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Adaptive Event-Triggering Consensus for Multi-Agent Systems with Linear Time-Varying Dynamics*

ZHANG Wenbing · ABUZAR HUSSEIN MOHAMMED Atitalla · BAO Jiatong · LIU Yurong

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Abstract In this paper, the authors study the fully distributed event-triggering consensus problem for multi-agent systems with linear time-varying dynamics, where each agent is described by a linear time-varying system. An adaptive event-triggering protocol is proposed for time-varying multi-agent systems under directed graph. Based on the Gramian matrix of linear time-varying systems, the design of control gain is done and sufficient conditions ensuring the consensus of linear time-varying multi-agent systems are obtained. It is shown that the coupling strength is closely related to the triggering condition. When it comes to undirected graph, it is shown that the coupling strength is independent on the triggering condition and thus the design procedure is of more freedom than the directed case. In addition, it is also proved that Zeno behaviours can be excluded in the proposed protocols. A numerical example is presented to demonstrate the effectiveness of the theoretical results.

Keywords Consensus, event-triggering, fully distributed control, multi-agent systems, time-varying systems.

1 Introduction

In the past decades, coordination of multi-agent systems is one of the hottest research topics in control system area[1–6] due to its wide applications in attitude of spacecraft alignment, smart grids, distributed optimization and so on. The basic problem of consensus is to design a distributed protocol such that all the agents can reach an agreement in which the distributed

ZHANG Wenbing · ABUZAR HUSSEIN MOHAMMED Atitalla
School of Mathematical Sciences, Yangzhou University, Yangzhou 225002, *China.* Email: zwb850506@126.com.
BAO Jiatong
School of Electrical, Energy and Power Engineering, Yangzhou University, Yangzhou 225009, *China.*
Email: jtbao@yzu.edu.cn.
LIU Yurong
School of Mathematical Sciences, Yangzhou University, Yangzhou 225002, *China.*
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protocol only depends on the local state of each agent and its neighbors. As a consequence, numerous results have been reported on how to design the distributed protocols in multi-agent systems[7, 8]. One common feature on consensus of multi-agent system is that it needs to know the eigenvalues of the Laplacian matrix of the graph. However, as the Laplacian matrix is a class of global information, in some situations, especially the network size is large, this global information may be unavailable. In this case, over the past ten years, some efforts have been devoted to designing adaptive protocols to study consensus of multi-agent system without using the Laplacian matrix[9–12], where the main idea is to design an adaptive mechanism to tune the coupling strengthes.

On the other hand, over the past decade, in order to reduce the resource communications caused by continuous communications, many event-triggering protocols have been proposed in multi-agent systems[13–18]. It should be noted in [13–18], they have used the eigenvalues of the Laplacian matrix. As mentioned above, such design encumbers their applications in large scale networks. Thus, very recently, there are some efforts reported on fully distributed event-triggering control of multi-agent systems[1, 19–22]. In [1], a fully distributed event-triggering protocol is designed to solve the consensus problem of multi-agent systems on undirected graphs. In [20], fully distributed event-triggering consensus problem is investigated for linear multi-agent systems on undirected graph and in [19], adaptive event-triggering consensus problem is investigated for linear multi-agent systems on directed graph.

Moreover, in some practical applications, owing to model uncertainties, performance degradation and external disturbances, many practical engineering systems are time-varying[23–26]. Thus, it would be more appropriate to model practical multi-agent systems by time-varying systems. The investigation on time-varying systems is more difficult than the time-invariant ones since some well-known methods such as linear matrix inequality and examining eigenvalues in time-invariant systems are no longer valid in time-varying systems. Actually, as mentioned in [27, 28], the asymptotic stability of linear time-varying systems has been listed as the first open problem in mathematical systems and control fields. Thus, several researches have been committed to address the stability problem of linear time-varying systems in the past few years[28–31]. Although some efforts have been devoted to investigating the stability problem of linear time-varying systems, there are few results being reported on consensus of multi-agent systems with linear time-varying dynamics[23, 24, 26]. Meanwhile, it is worth pointing out that the results in [23, 24, 26] are still restrictive. For instance, in [26], the control gain design problem has not fully been investigated. In [24], the matrix $B(t)$ is assumed to be an identity matrix and the transition matrix is assumed to be bounded by a constant that is greater than 1. Moreover, the communication graphs are assumed to be undirected and global information of the Laplacian matrix is required in [23, 24, 26]. In a word, it is challenging to derive a fully distributed event-triggering protocol for the multi-agent systems with time-varying dynamics under directed graph, since the design procedure of the fully event-triggering protocol not only depends on the agents' time-varying dynamics but also depends on the network topology.

In this paper, we aim to investigate the fully distributed consensus problem for multi-agent systems with linear time-varying dynamics under event-triggering control. Based on the

Gramian matrix of linear time-varying systems, the control gain design problem is presented and sufficient conditions are derived ensuring the consensus of time-varying multi-agent systems on directed graph, where coupling strengths are dependent on the triggering condition. In the case of undirected graph, it is shown that the design procedure of the fully distributed event-triggering protocol is of more freedom than in the directed case. The novelties of this paper can be concluded as follows: 1) Linear time-varying dynamics are considered in this paper. Unlike existing results on event-triggering consensus in [1, 13–16, 19, 21, 22, 32], each node's dynamics is described by a linear time-varying system in this paper, which makes our current research quite challenge since the classical methods in time-invariant systems such as Ricatti equation, linear matrix inequality and examining eigenvalues are no longer valid in time-varying systems. 2) The fully distributed event-triggering consensus problem is investigated for multi-agent systems with linear time-varying dynamics on directed graph. In addition to the linear time-varying dynamics, a fully distributed event-triggering protocol is designed on the directed graph, which is more involved than in the undirected graphs[1, 21, 22]. Moreover, the existence of the fully distributed event-triggering protocol is proved to be existed by using the Gramian matrix of linear time-varying systems and Zeno behaviours are proved to be excluded.

The organization of this paper is as follows. In Section 2, some preliminaries and problem statement are introduced. The fully distributed event-triggering protocols are proved to be existent in Section 3. Numerical examples are presented in Section 4 while Section 5 summarizes the results.

2 Preliminaries and Problem Statement

Throughout this paper, $\mathbb{R}^{n\times m}$ represents the set of $n \times m$ real matrices. I_n is the $n \times n$ identity matrix and $\otimes$ denotes the Kronecker product between two matrixes. For real symmetric matrices X and Y, $X \geq Y$ ($X > Y$) represents that $X - Y$ is semi-positive (positive) definite and $\lambda_M(X)$ and $\lambda_m(X)$ denote, respectively the maximum and minimum eigenvalues of X. $\|\cdot\|$ denotes the 2-norm and $\mathbb{N}$ stands for the set of nonnegative integers.

A directed graph $\mathcal{G}$ is represented by a pair $(\mathcal{V}, \mathcal{E})$ in which the nonempty finite set $\mathcal{V}$ is the node set and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the set of edges. A node j is said to be a neighbor of node i if $(j, i) \in \mathcal{E}$. The neighbor set of agent i is defined as $N_i = \{j|(j, i) \in \mathcal{E}\}$. In a graph, if $(j, i) \in \mathcal{E}$ implies $(i, j) \in \mathcal{E}$, then the graph is said to be undirected. A path from node j to node i is a sequence $v_{i_0}, v_{i_1}, \cdots, v_{i_l}$ of distinct node such that $(v_{i_k}, v_{i_{k+1}}) \in \mathcal{E}$, $k = 0, 1, \cdots, l-1$ with $v_{i_0} = j$ and $v_{i_l} = i$. A directed graph is said to contain a spanning tree if every other node can be reached from a node (usually call as the root node) through a directed path. A directed graph is strongly connected if for any two distinct nodes $v_i, v_j \in \mathcal{V}$ there exist directed paths from v_i to v_j and v_j to v_i. The adjacency matrix $\mathcal{A} = [a_{ij}]_{N\times N}$ associated with the graph $\mathcal{G}$ is defined as: $a_{ij} > 0$ if agent j can send information to agent i, otherwise $a_{ij} = 0$. The Laplacian matrix $\mathcal{L}$ is defined as $\mathcal{L} = D - \mathcal{A}$, where $D = \text{diag}\{\sum_{j=2}^{N} a_{1j}, \sum_{j=1,j\neq 2}^{N} a_{2j}, \cdots, \sum_{j=1}^{N-1} a_{Nj}\}$.

