\partial x_d}\right)\bigg|_{x=\widetilde{\varphi}(0)},$$

$$V_t(t,\widetilde{\varphi}(0),i) = \frac{\partial V(t,x,i)}{\partial t}\bigg|_{x=\widetilde{\varphi}(0)}, \quad V_{xx}(t,\widetilde{\varphi}(0),i) = \left(\frac{\partial^2 V(t,x,i)}{\partial x_j \partial x_k}\right)_{d\times d}\bigg|_{x=\widetilde{\varphi}(0)}.$$

Definition 2.1 (see [38]) The function $V : [t_0-\tau,+\infty)\times\mathbb{R}^d\times S \longrightarrow \mathbb{R}_0$ belongs to class v_0 if:

(i) for $k = 1, 2, \cdots$, the function V is twice differentiable in x on $[t_{k-1}, t_k)\times\mathbb{R}^d\times S$ and once continuously differentiable in t. In addition, $V(t,0,i)\equiv 0$ for all $t\geq 0$;

(ii) $V(t,x,r(t))$ is locally Lipschitzian in $x\in\mathbb{R}^d$;

(iii) for each $k = 1, 2, \cdots$, there exist finite limits

$$\lim_{(t,y,r(t))\to(t_k^-,x,r(t_k))} V(t,y,r(t)) = V(t_k^-,x,r(t_k)),$$

$$\lim_{(t,y,r(t))\to(t_k^+,x,r(t_k))} V(t,y,r(t)) = V(t_k^+,x,r(t_k)),$$

such that $V(t_k^+,x,r(t_k)) = V(t_k,x,r(t_k))$.

Definition 2.2 (see [32]) The trivial solution of System (1) is said to be pth ($p>0$) moment exponentially stable if there exist positive constants α and β such that, for every $\xi\in L^p_{\mathscr{F}_{t_0}}(\Omega, PC)$ following inequality holds

$$\mathrm{E}|x(t;\xi)|^p \leq \alpha\|\xi\|_0^p \mathrm{e}^{-\beta(t-t_0)}, \quad \forall t\geq t_0,$$

or equivalently, $\limsup_{t\to\infty}\frac{\ln \mathrm{E}|x(t;\xi)|^p}{t}\leq -\beta$.

Definition 2.3 (see [32]) The trivial solution of System (1) is said to be almost surely exponentially stable if there exists a positive constant γ such that

$$\limsup_{t\to\infty}\frac{\ln|x(t;\xi)|}{t}\leq -\gamma, \quad \text{a.s.}$$

for any initial $\xi\in L^p_{\mathscr{F}_{t_0}}(\Omega, PC)$.

3 Main Results

To study the problem that the trival solution of System (1) is exponentially stability, we first list the following assumption:

(A1) There exists a constant $l_0 \in (0,1)$ such that

$$\mathrm{E}|D(t,\varphi,r(t))|^p \le l_0^p \|\varphi\|_0^p,$$

for all $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$ and $\varphi \in L^p_{\mathscr{F}_t}(\Omega, PC)$.

Remark 3.1 Firstly, $x(t) - D(t, x_t, r(t))$ is regarded as a whole $\widetilde{x}(t)$, which the purpose is to be proved by the Razumikhin theorem of stochastic functional differential equations. Therefore, we only needs to estimate the value of $\widetilde{x}(t)$. Secondly, according to the characteristics of piecewise continuity of the solution, combined with proof by contradiction as well as mathematical induction method, and the result of following can be obtained.

Theorem 3.2 *Let $p > 1$, $\delta = \sup_{k\in\mathbb{N}}\{t_k - t_{k-1}\} < \infty$, and (A1) holds. Assume that there exist functions $V \in v_0$ and $b(t) \in PC([t_0-\tau,+\infty),\mathbb{R}_0)$ as well as several positive constants c_1, c_2, κ_1, q, ρ_1, ρ_2, $\widetilde{b}$ satisfying $\kappa_1 \in (l_0, 1)$, $q > c_1^{-1}c_2(1-\kappa_1)^{-p}$, and $e_k \ge 0$ $(k \in \mathbb{N})$, $\sum_{k=1}^{\infty} e_k < \infty$ and $\rho_1 + \rho_2 < 1$ such that*

(i) *for all $(t, x, r(t)) \in [t_0 - \tau, +\infty) \times \mathbb{R}^d \times S$,*

$$c_1|x|^p \le V(t,x,r(t)) \le c_2|x|^p; \tag{3}$$

(ii) *for all $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$, $\varphi \in L^p_{\mathscr{F}_{t_0}}(\Omega, PC)$ and $\theta \in [-\tau, 0]$,*

$$\mathrm{E}\mathcal{L}V(t,\varphi,r(t)) \le b(t)\mathrm{E}V(t,\widetilde{\varphi}(0),r(t)),$$

whenever $\mathrm{E}V(t+\theta,\varphi(\theta),r(t+\theta)) < q\mathrm{E}V(t,\widetilde{\varphi}(0),r(t))$;

(iii) *for all $k \in \mathbb{N}$, $\varphi \in L^p_{\mathscr{F}_{t_0}}(\Omega, PC)$,*

$$\begin{aligned}&\mathrm{E}V\big(t_k,\widetilde{\varphi}(0)+I_k(t_k,\widetilde{\varphi}(0),r(t_k)),r(t_k)\big)\\ \le\ &\rho_1(1+e_k)\mathrm{E}V\big(t_k^-,\widetilde{\varphi}(0),r(t_k^-)\big)\\ &+\rho_2(1+e_k)\sup_{\theta\in[-\tau,0]}\mathrm{E}V\big(t_k^-+\theta,\widetilde{\varphi}(\theta),r(t_k^-+\theta)\big);\end{aligned} \tag{4}$$

(iv) $\widetilde{b}\delta \ge \sup_{t\in[t_0,\infty)}\int_t^{t+\delta} b(s)ds$, $q > \frac{1}{\rho_1+\rho_2} > \mathrm{e}^{\widetilde{b}\delta}$.
Therefore, the trivial solution of System (1) *is pth moment exponentially stable for any bounded time delay $\tau \in (0,+\infty)$.*

Proof Set $M = \prod_{k=1}^{\infty}(1+e_k)$. From $e_k \ge 0$ and $\sum_{k=1}^{\infty} e_k < \infty$, we deduce that $1 \le M < \infty$. For any initial data $\xi \in L^p_{\mathscr{F}_{t_0}}(\Omega, PC)$, write $V(t,\widetilde{x}(t),r(t)) = V(t)$ for simplicity. From the condition (iv), one can choose a small enough constant $\gamma > 0$ such that

$$q > \frac{\mathrm{e}^{\gamma\tau}}{\rho_1+\rho_2\mathrm{e}^{\gamma\tau}} > \frac{1}{\rho_1+\rho_2\mathrm{e}^{\gamma\tau}} > \mathrm{e}^{(\widetilde{b}+\gamma)\delta}, \quad q\mathrm{e}^{-\gamma\tau} > 1. \tag{5}$$

Let $\widetilde{q} = q\mathrm{e}^{-\gamma\tau}c_1c_2^{-1}(1-\kappa_1)^p > 1$, and $N > 0$ is a large enough constant.

To prove the trivial solution of System (1) is pth moment exponentially stable, for any bounded initial value $\xi \in L^p_{\mathscr{F}_{t_0}}(\Omega, PC)$, it is sufficient to prove that

$$\mathrm{E}|x(t)|^p \le \alpha\|\xi\|_0^p \mathrm{e}^{-\gamma(t-t_0)}, \quad t \in [t_0,+\infty), \tag{6}$$

where α is a positive constant. Let $\beta_m = \prod_{k=1}^{m}(1+e_k)Nc_1^{-1}\|\xi\|_0^p$ for every $m \in \mathbb{N}$ and $\beta = MNc_1^{-1}\|\xi\|_0^p$, and let's first prove that

$$h(t) \triangleq \mathrm{e}^{\gamma(t-t_0)}\mathrm{E}|x(t)|^p \le (1-\kappa_1)^{-p}\beta_m, \quad t \in [t_0, t_m). \tag{7}$$

Note that if it holds

$$\widetilde{h}(t) \triangleq \mathrm{e}^{\gamma(t-t_0)}\mathrm{E}|\widetilde{x}(t)|^p \le \beta_m, \quad \forall t \in [t_0, t_m), \tag{8}$$

then by using the assumption (A1) it follows, for all $t \in [t_0, t_m)$, that

$$\begin{aligned} H(t) &\triangleq \sup_{t_0-\tau\le s\le t} h(s) \\ &\le \|\xi\|_0^p \vee \sup_{t_0\le s\le t} h(s) \\ &\le \|\xi\|_0^p \vee \sup_{t_0\le s\le t} \mathrm{e}^{\gamma(s-t_0)}[(1-\kappa_1)^{1-p}\mathrm{E}|\widetilde{x}(s)|^p + \kappa_1^{1-p}\mathrm{E}|D(s,x_s,r(s))|^p] \\ &\le \|\xi\|_0^p \vee \sup_{t_0\le s\le t} [(1-\kappa_1)^{1-p}\widetilde{h}(s) + \kappa_1^{1-p}l_0^p\mathrm{e}^{\gamma(s-t_0)}\|x_s\|_0^p] \\ &\le \beta_m(1-\kappa_1)^{1-p} + \kappa_1 H(t), \end{aligned}$$

which implies

$$H(t) \le (1-\kappa_1)^{-p}\beta_m. \tag{9}$$

Thus, one can obtain (9) from $h(t) \le H(t)$. Therefore, we only need to prove that (8) holds.

In the following, we shall prove that (8) holds. Let $W(t) = \mathrm{e}^{\gamma(t-t_0)}V(t)$ for all $t \in [t_0 - \tau, +\infty)$. By the condition (i) and the definition of γ, one can show that

$$\begin{aligned} \mathrm{E}W(t) &= \mathrm{e}^{\gamma(t-t_0)}\mathrm{E}V(t,\widetilde{x}(t),r(t)) \\ &\le \mathrm{e}^{\gamma(t-t_0)}c_2\mathrm{E}|\widetilde{x}(t)|^p \\ &\le \mathrm{e}^{\gamma(t-t_0)}c_2\mathrm{E}|x(t)-D(t,x_t,r(t))|^p \\ &\le 2^{p-1}c_2(1+l_0^p)\|\xi\|_0^p \\ &< N\|\xi\|_0^p, \quad t \in [t_0-\tau, t_0]. \end{aligned} \tag{10}$$

So it is only need to show

$$\mathrm{E}W(t) \le \beta c_1, \quad t \in [t_0, +\infty). \tag{11}$$

Firstly, it is

$$\mathrm{E}W(t) < N\|\xi\|_0^p, \quad t \in [t_0, t_1). \tag{12}$$

Suppose on the contrary that there exists a $\widetilde{t} \in (t_0, t_1)$ such that $\mathrm{E}W(\widetilde{t}) \ge N\|\xi\|_0^p$. Set

$$t^\dagger = \inf\{t \in [t_0, t_1) : \mathrm{E}W(t) \ge N\|\xi\|_0^p\}. \tag{13}$$

Notice that $\mathrm{E}W(t)$ is continuous on $t \in [t_0, t_1)$, then $t^\dagger \in (t_0, t_1)$ and

$$\mathrm{E}W(t^\dagger) = N\|\xi\|_0^p, \quad \mathrm{E}W(t) < N\|\xi\|_0^p, \quad t \in [t_0-\tau, t^\dagger). \tag{14}$$

We further define

$$t^{\ddagger} = \sup\left\{t \in [t_0, t^{\dagger}] : \mathrm{E}W(t) \leq \frac{1}{\widetilde{q}} N\|\xi\|_0^p\right\}. \tag{15}$$

then $t^{\ddagger} \in (t_0, t^{\dagger})$ and

$$\mathrm{E}W(t^{\ddagger}) = \frac{1}{\widetilde{q}} N\|\xi\|_0^p, \quad \mathrm{E}W(t) > \frac{1}{\widetilde{q}} N\|\xi\|_0^p, \quad t \in (t^{\ddagger}, t^{\dagger}]. \tag{16}$$

By the virtue of (14) and (16), one can derive for all $t \in [t^{\ddagger}, t^{\dagger}]$ that

$$\mathrm{E}W(t+\theta) \leq N\|\xi\|_0^p \leq \widetilde{q}\mathrm{E}W(t), \quad \forall \theta \in [-\tau, 0]. \tag{17}$$

Consequently,

$$\begin{aligned}
\mathrm{E}V(t+\theta, x(t+\theta), r(t+\theta)) &\leq c_2 \mathrm{E}|x(t+\theta)|^p \\
&= c_2 \mathrm{e}^{-\gamma(t+\theta-t_0)} h(t+\theta) \\
&< c_2 \mathrm{e}^{-\gamma(t+\theta-t_0)} H(t) \\
&< c_2 \mathrm{e}^{-\gamma(t+\theta-t_0)} (1-\kappa_1)^{-p} c_1^{-1} N\|\xi\|_0^p \\
&< c_2 \mathrm{e}^{-\gamma(t+\theta-t_0)} (1-\kappa_1)^{-p} c_1^{-1} \widetilde{q}\mathrm{E}W(t) \\
&< c_2 \mathrm{e}^{\gamma\tau} (1-\kappa_1)^{-p} c_1^{-1} \widetilde{q}\mathrm{E}V(t) \\
&\leq q\mathrm{E}V(t), \quad \forall \theta \in [-\tau, 0].
\end{aligned} \tag{18}$$

From the definition of $\mathcal{L}V$ and the condition (ii), we conclude that

$$\mathrm{E}\mathcal{L}W(t) = \mathrm{e}^{\gamma(t-t_0)}[\gamma\mathrm{E}V(t) + E\mathcal{L}V(t)] \leq (\gamma + b(t))\mathrm{E}W(t), \quad t \in [t^{\ddagger}, t^{\dagger}]. \tag{19}$$

On the other hand, applying Itô's formula to $\mathrm{e}^{\gamma(t-t_0)}V(t, \widetilde{x}(t), r(t))$ and by the well-known Gronwall inequality as well as the condition (iv), we can easily claim that

$$\mathrm{E}W(t^{\dagger}) \leq \mathrm{E}W(t^{\ddagger})\mathrm{e}^{\int_{t^{\ddagger}}^{t^{\dagger}}(\gamma+b(s))ds} \leq \mathrm{E}W(t^{\ddagger})\mathrm{e}^{(\gamma+\widetilde{b})\delta} = \frac{1}{\widetilde{q}} N\|\xi\|_0^p \mathrm{e}^{(\gamma+\widetilde{b})\delta} < N\|\xi\|_0^p, \tag{20}$$

which contradicts the definition of $t^{\dagger}$. Therefore, the relation (12) holds.

Now, one can assume that for some $n \in \mathbb{N}$, $n \geq 1$,

$$\mathrm{E}W(t) < N_n\|\xi\|_0^p, \quad t \in [t_0, t_n), \tag{21}$$

where $N_1 = N$, $N_n = N\prod_{1\leq i\leq n-1}(1+e_i)$ for $n \geq 2$. we proceed to prove that

$$\mathrm{E}W(t) < N_{n+1}\|\xi\|_0^p, \quad t \in [t_n, t_{n+1}). \tag{22}$$

Suppose the claim (22) is not true, there exists some $\overline{t} \in [t_n, t_{n+1})$ such that $\mathrm{E}W(\overline{t}) \geq N_{n+1}\|\xi\|_0^p$. Together with the inequality (21) and the condition (iv), it yields

$$\begin{aligned}
\mathrm{E}W(t_n) &\leq \rho_1(1+e_n)\mathrm{E}W(t_n^-) + \rho_2(1+e_n)\mathrm{e}^{\gamma\tau} \sup_{\theta\in[-\tau,0]} \mathrm{E}W(t_n^- + \theta) \\
&\leq (\rho_1 + \rho_2\mathrm{e}^{\gamma\tau})N_{n+1}\|\xi\|_0^p \\
&< N_{n+1}\|\xi\|_0^p.
\end{aligned} \tag{23}$$

Let

$$t' = \inf\{t \in [t_n, t_{n+1}) : \mathrm{E}W(t) \geq N_{n+1}\|\xi\|_0^p\}, \tag{24}$$

it has that $t' \in (t_n, t_{n+1})$ and

$$\mathrm{E}W(t') = N_{n+1}\|\xi\|_0^p, \quad \mathrm{E}W(t) < N_{n+1}\|\xi\|_0^p, \quad t \in [t_n, t'). \tag{25}$$

Moreover, denote

$$t'' = \sup\{t \in [t_n, t'] : \mathrm{E}W(t) \leq (\rho_1 + \rho_2 \mathrm{e}^{\gamma\tau})N_{n+1}\|\xi\|_0^p\}. \tag{26}$$

Hence, we can deduce that $t'' \in [t_n, t')$ and

$$\mathrm{E}W(t'') = (\rho_1 + \rho_2 \mathrm{e}^{\gamma\tau})N_{n+1}\|\xi\|_0^p, \quad \mathrm{E}W(t) > (\rho_1 + \rho_2 \mathrm{e}^{\gamma\tau})N_{n+1}\|\xi\|_0^p, \quad t \in (t'', t']. \tag{27}$$

Thus, it follows immediately, for every $t \in [t'', t']$, that

$$\mathrm{E}W(t+\theta) \leq N_{n+1}\|\xi\|_0^p \leq \frac{1}{\rho_1 + \rho_2 \mathrm{e}^{\gamma\tau}}\mathrm{E}W(t) < \widetilde{q}\mathrm{E}W(t), \quad \forall \theta \in [-\tau, 0]. \tag{28}$$

Now, in view of the assumption (A1), Itô's formula, and discussion similar to (17)–(19), we find that

$$\begin{aligned}
\mathrm{E}W(t') &\leq \mathrm{E}W(t'')\mathrm{e}^{\int_{t''}^{t'}(\gamma+b(s))ds} \\
&\leq \mathrm{E}W(t'')\mathrm{e}^{(\gamma+\widetilde{b})\delta} \\
&= \mathrm{e}^{(\gamma+\widetilde{b})\delta}(\rho_1 + \rho_2 \mathrm{e}^{\gamma\tau})N_{n+1}\|\xi\|_0^p \\
&< N_{n+1}\|\xi\|_0^p,
\end{aligned}$$

which contradicts the definition of t'. So the relation (22) is valid.