Consider N agents described by the following linear time-varying system:

$$\dot{x}_i(t) = A(t)x_i(t) + B(t)u_i(t), \quad i = 1, 2, \cdots, N, \tag{1}$$

where $x_i(t) \in \mathbb{R}^n$ and $u_i(t) \in \mathbb{R}^m$ are the state and the control input of the agent i, respectively. In this paper, we are interested in the following fully distributed event-triggering protocols:

$$u_i(t) = K(t) \sum_{j \in N_i} c_i(t) a_{ij} (x_i(t_k^i) - x_j(t_k^i)), \quad t \in [t_k^i, t_{k+1}^i), \tag{2}$$

where t_k^i with $t_0^i = 0, k \in \mathbb{N}$ is the k-th triggering instant of the node i, $i = 1, 2, \cdots, N$. $K(t)$ is the control gain matrix needs to be designed. $c_i(t) > 0, i = 1, 2, \cdots, N$ are adaptive coupling gains. The main purpose of this paper is to investigate the fully distributed event-triggering consensus problem of the multi-agent system in (1) with directed and undirected graphs. That is, we intend to design an event-triggering protocol without using the Laplapican matrix such that $\lim_{t \to \infty} \|x_i(t) - x_j(t)\| = 0, \forall i, j \in \mathcal{V}$.

Remark 2.1 In this paper, we aim to design a fully distributed event-triggering protocol without using the eigenvalues of the Laplacian matrix, where one needs to design an adaptive law for the coupling strengthes $c_i(t)$, which is different from the constant coupling considered in [13–16, 32]. In addition, as each agent's dynamics is time-varying, the well-known linear matrix inequalities and Riccati equation methods are no longer valid. Thus, in what follows, the main objective is to design fully distributed event-triggering protocols by using the Gramian matrix of linear time-varying systems.

Then, the main results will be presented by using the Lyapunov function method and the following lemmas will be used in our analysis.

Lemma 2.2 (see [33]) *If the graph contains a directed spanning tree, then zero is a simple eigenvalue of $\mathcal{L}$ and the other eigenvalues of $\mathcal{L}$ have positive real parts.*

Lemma 2.3 (see [34]) *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function on $[0, \infty)$. Suppose that $\lim_{t \to \infty} \int_0^t \phi(\tau) d\tau$ exists and is finite. Then*

$$\phi(t) \to 0 \quad as \quad t \to \infty.$$

3 Main Results

In this section, we will investigate the fully distributed consensus problem for multi-agent systems with linear time-varying dynamics. Firstly, the fully distributed event-triggering consensus problem will be investigated for multi-agent systems under directed graph. An adaptive distributed event-triggering protocol is proposed. Secondly, when it comes to undirected graph, we show that the design procedure is of more freedom than the directed graph. As each agent's dynamics is time-varying, the following notations on linear time-varying systems will be used to derive the main results.

Consider a linear time-varying system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \tag{3}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input and $A(t) \in \mathbb{R}^{n \times n}$ and $B(t) \in \mathbb{R}^{n \times m}$ are known matrix-valued functions. Let $\Phi(t_0, t)$ be the transition matrix of the time-varying system $\dot{x}(t) = A(t)x(t)$ with the initial value $x(t)|_{t=t_0} = x(t_0)$. Then the Gramian matrix $W_c(t_0, t_f)$ of the system in (3) is defined as

$$W_c(t_0, t_f) = \int_{t_0}^{t_f} \Phi(t_0, s)B(s)B^{\rm T}(s)\Phi^{\rm T}(t_0, s)ds.$$

In what follows, we will design fully distributed event-triggering protocols based on the Gramian matrix. The following assumption is made on the Gramian matrix.

Assumption 3.1 There exist three positive constants $\gamma_1 < \gamma_2$ and σ such that for all $t > t_0$, the following condition holds:

$$\gamma_1 I_n < W_c(t, t + \sigma) < \gamma_2 I_n. \tag{4}$$

Remark 3.1 Assumption 3.1 is inspired by [29, 35]. This kind of assumption is called as uniform controllability. It is shown in [35] that the linear time-varying system (3) is globally uniformly exponentially stable with $u(t) = K(t)x(t)$ if Assumption 3.1 holds, where $K(t)$ will be defined later.

Before presenting the main results, the following matrix function is given to construct the Lyapunov function:

$$P_1(t) = \int_t^{t+\sigma} {\rm e}^{4\varepsilon(t-s)}\,\Phi(t, s)B(s)B^{\rm T}(s)\Phi^{\rm T}(t, s)ds,$$

where $\varepsilon > 0$. According to Assumption 3.1, it can be seen that

$$\gamma_1 {\rm e}^{-4\varepsilon\sigma} I_n \le P_1(t) \le \gamma_2 I_n$$

and thus

$$\gamma_2^{-1} I_n \le P_1^{-1}(t) \le \gamma_1^{-1}{\rm e}^{4\varepsilon\sigma} I_n.$$

3.1 Directed Graph Case

In this subsection, we consider the fully distributed event-triggering consensus problem for multi-agent systems on directed graphs. Like [12], for the convenience of symbolic representation, we consider a group of N agents with linear time-varying dynamics, where the agent labeled by 1 is assumed to be the leader and the other $N - 1$ agents are assumed to be the followers and the leader's control input is assumed to be zero. In this case, the Laplacian matrix is in the following form:

$$\mathcal{L} = \begin{bmatrix} 0 & 0_{1\times(N-1)} \\ \mathcal{L}_2 & \mathcal{L}_1 \end{bmatrix}, \tag{5}$$

where $\mathcal{L}_2 \in \mathbb{R}^{N-1}$ and $\mathcal{L}_1 \in \mathbb{R}^{(N-1)\times(N-1)}$. The following lemma that plays an important role in deriving the main results will be used.

Lemma 3.2 (see [12]) *Suppose that the graph contains a directed spanning tree. Letting* $q = {\rm col}\{q_2, q_3, \cdots, q_N\} = (\mathcal{L}_1^{\rm T})^{-1}\mathbf{1}_{N-1}$ *and* $Q = {\rm diag}\{q_2, q_3, \cdots, q_N\}$*, then*

$$Q\mathcal{L}_1 + \mathcal{L}_1^{\rm T} Q > 0.$$

Letting $\eta_i(t) = \sum_{j\in N_i} a_{ij}(x_i(t) - x_j(t)), i = 2, 3, \cdots, N$ and $\eta(t) = [\eta_2^{\mathrm{T}}(t), \eta_3^{\mathrm{T}}(t), \cdots, \eta_N^{\mathrm{T}}(t)]^{\mathrm{T}}$, then we have

$$\eta(t) = (\mathcal{L}_1 \otimes I_n) \begin{bmatrix} x_2(t) - x_1(t) \\ x_3(t) - x_1(t) \\ \vdots \\ x_N(t) - x_1(t) \end{bmatrix}.$$