By mathematical induction, (22) holds for any $n \in \mathbb{N}$. Namely, the required (11) follows. Finally, what remains is to apply the condition (i) to conclude that

$$\mathrm{E}|x(t)|^p \leq \frac{MN(1-\kappa_1)^{-p}}{c_1}\|\xi\|_0^p \mathrm{e}^{-\gamma(t-t_0)}, \quad \forall t \in [t_0, +\infty), \tag{29}$$

which shows that the trivial solution of System (1) is pth moment exponentially stable. This completes the proof. ∎

Remark 3.3 Comparing to [32], in which uses impulses to stabilize unstable stochastic system, but we consider the interference of neutral terms and Markov switching to make it closer to the real situation. In [39], although Markov switching is considered, the influence of functionals' derivative on stability is ignored. What is more, we consider the case that the derivative of EV is limited by a function too, which is less conservative.

Remark 3.4 Theorem 3.2 implies that neutral stochastic functional differential systems can be stabilized by an appropriate perturbations of impulses, and the average growth rate of $\mathcal{L}V$ operator corresponding to the constructed V function or functional is bounded function that varies with time. But discrete dynamic underlying the impulsive inputs may be unstable in some cases. Therefore, it's necessary to give a new criterion that neutral stochastic functional differential systems with Markovian switching can still remain the exponential stability after disturbed by an unstable perturbations of impulses.

Remark 3.5 Due to the difference in the value of $\rho_1+\rho_2$, the value of $\widetilde{b}$ related to the function $b(t)$ will change, which leads to the proof of (21) being trickier than Theorem 3.2. Moreover, the results of the following theorem can be obtained by combining the idea of classification discussion.

Theorem 3.6 *Let $p>1$, $\delta=\inf_{k\in\mathbb{N}}\{t_k-t_{k-1}\}>0$, and* (A1) *holds. Suppose that there exist functions $V\in v_0$ and $b(t)\in PC([t_0-\tau,+\infty),\mathbb{R}_0)$ as well as several positive constants c_1, c_2, κ_1, q, ρ_1, ρ_2, $\widetilde{b}$ satisfying $\kappa_1\in(l_0,1)$, $q>c_1^{-1}c_2(1-\kappa_1)^{-p}$, and $e_k\geq 0$ $(k\in\mathbb{N})$, $\sum_{k=1}^{\infty}e_k<\infty$ and $\rho_1+\rho_2\geq 1$ such that* (3), (4) *and the following two assumptions hold:*

(v) *for all $t\in[t_k,t_{k+1})$, $k\in\mathbb{N}$, $\varphi\in L^p_{\mathscr{F}_{t_0}}(\Omega,PC)$ and $\theta\in[-\tau,0]$,*

$$\mathrm{E}\mathcal{L}V(t,\varphi,r(t))\leq -b(t)\mathrm{E}V(t,\widetilde{\varphi}(0),r(t)),$$

whenever $\mathrm{E}V(t+\theta,\varphi(\theta),r(t+\theta))<q\mathrm{E}V(t,\widetilde{\varphi}(0),r(t))$;

(vi) $\inf_{t\in[t_0,+\infty)}b(t)\geq\widetilde{b}$, $\rho_1+\rho_2\mathrm{e}^{\widetilde{b}\tau}<q<\mathrm{e}^{\widetilde{b}\delta}$.

Then, the trivial solution of System (1) *is pth moment exponentially stable for any bounded time delay $\tau\in(0,+\infty)$.*

Proof Set $M=\prod_{k=1}^{\infty}(1+e_k)$. Since $e_k\geq 0$ and $\sum_{k=1}^{\infty}e_k<\infty$ we claim that $1\leq M<\infty$. For any initial data $\xi\in L^p_{\mathscr{F}_{t_0}}(\Omega,PC)$, write $V(t,\widetilde{x}(t),r(t))=V(t)$ for simplicity. From the condition (vi), one can choose a small enough constant $\gamma>0$ such that

$$\mathrm{e}^{\gamma\tau}(\rho_1+\rho_2\mathrm{e}^{\widetilde{b}\tau})<q<\mathrm{e}^{(\widetilde{b}-\gamma)\delta},\quad \gamma<\widetilde{b}. \tag{30}$$

Let $\widetilde{q}=q\mathrm{e}^{-\gamma\tau}c_1c_2^{-1}(1-\kappa_1)^p>1$, and $N>0$ is a large enough constant.

To prove the trivial solution of System (1) is pth moment exponentially stable, for any bounded initial value $\xi\in L^p_{\mathscr{F}_{t_0}}(\Omega,PC)$, it is enough to prove the following

$$\mathrm{E}|x(t)|^p\leq\alpha\|\xi\|_0^p\mathrm{e}^{-\gamma(t-t_0)},\quad t\in[t_0,+\infty), \tag{31}$$

where α is a positive constant. Let $\beta_m=\prod_{k=1}^{m}(1+e_k)Nc_1^{-1}\|\xi\|_0^p$ for every $m\in\mathbb{N}$ and $\beta=MNc_1^{-1}\|\xi\|_0^p$, and we first prove

$$h(t)\triangleq\mathrm{e}^{\gamma(t-t_0)}\mathrm{E}|x(t)|^p\leq(1-\kappa_1)^{-p}\beta_m,\quad t\in[t_0,t_m). \tag{32}$$

Note that if it holds

$$\widetilde{h}(t)\triangleq\mathrm{e}^{\gamma(t-t_0)}\mathrm{E}|\widetilde{x}(t)|^p\leq\beta_m,\quad \forall t\in[t_0,t_m), \tag{33}$$

then by using the assumption (A1) it follows, for all $t \in [t_0, t_m)$, that

$$\begin{aligned} H(t) &\triangleq \sup_{t_0-\tau \le s \le t} h(s) \\ &\le \|\xi\|_0^p \vee \sup_{t_0 \le s \le t} h(s) \\ &\le \|\xi\|_0^p \vee \sup_{t_0 \le s \le t} \mathrm{e}^{\gamma(s-t_0)}[(1-\kappa_1)^{1-p}\mathrm{E}|\widetilde{x}(s)|^p + \kappa_1^{1-p}\mathrm{E}|D(s, x_s, r(s))|^p] \\ &\le \|\xi\|_0^p \vee \sup_{t_0 \le s \le t} [(1-\kappa_1)^{1-p}\widetilde{h}(s) + \kappa_1^{1-p} l_0^p \mathrm{e}^{\gamma(s-t_0)}\|x_s\|_0^p] \\ &\le \beta_m(1-\kappa_1)^{1-p} + \kappa_1 H(t), \end{aligned}$$

which implies

$$H(t) \le (1-\kappa_1)^{-p}\beta_m. \tag{34}$$

Thus, one can obtain (32) from $h(t) \le H(t)$. Therefore, we only need to prove that (33) holds.

In the following, we are going to prove that (33) holds. Let $W(t) = \mathrm{e}^{\gamma(t-t_0)}V(t)$ for all $t \in [t_0-\tau, +\infty)$. By the condition (3) and the definition of γ, it holds that

$$\begin{aligned} \mathrm{E}W(t) &= \mathrm{e}^{\gamma(t-t_0)}\mathrm{E}V(t, \widetilde{x}(t), r(t)) \\ &\le \mathrm{e}^{\gamma(t-t_0)}c_2\mathrm{E}|\widetilde{x}(t)|^p \\ &\le \mathrm{e}^{\gamma(t-t_0)}c_2\mathrm{E}|x(t) - D(t, x_t, r(t))|^p \\ &\le 2^{p-1}c_2(1+l_0^p)\|\xi\|_0^p \\ &< N\|\xi\|_0^p, \quad t \in [t_0-\tau, t_0]. \end{aligned} \tag{35}$$

So it is only need to show

$$\mathrm{E}W(t) \le \beta c_1, \quad t \in [t_0, +\infty). \tag{36}$$

Firstly, it is

$$\mathrm{E}W(t) < N\|\xi\|_0^p, \quad t \in [t_0, t_1). \tag{37}$$

We assume, on the contrary, there exists some $\widetilde{t} \in (t_0, t_1)$, such that $\mathrm{E}W(\widetilde{t}) \ge N\|\xi\|_0^p$. Define

$$t^\dagger = \inf\{t \in [t_0, t_1) : \mathrm{E}W(t) \ge N\|\xi\|_0^p\}. \tag{38}$$

Note that $\mathrm{E}W(t)$ is continuous on $t \in [t_0, t_1)$, therefore, $t^\dagger \in (t_0, t_1)$ and

$$\mathrm{E}W(t^\dagger) = N\|\xi\|_0^p, \quad \mathrm{E}W(t) < N\|\xi\|_0^p, \quad t \in [t_0-\tau, t^\dagger). \tag{39}$$

We further define

$$t^\ddagger = \sup\left\{t \in [t_0, t^\dagger] : \mathrm{E}W(t) \le \frac{1}{\widetilde{q}}N\|\xi\|_0^p\right\}, \tag{40}$$

then $t^\ddagger \in (t_0, t^\dagger)$ and

$$\mathrm{E}W(t^\ddagger) = \frac{1}{\widetilde{q}}N\|\xi\|_0^p, \quad \mathrm{E}W(t) > \frac{1}{\widetilde{q}}N\|\xi\|_0^p, \quad t \in (t^\ddagger, t^\dagger]. \tag{41}$$

By the virtue of (39) and (41), one can derive for all $t \in [t^{\ddagger}, t^{\dagger}]$ that

$$\mathrm{E}W(t+\theta) \le N\|\xi\|_0^p \le \widetilde{q}\mathrm{E}W(t), \quad \forall \theta \in [-\tau, 0]. \tag{42}$$

Consequently,

$$\begin{aligned} \mathrm{E}V(t+\theta, x(t+\theta), r(t+\theta)) &\le c_2 \mathrm{E}|x(t+\theta)|^p \\ &= c_2 \mathrm{e}^{-\gamma(t+\theta-t_0)} h(t+\theta) \\ &< c_2 \mathrm{e}^{-\gamma(t+\theta-t_0)} H(t) \\ &< c_2 \mathrm{e}^{-\gamma(t+\theta-t_0)} (1-\kappa_1)^{-p} c_1^{-1} N\|\xi\|_0^p \\ &< c_2 \mathrm{e}^{-\gamma(t+\theta-t_0)} (1-\kappa_1)^{-p} c_1^{-1} \widetilde{q}\mathrm{E}W(t) \\ &< c_2 \mathrm{e}^{\gamma\tau} (1-\kappa_1)^{-p} c_1^{-1} \widetilde{q}\mathrm{E}V(t) \\ &\le q\mathrm{E}V(t), \quad \forall \theta \in [-\tau, 0]. \end{aligned} \tag{43}$$

From the definition of $\mathcal{L}V$ and the condition (v), we conclude that

$$\mathrm{E}\mathcal{L}W(t) = \mathrm{e}^{\gamma(t-t_0)}[\gamma \mathrm{E}V(t) + E\mathcal{L}V(t)] \le (\gamma - b(t))\mathrm{E}W(t), \quad t \in [t^{\ddagger}, t^{\dagger}]. \tag{44}$$

Meanwhile, applying Itô's formula to $\mathrm{e}^{\gamma(t-t_0)}V(t, \widetilde{x}(t), r(t))$ and by the well-known Gronwall inequality (see [11]) as well as the condition (vi), we can easily claim that

$$\begin{aligned} \mathrm{E}W(t^{\dagger}) &\le \mathrm{E}W(t^{\ddagger}) \mathrm{e}^{\int_{t^{\ddagger}}^{t^{\dagger}} (\gamma - b(s)) ds} \\ &\le \mathrm{E}W(t^{\ddagger}) \mathrm{e}^{(\gamma - \widetilde{b})(t^{\dagger} - t^{\ddagger})} \\ &< \mathrm{E}W(t^{\ddagger}) \\ &= \frac{1}{\widetilde{q}} N\|\xi\|_0^p \\ &< N\|\xi\|_0^p, \end{aligned} \tag{45}$$

which contradicts the definition of $t^{\dagger}$. Therefore, the relation (37) holds.

Now, we assume that for some $n \in \mathbb{N}$, $n \ge 1$,

$$\mathrm{E}W(t) < N_n \|\xi\|_0^p, \quad t \in [t_0, t_n), \tag{46}$$

where $N_1 = N$, $N_n = N \prod_{1 \le i \le n-1} (1 + e_i)$ for $n \ge 2$. we proceed to show that

$$\mathrm{E}W(t) < N_{n+1} \|\xi\|_0^p, \quad t \in [t_n, t_{n+1}). \tag{47}$$

To do this, we first prove

$$\mathrm{E}W(t_n^- + \theta) \le \frac{\mathrm{e}^{(\widetilde{b} - \gamma)\tau}}{\widetilde{q}} N_n \|\xi\|_0^p, \quad \theta \in [-\tau, 0). \tag{48}$$

Suppose not, then there exists $\theta_1 \in [-\tau, 0)$ such that $\mathrm{E}W(t_n^- + \theta_1) > \frac{\mathrm{e}^{(\widetilde{b}-\gamma)\tau}}{\widetilde{q}} N_n \|\xi\|_0^p$. Without lose generality, we assume $t_n + \theta_1 \in (t_{m-1}, t_m]$, $m \in \mathbb{N}$, $m \le n$.

Case 1 $\mathrm{E}W(t) > \frac{\mathrm{e}^{(\widetilde{b}-\gamma)\tau}}{\widetilde{q}} N_n\|\xi\|_0^p$ over $t \in [t_{m-1}, t_n + \theta_1)$.

By the assumption (46), for all $t \in [t_{m-1}, t_n + \theta_1)$, we see that

$$\mathrm{E}W(t+\theta) < N_n\|\xi\|_0^p < \mathrm{e}^{(\widetilde{b}-\gamma)\tau} N_n\|\xi\|_0^p < \widetilde{q}\mathrm{E}W(t), \quad \theta \in [-\tau, 0].$$

Hence, it follows from Itô's formula, the conditions (v), (4) and discussion similar to (43)–(45), it has

$$\begin{aligned}\mathrm{E}W(t_n^- + \theta_1) &\le \mathrm{E}W(t_{m-1})\mathrm{e}^{(\gamma-\widetilde{b})(t_n^- + \theta_1 - t_{m-1})} \\ &< N_n\|\xi\|_0^p \mathrm{e}^{(\widetilde{b}-\gamma)\tau}\mathrm{e}^{(\gamma-\widetilde{b})(t_n - t_{m-1})} \\ &\le \frac{\mathrm{e}^{(\widetilde{b}-\gamma)\tau}}{q^{n-m+1}} N_n\|\xi\|_0^p \\ &< \frac{\mathrm{e}^{(\widetilde{b}-\gamma)\tau}}{\widetilde{q}} N_n\|\xi\|_0^p,\end{aligned}$$

which contradicts the definition of θ_1.

Case 2 There is some $t \in [t_{m-1}, t_n + \theta_1)$ such that $\mathrm{E}W(t) > \frac{\mathrm{e}^{(\widetilde{b}-\gamma)\tau}}{\widetilde{q}} N_n\|\xi\|_0^p$.

In this case, we denote

$$\check{t} = \sup\left\{t \in [t_{m-1}, t_n + \theta_1] : \mathrm{E}W(t) \le \frac{\mathrm{e}^{(\widetilde{b}-\gamma)\tau}}{\widetilde{q}} N_n\|\xi\|_0^p\right\}.$$

Thus $\check{t} \in [t_{m-1}, t_n + \theta_1)$ and

$$\mathrm{E}W(\check{t}) = \frac{\mathrm{e}^{(\widetilde{b}-\gamma)\tau}}{\widetilde{q}} N_n\|\xi\|_0^p, \quad \mathrm{E}W(t) > \frac{\mathrm{e}^{(\widetilde{b}-\gamma)\tau}}{\widetilde{q}} N_n\|\xi\|_0^p, \quad \forall t \in (\check{t}, t_n + \theta_1).$$

So for all $t \in [\check{t}, t_n + \theta_1)$, it has

$$\mathrm{E}W(t+\theta) < N_n\|\xi\|_0^p < \mathrm{e}^{(\widetilde{b}-\gamma)\tau} N_n\|\xi\|_0^p \le \widetilde{q}\mathrm{E}W(t), \quad \forall \theta \in [-\tau, 0].$$

In view of Itô's formula, the well-known Gronwall inequality, the conditions (v), (4) and discussion similar to (43)–(45), we then derive that

$$\mathrm{E}W(t_n^- + \theta_1) \le \mathrm{E}W(\check{t})\mathrm{e}^{(\gamma-\widetilde{b})(t_n + \theta_1 - \check{t})} < \frac{\mathrm{e}^{(\widetilde{b}-\gamma)\tau}}{\widetilde{q}} N_n\|\xi\|_0^p.$$

The inequality contradicts the definition of θ_1. Therefore, (48) holds.

Similarly, one can prove

$$\mathrm{E}W(t_n^-) \le \frac{1}{\widetilde{q}} N_n\|\xi\|_0^p. \tag{49}$$

By virtue of (48) and (49) and the inequality (4), we have

$$\begin{aligned}\mathrm{E}W(t_n) &\le \rho_1(1+e_n)\mathrm{E}W(t_n^-) + \rho_2(1+e_n)\mathrm{e}^{\gamma\tau}\sup_{\theta\in[-\tau,0]}\mathrm{E}W(t_n^- + \theta) \\ &\le \frac{\rho_1 + \rho_2\mathrm{e}^{\widetilde{b}\tau}}{\widetilde{q}} N_{n+1}\|\xi\|_0^p \\ &< N_{n+1}\|\xi\|_0^p.\end{aligned} \tag{50}$$

Now suppose (47) is not true, then, there exists $\bar{t} \in [t_n, t_{n+1})$ such that $\mathrm{E}W(\bar{t}) > N_{n+1}\|\xi\|_0^p$. Let $t' = \inf\{t \in [t_n, t_{n+1}) : \mathrm{E}W(t) \geq N_{n+1}\|\xi\|_0^p\}$. Then $t' \in (t_n, t_{n+1})$ and

$$\mathrm{E}W(t') = N_{n+1}\|\xi\|_0^p, \quad \mathrm{E}W(t) < N_{n+1}\|\xi\|_0^p, \quad t \in [t_n, t').$$

We further denote

$$t'' = \sup\left\{t \in [t_n, t'] : \mathrm{E}W(t) \leq \frac{1}{\widetilde{q}} N_{n+1}\|\xi\|_0^p\right\},$$

which implies $t'' \in [t_n, t')$ and

$$\mathrm{E}W(t'') = \frac{1}{\widetilde{q}} N_{n+1}\|\xi\|_0^p, \quad \mathrm{E}W(t) > \frac{1}{\widetilde{q}} N_{n+1}\|\xi\|_0^p, \quad t \in (t'', t'].$$

Hence, for every $t \in [t'', t']$, it is easy to obtain that

$$\mathrm{E}W(t+\theta) \leq N_{n+1}\|\xi\|_0^p \leq \widetilde{q}\,\mathrm{E}W(t), \quad \theta \in [-\tau, 0].$$

Applying Itô's formula to $\mathrm{e}^{\gamma(t-t_0)} V(t, \widetilde{x}(t), r(t))$ and by the well-known Gronwall inequality (see [11]) as well as the condition (v), (4) and discussion similar to (43)–(45), we can easily claim that

$$\mathrm{E}W(t') \leq \mathrm{E}W(t'')\mathrm{e}^{(\gamma-\widetilde{b})(t'-t'')} \leq \mathrm{E}W(t'') < N_{n+1}\|\xi\|_0^p.$$

This inequality clearly contradicts the definition of t', therefore, (47) holds.