So

$$\dot{\eta}(t) = [I_{N-1} \otimes A(t) + \mathcal{L}_1\mathcal{C} \otimes B(t)K(t)]\eta(t) + [\mathcal{L}_1\mathcal{C} \otimes B(t)K(t)]\varepsilon(t), \tag{6}$$

where $\varepsilon(t) = \mathrm{col}\{\varepsilon_2(t), \varepsilon_3(t), \cdots, \varepsilon_N(t)\}$ with $\varepsilon_i(t) = \sum_{j\in N_i} a_{ij}[x_i(t_k^i) - x_j(t_k^i) - (x_i(t) - x_j(t))], i = 2, 3, \cdots, N$, and $\mathcal{C} = \mathrm{diag}\{c_2(t), c_3(t), \cdots, c_N(t)\}$. Thus, the consensus problem is equivalent to the stability problem of (6) due to the fact that $\mathcal{L}_1$ is nonsingular. Letting $r_i(t) = \eta_i(t)P_1^{-1}(t)\eta_i(t), t \in [t_k^i, t_{k+1}^i)$, then the adaptive coupling laws for $c_i(t), i = 2, 3, \cdots, N$ are designed as follows:

$$\begin{aligned} c_i(t) &= p_i(t) + r_i(t_k^i), \quad t \in [t_k^i, \ t_{k+1}^i), \\ \dot{p}_i(t) &= \left\| K(t) \sum_{j\in N_i} a_{ij}[x_i(t_k^i) - x_j(t_k^i)] \right\|^2, \quad t \in [t_k^i, t_{k+1}^i). \end{aligned} \tag{7}$$

Letting ρ_1, γ and β be three positive constants, then the following triggering condition is used to determine the triggering instants:

$$t_{k+1}^i = \inf\{t > t_k^i | f_i(t) = 0\}, \quad k \in \mathbb{N}, \quad i = 2, 3, \cdots, N, \tag{8}$$

and the triggering function is defined as

$$\begin{aligned} f_i(t) =& \max\left\{ \overline{c}_i^2(t)\|K(t)\varepsilon_i(t)\|^2, (r_i(t) - r_i(t_k^i))^2\|K(t)\delta_i(t_k^i)\|^2 \right\} - \gamma \mathrm{e}^{-\beta t} \\ & - \rho_1 \left\| K(t) \sum_{j\in N_i} a_{ij}(x_i(t) - x_j(t)) \right\|^2, \end{aligned} \tag{9}$$

where $\delta_i(t_k^i) = \sum_{j\in N_i} a_{ij}[x_i(t_k^i) - x_j(t_k^i)]$ and $\overline{c}_i(t) = p_i(t) + r_i(t)$. In what follows, we will present a sufficient condition to ensure the consensus of the multi-agent system in (1) under directed graph, in which the existence of the event-triggering protocol is proved under Assumption 3.1. Moreover, the Zeno behavior will be proved to be excluded.

Theorem 3.3 *Consider the multi-agent network in* (1) *with the protocol* (2) *on a directed graph. Assume that the graph contains a directed spanning tree, the adaptive gains* $c_i(t)$ *satisfy* (7) *and* ρ_1 *in* (9) *is chosen such that* $p_i(0) > \sqrt{2\rho_1}, i = 2, 3, \cdots, N$. *Then the multi-agent system reaches consensus if the triggering function is defined as in* (9) *with* $K(t) = -B^{\mathrm{T}}(t)P_1^{-1}(t)$. *Moreover, no Zeno behavior exists and the adaptive gains* $c_i(t)$ *will converge to some positive constants.*

Proof Consider the following Lyapunov function

$$V_1(t) = \sum_{i=2}^{N} q_i \left(p_i(t) r_i(t) + \frac{1}{2} r_i(t)^2 \right) + \frac{1}{2} \sum_{i=2}^{N} q_i (p_i(t) - \alpha)^2, \tag{10}$$

where α is a positive constant that will be defined later. The derivative of $V_1(t)$ is given by

$$\begin{aligned}
\dot{V}_1(t) &= \sum_{i=2}^{N} q_i \Big(\dot{p}_i(t) r_i(t) + p_i(t) \dot{r}_i(t) + r_i(t) \dot{r}_i(t) + (p_i(t) - \alpha) \dot{p}_i(t) \Big) \\
&= \sum_{i=2}^{N} q_i \Big(\overline{c}_i(t) \dot{r}_i(t) + (\overline{c}_i(t) - \alpha) \dot{p}_i(t) \Big) \\
&= \sum_{i=2}^{N} q_i \Big(2\overline{c}_i(t) \eta_i^{\mathrm{T}}(t) P_1^{-1}(t) \dot{\eta}_i(t) + \overline{c}_i(t) \eta_i^{\mathrm{T}}(t) \dot{P}_1^{-1}(t) \eta_i(t) + (\overline{c}_i(t) - \alpha) \dot{p}_i(t) \Big).
\end{aligned} \tag{11}$$

Noting $K(t) = -B^{\mathrm{T}}(t) P_1^{-1}(t)$, one can obtain from (6) that

$$\begin{aligned}
& 2 \sum_{i=2}^{N} \overline{c}_i(t) q_i \eta_i^{\mathrm{T}}(t) P_1^{-1}(t) \dot{\eta}_i(t) \\
=& 2\eta^{\mathrm{T}}(t) (\overline{\mathcal{C}} Q \otimes P_1^{-1}(t)) \dot{\eta}(t) \\
=& 2\eta^{\mathrm{T}}(t) (\overline{\mathcal{C}} Q \otimes P_1^{-1}(t)) \{ [\mathcal{L}_1 \mathcal{C} \otimes B(t) K(t)] \varepsilon(t) + [I_{N-1} \otimes A(t) + \mathcal{L}_1 \mathcal{C} \otimes B(t) K(t)] \eta(t) \} \\
=& \eta^{\mathrm{T}}(t) \{ \overline{\mathcal{C}} Q \otimes (P_1^{-1}(t) A(t) + A^{\mathrm{T}}(t) P_1^{-1}(t)) \} \eta(t) - 2\eta^{\mathrm{T}}(t) [\overline{\mathcal{C}} Q \mathcal{L}_1 \mathcal{C} \otimes K^{\mathrm{T}}(t) K(t)] \eta(t) \\
& - 2\eta^{\mathrm{T}}(t) [\overline{\mathcal{C}} Q \mathcal{L}_1 \mathcal{C} \otimes K^{\mathrm{T}}(t) K(t)] \varepsilon(t).
\end{aligned} \tag{12}$$

It follows from Lemma 3.2 that

$$\begin{aligned}
& - 2\eta^{\mathrm{T}}(t) [\overline{\mathcal{C}} Q \mathcal{L}_1 \mathcal{C} \otimes K^{\mathrm{T}}(t) K(t)] \eta(t) \\
=& - 2\eta^{\mathrm{T}}(t) [\overline{\mathcal{C}} Q \mathcal{L}_1 \overline{\mathcal{C}} \otimes K^{\mathrm{T}}(t) K(t)] \eta(t) - 2\eta^{\mathrm{T}}(t) [\overline{\mathcal{C}} Q \mathcal{L}_1 (\mathcal{C} - \overline{\mathcal{C}}) \otimes K^{\mathrm{T}}(t) K(t)] \eta(t) \\
=& - \eta^{\mathrm{T}}(t) (\overline{\mathcal{C}} \otimes K^{\mathrm{T}}(t)) [(Q \mathcal{L}_1 + \mathcal{L}_1^{\mathrm{T}} Q) \otimes I_{N-1}] (\overline{\mathcal{C}} \otimes K(t)) \eta(t) \\
& - 2\eta^{\mathrm{T}}(t) [\overline{\mathcal{C}} Q \mathcal{L}_1 (\mathcal{C} - \overline{\mathcal{C}}) \otimes K^{\mathrm{T}}(t) K(t)] \eta(t) \\
\leq& - \lambda_{\min}(Q \mathcal{L}_1 + \mathcal{L}_1^{\mathrm{T}} Q) \eta^{\mathrm{T}}(t) [\overline{\mathcal{C}}^2 \otimes K^{\mathrm{T}}(t) K(t)) \eta(t) \\
& - 2\eta^{\mathrm{T}}(t) [\overline{\mathcal{C}} Q \mathcal{L}_1 (\mathcal{C} - \overline{\mathcal{C}}) \otimes K^{\mathrm{T}}(t) K(t)] \eta(t).
\end{aligned} \tag{13}$$

Let ς_1 and ς_2 be two positive constants. Then the following derivations can be made from the triggering condition (9):

$$\begin{aligned}
& - 2\eta^{\mathrm{T}}(t) [\overline{\mathcal{C}} Q \mathcal{L}_1 \mathcal{C} \otimes K^{\mathrm{T}}(t) K(t)] \varepsilon(t) \\
=& 2\eta^{\mathrm{T}}(t) [\overline{\mathcal{C}} Q \mathcal{L}_1 (\overline{\mathcal{C}} - \mathcal{C}) \otimes K^{\mathrm{T}}(t) K(t)] \varepsilon(t) \\
& - 2\eta^{\mathrm{T}}(t) [\overline{\mathcal{C}} Q \mathcal{L}_1 \overline{\mathcal{C}} \otimes K^{\mathrm{T}}(t) K(t)] \varepsilon(t) \\
\leq& \varsigma_1 \eta^{\mathrm{T}}(t) [\overline{\mathcal{C}} Q \mathcal{L}_1 \mathcal{L}_1^{\mathrm{T}} Q \overline{\mathcal{C}} \otimes K^{\mathrm{T}}(t) K(t)] \eta(t) \\
& + \frac{1}{\varsigma_1} \varepsilon^{\mathrm{T}}(t) (\overline{\mathcal{C}}^2 \otimes K^{\mathrm{T}}(t) K(t)) \varepsilon(t)
\end{aligned}$$

$$
\begin{aligned}
&+2\eta^{\mathrm{T}}(t)[\overline{\mathcal{C}}Q\mathcal{L}_1(\overline{\mathcal{C}}-\mathcal{C})\otimes K^{\mathrm{T}}(t)K(t)]\varepsilon(t)\\
\leq&\varsigma_1\lambda_{\max}(Q\mathcal{L}_1\mathcal{L}_1^{\mathrm{T}}Q)\eta^{\mathrm{T}}(t)[\overline{\mathcal{C}}^2\otimes K^{\mathrm{T}}(t)K(t)]\eta(t)\\
&+\frac{\rho_1}{\varsigma_1}\eta^{\mathrm{T}}(t)(I\otimes K^{\mathrm{T}}(t)K(t))\eta(t)+N\frac{\gamma\mathrm{e}^{-\beta t}}{\varsigma_1}\\
&+2\eta^{\mathrm{T}}(t)[\overline{\mathcal{C}}Q\mathcal{L}_1(\overline{\mathcal{C}}-\mathcal{C})\otimes K^{\mathrm{T}}(t)K(t)]\varepsilon(t)
\end{aligned}\tag{14}
$$

and

$$
\begin{aligned}
&2\eta^{\mathrm{T}}(t)[\overline{\mathcal{C}}Q\mathcal{L}_1(\overline{\mathcal{C}}-\mathcal{C})\otimes K^{\mathrm{T}}(t)K(t)]\varepsilon(t)-2\eta^{\mathrm{T}}(t)[\overline{\mathcal{C}}Q\mathcal{L}_1(\mathcal{C}-\overline{\mathcal{C}})\otimes K^{\mathrm{T}}(t)K(t)]\eta(t)\\
=&2\eta^{\mathrm{T}}(t)[\overline{\mathcal{C}}Q\mathcal{L}_1(\overline{\mathcal{C}}-\mathcal{C})\otimes K^{\mathrm{T}}(t)K(t)](\varepsilon(t)+\eta(t))\\
\leq&\varsigma_2\eta^{\mathrm{T}}(t)[\overline{\mathcal{C}}Q\mathcal{L}_1\mathcal{L}_1^{\mathrm{T}}Q\overline{\mathcal{C}}\otimes K^{\mathrm{T}}(t)K(t)]\eta(t)\\
&+\frac{1}{\varsigma_2}(\varepsilon(t)+\eta(t))^{\mathrm{T}}((\overline{\mathcal{C}}-\mathcal{C})^2\otimes K^{\mathrm{T}}(t)K(t))(\varepsilon(t)+\eta(t))\\
\leq&\varsigma_2\eta^{\mathrm{T}}(t)\lambda_{\max}(Q\mathcal{L}_1\mathcal{L}_1^{\mathrm{T}}Q)[\overline{\mathcal{C}}^2\otimes K^{\mathrm{T}}(t)K(t)]\eta(t)\\
&+\frac{1}{\varsigma_2}\rho_1\eta^{\mathrm{T}}(t)(I\otimes K^{\mathrm{T}}(t)K(t))\eta(t)+N\frac{\gamma}{\varsigma_2}\mathrm{e}^{-\beta t}.
\end{aligned}\tag{15}
$$

In addition, according to (7), one has

$$
(\overline{c}_i(t)-\alpha)\dot{p}_i=(\overline{c}_i(t)-\alpha)\bigg\|K(t)\sum_{j\in N_i}a_{ij}[x_i(t_k^i)-x_j(t_k^i)]\bigg\|^2.\tag{16}
$$

Then, in view of the triggering condition (9), it follows that

$$
\begin{aligned}
&\overline{c}_i(t)\bigg\|K(t)\sum_{j\in N_i}a_{ij}[x_i(t_k^i)-x_j(t_k^i)]\bigg\|^2\\
=&\overline{c}_i(t)\|K(t)\eta_i(t)+K(t)\varepsilon_i(t)\|^2\\
\leq&2\overline{c}_i(t)\|K(t)\eta_i(t)\|^2+2\overline{c}_i(t)\|K(t)\varepsilon_i(t)\|^2\\
\leq&2\overline{c}_i(t)\|K(t)\eta_i(t)\|^2+2\frac{\rho_1}{\overline{c}_i(t)}\|K(t)\eta_i(t)\|^2+\frac{2\gamma\mathrm{e}^{-\beta t}}{\overline{c}_i(t)}
\end{aligned}\tag{17}
$$

and

$$
\begin{aligned}
&-\alpha\bigg\|K(t)\sum_{j\in N_i}a_{ij}[x_i(t_k^i)-x_j(t_k^i)]\bigg\|^2\\
=&\frac{\alpha}{2}\bigg\|K(t)\sum_{j\in N_i}a_{ij}[x_i(t_k^i)-x_j(t_k^i)]-K(t)\varepsilon_i(t)\bigg\|^2\\
&-\frac{\alpha}{2}\bigg\|K(t)\sum_{j\in N_i}a_{ij}[x_i(t)-x_j(t)]\bigg\|^2\\
&-\alpha\bigg\|K(t)\sum_{j\in N_i}a_{ij}[x_i(t_k^i)-x_j(t_k^i)]\bigg\|^2
\end{aligned}
$$

$$
\begin{aligned}
&\le \alpha\Big\|K(t)\sum_{j\in N_i} a_{ij}[x_i(t_k^i)-x_j(t_k^i)]\Big\|^2+\alpha\|K(t)\varepsilon_i(t)\|^2\\
&\quad -\frac{\alpha}{2}\Big\|K(t)\sum_{j\in N_i} a_{ij}[x_i(t)-x_j(t)]\Big\|^2\\
&\quad -\alpha\Big\|K(t)\sum_{j\in N_i} a_{ij}[x_i(t_k^i)-x_j(t_k^i)]\Big\|^2\\
&=\alpha\|K(t)\varepsilon_i(t)\|^2-\frac{\alpha}{2}\Big\|K(t)\sum_{j\in N_i} a_{ij}[x_i(t)-x_j(t)]\Big\|^2\\
&\le\frac{\alpha\rho_1}{\overline{c}_i^2(t)}\|K(t)\eta_i(t)\|^2+\frac{\alpha\gamma e^{-\beta t}}{\overline{c}_i^2(t)}-\frac{\alpha}{2}\|K(t)\eta_i(t)\|^2. \qquad (18)
\end{aligned}
$$