By mathematical induction, (47) holds for any $n \in \mathbb{N}$. Namely, (36) is valid. Finally, what remains is to apply the assumption (A1) to conclude that

$$\mathrm{E}|x(t)|^p \leq \frac{MN(1-\kappa_1)^{-p}}{c_1}\|\xi\|_0^p \mathrm{e}^{-\gamma(t-t_0)}, \quad \forall t \in [t_0, +\infty), \tag{51}$$

which shows that the trivial solution of System (1) is pth moment exponentially stable. The proof is completed. ∎

Remark 3.7 To conclude this section, under an irrestrictive condition, we shall present a theorem about the almost surely exponentially stable of System (1). As a result, it is necessary to use the method of moment estimation, and combine with some elementary inequalities such as $(x+y)^p \leq 2^{p-1}(x^p + y^p)$ (see [10]), and the basic inequalities such as Hölder and Burkholder-Davis-Gundy as well as the Borel-Cantelli lemma to obtain the conclusion of Corollary 3.8.

Firstly, the necessary assumption is given as follows:

(A2) Suppose the impulsive instances t_k satisfy

$$\Delta_{\sup} = \sup_{k\in\mathbb{N}}\{t_k - t_{k-1}\} < \infty, \quad \Delta_{\inf} = \inf_{k\in\mathbb{N}}\{t_k - t_{k-1}\} > 0.$$

Corollary 3.8 *Set $p \geq 1$, and the assumption* (A2) *holds. Suppose there is a constant $L > 0$ such that for all $(t, \varphi, r(t)) \in [t_0, +\infty) \times L^p_{\mathscr{F}_t}(\Omega, PC) \times S$,*

$$\mathrm{E}(|f(t,\varphi,r(t))|^p + |\mathrm{g}(t,\varphi,r(t))|^p) \leq L \sup_{-\tau\leq\theta\leq 0} \mathrm{E}|\varphi|^p. \tag{52}$$

Therefore, (29) *and* (51) *imply that for all* $t \in [t_0, +\infty)$

$$|x(t)| \le \overline{C}\mathrm{e}^{-(\gamma/p)(t-t_0)}\|\xi\|_0, \quad a.s.,$$

where $\overline{C}$ *is a positive constant. Namely, under the condition* (52)*, pth moment exponential stability implies almost surely exponential stability for System* (1).

Proof Let δ with $0 < \delta < \Delta_{\inf}$ sufficiently small. For the fixed $\delta > 0$, choose $k_\delta = [\frac{t_k - t_{k-1}}{\delta}] \in \mathbb{N}$. Here $[X]$ is the maximum integer not more than X. Therefore, $k_\delta \le [\Delta_{\sup}/\delta] < \infty$ and for any $t \in [t_{k-1}, t_k)$, there exist some constant i with $1 \le i \le k_\delta + 1$ such that $t_{k-1} + (i-1)\delta \le t < t_{k-1} + i\delta$. Thus, for every $t \in [t_{k-1}, t_k)$, $k \in \mathbb{N}$, one has

$$\mathrm{E}\Big[\sup_{t_{k-1}\le t<t_k}|x(t)|^p\Big] \le \sum_{i=1}^{k_\delta+1}\mathrm{E}\Big[\sup_{t_{k-1}+(i-1)\delta\le t<t_{k-1}+i\delta}|x(t)|^p\Big]. \tag{53}$$

For each i satisfying $1 \le i \le k_\delta + 1$, $k \in \mathbb{N}$,

$$\begin{aligned}
&\mathrm{E}\Big[\sup_{t_{k-1}+(i-1)\delta\le t<t_{k-1}+i\delta}|x(t)|^p\Big]\\
&\le 4^p\mathrm{E}\big|x(t_{k-1}+(i-1)\delta) - D\big(t_{k-1}+(i-1)\delta, x_{t_{k-1}+(i-1)\delta}, r(t_{k-1}+(i-1)\delta)\big)\big|^p\\
&\quad + 4^p\mathrm{E}\Big[\sup_{t_{k-1}+(i-1)\delta\le t<t_{k-1}+i\delta}\big|D\big(t,x_t,r(t)\big)\big|^p\Big] + 4^p\mathrm{E}\Big[\Big(\int_{t_{k-1}+(i-1)\delta}^{t_{k-1}+i\delta}|f(s,x_s,r(s))|ds\Big)^p\Big]\\
&\quad + 4^p\mathrm{E}\Big[\sup_{t_{k-1}+(i-1)\delta\le t<t_{k-1}+i\delta}\Big|\int_{t_{k-1}+(i-1)\delta}^{t} g(s,x_s,r(s))dw(s)\Big|^p\Big]\\
&\triangleq I_1 + I_2 + I_3 + I_4,
\end{aligned} \tag{54}$$

applying the assumption (A1), we find that

$$\begin{aligned}
I_1 &\le 4^p\cdot 2^p\big[\mathrm{E}|x(t_{k-1}+(i-1)\delta)\big|^p + l_0^p\mathrm{E}\big|x(t_{k-1}+(i-1)\delta)\big|^p\big]\\
&= 4^p\cdot 2^p(1+l_0^p)\mathrm{E}\big|x(t_{k-1}+(i-1)\delta)\big|^p.
\end{aligned} \tag{55}$$

Note that $D\big(t,x_t,r(t)\big)$ is continuous on $[t_{k-1}+(i-1)\delta, t_{k-1}+i\delta]$, thus, there exists some $t' \in [t_{k-1}+(i-1)\delta, t_{k-1}+i\delta]$ such that

$$\begin{aligned}
\mathrm{E}\Big[\sup_{t_{k-1}+(i-1)\delta\le t<t_{k-1}+i\delta}\big|D\big(t,x_t,r(t)\big)\big|^p\Big] &\le \mathrm{E}\Big[\sup_{t_{k-1}+(i-1)\delta\le t\le t_{k-1}+i\delta}\big|D\big(t,x_t,r(t)\big)\big|^p\Big]\\
&= \mathrm{E}|D(t',x_{t'},r(t'))|^p,
\end{aligned}$$

in view of the assumption (A1) and (51), we conclude that

$$\begin{aligned}
I_2 &\le 4^p\mathrm{E}|D(t',x_{t'},r(t'))|^p\\
&\le 4^p\cdot l_0^p\mathrm{E}\big|x(t')\big|^p\\
&\le 4^p\cdot l_0^p\frac{MN(1-\kappa_1)^{-p}}{c_1}\|\xi\|_0^p\mathrm{e}^{-\gamma(t'-t_0)}\\
&\le 4^p\cdot l_0^p\frac{MN(1-\kappa_1)^{-p}}{c_1}\|\xi\|_0^p\mathrm{e}^{-\gamma(t_{k-1}-t_0)}.
\end{aligned} \tag{56}$$

On the one hand, from the condition (51), (52) and the Hölder inequality (see [10]), one can obtain that

$$\begin{aligned} I_3 &\le 4^p\delta^{p-1}L\cdot\int_{t_{k-1}+(i-1)\delta}^{t_{k-1}+i\delta}\sup_{-\tau\le\theta\le 0}\mathrm{E}|x(s+\theta)|^p ds\\ &\le 4^p\delta^{p-1}L\cdot\frac{MN(1-\kappa_1)^{-p}}{c_1}\|\xi\|_0^p\int_{t_{k-1}+(i-1)\delta}^{t_{k-1}+i\delta}\mathrm{e}^{-\gamma(s-\tau-t_0)}ds\\ &\le 4^p\delta^{p}L\cdot\frac{MN(1-\kappa_1)^{-p}}{c_1}\|\xi\|_0^p\mathrm{e}^{\gamma\tau}\cdot\mathrm{e}^{-\gamma(t_{k-1}-t_0)}. \end{aligned} \tag{57}$$

On the other hand, by the Burkholder-Davis-Gundy inequality (see [10]), (51) and the Hölder inequality, one can get

$$\begin{aligned} I_4 &\le 4^pC_p\mathrm{E}\Big[\int_{t_{k-1}+(i-1)\delta}^{t_{k-1}+i\delta}\big|g(s,x_s,r(s))\big|^2ds\Big]^{p/2}\\ &\le 4^pC_p\delta^{(p/2)-1}\mathrm{E}\Big[\int_{t_{k-1}+(i-1)\delta}^{t_{k-1}+i\delta}\big|g(s,x_s,r(s))\big|^pds\Big]\\ &\le 4^pC_p\delta^{(p/2)-1}L\cdot\int_{t_{k-1}+(i-1)\delta}^{t_{k-1}+i\delta}\sup_{-\tau\le\theta\le 0}\mathrm{E}|x(s+\theta)|^p ds\\ &\le 4^pC_p\delta^{p/2}L\cdot\frac{MN(1-\kappa_1)^{-p}}{c_1}\|\xi\|_0^p\mathrm{e}^{\gamma\tau}\cdot\mathrm{e}^{-\gamma(t_{k-1}-t_0)}, \end{aligned} \tag{58}$$

where C_p is a positive constant dependent of p only. Substituting (51) and (55)–(58) into (54) yields

$$\begin{aligned} &\mathrm{E}\Big[\sup_{t_{k-1}+(i-1)\delta\le t<t_{k-1}+i\delta}|x(t)|^p\Big]\\ &\le 4^p\big[2^p(1+l_0^p)+l_0^p+(\delta^p+C_p\delta^{p/2})L\mathrm{e}^{\gamma\tau}\big]\frac{MN(1-\kappa_1)^{-p}}{c_1}\|\xi\|_0^p\mathrm{e}^{-\gamma(t_{k-1}-t_0)}. \end{aligned} \tag{59}$$

Thus, it follows from (53) and (59), we have

$$\mathrm{E}\Big[\sup_{t_{k-1}\le t<t_k}|x(t)|^p\Big]\le\widetilde{C}^p\|\xi\|_0^p\mathrm{e}^{-\gamma(t_{k-1}-t_0)}, \tag{60}$$

where

$$\widetilde{C}^p=4^p\big[2^p(1+l_0^p)+l_0^p+(\delta^p+C_p\delta^{p/2})L\mathrm{e}^{\gamma\tau}\big]\frac{MN(1-\kappa_1)^{-p}(k_\delta+1)}{c_1}.$$

Using Chebyshev's inequality (see [10]), one can see that for any $\varepsilon\in(0,\gamma)$,

$$P\Big\{\sup_{t_{k-1}\le t<t_k}|x(t)|^p>\widetilde{C}^p\|\xi\|_0^p\mathrm{e}^{-(\gamma-\varepsilon)(t_{k-1}-t_0)}\Big\}\le\mathrm{e}^{-\varepsilon(t_{k-1}-t_0)}.$$

Note that $t_k\to\infty$ as $k\to\infty$, and in view of the well-known Borel-Cantelli lemma, it follows that

$$\sup_{t_{k-1}\le t<t_k}|x(t)|^p\le\widetilde{C}^p\|\xi\|_0^p\mathrm{e}^{-(\gamma-\varepsilon)(t_{k-1}-t_0)},\quad \text{a.s.},$$

which implies

$$|x(t)| \le \widetilde{C}\|\xi\|_0 \mathrm{e}^{-(\gamma-\varepsilon)(t_{k-1}-t_0)/p}, \quad \text{a.s.},$$

for all $t \in [t_{k-1}, t_k)$. Letting $\varepsilon \to 0$, one can get that for any $t \in [t_{k-1}, t_k)$, $k \in \mathbb{N}$

$$|x(t)| \le \overline{C}\|\xi\|_0 \mathrm{e}^{-(\gamma/p)(t-t_0)}, \quad \text{a.s.},$$

where $\overline{C} = \widetilde{C}\mathrm{e}^{(\gamma/p)\Delta_{\sup}}$. Therefore, the trivial solution of System (1) is almost surely exponentially stable. ∎

4 Example

Example 4.1 Consider INSFDEs-MS as follows:

$$\begin{cases} d[x(t) - D(t, x_t, r(t))] \\ = f(t, x_t, r(t))dt + g(t, x_t, r(t))dw(t), \quad t \ge t_0, t \ne t_k, \quad k \in \mathbb{N}, \\ \triangle x(t_k) = \dfrac{1}{k^2} x(t_k^- - \tau), \quad k \in \mathbb{N}, \\ x_{t_0} \triangleq \xi = \{\xi(\theta) : -\tau \le \theta \le 0\} \in L^p_{\mathscr{F}_{t_0}}(\Omega, PC), \end{cases} \tag{61}$$

where $x(t) = (x_1(t), x_2(t))^{\mathrm{T}}$, $\{r(t),\ t \ge 0\}$ is a right-continuous Markov chain taking values in $S = \{1, 2\}$ with generator $Q = \begin{pmatrix} -1.5 & 1.5 \\ 1 & -1 \end{pmatrix}$ and dependent of standard two-dimensional Brownian motion $\{w(t),\ t \ge 0\}$ defined on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t \ge t_0}, P)$. $f(t, x_t, 1) = -(1 - 0.5\cos t)x(t) + 0.5x(t-\tau)$, $f(t, x_t, 2) = -(0.5 - \frac{4}{9}\cos t)x(t) + 0.3x(t-\tau)$, $g(t, x_t, 1) = 0.5x(t-\tau)$, $g(t, x_t, 2) = 0.7x(t-\tau)$, $D(t, x_t, 1) = 0.2x(t-\tau)$, $D(t, x_t, 2) = 0.1x(t-\tau)$, $t_k - t_{k-1} = 0.1\pi$ $(k \in \mathbb{N})$.

Take Lyapunov function $V(t, x(t), r(t)) = 0.1x(t)^2$, then we obtain that the assumption (A1) holds with $c_1 = c_2 = 0.1$, $p = 2$. On the other hand, set $l_0 = 0.25$, $\kappa_1 \in (0.25, 0.29)$, $\rho_1 = 0.23$, $\rho_2 = 0.3$, $e_k = \frac{1}{k^2}$ $(k \in \mathbb{N})$. By calculation, we can find that $q > 1.9837$. Choose $q = 2$. According to $\mathrm{E}|x(t)|^2 \le (1-\kappa_1)^{-2}\mathrm{E}|\widetilde{x}(t)|^2 < 2\mathrm{E}|\widetilde{x}(t)|^2$ and $\mathrm{E}|x(t-\tau)|^2 \le 2\mathrm{E}|\widetilde{x}(t)|^2$, we have

$$\begin{aligned} &\mathrm{E}\mathcal{L}V(t, x_t, 1) \\ &= 0.2\mathrm{E}[x(t) - 0.2x(t-\tau)][-(1 - 0.5\cos t)x(t) + 0.5x(t-\tau)] + 0.025\mathrm{E}|x(t-\tau)|^2 \\ &\quad -0.02\mathrm{E}|x(t-\tau)|^2 + 0.025\mathrm{E}|x(t-\tau)|^2 \\ &\le -(0.2 - 0.1\cos t)\mathrm{E}|x(t)|^2 + 0.05\mathrm{E}|x(t)|^2 + 0.05\mathrm{E}|x(t-\tau)|^2 \\ &\quad +(0.02 - 0.01\cos t)\mathrm{E}|x(t-\tau)|^2 + (0.02 - 0.01\cos t)\mathrm{E}|x(t)|^2 + 0.005\mathrm{E}|x(t-\tau)|^2 \\ &= (-0.13 + 0.09\cos t)\mathrm{E}|x(t)|^2 + (0.075 - 0.01\cos t)\mathrm{E}|x(t-\tau)|^2 \\ &\le (-0.11 + 0.16\cos t)\mathrm{E}|\widetilde{x}(t)|^2 \\ &< (1.1 + 0.16\cos t)\mathrm{E}|\widetilde{x}(t)|^2, \end{aligned}$$

$$\begin{aligned}
&\mathrm{E}\mathcal{L}V(t,x_t,2)\\
=\ &0.2\mathrm{E}[x(t)-0.1x(t-\tau)]\left[-\left(0.5-\frac{4}{9}\cos t\right)x(t)+0.3x(t-\tau)\right]+0.049\mathrm{E}|x(t-\tau)|^2\\
&-0.006\mathrm{E}|x(t-\tau)|^2+0.049\mathrm{E}|x(t-\tau)|^2\\
\le\ &-\left(0.1-\frac{4}{45}\cos t\right)\mathrm{E}|x(t)|^2+0.03\mathrm{E}|x(t)|^2+0.03\mathrm{E}|x(t-\tau)|^2\\
&+\left(0.005-\frac{1}{225}\cos t\right)\mathrm{E}|x(t-\tau)|^2+\left(0.005-\frac{1}{225}\cos t\right)\mathrm{E}|x(t)|^2+0.049\mathrm{E}|x(t-\tau)|^2\\
=\ &\left(-0.065+\frac{19}{225}\cos t\right)\mathrm{E}|x(t)|^2+\left(0.078-\frac{1}{225}\cos t\right)\mathrm{E}|x(t-\tau)|^2\\
\le\ &(-0.026+0.16\cos t)\mathrm{E}|\widetilde{x}(t)|^2\\
<\ &(1.1+0.16\cos t)\mathrm{E}|\widetilde{x}(t)|^2.
\end{aligned}$$

Let $b(t)=1.1+0.16\cos t$, $\widetilde{b}=2$, therefore, $\mathrm{E}\mathcal{L}V(t,x_t,r(t))\le b(t)\mathrm{E}V(t,\widetilde{x}(t),r(t))$. Obviously

$$\begin{aligned}
\sup_{t\ge t_0}\int_t^{t+\delta}(1.1+0.16\cos s)ds &= 1.1\delta+\sup_{t\ge t_0}\int_t^{t+\delta}0.16\cos s ds\\
&=0.11\pi-0.16\sin 1.9\pi\\
&\approx 0.3950\\
&<\widetilde{b}\delta\\
&\approx 0.6283.
\end{aligned}$$

It is easy to check that the conditions (iii) and (iv) of Theorem 3.2 are also satisfied, which means System (61) is mean square exponentially stable. In addition, Corollary 3.8 guarantees that the trivial solution of System (61) is almost surely exponentially stable. The numerical simulations for System (61) is shown in Figure 1 and Figure 2 with the initial value $\xi=(\cos t,\sin t)^{\mathrm{T}}$ $(-\tau\le t\le 0)$ and $\tau=0.5$.