On the other hand, noting $\dot{P}_1(t)=-B(t)B^{\mathrm T}(t)+4\varepsilon P_1(t)+A(t)P_1(t)+P_1(t)A^{\mathrm T}(t)+e^{4\varepsilon\sigma}\varPhi(t,t+\sigma)B(t+\sigma)B^{\mathrm T}(t+\sigma)\varPhi^{\mathrm T}(t,t+\sigma)$, it can be obtained from the fact $\dot{P}_1^{-1}(t)=-P_1^{-1}(t)\dot{P}_1(t)P_1^{-1}(t)$ that

$$
\begin{aligned}
&\sum_{i=2}^N \overline{c}_i(t)q_i\eta_i^{\mathrm T}(t)\dot{P}_1^{-1}(t)\eta_i(t)\\
=&\sum_{i=2}^N \overline{c}_i(t)q_i\eta_i^{\mathrm T}(t)\big\{P_1^{-1}(t)B(t)B^{\mathrm T}(t)P_1^{-1}(t)\\
&\qquad -4\varepsilon P_1^{-1}(t)-P_1^{-1}(t)A(t)-A^{\mathrm T}(t)P_1^{-1}(t)-e^{4\varepsilon\sigma}P_1^{-1}(t)\\
&\qquad \times\varPhi(t,t+\sigma)B(t+\sigma)B^{\mathrm T}(t+\sigma)\varPhi^{\mathrm T}(t,t+\sigma)P_1^{-1}(t)\big\}\eta_i(t)\\
\le&\eta^{\mathrm T}(t)(\overline{\mathcal{C}}Q\otimes P_1^{-1}(t)B(t)B^{\mathrm T}(t)P_1^{-1}(t))\eta(t)\\
&-\eta^{\mathrm T}(t)\{\overline{\mathcal{C}}Q\otimes(P_1^{-1}(t)A(t)+A^{\mathrm T}(t)P_1^{-1}(t))\}\eta(t)\\
&-4\varepsilon\eta^{\mathrm T}(t)(\overline{\mathcal{C}}Q\otimes P_1^{-1}(t))\eta(t). \qquad (19)
\end{aligned}
$$

Let $\varsigma_3=\lambda_{\min}(Q\mathcal{L}_1+\mathcal{L}_1^{\mathrm T}Q)-(\varsigma_1+\varsigma_2)\lambda_{\max}(Q\mathcal{L}_1\mathcal{L}_1^{\mathrm T}Q)$ and ς_1 and ς_2 are chosen such that $\varsigma_3>0$. Then, it can be derived from (11)–(19) and $\overline{c}_i\ge p_i(t)\ge p_i(0)$ that

$$
\begin{aligned}
\dot{V}_1(t)\le&-4\varepsilon\eta^{\mathrm T}(t)(\overline{\mathcal{C}}Q\otimes P_1^{-1}(t))\eta(t)\\
&+\eta^{\mathrm T}(t)\bigg\{\bigg[-\varsigma_3\overline{\mathcal{C}}^2+\frac{\rho_1}{\varsigma_1}I+\frac{\rho_1}{\varsigma_2}I+4\overline{\mathcal{C}}Q+\bigg(2\frac{\rho_1}{p(0)}+\frac{\alpha\rho_1}{p^2(0)}-\frac{\alpha}{2}\bigg)Q\bigg]\otimes K^{\mathrm T}(t)K(t)\bigg\}\eta(t)\\
&+\bigg(\frac{1}{\varsigma_1}+\frac{1}{\varsigma_2}\bigg)\times N\gamma e^{-\beta t}+\gamma e^{-\beta t}\sum_{i=2}^N\bigg(\frac{2q_i}{p(0)}+\frac{\alpha q_i}{p^2(0)}\bigg)\\
\le&-4\varepsilon\eta^{\mathrm T}(t)(\overline{\mathcal{C}}Q\otimes P_1^{-1}(t))\eta(t)-\varsigma_3\sum_{i=2}^N\eta_i^{\mathrm T}(t)\bigg[\overline{c}_i-\frac{2q_i}{\varsigma_3}\bigg]^2\eta_i(t)\\
&+\eta^{\mathrm T}(t)\bigg\{\bigg[\frac{4Q^2}{\varsigma_3}+\frac{\rho_1}{\varsigma_1}I+\frac{\rho_1}{\varsigma_2}I+\bigg(2\frac{\rho_1}{p(0)}+\frac{\alpha\rho_1}{p^2(0)}-\frac{\alpha}{2}\bigg)Q\bigg]\otimes K^{\mathrm T}(t)K(t)\bigg\}\eta(t)\\
&+\bigg(\frac{1}{\varsigma_1}+\frac{1}{\varsigma_2}\bigg)N\gamma e^{-\beta t}+\gamma e^{-\beta t}\sum_{i=2}^N\bigg(\frac{2q_i}{p(0)}+\frac{\alpha q_i}{p^2(0)}\bigg), \qquad (20)
\end{aligned}
$$

where $p(0) = \max_{i\in\mathcal{V}}\{p_i(0)\}$. Thus, according to the initial condition that $p_i(0) > \sqrt{2\rho_1}$, one can take a sufficient large α such that

$$\frac{4Q^2}{\varsigma_3} + \frac{\rho_1}{\varsigma_1}I + \frac{2\rho_1}{\varsigma_2}I + \left(2\frac{\rho_1}{p(0)} + \frac{\alpha\rho_1}{p^2(0)} - \frac{\alpha}{2}\right)Q \leq 0. \tag{21}$$

Letting $\theta = [(\frac{N}{\varsigma_1} + \frac{N}{\varsigma_2}) + \sum_{i=2}^{N}(\frac{2q_i}{p(0)} + \frac{\alpha q_i}{p^2(0)})]$ and substituting (21) into (20) yields

$$\dot{V}_1(t) \leq -4\varepsilon\eta^{\mathrm{T}}(t)(\overline{\mathcal{C}}Q \otimes P_1^{-1}(t))\eta(t) + \theta\gamma \mathrm{e}^{-\beta t}. \tag{22}$$

Integrating (22) further gives

$$\begin{aligned}0 \leq V_1(t) = \int_0^t \dot{V}_1(s)ds + V_1(0) &\leq \int_0^t \theta\gamma\mathrm{e}^{-\beta s}ds + V_1(0)\\ &\leq \frac{\theta\gamma}{\beta} + V_1(0), \quad t \geq 0.\end{aligned} \tag{23}$$

This implies that $V_1(t)$ is bounded and so is $c_i(t)$. In addition, we can conclude that $c_i(t)$ converge to some positive constants since $c_i(t)$ are monotonically increasing functions. By means of (22) and (23), one has

$$\lim_{t\to\infty} 4\varepsilon \int_0^t \eta^{\mathrm{T}}(s)(\overline{\mathcal{C}}Q \otimes P_1^{-1}(s))\eta(s)ds \leq \frac{\theta\gamma}{\beta} + V_1(0) - V_1(\infty) < \infty. \tag{24}$$

Then, according to Lemma 2.3, one can conclude that $\eta_i(t) \to 0$ as $t \to \infty$, $i = 1, 2, \cdots, N$.

In what follows, we will show that the Zeno behavior of the protocol (2) can be excluded by using the contradiction method. Assuming that $\lim_{k\to\infty} t_k^i < \infty$ and then due to the continuity of $\varepsilon_i(t)$ and $K(t)$ for $t \in [t_k^i, t_{k+1}^i)$, we have

$$\|K(t_{k+1}^i)\varepsilon_i(t_{k+1}^i)\| \to 0, \quad k \to \infty \tag{25}$$

and

$$(r_i(t_{k+1}^i) - r_i(t_k^i))^2 \to 0, \quad k \to \infty. \tag{26}$$

Moreover, according to the triggering condition (8), we can conclude that

$$\|K(t_{k+1}^i)\varepsilon_i(t_{k+1}^i)\|^2 \geq \frac{\gamma}{\overline{c}_i^2(t_{k+1}^i)}\mathrm{e}^{-\beta t_{k+1}^i} \tag{27}$$

or

$$(r_i(t_{k+1}^i) - r_i(t_k^i))^2\|K(t_{k+1}^i)\delta_i(t_{k+1}^i)\|^2 \geq \gamma\mathrm{e}^{-\beta t_{k+1}^i}. \tag{28}$$

This further means that

$$0 < \frac{\gamma}{\overline{c}_i^2(t_\infty^i)}\mathrm{e}^{-\beta t_\infty^i} = \lim_{k\to\infty} \frac{\gamma}{\overline{c}_i^2(t_{k+1}^i)}\mathrm{e}^{-\beta t_{k+1}^i} \leq \lim_{k\to\infty} \|K(t_{k+1}^i)\varepsilon_i(t_{k+1}^i)\|^2 = 0 \tag{29}$$

or

$$0 < \gamma\mathrm{e}^{-\beta t_\infty^i} = \lim_{k\to\infty} \gamma\mathrm{e}^{-\beta t_{k+1}^i} \leq \lim_{k\to\infty} (r_i(t_{k+1}^i) - r_i(t_k^i))^2\|K(t_{k+1}^i)\delta_i(t_{k+1}^i)\|^2 = 0. \tag{30}$$

This contradiction shows that the Zeno behaviour can be excluded. ∎

Remark 3.4 First, different from the existing results on consensus of multi-agent systems (see [1, 19, 21, 22, 32, 36, 37]), linear time-varying dynamics is considered in this paper and the controller $K(t)$ is designed by using the Gramian matrix of linear time-varying systems. Second, in addition to the linear time-varying dynamics, different from the fully distributed event-triggering consensus problem of multi-agent systems in [1, 21, 22], directed graph is considered in this paper, while the above mentioned results only consider undirected cases. In [1], when designing an event-triggering protocol, the absolute measurement error $x_i(t) - x_i(t_k^i)$ is used to design the triggering instants. However, it is shown in [38] that this absolute information cannot be obtained precisely in some situations. In this paper, it can be seen from the triggering condition (9) that only relative information between agents is utilized. Thus, not only the model and network topology considered in this paper are more general than those in [1], but also the event-triggering condition is also more concise than the one in [1]. It is worth pointing out that in this paper, the control gain matrix $K(t)$ is solved as $K(t) = -B^{\mathrm{T}}(t)P_1^{-1}(t)$, which needs to solve the transition matrix $\Phi(t_0, t)$. Different from time-invariant systems, in time-varying systems, it is not easy to compute $\Phi(t_0, t)$. In the following, we present two simple methods to compute $\Phi(t_0, t)$: 1) if $A(t)\int_{t_0}^{t} A(\tau)d\tau = \int_{t_0}^{t} A(\tau)d\tau A(t)$, then $\Phi(t_0, t) = \exp(\int_{t_0}^{t} A(\tau)d\tau)$ and 2) if $A(t)\int_{t_0}^{t} A(\tau)d\tau \neq \int_{t_0}^{t} A(\tau)d\tau A(t)$, then $\Phi(t_0, t) = I + \int_{t_0}^{t} A(\tau_0)d\tau_0 + \int_{t_0}^{t} A(\tau_0)\int_{t_0}^{\tau_1} A(\tau_1)d\tau_1 d\tau_0 + \int_{t_0}^{t} A(\tau_0)\int_{t_0}^{\tau_0} A(\tau_1)\int_{t_0}^{\tau_1} A(\tau_2)d\tau_2\tau_1 d\tau_0 + \cdots$.

3.2 Undirected Graph Case

In this subsection, we will show that when the graph is undirected, the condition that the fully distributed event-triggering protocol is dependent on the initial value of the coupling strength ($p_i(0) > \sqrt{2\rho_1}$) can be removed. The graph in this subsection is assumed to be connected. In order to do this, a new Lyapunov function will be used to derive the main results.

Let $\overline{x}(t) = \frac{\sum_{j=1}^{N} x_j(t)}{N}$, $e_i(t) = x_i(t) - \overline{x}(t)$ and $e(t) = \mathrm{col}\{e_1(t), e_2(t), \cdots, e_N(t)\}$. Then the following error dynamical system can be obtained from (1) and (2):

$$\begin{aligned}\dot{e}(t) =& (I_N \otimes A(t))e(t) + (\mathcal{CL} \otimes B(t)K(t))e(t)\\ &+ (U\mathcal{C} \otimes B(t)K(t))\varepsilon(t),\end{aligned} \tag{31}$$

where

$$U = \frac{1}{N}\begin{bmatrix} N-1 & -1 & \cdots & -1 \\ -1 & N-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & N-1 \end{bmatrix}.$$

The other parameters are the same as the ones in Theorem 3.3. For $i \in \mathcal{V}$, the adaptive coupling $c_i(t)$ is defined as

$$\dot{c}_i(t) = \kappa_i \left\| K(t) \sum_{j \in N_i} a_{ij}[x_i(t_k^i) - x_j(t_k^i)] \right\|^2, \quad t \in [t_k^i, t_{k+1}^i), \tag{32}$$

where $\kappa_i, i \in \mathcal{V}$ are positive constants. Letting ρ, γ, and β be three positive constants, then the following triggering function is used to determine the triggering instants:

$$f_i(t) = \|K(t)\varepsilon_i(t)\|^2 - \rho\left\|K(t)\sum_{j\in N_i} a_{ij}(x_i(t) - x_j(t))\right\|^2 - \frac{\gamma}{c_i(t)}\mathrm{e}^{-\beta t}. \tag{33}$$

Theorem 3.5 *Consider the multi-agent system in* (1) *with the event-triggering protocol* (2) *under an undirected connected graph. The multi-agent system in* (1) *reaches consensus if the adaptive coupling gains* $c_i(t)$ *satisfy* (32) *and* ρ *in* (33) *is chosen such that* $\rho \in (0, \frac{1}{2})$. *Moreover, the multi-agent system in* (1) *with the protocol* (2) *reaches consensus with* $K(t) = -B^{\mathrm{T}}(t)P_1^{-1}(t)$ *and* $c_i(t), i \in \mathcal{V}$ *converge to some positive constants. Besides, the protocol* (2) *does not exhibit Zeno behaviour.*

Proof Construct the following Lyapunov function candidate:

$$V(t) = e^{\mathrm{T}}(t)(\mathcal{L}\otimes P_1^{-1}(t))e(t) + \sum_{i=1}^{N}\frac{(c_i(t)-\alpha)^2}{16\kappa_i}. \tag{34}$$

Then the derivative of $V(t)$ is given by

$$\begin{aligned}\dot{V}(t) =& 2e^{\mathrm{T}}(t)(\mathcal{L}\otimes P_1^{-1}(t))\dot{e}(t) + \sum_{i=1}^{N}\frac{(c_i(t)-\alpha)\dot{c}_i}{8\kappa_i} + e^{\mathrm{T}}(t)(\mathcal{L}\otimes \dot{P}_1^{-1}(t))e(t)\\ =& 2e^{\mathrm{T}}(t)(\mathcal{L}\otimes P_1^{-1}(t)A(t))e(t) + 2e^{\mathrm{T}}(t)(\mathcal{LCL}\otimes P_1^{-1}(t)B(t)K(t))e(t)\\ &+ 2e^{\mathrm{T}}(t)(\mathcal{LC}\otimes P_1^{-1}(t)B(t)K(t))\varepsilon(t) + \frac{1}{8}(c_i(t)-\alpha)\left\|K(t)\sum_{j\in N_i} a_{ij}[x_i(t_k^i) - x_j(t_k^i)]\right\|^2\\ &+ e^{\mathrm{T}}(t)(\mathcal{L}\otimes \dot{P}_1^{-1}(t))e(t),\end{aligned} \tag{35}$$

where $\mathcal{L}U = \mathcal{L}$ is used to get the second equality.