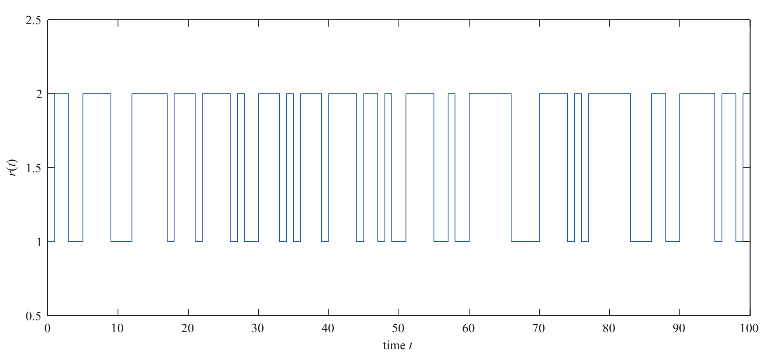

Figure 1 State curve of the Markovian switching $r(t)$ in Example 4.1

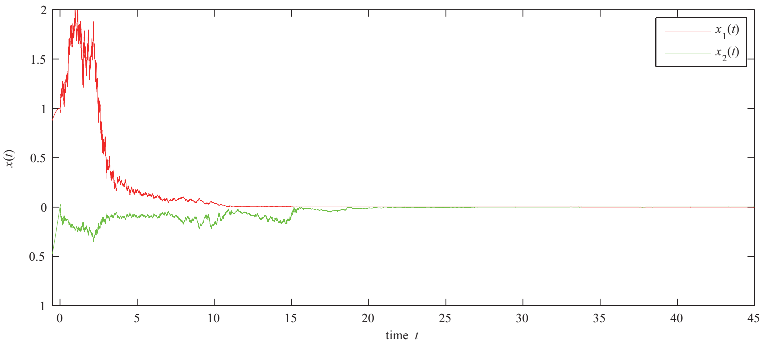

Figure 2 The state response of System (61) in Example 4.1

Example 4.2 Consider INSFDEs-MS as follows:

$$\begin{cases} d[x(t)-D(t,x_t,r(t))] \\ =f(t,x_t,r(t))dt+g(t,x_t,r(t))dw(t), \quad t\geq t_0, t\neq t_k, \quad k\in\mathbb{N}, \\ x(t_k)=0.8x(t_k^-)-0.5x(t_k^- -0.2), \qquad k\in\mathbb{N}, \\ x_{t_0}\triangleq\xi=\{\xi(\theta):-0.2\leq\theta\leq 0\}\in L^p_{\mathscr{F}_{t_0}}(\Omega,PC), \end{cases} \tag{62}$$

where $x(t)=(x_1(t),x_2(t))^{\mathrm{T}}$, $\{r(t),\ t\geq 0\}$ is a right-continuous Markov chain taking values in $S=\{1,2\}$ with generator $Q=\begin{pmatrix} -4 & 4 \\ 5 & -5\end{pmatrix}$ and dependent of standard two-dimensional Brownian motion $\{w(t),\ t\geq 0\}$ defined on $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\geq t_0},P)$. $f(t,x_t,1)=0.4x(t)+0.2x(t-0.2)$, $f(t,x_t,2)=0.2x(t)+0.3x(t-0.2)$, $g(t,x_t,1)=-0.5x(t-0.2)$, $g(t,x_t,2)=0.4x(t-0.2)$, $D(t,x_t,1)=-0.25x(t-0.2)$, $D(t,x_t,2)=-0.2x(t-0.2)$, $t_k-t_{k-1}=0.2$ $(k\in\mathbb{N})$.

Take Lyapunov function $V(t,x(t),r(t))=|x(t)|^2$. Set $l_0=0.25$, $\kappa_1\in(0.25,0.29)$, $\rho_1=1.5$, $\rho_2=0.002$, $e_k=\frac{1}{k^2}$ $(k\in\mathbb{N})$ and $\tau=0.2$. By the Young inequality, we can get

$$\begin{aligned}
&\mathrm{E}\mathcal{L}V(t,x_t,1)\\
=&\mathrm{E}\big\{p|x(t)+0.25x(t-0.2)|^{p-1}\cdot[0.4x(t)+0.2x(t-0.2)]\\
&+\frac{p(p-1)}{2}|x(t)+0.25x(t-0.2)|^{p-2}\cdot|x(t-0.2)|^2\big\}\\
\leq&\mathrm{E}\big\{px^{p-1}[|x(t)^{p-1}+0.25^{p-1}|x(t-0.2)|^{p-1}][0.4x(t)+0.2x(t-0.2)]\\
&+\frac{p(p-1)}{2}2^{p-2}[|x(t)^{p-2}+0.25^{p-2}|x(t-0.2)|^{p-2}]\cdot 0.25|x(t-0.2)|^2\big\}\\
\leq&\mathrm{E}\big\{0.4p2^{p-1}|x(t)|^p+0.2(p-1)2^{p-1}|x(t)|^px(t-0.2)+0.2\cdot 2^{p-1}|x(t-0.2)|^p\\
&+0.4px^{p-1}0.5^{p-1}|x(t-0.2)|^p+0.4\cdot 0.5^{p-1}|x(t)|^p+0.2p\cdot 0.5^{p-1}|x(t-0.2)|^p\\
&+(p-1)(p-2)2^{p-5}|x(t)|^p+(p-1)2^{p-5}|x(t-0.2)|^p+p(p-1)0.5^{p+1}|x(t-0.2)|^p\big\}\\
=&[0.4p2^{p-1}+0.2(p-1)2^{p-1}+0.4\cdot 0.5^{p-1}+(p-1)(p-2)2^{p-5}+p(p-1)0.5^{p+1}]\mathrm{E}|x(t)|^p\\
&+[0.2\cdot 2^{p-1}+0.4(p-1)0.5^{p-1}+0.2p0.5^{p-1}+(p-1)2^{p-5}+p(p-1)0.5^{p+1}]\mathrm{E}|x(t-0.2)|^p.
\end{aligned}$$

Let $p = 3$ and $q = 2$, we can get $\mathrm{E}\mathcal{L}V(t, x_t, 1) \leq 7.375\mathrm{E}|x(t)|^3 + 2.025\mathrm{E}|x(t-0.2)|^3 \leq 11.425\mathrm{E}|x(t)|^3$. Similarly, it has $\mathrm{E}\mathcal{L}V(t, x_t, 2) \leq 8.992\mathrm{E}|x(t)|^3$. Set $b(t) \equiv \widetilde{b} = 12$, $\rho_1 = 1.5$ and $\rho_2 = 0.002$, obviously, $\rho_1 + \rho_2 \mathrm{e}^{\widetilde{b}\tau} = 1.5 + 0.002\mathrm{e}^{12 \cdot 0.2} \approx 1.75 < q < \mathrm{e}^{\widetilde{b}\delta}$. Thus, System (62) is 3th moment exponentially stable by Theorem 3.6. Numerical simulation for System (62) are shown in Figure 3 with the initial value $\xi = (\cos t, -\cos t)$ $(-0.2 \leq t \leq 0)$.

Remark 4.3 Obviously, the conclusion obtained in this paper is less conservative than that obtained in [32] in the sense that the coefficients of the estimated upper bound for the diffusion operator of Lyapunov functionals can be allowed sign-changing as the time-varying.

Remark 4.4 From Figure 3 and Figure 4 show that an unstable stochastic system can be exponentially stabilized by an suitable sequence of impulses, and impulses may change the asymptotic behavior of the given systems.