Noting that $K(t) = -B^{\mathrm{T}}(t)P_1^{-1}(t)$, we can make the following derivations:

$$2e^{\mathrm{T}}(t)(\mathcal{LCL}\otimes P_1^{-1}(t)B(t)K(t))e(t) = -2e^{\mathrm{T}}(t)[\mathcal{LCL}\otimes K^{\mathrm{T}}(t)K(t)]e(t) \tag{36}$$

and

$$\begin{aligned}&2e^{\mathrm{T}}(t)(\mathcal{LC}\otimes P_1^{-1}(t)B(t)K(t))\varepsilon(t)\\ =& -2e^{\mathrm{T}}(t)(\mathcal{LC}\otimes K^{\mathrm{T}}(t)K(t))\varepsilon(t)\\ \le& e^{\mathrm{T}}(t)(\mathcal{LCL}\otimes K^{\mathrm{T}}(t)K(t))e(t) + \varepsilon^{\mathrm{T}}(t)(\mathcal{C}\otimes K^{\mathrm{T}}(t)K(t))\varepsilon(t).\end{aligned} \tag{37}$$

From the triggering condition in (33), one has

$$\begin{aligned}\sum_{i=1}^{N}[c_i(t)\|K(t)\varepsilon_i(t)\|^2] =& \varepsilon^{\mathrm{T}}(t)(\mathcal{C}\otimes K^{\mathrm{T}}(t)K(t))\varepsilon(t)\\ \le& \rho e^{\mathrm{T}}(t)(\mathcal{L}^{\mathrm{T}}\mathcal{CL}\otimes K^{\mathrm{T}}(t)K(t))e(t) + N\gamma\mathrm{e}^{-\beta t}\end{aligned} \tag{38}$$

and

$$\begin{aligned}\sum_{i=1}^{N}[\|K(t)\varepsilon_i(t)\|^2] =&\varepsilon^{\mathrm{T}}(t)(I_N \otimes K^{\mathrm{T}}(t)K(t))\varepsilon(t)\\ &\le\rho e^{\mathrm{T}}(t)(\mathcal{L}^{\mathrm{T}}\mathcal{L} \otimes K^{\mathrm{T}}(t)K(t))e(t) + \sum_{i=1}^{N}\frac{\gamma}{c_i(t)}\mathrm{e}^{-\beta t}.\end{aligned} \tag{39}$$

Moreover,

$$\frac{(c_i(t)-\alpha)}{\kappa_i}\dot{c}_i = (c_i(t)-\alpha)\bigg\|K(t)\sum_{j\in N_i}a_{ij}[x_i(t_k^i) - x_j(t_k^i)]\bigg\|^2. \tag{40}$$

In view of the triggering condition (33), we have

$$\begin{aligned}&\sum_{i=1}^{N}\bigg[c_i(t)\bigg\|K(t)\sum_{j\in N_i}a_{ij}[x_i(t_k^i) - x_j(t_k^i)]\bigg\|^2\bigg]\\ =&\sum_{i=1}^{N}\bigg[c_i(t)\bigg\|K(t)\varepsilon_i(t) + K(t)\sum_{j\in N_i}a_{ij}[x_i(t) - x_j(t)]\bigg\|^2\bigg]\\ \le&2\sum_{i=1}^{N}c_i(t)\bigg[\|K(t)\varepsilon_i(t)\|^2 + \bigg\|K(t)\sum_{j\in N_i}a_{ij}[x_i(t) - x_j(t)]\bigg\|^2\bigg]\\ \le&(2\rho+2)e^{\mathrm{T}}(t)(\mathcal{L}^{\mathrm{T}}\mathcal{C}\mathcal{L} \otimes K^{\mathrm{T}}(t)K(t))e(t) + N\gamma\mathrm{e}^{-\beta t}.\end{aligned} \tag{41}$$

On the other hand, similar to (18), we get

$$\begin{aligned}&-\alpha\bigg\|K(t)\sum_{j\in N_i}a_{ij}[x_i(t_k^i) - x_j(t_k^i)]\bigg\|^2\\ \le&\alpha\|K(t)\varepsilon_i(t)\|^2 - \frac{\alpha}{2}\bigg\|K(t)\sum_{j\in N_i}a_{ij}[x_i(t) - x_j(t)]\bigg\|^2.\end{aligned} \tag{42}$$

Thus, we arrive at

$$\begin{aligned}&\sum_{i=1}^{N}\bigg[-\alpha\bigg\|K(t)\sum_{j\in N_i}a_{ij}[x_i(t_k^i) - x_j(t_k^i)]\bigg\|^2\bigg]\\ \le&\sum_{i=1}^{N}\bigg[\alpha\|K(t)\varepsilon_i(t)\|^2 - \frac{\alpha}{2}\bigg\|K(t)\sum_{j\in N_i}a_{ij}[x_i(t) - x_j(t)]\bigg\|^2\bigg]\\ \le&\Big(\rho - \frac{1}{2}\Big)\alpha e^{\mathrm{T}}(t)(\mathcal{L}^{\mathrm{T}}\mathcal{L} \otimes K^{\mathrm{T}}(t)K(t))e(t) + \sum_{i=1}^{N}\frac{\alpha\gamma}{c_i(t)}\mathrm{e}^{-\beta t}.\end{aligned} \tag{43}$$

From (35)–(43) and the definition of $P_1(t)$, the following statements can be obtained.

$$\dot{V}(t) \le -4\varepsilon e^{\mathrm{T}}(t)(\mathcal{L} \otimes P_1^{-1}(t))e(t)$$

$$
\begin{aligned}
&+ e^{\mathrm{T}}(t)\left(\frac{5\rho}{4}-\frac{3}{4}\right)(\mathcal{LCL}\otimes K^{\mathrm{T}}(t)K(t))e(t)\\
&+\frac{(\rho-\frac{1}{2})\alpha}{8}e^{\mathrm{T}}(t)(\mathcal{LL}\otimes K^{\mathrm{T}}(t)K(t))e(t)\\
&+4e^{\mathrm{T}}(t)(\mathcal{L}\otimes K^{\mathrm{T}}(t)K(t))e(t)+\upsilon(t)\mathrm{e}^{-\beta t},
\end{aligned}\tag{44}
$$

where $\upsilon(t)=\frac{9N\gamma}{8}+\frac{1}{8}\sum_{j=1}^{N}\frac{\alpha\gamma}{c_i(t)}$. Let $\xi(t)=(\mathcal{U}\otimes I_n)e(t)$, where $\mathcal{U}$ is the unitary matrix such that $\mathcal{U}^{\mathrm{T}}\mathcal{L}\mathcal{U}=\mathrm{diag}\{0,\lambda_2,\lambda_3,\cdots,\lambda_N\}$ with $0<\lambda_2\le\lambda_3\le\cdots\le\lambda_N$. Then, we have

$$
\begin{aligned}
\dot V(t)=&-4\varepsilon\xi^{\mathrm{T}}(t)(\Lambda\otimes P_1^{-1}(t))\xi(t)\\
&+\frac{(\rho-\frac{1}{2})\alpha}{8}\xi^{\mathrm{T}}(t)(\Lambda^2\otimes K^{\mathrm{T}}(t)K(t))\xi(t)\\
&+4\xi^{\mathrm{T}}(t)(\Lambda\otimes K^{\mathrm{T}}(t)K(t))\xi(t)+\upsilon e^{-\beta t}\\
=&\sum_{i=2}^{N}\xi_i^{\mathrm{T}}\bigg[-4\varepsilon\lambda_iP_1^{-1}(t)+\left(\frac{(\rho-\frac{1}{2})\alpha}{8}\lambda_i^2+4\lambda_i\right)\\
&\qquad\times K^{\mathrm{T}}(t)K(t)\bigg]\xi_i(t)+\upsilon(t)\mathrm{e}^{-\beta t}.
\end{aligned}\tag{45}
$$