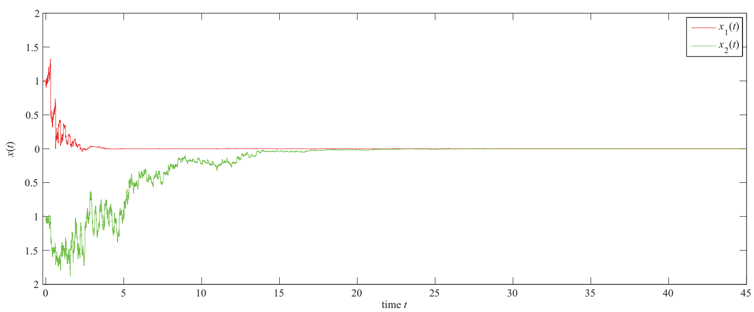

Figure 3 The state response of System (62) in Example 4.2 under impulsive control

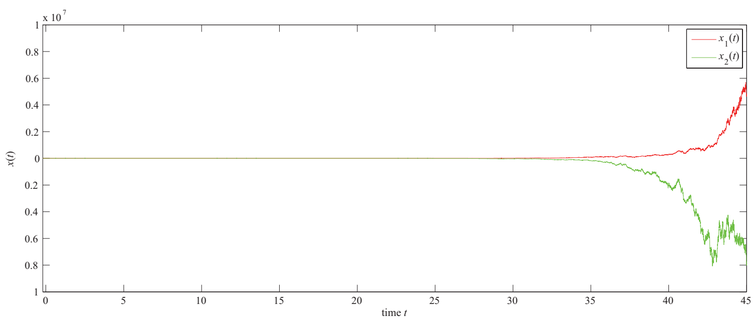

Figure 4 The state response of System (62) in Example 4.2 without impulses

5 Conclusion

In this work, we investigated the exponential stability for INSFDEs-MS by applying the Razumikhin technique and stochastic analysis approaches. Some criteria on the pth moment and almost exponential stability are proposed, and the results show that impulses do contribute to exponential stability of NSFDEs-MS. In fact, we extend some previous findings to the exponential stability of impulsive stochastic functional equations with Markov switching involving derivatives with delays, and the obtained results are verified to be more general than the existing results. Furthermore, two examples are provided in the end to illustrate the applications of the obtained results. In our future work, it is worth considering the following two aspects:

1) Due to the exponential stability is too restrictive to be easily obtained in practical situations, perhaps we can consider the general decay stability theorem or stability in distribution criterions for the trivial solutions of System (1).

2) Furthermore, before considering the exponential stability of the system (1), the linear growth condition (similar to H2 in reference [32]) needs to be satisfied to ensure the existence and uniqueness of the solution. However, this condition is usually violated in practical systems, so it is an interesting idea to consider reducing the constraint of this condition to obtain more satisfactory results.

Conflict of Interest

The authors declare no conflict of interest.

References

[1] Hale J K and Meyer K R, A class of functional equations of neutral type, *Memoirs of the American Mathematical Society*, 1967, **76**: 1–65.

[2] Kolmanovskii V B and Nosov V R, *Stability and Periodic Modes of Control Systems with Aftereffect*, Nauka, Moscow, 1981.

[3] Hale J K and Lunel S M V, *Introduction to Functional Differential Equations*, Springer-Verlag, New York, 1993.

[4] Yang X T, On the periodic solution of neutral functional differential equations, *Journal of Systems Science & Complexity*, 2006, **26**(6): 684–692.

[5] Kolmanovskii V B and Nosov V R, *Stability of Functional Differential Equations*, Academic Press, New York, 1986.

[6] Pavlović G and Janković S, The Razumikhin approach on general decay stability for neutral stochastic functional differential equations, *Journal of the Franklin Institute*, 2013, **350**(8): 2124–2145.

[7] Huang L R and Deng F Q, Razumikhin-type theorems on stability of neutral stochastic functional differential equations, *IEEE Transactions on Automatic Control*, 2008, **53**(7): 1718–1723.

[8] Chen H B and Wang L, New result on exponential stability for neutral stochastic linear system with time-varying delay, *Applied Mathematics and Computation*, 2014, **239**: 320–325.

[9] Mao X R, Exponential stability in mean square of neutral stochastic differential functional equations, *Systems & Control Letters*, 1995, **26**: 245–251.

[10] Mao X R, *Stochastic Differential Equations and Applications*, 2nd Edition, Horwood, England, 2007.

[11] Luo Q, Mao X R, and Shen Y, New criteria on exponential stability of neutral stochastic differential delay equations, *Systems & Control Letters*, 2006, **55**(10): 826–834.

[12] Randjelović J and Janković S, On the pth moment exponential stability criteria of neutral stochastic functional differential equations, *Journal of Mathematical Analysis and Applications*, 2007, **326**: 266–280.

[13] Janković S, Randjelović J, and Janković M, Razumikhin-type exponential stability criteria of neutral stochastic functional differential equations, *Journal of Mathematical Analysis and Applications*, 2009, **355**: 811–820.

[14] Chen Y, Zheng W X, and Xue A, A new result on stability analysis for stochastic neutral systems, *Automatica*, 2010, **46**(12): 2100–2104.

[15] Cheng P, He S P, Luan X L, et al., Finite-region asynchronous H_∞ control for 2D Markov jump systems, *Automatica*, 2021, **129**: 109590.

[16] Ren C C, He S P, Luan X L, et al., Finite-time L_2-gain asynchronous control for continuous-time positive hidden Markov jump systems via T–S fuzzy model approach, *IEEE Transactions on Cybernetics*, 2021, **51**(1): 77–87.

[17] Kolmanovskii V, Koroleva N, Maizenberg T, et al., Neutral stochastic differential delay equations with Markovian switching, *Stochastic Analysis and Applications*, 2003, **21**(4): 839–867.

[18] Zhou S B and Hu S G, Razumikhin-type theorems of neutral stochastic functional differential equations, *Acta Mathematica Scientia*, 2009, **29B**(1): 181–190.

[19] Rakkiyappan R, Zhu Q X, and Chandrasekar A, Stability of stochastic neural networks of neutral type with Markovian jumping parameters: A delay-fractioning approach, *Journal of the Franklin Institute*, 2014, **351**(3): 1553–1570.

[20] Xu Y and He Z M, Exponential stability of neutral stochastic delay differential equations with Markovian switching, *Applied Mathematics Letters*, 2016, **52**: 64–73.

[21] Caraballo T, Mchiri L, and Rhaima M, Partial practical exponential stability of neutral stochastic functional differential equations with Markovian switching, *Mediterranean Journal of Mathematics*, 2021, **18**(4): 1–26.

[22] Song R L, Lu B L, and Zhu Q X, Stability of a class of neutral stochastic functional differential equations with Markovian switching, *IET Control Theory & Applications*, 2018, **12**(15): 2043–2054.

[23] Xu D Y and Yang Z C, Impulsive delay differential inequality and stability of neural networks, *Journal of Mathematical Analysis and Applications*, 2005, **305**(1): 107–120.

[24] Li C G, Chen L N, and Aihara K, Impulsive control of stochastic systems with applications in chaos control, chaos synchronization, and neural networks, *Chaos*, 2008, **18**(2): 023132.

[25] Stamova I M and Stamov T G, Asymptotic stability of impulsive control neutral-type systems, *International Journal of Control*, 2014, **87**(1): 25–31.

[26] Peng S G and Zhang Y, Razumikhin-type theorems on pth moment exponential stability of impulsive stochastic delay differential equations, *IEEE Transactions on Automatic Control*, 2010, **55**(8): 1917–1922.

[27] Hu W and Zhu Q X, Exponential stability of stochastic differential equations with impulse effects at random times, *Asian Journal of Control*, 2020, **22**(2): 779–787.

[28] Guo Y X, Zhu Q X, and Wang F, Stability analysis of impulsive stochastic functional differential equations, *Communications in Nonlinear Science and Numerical Simulation*, 2020, **82**: 105013.

[29] Yu G S, On the pth moment exponential stability criteria of neutral stochastic functional differential equations with jumps, *Acta Mathematicae Applicatae Sinica-English Series*, 2009, **22**(1): 148–154.

[30] Benhadri M, Caraballo T, and Halim Z, Existence of solutions and stability for impulsive neutral stochastic functional differential equations, *Stochastic Analysis and Applications*, 2019, **37**(5): 777–798.

[31] Fu X Z and Zhu Q X, Exponential stability of neutral stochastic delay differential equation with delay-dependent impulses, *Applied Mathematics and Computation*, 2020, **377**: 125146.

[32] Yu G S, Yang W Q, Xu L, et al., Pth moment and almost sure exponential stability of impulsive neutral stochastic functional differential equations with Markovian switching, *International Journal of Systems Science*, 2018, **49**(6): 1164–1177.

[33] Mchiri L and Rhaima M, β-stability in q-th moment of neutral impulsive stochastic functional differential equations with Markovian switching, *Mathematical Problems in Engineering*, 2021, **2021**(1): 1–11.

[34] Wu X T, Zhang W B, and Tang Y, Pth moment stability of impulsive stochastic delay differential systems with Markovian switching, *Communications in Nonlinear Science and Numerical Simulation*, 2013, **18**(7): 1870–1879.

[35] Cheng P, Deng F Q, and Yao F Q, Exponential stability analysis of impulsive stochastic functional differential systems with delayed impulses, *Communications in Nonlinear Science and Numerical Simulation*, 2014, **19**(6): 2104–2114.

[36] Gao L J, Wang D D, and Zong G D, Exponential stability for generalized stochastic impulsive functional differential equations with delayed impulses and Markovian switching, *Nonlinear Analysis: Hybrid Systems*, 2018, **30**: 199–212.

[37] Skorohod A V, *Asymptotic Methods in the Theory of Stochastic Differential Equations, American Mathematical Society*, Providence, RI, 1989.

[38] Zhu Q X, Pth moment exponential stability of impulsive stochastic functional differential equations with Markovian switching, *Journal of the Franklin Institute*, 2014, **351**(7): 3965–3986.

[39] Kao Y G, Zhu Q X, and Qi W H, Exponential stability and instability of impulsive stochastic functional differential equations with Markovian switching, *Applied Mathematics and Computation*, 2015, **271**: 795–804.