We can choose $\alpha\ge\frac{64}{\lambda_2(\frac{1}{2}-\rho)}$, thus $(\frac{(\rho-\frac{1}{2})\alpha}{8}\lambda_i^2+4\lambda_i)\le 0$. Therefore, we obtain

$$
\dot V(t)\le -4\varepsilon\sum_{i=2}^{N}\lambda_i\xi_i^{\mathrm{T}}(t)P_1^{-1}(t)\xi_i(t)+\upsilon(t)\mathrm{e}^{-\beta t}.\tag{46}
$$

Moreover, it can be seen from (32) that $c_i(t)$ are monotonically increasing functions, so it can be concluded that $\upsilon(t)$ is bounded by a positive constant $\overline{\upsilon}$. Thus, we get

$$
\begin{aligned}
0\le V(t)=\int_0^t\dot V(s)ds+V(0)\le&\int_0^t\overline{\upsilon}\mathrm{e}^{-\beta s}ds+V(0)\\
\le&\frac{\overline{\upsilon}}{\beta}+V(0),\quad t\ge 0.
\end{aligned}\tag{47}
$$

This implies that $V(t)$ is bounded and so are $c_i(t)$. In addition, we can conclude that $c_i(t)$ converge to some positive constants since $c_i(t)$ are monotonically increasing functions. Then, similar to the proof in Theorem 3.3, we can show that the consensus can be achieved and the Zeno behaviour can be excluded. Thus, the proof is completed. ▮

4 Numerical Simulations

In this section, numerical simulations are provided to illustrate the effectiveness of the theoretical results.

Example 4.1 Consider the multi-agent system in (1) on directed graph in which each agent's dynamics is described by the following linear time-varying system:

$$
\dot x_i(t)=A(t)x_i(t)+B(t)u_i(t),\quad i=1,2,\cdots,6,\tag{48}
$$

where

$$A(t) = \begin{bmatrix} 0 & 12\cos(15t) + 0.8\sin(15t) \\ 0 & 1 \end{bmatrix}, \quad B(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

According to the definition of $A(t)$, we can get

$$\varPhi(t,s) = \begin{bmatrix} 1 & 0.8\mathrm{e}^{t-s}\sin(15t) - \sin(15s) \\ 0 & \mathrm{e}^{t-s} \end{bmatrix}.$$

Letting $\sigma = 0.1$, then we have $\gamma_1 = 0.0003$ and $\gamma_2 = 0.15$.

The communication graph is shown in Figure 1. Let $\rho_1 = 0.2$, $\gamma = 0.1$, $\beta = 0.3$ and the initial value of $p_i(t)$ is chosen as $\mathrm{col}\{p_1(t), p_2(t), \cdots, p_6(t)\} = \mathrm{col}\{0.65, 0.75, 0.8, 0.85, 0.95, 1\}$. The state trajectories of the multi-agent system in Example 4.1 are displayed in Figures 2 and 3 and the coupling gains and triggering instants of each agent are shown in Figures 4 and 5, respectively. Form these figures, we can see that the consensus can be achieved since $x_i(t) - x_j(t) \to 0, i, j = 1, 2, \cdots, N$ and coupling strengths $c_i(t)$ converge to some constants. In addition, it can be seen from Figure 5 that the protocol proposed in this paper does not need to update all the time, which can save communication cost in continuous-time communication[11, 12].

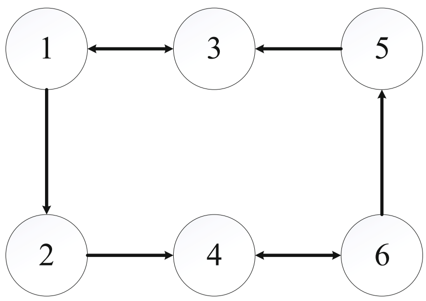

Figure 1 Communication graph of Example 4.1

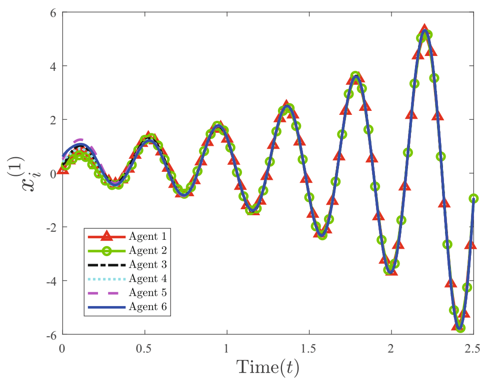

Figure 2 The state trajectories of x_{i1} of Example 4.1

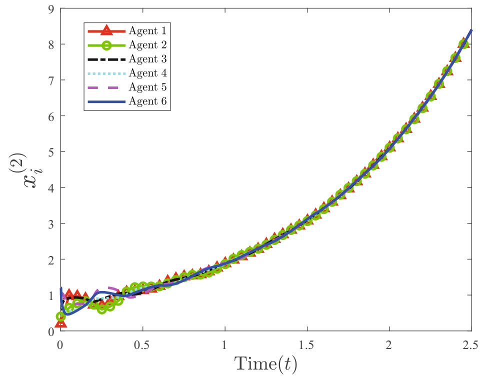

Figure 3 The state trajectories of x_{i2} of Example 4.1

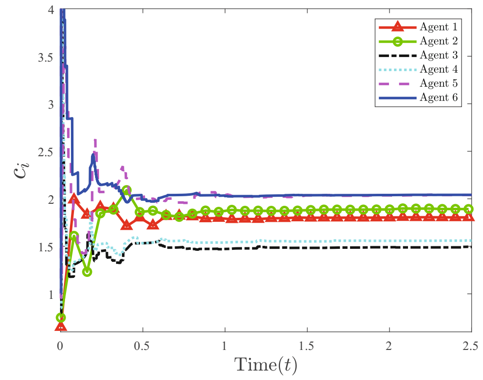

Figure 4 Coupling gains $c_i(t)$ of Example 4.1

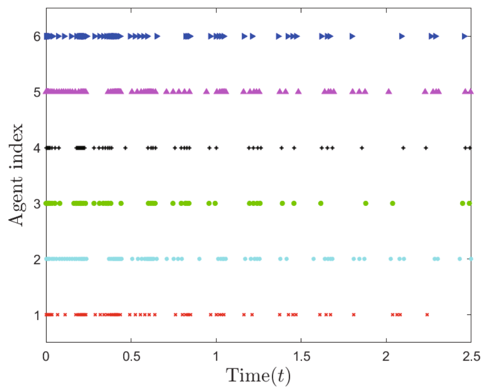

Figure 5 Triggering instants of Example 4.1

5 Conclusions

This paper has investigated the fully distributed event-triggering consensus problem of multi-agent systems with linear time-varying dynamics. First of all, an adaptive event-triggering protocol has been proposed for the time-varying multi-agent system under directed graphs. Then, by assuming that the linear time-varying system is of uniform controllability and using the Lyapunov function method, sufficient conditions have been obtained to ensure the solvability of the consensus problem of the multi-agent system in which the global information of the Laplacian matrix is not used and thus our event-triggering protocol can be implemented in a fully distributed way. When it comes to undirected graphs, we have shown that the fully distributed protocol can be designed with more freedom than in the directed case. In both cases, it has been proved that the Zeno behaviour is excluded. The effectiveness of the theoretical results has been verified by a numerical example. Future works will include an extension of our current research into more complicated models such as time-varying coupling, nonlinear time-varying dynamics[39] and multi-agent systems with cyber-attacks[40, 41].

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